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Some properties of second-order weak subdifferentials

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Abstract: This article deals with second-order weak subdifferential. Firstly, the concept of second-order weak subdifferential is defined. Next, some of its properties are investigated. The necessary and sufficient condition for a second-order weakly subdifferentiable function to have a global minimum has been proved. It has been proved that a second-order weakly subdifferentiable function is both lower semicontinuous and lower Lipschitz.

Key words: Second-order weak subgradient, second-order weak subdifferential, lower semicontinuity, lower Lipschitz function

1. Introduction

Let $(X, \|.\|_X)$ be a real normed space and let X^* be the topological dual of X. Let $(x^*, c) \in X^* \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of nonnegative real numbers.

Definition 1.1 *[11, 15] Let $F : X \to \mathbb{R} \cup \{\mp\infty\}$ be a function and let $\bar{x} \in X$ be given. The set*

$$\partial F(\bar{x}) = \{x^* \in X : \langle x^*, x - \bar{x}\rangle \leq F(x) - F(\bar{x}), \text{ for all } x \in X\}$$

is called the subdifferential of F at $\bar{x} \in X$.

Definition 1.2 *[3, 4, 11] Let $F : X \to \mathbb{R}$ be a single-valued function and $\bar{x} \in X$ be given, where $F(\bar{x})$ is finite. A pair $(x^*, c) \in X^* \times \mathbb{R}_+$ is called the weak subgradient of F at $\bar{x}$ if*

$$F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|, \text{ for all } x \in X. \tag{1.1}$$

The set

$$\partial^w F(\bar{x}) = \{ (x^*, c) \in X^* \times \mathbb{R}_+ : F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|, \forall x \in X \} \tag{1.2}$$

of all weak subgradients of F at $\bar{x}$ is called the weak subdifferential of F at $\bar{x}$. If $\partial^w F(\bar{x}) \neq \emptyset$, then F is called weakly subdifferentiable at $\bar{x}$.

The concept of subgradient has an important place in convex and nonsmooth analysis. A nonconvex set has no supporting hyperplane at each boundary point. For this reason, most researches have generalized the concept of subgradient for nonconvex optimality problems [6, 7, 16, 17]. The concept of weak subdifferential,

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which is a generalization of the classical subdifferential, was first introduced by Azimov and Gasimov [3]. Kasımbeyli and İnceoğlu [11] examined the properties of weak subdifferentials. Kasımbeyli and Mammadov proposed relations between the directional derivatives, the weak subdifferentials, and the radial epiderivatives for nonconvex real-valued functions [12]. Kasımbeyli and Mammadov generalized the well-known necessary and sufficient optimality condition of nonsmooth convex optimization to the nonconvex case by using the notion of weak subdifferentials [13]. Cheraghi et al. presented the necessary and sufficient conditions for a weakly subdifferentiable function with global minimum [5]. In [9], Farajzadeh and Cheraghi investigated the relation between weak subdifferential and augmented normal cone. Meherrem and Polat proposed necessary optimality conditions by using the weak sunbdifferentials [14]. Anh introduced higher-order weak subdifferential and higher radial epiderivative concepts and investigated the relation between the higher-order weak subdifferential and higher-order radial epiderivative [2]. Motivated by the work in [2], we propose the second-order weak subdifferential and examine some important properties of the weak subdifferentials.

2. Second-order weak subdifferentials

In this section the second-order weak subdifferential concept was defined and some of its properties were investigated. The relationship between with lower Lipschitz function and second-order weak subdifferential was proved. Firstly, we introduce the second-order weak subdifferential and give an example.

Definition 2.1 *Let $F : X \to \mathbb{R}$ be a single-valued function and $\bar{x} \in X$ be given, where $F(\bar{x})$ is finite. A pair $(x^*, c) \in X^* \times \mathbb{R}_+$ is called the second-order weak subgradient of F at $\bar{x}$ if*

$$F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle^2 - c \|x - \bar{x}\|^2, \text{ for all } x \in X. \tag{2.1}$$

The set

$$\partial_w^2 F(\bar{x}) = \left\{ \ (x^*, c) \in X^* \times \mathbb{R}_+ : F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle^2 - c \|x - \bar{x}\|^2, \forall x \in X \ \right\} \tag{2.2}$$

of all second-order weak subgradients of F at $\bar{x}$ is called the second-order weak subdifferential of F at $\bar{x}$. If $\partial_w^2 F(\bar{x}) \neq \emptyset$, then F is called second-order weakly subdifferentiable at $\bar{x}$.

Example 2.2 *Let $F : \mathbb{R} \to \mathbb{R}$ and $F(x) = x^2$. Then it follows from definition of the second-order weakly subdifferentiable that*

$$(a, c) \in \partial_w^2 F(0) \Leftrightarrow (a, c) \in \mathbb{R} \times \mathbb{R}_+ \text{and } x^2 \geq a^2 x^2 - c x^2, \text{for all } x \in \mathbb{R}.$$

Hence, the second-order weak subdifferential can be written as

$$\partial_w^2 F(0) = \left\{ (a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 1 \right\}$$

The following theorem is a version of Theorem 3 given in [11], formulated for the second-order weak subdifferential.

Theorem 2.3 *Let the second-order weak subdifferential $\partial_w^2 F(\bar{x})$ of the function $F : X \to \mathbb{R}$ be not empty. Then the set $\partial_w^2 F(\bar{x})$ is closed and convex.*

Proof Firstly, we show that $\partial_w^2 F(\bar{x})$ is closed. Let $\{(x_n^*, c_n)\} \subset \partial_w^2 F(\bar{x})$ and let $(x_n^*, c_n) \to (x^*, c)$. We have to prove that $(x^*, c) \in \partial_w^2 F(\bar{x})$. Suppose to the contrary that $(x^*, c) \notin \partial_w^2 F(\bar{x})$. Then

$$F(x) - F(\bar{x}) \leq -c\|x - \bar{x}\|^2 + \langle x - \bar{x}, x^* \rangle^2, \text{ for some } x \in X \tag{2.3}$$

and by the inclusion $\{(x_n^*, c_n)\} \subset \partial_w^2 F(\bar{x})$,

$$F(x) - F(\bar{x}) \geq -c_n\|x - \bar{x}\|^2 + \langle x - \bar{x}, x_n^* \rangle^2, \quad \text{for all } x \in X. \tag{2.4}$$

In inequality (2.4) by letting to the limit as $n \to \infty$, we obtain

$$F(x) - F(\bar{x}) \geq -c\|x - \bar{x}\|^2 + \langle x - \bar{x}, x^* \rangle^2, \quad \text{for all } x \in X. \tag{2.5}$$

However, inequality (2.5) contradicts with inequality (2.3).

Now we prove the convexity condition. For $(x_1^*, c_1) \in \partial_w^2 F(\bar{x})$, $(x_2^*, c_2) \in \partial_w^2 F(\bar{x})$ and $\lambda \in [0,1]$, we have

$$F(x) - F(\bar{x}) \geq -c_1\|x - \bar{x}\|^2 + \langle x - \bar{x}, x_1^* \rangle^2, \forall x \in X \tag{2.6}$$

and

$$F(x) - F(\bar{x}) \geq -c_2\|x - \bar{x}\|^2 + \langle x - \bar{x}, x_2^* \rangle^2, \forall x \in X \tag{2.7}$$

Since $\lambda \geq \lambda^2$ and $(1-\lambda) \geq (1-\lambda)^2$, we have from the inequalities (2.6) and (2.7)

$$\begin{aligned} \lambda(F(x) - F(\bar{x})) &\geq -c_1\lambda\|x - \bar{x}\|^2 + \lambda\langle x - \bar{x}, x_1^* \rangle^2, \forall x \in X \\ &\geq -c_1\lambda\|x - \bar{x}\|^2 + \lambda^2\langle x - \bar{x}, x_1^* \rangle^2, \forall x \in X \\ &= -c_1\lambda\|x - \bar{x}\|^2 + \langle x - \bar{x}, \lambda x_1^* \rangle^2, \forall x \in X \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} (1-\lambda)(F(x) - F(\bar{x})) &\geq -c_2(1-\lambda)\|x - \bar{x}\|^2 + (1-\lambda)\langle x - \bar{x}, x_2^* \rangle^2, \forall x \in X \\ &\geq -c_2(1-\lambda)\|x - \bar{x}\|^2 + (1-\lambda)^2\langle x - \bar{x}, x_2^* \rangle^2, \forall x \in X \\ &= -c_2(1-\lambda)\|x - \bar{x}\|^2 + \langle x - \bar{x}, (1-\lambda)x_2^* \rangle^2, \forall x \in X \end{aligned} \tag{2.9}$$

By collecting side by side the inequalities (2.8) and (2.9), we obtain

$$F(x) - F(\bar{x}) \geq (-c_1\lambda + c_2(1-\lambda))\|x - \bar{x}\|^2 + \langle x - \bar{x}, \lambda x_1^* + (1-\lambda)x_2^* \rangle^2, \forall x \in X.$$

It follows from that

$$\lambda(x_1^*, c_1) + (1-\lambda)(x_2^*, c_2) \in \partial_w^2 F(\bar{x})$$

This completes the proof. □

Proposition 2.4 *Let $F, G : X \to \mathbb{R}$ and $F + G : X \to \mathbb{R}$ single-valued functions being second-order weakly subdifferentiable at $\bar{x} \in X$. Then $\partial_w^2 F(\bar{x}) + \partial_w^2 G(\bar{x}) \subset \partial_w^2 (F+G)(\bar{x})$.*

Proof Take arbitrary $(x_1^*, c_1) \in \partial_w^2 F(\bar{x})$, $(x_2^*, c_2) \in \partial_w^2 G(\bar{x})$. Since $(x_1^*, c_1) \in \partial_w^2 F(\bar{x})$, $(x_2^*, c_2) \in \partial_w^2 G(\bar{x})$, we have, by the definition of the second-order weak subgradient,

$$F(x) - F(\bar{x}) \geq -c_1 \|x - \bar{x}\|^2 + \langle x - \bar{x}, x_1^* \rangle^2, \forall x \in X \tag{2.10}$$

and

$$G(x) - G(\bar{x}) \geq -c_2 \|x - \bar{x}\|^2 + \langle x - \bar{x}, x_2^* \rangle^2, \forall x \in X \tag{2.11}$$

By collecting side by side inequalities (2.10) and (2.11), we obtain

$$\begin{aligned}
&(F(x) + G(x)) - (F(\bar{x}) - G(\bar{x})) \geq -c_1 \|x - \bar{x}\|^2 + \langle x - \bar{x}, x_1^* \rangle^2 - c_2 \|x - \bar{x}\|^2 + \langle x - \bar{x}, x_2^* \rangle^2, \ \forall x \in X \\
&(F(x) + G(x)) - (F(\bar{x}) - G(\bar{x})) \geq -(c_1 + c_2) \|x - \bar{x}\|^2 + \langle x - \bar{x}, x_1^* + x_2^* \rangle^2, \forall x \in X
\end{aligned}$$

Thus, $(x_1^* + x_2^*, c_1 + c_2) \in \partial_w^2 (F + G)(\bar{x})$, and then we obtain $\partial_w^2 F(\bar{x}) + \partial_w^2 G(\bar{x}) \subset \partial_w^2 (F + G)(\bar{x})$. □

If F is second-order weakly subdifferentiable and second-order positively homogeneous function, the following equality conditions hold.

Proposition 2.5 *Let $F : X \to (-\infty, +\infty]$ be second-order weakly subdifferentiable at $\bar{x} \in X$ and $\alpha\bar{x} \in X$ and second-order positively homogeneous function. Then $\partial_w^2 F(\alpha\bar{x}) = \partial_w^2 F(\bar{x})$.*

Proof Since $F : X \to \mathbb{R}$ is second-order weakly subdifferentiable at $\bar{x} \in X$ and $\alpha\bar{x} \in X$ and second-order positively homogeneous function

$$\begin{aligned}
(x^*, c) \in \partial_w^2 F(\alpha\bar{x}) \quad &\Leftrightarrow \quad F(\alpha x) - F(\alpha\bar{x}) \geq -c \|\alpha x - \alpha\bar{x}\|^2 + \langle \alpha x - \alpha\bar{x}, x^* \rangle^2, \ \forall x \in X \\
&\Leftrightarrow \quad \alpha^2 (F(x) - F(\bar{x})) \geq \alpha^2 \left(-c \|x - \bar{x}\|^2 + \langle x - \bar{x}, x^* \rangle^2 \right), \ \forall x \in X \\
&\Leftrightarrow \quad (x^*, c) \in \partial_w^2 (F)(\bar{x})
\end{aligned}$$

This proves the proposition. □

Proposition 2.6 *Let $F : X \to (-\infty, +\infty]$ be second-order weakly subdifferentiable at $\bar{x} \in X$. F has a global minimum at $\bar{x} \in X$ if and only if $(0, c) \in \partial_w^2 F(\bar{x})$, for all $c \geq 0$.*

Proof F has a global minimum at $\bar{x} \in X$

$$\begin{aligned}
&\Leftrightarrow \quad F(x) \geq F(\bar{x}), \forall x \in X \\
&\Leftrightarrow \quad F(x) - F(\bar{x}) \geq 0, \forall x \in X \\
&\Leftrightarrow \quad F(x) \geq F(\bar{x}) - c \|x - \bar{x}\|^2 + \langle x - \bar{x}, 0 \rangle^2, \forall x \in X \\
&\Leftrightarrow \quad (0, c) \in \partial_w^2 F(\bar{x}).
\end{aligned}$$

□

Proposition 2.7 *Let $F, G : X \to \mathbb{R}$ and F be second-order weakly subdifferentiable at $\bar{x} \in X$, $G - F$ attains a global minimum at $\bar{x}$. Then G is second-order weakly subdifferentiable at $\bar{x} \in X$ and $\partial_w^2 F(\bar{x}) \subset \partial_w^2 G(\bar{x})$.*

Proof Since F is second-order weakly subdifferentiable at $\bar{x} \in X$,

$$F(x) - F(\bar{x}) \geq -c\|x - \bar{x}\|^2 + \langle x - \bar{x}, x^* \rangle^2, \ \forall x \in X. \tag{2.12}$$

By the assumption, we have

$$(G - F)(x) \geq (G - F)(\bar{x}), \forall x \in X \tag{2.13}$$
$$G(x) - G(\bar{x}) \geq F(x) - F(\bar{x}), \forall x \in X. \tag{2.14}$$

By inequalities (2.12) and (2.14), we have

$$G(x) - G(\bar{x}) \geq -c\|x - \bar{x}\|^2 + \langle x - \bar{x}, x^* \rangle^2, \ \forall x \in X$$

Hence, G is second-order weakly subdifferentiable at $\bar{x} \in X$ and

$$\partial_w^2 F(\bar{x}) \subset \partial_w^2 G(\bar{x}).$$

□

Now, we recall the definition of lower Lipschitz function.

Definition 2.8 *[4, 11] A function $F : X \to (-\infty, +\infty]$ is called lower locally Lipschitz at $\bar{x} \in X$, if there exist a nonnegative number L (Lipschitz constant) and a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ such that*

$$F(x) - F(\bar{x}) \geq -L\|x - \bar{x}\|, \quad \text{for all } x \in \mathcal{N}(\bar{x}). \tag{2.15}$$

If the above inequality holds true for all $x \in X$ then F is called lower Lipschitz at $\bar{x}$ with the Lipschitz constant L.

The following theorem describes the relationship between the function a and the second-order weak subdifferentiable.

Theorem 2.9 *Let $F : X \to (-\infty, +\infty]$ function and let $\bar{x} \in X$ be given where $F(\bar{x})$ is finite. If F is second-order weakly subdifferentiable at $\bar{x}$, then F is lower Lipschitz at $\bar{x}$.*

Proof Let F be lower Lipschitz at $\bar{x}$ with the Lipschitz constant c. By the definition of Lipschitz function, we have

$$F(x) - F(\bar{x}) \geq -c\|x - \bar{x}\| \geq -c\|x - \bar{x}\|^2 + \langle 0, x - \bar{x} \rangle^2, \text{for all } x \in X$$

□

Before giving the relationship between the lower semicontinuous function and the second-order weakly differentiability, we recall the definition of the semicontinuous function.

Definition 2.10 *[11, 15] A function $F : X \to (-\infty, +\infty]$ is lower semicontinuous at $\bar{x} \in X$ if*

$$\lim_{x \to \bar{x}} inf F(x) \geq F(\bar{x}).$$

Proposition 2.11 *Let $F : X \to (-\infty, +\infty]$ function be second-order weak subdifferentiable at $\bar{x} \in X$. Then F is lower semicontinuous at $\bar{x} \in X$.*

Proof Since $F : X \to (-\infty, +\infty]$ function is second-order weak subdifferentiable at $\bar{x} \in X$, $\partial_w^2 F \neq \emptyset$. Then there exists the pair $(x^*, c) \in X^* \times \mathbb{R}$ such that

$$F(x) - F(\bar{x}) \geq -c \|x - \bar{x}\|^2 + \langle x - \bar{x}, x^* \rangle^2, \quad \text{for all } x \in X. \tag{2.16}$$

In both sides of inequality (2.16) by letting to the limit inferior as $x \to \bar{x}$, we obtain

$$\lim_{x \to \bar{x}} inf F(x) \geq F(\bar{x}).$$

This completes the proof. □

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