Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 943 – 954
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doi:10.3906/mat-2101-2

Oscillation tests for nonlinear differential equations with nonmonotone delays

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Received: 01.01.2021 • **Accepted/Published Online:** 16.02.2021 • **Final Version:** 26.03.2021

Abstract: In this paper, our aim is to investigate a class of first-order nonlinear delay differential equations with several deviating arguments. In addition, we present some sufficient conditions for the oscillatory solutions of these equations. Differing from other studies in the literature, delay terms are not necessarily monotone. Finally, we give examples to demonstrate the results.

Key words: Delay equations, nonlinear, nonmonotone arguments, nonoscillatory solution, oscillatory solution

1. Introduction

The theory of delay differential equations is a remarkable research area for modern applied mathematics. In recent years, significant concern has been dedicated to the oscillatory and nonoscillatory solutions of these equations. Besides, the question of obtaining new sufficient criteria for the oscillatory behavior of these equations has attracted the attention of many scientists. See, for example [1–26], and the references cited therein. Moreover, oscillations of first-order delay differential equations have numerous applications in the study of oscillation and asymptotic behavior of higher-order differential/dynamic equations. See the studies in [1,2,13,17,23,24] for more detail. The reader is referred to monograph [20] for the general information about oscillation theory.

Consider a class of first-order nonlinear delay differential equations

$$x'(t) + p(t) f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t))) = 0, \ t \geq t_0, \tag{1.1}$$

where the functions $p, \ \tau_i \in C([t_0, \infty), \mathbb{R}^+)$ and $\tau_i(t)$ are not necessarily monotone for $1 \leq i \leq n$ such that

$$\tau_i(t) \leq t \ \text{ for } t \geq t_0, \ \lim_{t\to\infty} \tau_i(t) = \infty, \ 1 \leq i \leq n \tag{1.2}$$

and $f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t)))$ is a continuous function on $\mathbb{R}^n$ such that

$$x f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t))) > 0 \ \text{ for } x \neq 0. \tag{1.3}$$

By a solution of (1.1), we mean continuously differentiable function defined on $[\tau_i(T_0), \infty)$ for some $T_0 \geq t_0$ such that (1.1) holds for $t \geq T_0, \ 1 \leq i \leq n$. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory.

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2010 *AMS Mathematics Subject Classification:* 34C10, 34K06, 34K11.

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For $n = 1$, (1.1) turns into the equation

$$x'(t) + p(t)f(x(\tau_1(t))) = 0, \quad t \geq t_0. \tag{1.4}$$

When $f(x) = x$, we have the linear form of (1.4)

$$x'(t) + p(t)x(\tau_1(t)) = 0, \quad t \geq t_0. \tag{1.5}$$

Establishing sufficient conditions for the oscillation of all solutions of (1.5) has been the subject field of many examinations. See, for example, [3–12,14,15,18–22].

Ladde et al. [22] established the following result.

Suppose that p, τ_1, and f in (1.4) satisfy the following conditions.

(i) $\tau_1(t) \leq t$ for $t \geq t_0$, $\lim_{t\to\infty} \tau_1(t) = \infty$ and $\tau_1(t)$ is strictly increasing on $\mathbb{R}^+$.

(ii) $p(t)$ is locally integrable and $p(t) \geq 0$.

(iii) $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ for $x \neq 0$, f is nondecreasing and $\lim\limits_{x\to 0} \frac{x}{f(x)} = N_1 < \infty$.

If

$$\limsup_{t\to\infty} \int_{\tau_1(t)}^{t} p(s)ds > N_1$$

or

$$\liminf_{t\to\infty} \int_{\tau_1(t)}^{t} p(s)ds > \frac{N_1}{e},$$

then all solutions of (1.4) oscillate.

In 1984, Fukagai and Kusano [19] obtained the following result.

Suppose that (1.2) holds,

$$f \in C(\mathbb{R}, \mathbb{R}), \ xf(x) > 0 \text{ for } x \neq 0 \tag{1.6}$$

and

$$\limsup_{x\to 0} \frac{|x|}{|f(x)|} = N_2 < \infty. \tag{1.7}$$

If $\tau_1(t)$ is nondecreasing and

$$\liminf_{t\to\infty} \int_{\tau_1(t)}^{t} p(s)ds > \frac{N_2}{e},$$

then all solutions of (1.4) oscillate.

In 2017 and 2020, Öcalan et al. [25,26] proved the following result.

Assume that (1.2) and (1.6) hold, and $\limsup\limits_{x\to 0} \frac{x}{f(x)} = N_3$. If $\tau_1(t)$ is not necessarily monotone and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds > N_3, \ 0 < N_3 < \infty$$

or

$$\liminf_{t\to\infty}\int_{\tau_1(t)}^{t} p(s)ds > \frac{N_3}{e},\ 0 \le N_3 < \infty,$$

where $h(t) := \sup_{s\le t}\{\tau_1(s)\},\ t \ge 0,$ then all solutions of (1.4) oscillate.

Now, consider again (1.1). The following theorem was given by Fukagai and Kusano in 1984 [19].

Theorem 1.1 *Suppose that* (1.2) *and* (1.3) *hold,* $\tau_i(t)$ *are nondecreaing for* $1 \le i \le n$ *and*

$$N_4 = \limsup_{x\to 0}\frac{|x(\tau_1(t))|^{\alpha_1}|x(\tau_2(t))|^{\alpha_2}\cdots|x(\tau_n(t))|^{\alpha_n}}{|f(x(\tau_1(t)),x(\tau_2(t)),\ldots,x(\tau_n(t)))|} < \infty, \tag{1.8}$$

where α_i *are nonnegative constants with* $\sum_{i=1}^{n}\alpha_i = 1$*. If there is a continuous nondecreasing function* $\tau^*(t)$ *such that* $\tau_i(t) \le \tau^*(t) \le t$ *for* $1 \le i \le n,\ t \ge a$ *and*

$$\liminf_{t\to\infty}\int_{\tau^*(t)}^{t} p(s)ds > \frac{N_4}{e},$$

then all solutions of (1.1) *oscillate.*

Thus, in this paper, our aim is to essentially develop these results under the assumption that $\tau_i(t)$ are not necessarily monotone arguments for $1 \le i \le n$ and to obtain new criteria for the oscillation of (1.1).

2. Main results

In this section, we present some new sufficient conditions for the oscillation of all solutions of (1.1), under the assumption that delay arguments $\tau_i(t)$ are not necessarily monotone for $1 \le i \le n$. Set

$$h_i(t) := \sup_{s\le t}\{\tau_i(s)\} \text{ and } h(t) = \max_{1\le i\le n}\{h_i(t)\},\quad t \ge 0 \tag{2.1}$$

and

$$\limsup_{x\to 0}\frac{x(\tau(t))}{f(x(\tau_1(t)),x(\tau_2(t)),\ldots,x(\tau_n(t)))} = N,\quad \tau(t) = \max_{1\le i\le n}\{\tau_i(t)\}. \tag{2.2}$$

Clearly, $h_i(t)$ are nondecreasing and $\tau_i(t) \le \tau(t) \le h_i(t) \le h(t)$ for all $1 \le i \le n,\ t \ge 0.$

The following result was given in [18].

Lemma 2.1 *Assume that* (2.1) *holds and*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds = L.$$

Then, we have

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds = \liminf_{t\to\infty}\int_{h(t)}^{t} p(s)ds = L, \tag{2.3}$$

where $\tau(t) = \max_{1\leq i\leq n}\{\tau_i(t)\}$.

Lemma 2.2 *Assume that* $x(t)$ *is an eventually positive solution of* (1.1)*. If*

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)ds > 0, \tag{2.4}$$

where $h(t)$ *is defined by* (2.1)*, then* $\lim_{t\to\infty} x(t) = 0$.

Moreover, assume that $x(t)$ *is an eventually negative solution of* (1.1)*. If* (2.4) *holds, then* $\lim_{t\to\infty} x(t) = 0$.

Proof Assume that (2.4) holds. Let $x(t)$ be an eventually positive solution of (1.1). Then, there exists a $t_1 > t_0$ such that $x(t),\ x(\tau(t)),\ x(h(t)) > 0$ for all $t \geq t_1$. Thus, from (1.1), we get

$$x'(t) = -p(t)f(x(\tau_1(t)), x(\tau_2(t)), \dots, x(\tau_n(t))) \leq 0$$

for all $t \geq t_1$, which means that $x(t)$ is nonincreasing and has a limit $l > 0$ or $l = 0$. Now, we claim that $\lim_{t\to\infty} x(t) = 0$. Otherwise, $\lim_{t\to\infty} x(t) = l > 0$. Then, integrating (1.1) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + \int_{h(t)}^{t} p(s)f(x(\tau_1(s)), x(\tau_2(s)), \dots, x(\tau_n(s)))ds = 0. \tag{2.5}$$

Moreover, since f is continuous, then it has a limit, so there exists a t_2 such that $f(x(\tau_1(t)), x(\tau_2(t)), \dots, x(\tau_n(t))) \geq d > 0$ for $t \geq t_2$. By using this and (2.5), we have the inequality

$$x(t) - x(h(t)) + d\int_{h(t)}^{t} p(s)ds \leq 0. \tag{2.6}$$

Then, (2.4) implies that there exists at least one sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty}\int_{h(t_n)}^{t_n} p(s)ds > 0. \tag{2.7}$$

By writing $t \to t_n$ and taking limit as $n \to \infty$ in (2.6), we have

$$\lim_{n\to\infty}(x(t_n) - x(h(t_n))) + d\lim_{n\to\infty}\int_{h(t_n)}^{t_n} p(s)ds \leq 0$$

or

$$d \lim_{n\to\infty} \int_{h(t_n)}^{t_n} p(s)ds \leq 0$$

but this contradicts with (2.7). Thus, the proof of the lemma is completed.

By using same process, it is easy to see that when $x(t)$ is an eventually negative solution of (1.1) under the assumption that (2.4) holds, $\lim_{t\to\infty} x(t) = 0$. □

Theorem 2.3 *Assume that* (1.2), (1.3), (2.1), *and* (2.2) *hold. If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{N}{e}, \ 0 \leq N < \infty, \tag{2.8}$$

where $\tau(t) = \max_{1\leq i\leq n} \{\tau_i(t)\}$, *then all solutions of* (1.1) *oscillate.*

Proof Assume, for the sake of contradiction, that there exists an eventually positive solution $x(t)$ of (1.1). If there exists an eventually negative solution $x(t)$ of (1.1), then the proof can be done similarly as below. Then, there exists a $t_1 > t_0$ such that $x(t),\ x(\tau_i(t)),\ x(h_i(t)) > 0$ for all $1 \leq i \leq n,\ t \geq t_1$. Thus, from (1.1), we have

$$x'(t) = -p(t) f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t))) \leq 0$$

for all $t \geq t_1$, which means that $x(t)$ is an eventually nonincreasing function. Condition (2.8) and Lemma 2.2 imply that $\lim_{t\to\infty} x(t) = 0$.

Case I: Let $N > 0$. Then, by (2.2), we can choose $t_2 \geq t_1$ so large that

$$f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t))) \geq \frac{1}{2N} x(\tau(t)) \tag{2.9}$$

for $t \geq t_2$. Since $\tau(t) \leq h(t)$, $x(t)$ is nonincreasing and using (2.9), we have from (1.1)

$$x'(t) + \frac{1}{2N} p(t) x(\tau(t)) \leq 0$$

or

$$x'(t) + \frac{1}{2N} p(t) x(h(t)) \leq 0. \tag{2.10}$$

Moreover, from (2.8) and Lemma 2.1, there exists a constant $c > 0$ such that

$$\int_{h(t)}^{t} p(s)ds \geq c > \frac{N}{e}, \ t \geq t_3 \geq t_2. \tag{2.11}$$

Furthermore, from (2.8) there exists a real number $t^* \in (h(t), t)$ for all $t \geq t_3$ such that

$$\int_{h(t)}^{t^*} p(s)ds > \frac{N}{2e} \ \text{ and } \ \int_{t^*}^{t} p(s)ds > \frac{N}{2e}. \tag{2.12}$$

Integrating (2.10) from $h(t)$ to t^*, by taking into account that $x(t)$ is nonincreasing, $h(t)$ is nondecreasing, and (2.12), we have

$$x(t^*) - x(h(t)) + \frac{1}{2N} \int_{h(t)}^{t^*} p(s)x(h(s))ds \leq 0$$

and

$$x(t^*) - x(h(t)) + \frac{1}{2N} x(h(t^*)) \frac{N}{2e} < 0$$

or

$$x(h(t)) > \frac{1}{4e} x(h(t^*)). \tag{2.13}$$

Integrating (2.10) from t^* to t, by using the same facts, we obtain

$$x(t) - x(t^*) + \frac{1}{2N} \int_{t^*}^{t} p(s)x(h(s))ds \leq 0$$

and

$$x(t) - x(t^*) + \frac{1}{2N} x(h(t)) \frac{N}{2e} < 0$$

or

$$x(t^*) > \frac{1}{4e} x(h(t)). \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$x(t^*) > \frac{1}{4e} x(h(t)) > \frac{1}{(4e)^2} x(h(t^*)),$$

$$\frac{x(h(t^*))}{x(t^*)} < (4e)^2, \ t \geq t_4. \tag{2.15}$$

Let

$$u = \liminf_{t \to \infty} \frac{x(h(t))}{x(t)} \geq 1 \tag{2.16}$$

and because of $1 \leq u \leq (4e)^2$, u is finite.

Now, dividing (1.1) with $x(t)$ and integrating from $h(t)$ to t, we have

$$\int_{h(t)}^{t} \frac{x'(s)}{x(s)} ds + \int_{h(t)}^{t} p(s) \frac{f(x(\tau_1(s)), x(\tau_2(s)), \ldots, x(\tau_n(s)))}{x(s)} ds = 0$$

or

$$\ln \frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} p(s) \frac{f(x(\tau_1(s)), x(\tau_2(s)), \ldots, x(\tau_n(s)))}{x(\tau(s))} \frac{x(\tau(s))}{x(s)} ds = 0.$$

By using the facts that $x(t)$ is nonincreasing and $\tau(t) \leq h(t)$, we get

$$\ln \frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} p(s) \frac{f(x(\tau_1(s)), x(\tau_2(s)), \ldots, x(\tau_n(s)))}{x(\tau(s))} \frac{x(h(s))}{x(s)} ds \leq 0.$$

Moreover, there exists a ζ such that $h(t) \leq \zeta \leq t$. Then, we have

$$\ln \frac{x(h(t))}{x(t)} \geq \frac{f(x(\tau_1(\zeta)), x(\tau_2(\zeta)), \ldots, x(\tau_n(\zeta)))}{x(\tau(\zeta))} \frac{x(h(\zeta))}{x(\zeta)} \int_{h(t)}^{t} p(s) ds. \tag{2.17}$$

Then, taking lower limit on both side of (2.17), we obtain $\ln u > \frac{u}{e}$. Since $\ln x \leq \frac{x}{e}$ for all $x > 0$, it is impossible.

Case II: Let $N = 0$. It is obvious that $\frac{x(\tau(t))}{f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t)))} > 0$ and

$$\lim_{x \to 0} \frac{x(\tau(t))}{f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t)))} = 0. \tag{2.18}$$

By (2.18), we have

$$\frac{x(\tau(t))}{f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t)))} < \varepsilon \tag{2.19}$$

or

$$\frac{f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t)))}{x(\tau(t))} > \frac{1}{\varepsilon} \tag{2.20}$$

where ε is an arbitrary real number. Because of this, $\tau(t) \leq h(t)$ and $x(t)$ is nonincreasing and using (2.20), we have from (1.1)

$$x'(t) + \frac{1}{\varepsilon} p(t) x(\tau(t)) < 0$$

or

$$x'(t) + \frac{1}{\varepsilon} p(t) x(h(t)) < 0. \tag{2.21}$$

Integrating the last inequality from $h(t)$ to t, we get

$$x(t) - x(h(t)) + \frac{1}{\varepsilon} \int_{h(t)}^{t} p(s) x(h(s)) ds < 0,$$

$$x(h(t)) \left[\frac{1}{\varepsilon} \int_{h(t)}^{t} p(s) ds - 1 \right] < 0.$$

Then, using (2.11), we obtain

$$1 > \frac{c}{\varepsilon}$$

or

$$\varepsilon > c \tag{2.22}$$

but this contradicts with (2.18); hence, the proof of the theorem is completed. □

Theorem 2.4 *Assume that* (1.2), (1.3), (2.1), *and* (2.2) *hold. If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds > N, \ 0 < N < \infty, \tag{2.23}$$

then all solutions of (1.1) *oscillate.*

Proof Assume, for the sake of contradiction, that there exists an eventually positive solution $x(t)$ of (1.1). If there exists an eventually negative solution $x(t)$ of (1.1), then the proof can be done similarly as below. Then, there exists a $t_1 \geq t_0$ such that $x(t), \ x(\tau_i(t)), \ x(h_i(t)) > 0$ for all $1 \leq i \leq n, \ t \geq t_1$. From Theorem 2.3, $x(t)$ is an eventually nonincreasing, also from (2.23) and Lemma 2.2, $\lim_{t\to\infty} x(t) = 0$. By taking into account of (2.2) for $\theta > 1$, we get the inequality

$$f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_n(t))) \geq \frac{1}{\theta N} x(\tau(t)). \tag{2.24}$$

From, (2.23), there exists a constant $K > 0$ such that

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds = K > N. \tag{2.25}$$

Since $K > N$, we have $N < \frac{K+N}{2} < K$. Moreover, by (2.24) and using $\tau(t) \leq h(t)$ and $x(t)$ is nonincreasing from (1.1), we have

$$x'(t) + \frac{1}{\theta N} p(t) x(\tau(t)) \leq 0$$

or

$$x'(t) + \frac{1}{\theta N} p(t) x(h(t)) \leq 0. \tag{2.26}$$

Integrating (2.26) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + \frac{1}{\theta N} \int_{h(t)}^{t} p(s) x(h(s)) ds \leq 0$$

or

$$x(h(t)) \left[\frac{1}{\theta N} \int_{h(t)}^{t} p(s)ds - 1 \right] \leq 0;$$

hence,

$$\int_{h(t)}^{t} p(s)ds < \theta N$$

for sufficiently large t. Therefore,

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds \leq \theta N.$$

Since $\theta > 1$ and $\frac{K+N}{2N} > 1$, we can choose this term instead of θ. If the term $\theta = \frac{K+N}{2N} > 1$ is replaced in the last inequality, then we obtain

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds = K \leq \frac{K+N}{2},$$

which contradicts with $K > \frac{K+N}{2}$, so the proof is completed. □

Example 2.5 *Consider the first-order nonlinear delay differential equation*

$$x'(t) + \frac{2}{e} x(\tau_1(t)) \ln(|x(\tau_1(t))\, x(\tau_2(t))| + 3) = 0, \; t \geq 0, \tag{2.27}$$

where

$$\tau_1(t) = \begin{cases} t-1, & t \in [3k,\; 3k+1] \\ -3t+12k+3, & t \in [3k+1,\; 3k+2] \\ 5t-12k-13, & t \in [3k+2,\; 3k+3] \end{cases}, \quad k \in \mathbb{N}_0$$

$$\tau_2(t) = \tau_1(t) - 2$$

and with the help of (2.1)

$$h_1(t) := \sup_{s\leq t}\{\tau_1(s)\} = \begin{cases} t-1, & t \in [3k,\; 3k+1] \\ 3k, & t \in [3k+1,\; 3k+2.6] \\ 5t-12k-13, & t \in [3k+2.6,\; 3k+3] \end{cases}, \qquad k \in \mathbb{N}_0,$$

$$h_2(t) = h_1(t) - 2,$$

then, we have

$$\tau(t) = \max_{1\leq i\leq 2}\{\tau_i(t)\} = \tau_1(t).$$

Moreover, we find

$$N = \limsup_{x\to 0} \frac{x(\tau_1(t))}{x(\tau_1(t)) \ln(|x(\tau_1(t))\, x(\tau_2(t))| + 3)} = \frac{1}{\ln 3} \cong 0.91023.$$

Since

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \liminf_{t\to\infty} \int_{t-1}^{t} p(s)ds = \frac{2}{e} \cong 0.73575 > \frac{N}{e} = \frac{1}{e\ln 3} \cong 0.33485,$$

all solutions of (2.27) *are oscillatory.*

Example 2.6 *Consider the first-order nonlinear delay differential equation*

$$x'(t) + \frac{1}{e} x(\tau_1(t)) \ln(e^{x(\tau_2(t))} + 1.1) = 0, \; t \geq 0, \tag{2.28}$$

where

$$\tau_1(t) = \begin{cases} -t + 12k - 2, & t \in [6k, \; 6k+1] \\ 4t - 18k - 7, & t \in [6k+1, \; 6k+2] \\ -t + 12k + 3, & t \in [6k+2, \; 6k+3] \\ t - 3, & t \in [6k+3, \; 6k+4] \\ -2t + 18k + 9, & t \in [6k+4, \; 6k+5] \\ 5t - 24k - 26, & t \in [6k+5, \; 6k+6] \end{cases}, \quad k \in \mathbb{N}_0,$$

$$\tau_2(t) = \tau_1(t) - 1$$

and with the help of (2.1)

$$h_1(t) := \sup_{s \leq t} \{\tau_1(s)\} = \begin{cases} 6k - 2, & t \in [6k, \; 6k+1.25] \\ 4t - 18k - 7, & t \in [6k+1.25, \; 6k+2] \\ 6k+1, & t \in [6k+2, \; 6k+5.4] \\ 5t - 24k - 26, & t \in [6k+5.4, \; 6k+6] \end{cases}, \qquad k \in \mathbb{N}_0,$$

$$h_2(t) := \sup_{s \leq t} \{\tau_2(s)\} = h_1(t) - 1,$$

$\mathbb{N}_0$ *is the set of nonnegative integers, then we have*

$$h(t) = \max_{1 \leq i \leq 2} \{h_i(t)\} = h_1(t).$$

Moreover, we find

$$N = \limsup_{x \to 0} \frac{x(\tau_1(t))}{x(\tau_1(t)) \ln(e^{x(\tau_2(t))} + 1.1)} = \frac{1}{\ln(2.1)} \cong 1.34782.$$

Finally, for t=6k+5.4, we observe that

$$\limsup_{t \to \infty} \int_{h(t)}^{t} p(s) ds = \limsup_{t \to \infty} \int_{6k+1}^{6k+5.4} \frac{1}{e} ds \cong 1.61866 > N \cong 1.34782.$$

Thus, all solutions of (2.28) *are oscillatory.*

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