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Almost quasi clean rings

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Abstract: The element q of a ring R is called quasi-idempotent element if $q^2 = uq$ for some central unit u of R, or equivalently $q = ue$, where u is a central unit and e is an idempotent of R. In this paper, we define that the ring R is almost quasi-clean if each element of R is the sum of a regular element and a quasi-idempotent element. Several properties of almost-quasi clean rings are investigated. We prove that every quasi-continuous and nonsingular ring is almost quasi-clean. Finally, it is determined that the conditions under which the idealization of an R-module M is almost quasi clean.

Key words: Quasi-idempotent, almost quasi-clean ring

1. Introduction

For a ring R, an element a of R is called clean if it can be written as the sum of a unit and an idempotent. A ring R is clean if each element of R is clean. Clean rings were defined by W. K. Nicholson in [13] in relation to exchange rings. In [11] McGovern introduced almost clean ring as each element of its is the sum of regular element (neither a left nor a right zero divisor) and an idempotent. Clearly a clean ring is almost clean ring. Exchange rings and clean rings have been extensively studied by many authors (for example, see [7, 9, 10]).

An element a in a ring R is called strongly regular if $a \in a^2R \cap Ra^2$. By [14], $a \in R$ is strongly regular if and only if $a = ue = eu$ where $e^2 = e$ and u is a unit in R if and only if $a^2 = ua = au$ for a unit u in R. Tang and Su in [Quasi-clean rings and strongly quasi-clean rings, preprint] introduced the notion of a quasi-idempotent element in which a special kind of strongly regular elements is a natural generalization of idempotent: An element q of a ring R is called quasi-idempotent if $q^2 = uq$ for some central unit u of R, or equivalently, $q = ue$, where $e^2 = e \in R$ and u is a central unit of R.

The paper consists of three parts. In section 2, we defined a ring R as an almost quasi-clean ring if each of its element can be written as the sum of a regular element (neither a left nor a right zero-divisor) and a quasi-idempotent. We give characterizations of almost quasi-clean rings and discuss various consequences. For example, we show that all right quasi-continuous and right nonsingular rings are almost quasi-clean, also an abelian Rickart ring has an almost quasi-clean decomposition. In the last section, we give some conditions in which the idealization and the trivial extension of an R-module M is almost quasi-clean.

In this paper, a ring R is an associative ring with identity. We write $Z(R)$, $U(R)$, $UC(R)$, $Reg(R)$, $Id(R)$ and $nil(R)$ to denote the set of all zero divisors, the set of all unit elements, the set of central unit

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elements, the set of regular elements, and the set of idempotent elements of R, respectively. Also we denote Jacobson radical of R by $J(R)$.

Let R be a ring. In this paper, we also use $ann_r(x)$ and $ann_l(x)$ to denote the right and left annihilator of an element x of R. The maximal right ring of quotients are denoted by $Q^r_{max}(R)$. In this paper, every module is the right module.

2. Almost quasi-clean rings

Definition 2.1 *An element q of a ring R is called quasi-idempotent if $q^2 = uq$ for some central unit u of R. The element q is called a u-idempotent. The set of quasi-idempotent of R is denoted by $QId(R)$.*

A quasi-idempotent is a strongly regular element, but the converse is clearly not true. If q is a quasi-idempotent, then $(u^{-1}q)^2 = u^{-1}q$ is an idempotent. So we have that an element q is a quasi-idempotent if and only if $q = ue$, where e is an idempotent and u is a unit in R.

If q is a u-idempotent, i.e. $q^2 = uq$ and $q = ue$ where e is an idempotent, we have a direct sum decomposition of R:

$$R = qR \oplus (u-q)R = eR \oplus (1-e)R,$$

where $qR = eR$, $(u-q)R = (1-e)R$ and $u - q = u(1-e)$ is also a $u-$idempotent.

Definition 2.2 *An element a of a ring R is called almost quasi-clean if a is a sum of a quasi-idempotent and a regular element. A ring R is called almost quasi-clean if each of its elements is almost quasi-clean.*

A ring R is called a *quasi-Boolean* ring if every element of R is quasi-idempotent. Boolean rings and any direct product of fields are quasi-Boolean. Note that the class of quasi-Boolean rings is closed under homomorphic images and direct products.

Example 2.3 *Clean rings, almost clean rings, quasi-Boolean rings, weakly clean rings, and quasi-clean rings are almost quasi-clean.*

If R is quasi-clean, then each of its homomorphic image is also quasi-clean. We next show that the homomorphic image of an almost quasi-clean ring does not need to be almost quasi-clean.

Example 2.4 *Let $R = K[x,y]$ where K is any field. Certainly, $K[x,y]$ being a domain is almost quasi-clean. Let $\overline{R} = R/((x) \cap (x-1) \cap (y))$. Here, $\overline{x}$ and $\overline{x-1}$ are both zero divisor elements of $\overline{R}$, so $\overline{R}$ is not almost quasi-clean.*

In the following proposition, we collect some basic properties of almost quasi-clean rings.

Proposition 2.5 *Let R and R_α denote rings.*

1. *$R = \Pi_{\alpha \in I} R_\alpha$ is almost quasi-clean if and only if each R_α is almost quasi-clean.*
2. *R is an almost quasi-clean ring if and only if the power series $R[[x]]$ is almost quasi-clean.*
3. *If R is an almost quasi-clean ring, then $R[[\{X_\alpha\}]]$ an is almost quasi-clean ring.*

Proof (1) $(:\Rightarrow)$ For $\alpha \in I$, let $x_\alpha \in R_\alpha$. Then $x := (x_1, \cdots, x_\alpha, \cdots) \in R$ can be written as $x = r + q$ where $r \in Reg(R)$ and $q \in QId(R)$. Since $Reg(R) = \Pi_{\alpha\in I} Reg(R_\alpha)$, we get $r_\alpha \in Reg(R_\alpha)$ for each $\alpha \in I$. Also, if $q \in QId(R)$ then $q = ue$ where $u \in UC(R)$ and $e \in Id(R)$. Because of $u \in UC(R)$ there exist $v \in R$ such that $uv = 1_R$. Then $(u_1, \cdots, u_\alpha, \cdots)(v_1, \cdots, v_\alpha, \cdots) = (1_{R_1}, \cdots, 1_{R_\alpha}, \cdots)$ which implies, for each $\alpha \in I$, $u_\alpha v_\alpha = 1_{R_\alpha}$. Hence $u_\alpha \in U(R_\alpha)$. Similarly, $u_\alpha \in C(R_\alpha)$ for each $\alpha \in I$. Thus, $u_\alpha \in UC(R_\alpha)$ for each $\alpha \in I$. In the same way, if $e = (e_1, \cdots, e_\alpha, \cdots) \in Id(R)$ then, for each $\alpha \in I$, $e_\alpha \in Id(R_\alpha)$. Consequently, we get $x_\alpha = r_\alpha + u_\alpha e_\alpha = r_\alpha + q_\alpha$ where $r_\alpha \in Reg(R_\alpha)$ and $q_\alpha \in QId(R_\alpha)$ for each $\alpha \in I$. So R_α is almost quasi-clean.
$(\Leftarrow:)$ Let $x := (x_\alpha) \in R = \Pi_{\alpha\in I} R_\alpha$. For each $\alpha \in I$, write $x_\alpha = r_\alpha + q_\alpha$ where $r_\alpha \in Reg(R_\alpha)$ and $q_\alpha \in QId(R_\alpha)$. Let $q_\alpha \in QId(R_\alpha) = u_\alpha e_\alpha$, where $u_\alpha \in UC(R_\alpha)$ and $e_\alpha \in Id(R_\alpha)$. Then $x_\alpha = r_\alpha + u_\alpha e_\alpha$. Since $r_\alpha \in Reg(R_\alpha)$, there exist $0 \neq y_\alpha \in R_\alpha$ such that $r_\alpha y_\alpha = 0$. So $r := (r_\alpha) \in Reg(R)$. On the other hand, $u := (u_\alpha) \in UC(R)$ and $e := (e_\alpha) \in Id(R)$. Hence we have an almost quasi-clean representation $x = r + ue$, as desired.
(2) $(:\Rightarrow)$ Let $f(x) \in R[[x]]$. Then $f(x) = \sum_{i=0}^{\infty} r_i x^i = r_0 + r_1 x + r_2 x^2 + \cdots$. Since R is an almost quasi-clean ring, we have the representation $r_0 = r + q$ where $r \in Reg(R)$ and $q \in QId(R)$. Hence $r_0 = r + ue$ where $u \in UC(R)$ and $e \in Id(R)$. Now

$$f(x) = \sum_{i=0}^{\infty} r_i x^i = r_0 + r_1 x + r_2 x^2 + \cdots = r + ue + r_1 x + r_2 x^2 + ... = ue + g(x),$$

where $g(x) = r + r_1 x + r_2 x^2 + \ldots$. If $g(x) \notin Reg(R[[x]])$, then there exist $h(x) = \sum_{i=0}^{\infty} h_i x^i \neq 0$ such that $g(x)h(x) = 0$. Thus $rh(x) = 0$ and then $rh_i = 0$ for all i. Since $r \in Reg(R)$, for all i, h_i must be zero, which is a contradiction. So $g(x) \in Reg(R[[x]])$. Since $u \in UC(R) \subseteq UC(R[[x]])$ and $e \in Id(R) \subseteq Id(R[[x]])$, we obtain that $R[[x]]$ is almost quasi-clean.
$(\Leftarrow:)$ Since $R[[x]] \cong \{(a_0, a_1, ...) \lfloor a_i \in R\} = \Pi_{i\geq 0} R$, we obtain R is almost quasi-clean by (1).
(3) Let $f(x) \in R[[\{X_\alpha\}]]$. Then $f = f_0 + f'$ where $f_0 \in R$ and $f' \in (\{X_\alpha\})$. Since $f_0 \in R$, we can write $f_0 = r + q$ where $r \in Reg(R)$ and $q \in QId(R)$. Hence $f_0 = r + q = r + ue$ where $u \in UC(R)$ and $e \in Id(R)$. Now

$$f = r + ue + f' = (r + f') + ue$$

where $r + f' \in Reg(R[[\{X_\alpha\}]])$. Since $u \in UC(R) \subseteq UC(R[[\{X_\alpha\}]])$ and $e \in Id(R) \subseteq Id(R[[\{X_\alpha\}]])$, we obtain that $R[[\{X_\alpha\}]]$ is almost quasi-clean. □

Let us recall a few definitions in module theory. Let R be a ring. Recall the following conditions for a right R module M:

(C1) Every submodule of M is essential inside a (direct) summand of M.

(C2) Every submodule of M that is isomorphic to a summand of M is itself a summand of M.

(C3) If A and B are summands of M with $A \cap B = 0$, then $A \oplus B$ is also a summand of M

M is called a *CS* module if it satisfies the condition $(C1)$. M is called *continuous* if it satisfies $(C1)$ and $(C2)$. M is called a *quasi-continuous module* if it satisfies $(C1)$ and $(C3)$. A ring R is right *CS (right quasi-continuous or right continuous)* if R_R is *CS (quasi-continuous or continuous)*. It is well known that a continuous module is quasi-continuous.

Proposition 2.6 *Let R be a ring that can be embedded in a quasi clean-ring has the same quasi idempotents as R, then R is almost quasi clean.*

Proof Let S be denote a quasi-clean ring with the same quasi idempotent as R in which R embeds. Let $a \in R$. Then $a \in S$. Thus, $a = u + q$ where $u \in U(S)$ and $q \in QId(S)$. By hypothesis, $q \in QId(R)$. Thus, $u = a - q \in R$. Now we show that $u \in Reg(R)$. Suppose that $u \notin Reg(R)$. Then there exist a nonzero element x in R such that $xu = 0$. Since x and u in R, $x, u \in S$ and $xu \in S$ as well. Also $u \in U(S)$ there exist $v \in S$ such that $uv = 1$. Then

$$xu = 0 \Rightarrow (xu)v = 0 \Rightarrow x = 0$$

a contradiction. So, $u \in Reg(R)$. Hence R is almost quasi-clean. □

Proposition 2.7 *If R is quasi-continuous and nonsingular, then R is almost quasi-clean.*

Proof If R is nonsingular, then $Q^r_{max}(R)$ is regular and self injective. Thus, $Q^r_{max}(R)$ is quasi-clean by [5, Corollary 3.2] and by Example 2.3. Then, R is almost quasi-clean by Proposition 2.6. □

In [4, Theorem 1], Camillo and Khurana showed that a ring a R is unit regular if and only if every element a of R has a clean decomposition $a = e + u$ such that $aR \cap eR = 0$. The following theorem was given by Tang and Su in [Quasi-clean rings and strongly quasi-clean rings, preprint] as a theorem.

Theorem 2.8 *A ring R is unit regular if and only if every element of a of R has a quasi-clean decomposition $a = u + q$ such that $aR \cap qR = 0$.*

Corollary 2.9 *If R is quasi-continuous and $Q^r_{max}(R)$ is unit-regular ring, then every element a of R has almost quasi-clean decomposition $a = r + q$ such that $aR \cap qR = 0$.*

Proof Since $Q^r_{max}(R)$ is unit-regular ring, then each element a of $Q^r_{max}(R)$ has a quasi-clean decomposition $a = u + q$ such that $aQ^r_{max}(R) \cap qQ^r_{max}(R) = 0$ where $u \in U(Q^r_{max}(R))$ and $q \in QId(Q^r_{max}(R))$. Then, $aR \cap qR = 0$ as well. Since R is quasi-continuous, $q \in QId(R)$. Thus, $u \in R$ and it has to be regular just like proof of Proposition 2.6. Thus $a = u + q$ is almost a quasi-clean decomposition such that $aR \cap qR = 0$. □

Recall that a ring R is abelian ring if every idempotent is central. An element in an abelian Rickart ring, which has not to be commutative, is a product of a regular element and an idempotent.

Theorem 2.10 *Let R be an abelian ring. Then R is Rickart if and only if every element a of R has an almost quasi-clean decomposition $a = r + q$ such that $aR \cap qR = 0$.*

Proof ($\Rightarrow$:) Let R be an abelian Rickart ring. If $a \in R$, then $ann_r(a) = eR$ for some idempotent element $e \in R$. Let us take $q = ue$ for some $u \in UC(R)$. In this case, $a = q + (a - q)$. We claim that $a - q$ is regular element in R. Let $(a - q)r = 0$ for $r \in R$. Since $e \in ann_r(a)$, $ae = 0 \Rightarrow aq = 0$ and so $a(u - q) = au$. Thus, $0 = (u - q)(a - q)r = (u - q)ar = a(u - q)r = aur$. Since $u \in UC(R)$ there exist $v \in R$ such that $uv = 1 = vu$. Then, $ar = 0$, which implies $r \in ann_r(a) = eR \subseteq qR$. Otherwise, we have $0 = q(a - q)r = qar - q^2r = aqr - uqr$, then $uqr = 0$ and $qr = 0$. So $(u - q)r = ur$ then $r \in (1 - e)R$. Therefore, $r \in eR \cap (1 - e)R = 0$. Hence, $a - q$ is right regular element in R. Since R is abelian Rickart ring, $a - q$ is left regular element in R.
We also claim that $aR \cap qR = 0$. Let $x \in aR \cap qR$ Then there exist $r_1, r_2 \in R$ such that $x = ar_1 = qr_1$.

Since $q \in QId(R)$, $q = ue$ where $u \in UC(R)$ and $e \in Id(R)$. Also we have $xe = ar_1e = aer_1 = 0$ and $xe = qr_2e = uer_2e = ue^2r_2 = uer_2 = qe = x$. Hence, $xe = x = 0$.

($\Leftarrow$:) Suppose that a of R has a almost quasi-clean decomposition $a = r + q$ such that $aR \cap qR = 0$ where $q \in QId(R)$ and $r \in Reg(R)$. Since $q \in QId(R)$, there exist a central unit u and an idempotent e in R such that $q = ue$. Now we show that $ann_r(a) = eR$. Let $x \in ann_r(a)$. Then $0 = ax = rx + qx$ and so $erx + eqx = 0 = erx + qx$, since $eq = eue = ue^2 = ue = q$. Thus we have that $rx + qx = erx + qx$, which implies that $rx = erx$. Because of R is abelian we get $rex = rx$. So $x = ex$, since $r \in Reg(R)$. Thus, we have $x \in eR$ and so $ann_r(a) \subseteq eR$.

On the other hand, let $x \in eR$. Then for some $y \in R$, $x = ey$. We have, $ax = aey = ae(uu^{-1})y = (r+q)e(uu^{-1})y = reuu^{-1}y + qeuu^{-1}y = rqu^{-1}y + qy = q(ru^{-1}y + y) \in aR \cap qR = 0$. Thus $ax = 0$ and so $x \in ann_r(a)$. Therefore, $ann_r(a) = eR$ and so R is right Rickart. Because of R is abelian, R is left Rickart. □

We notice that let R be presimplifiable, if $1 \neq e \in Id(R)$, then $e \in Z(R) \subseteq J(R)$ which implies $e = 0$. Hence, such rings R are indecomposable.

Proposition 2.11 *Let R be a ring. If $Z(R) \subseteq J(R)$, then R is an indecomposable almost quasi-clean ring.*

Proof Suppose $x \in Reg(R)$. Then $x = x + 0 = x + u0$ where $u \in UC(R)$ and $0 \in Id(R)$ (that is, $u0 = q \in QId(R)$). Thus R is an indecomposable almost quasi-clean ring by the previous sentence and the almost clean decomposition.

Suppose that x is not regular element of R. Then $x \in Z(R) \subseteq J(R)$. Hence $x - 1 := u \in U(R)$. If we take $v = 1$ and $e = 1$ then we can write $x = u + 1 = u + 1.1 = u + ve$ where $u \in U(R) \subseteq Reg(R)$, $v = 1 \in UC(R)$ (and hence $q = ve \in QId(R)$) and $1 \in Id(R)$. □

3. The almost quasi-clean property over commutative rings

In [3, Theorem 3], the authors proved that a commutative indecomposable ring R is clean if and only if R is a local ring. Ahn and Anderson [1, Corollary 1.4] obtained that a commutative indecomposable ring R is weakly clean if and only if R is a local ring or an indecomposable ring with exactly two maximal ideals in which 2 is a unit.

Proposition 3.1 *If R is Rickart ring, then R is almost quasi-clean ring.*

Proof Let R be a Rickart ring. By [11, Proposition 15], for a regular element r in R and an idempotent e in R, any element x in R can be written as $x = re$. To take, for some $u \in UC(R)$, $q = u(1-e)$. Let $v = re - q$, then $x = v + q$. Now we will show that $v \in Reg(R)$. Suppose that $sv = 0$ for some nonzero element $s \in R$. Then, we obtain that

$$sx = sv + sq \Rightarrow sre = sq$$

; hence, $sre = 0 = sq$. Since $r \in Reg(R)$, we get $se = 0$. Thus, $s \in Ann(1-e) = (e)$ and we can write $s = te$ for some $t \in R$. So, $0 = se = (te)e = te^2 = te = s$ is a contradiction. Hence, v is a regular element in R. Then, $x = v + q$ where $v \in Reg(R)$ and $q \in QId(R)$, R is almost quasi-clean ring. □

Theorem 3.2 *The following conditions are equivalent for a commutative ring R:*

(1) R is an indecomposable almost quasi-clean ring;

(2) For $x \in R$, either x or $x-u$ is regular where $u \in UC(R)$.

(3) For ideals I and J of R consisting of zero divisor, $I + J \neq R$

Proof (1) $\Rightarrow$ (2) Let R be an indecomposable almost quasi-clean ring. Then, every element x in R can be written $x = r + q$, where $r \in Reg(R)$ and $q \in QId(R)$. Let $q = ue$ where $u \in UC(R)$ and $e \in Id(R)$. Then $x = r + ue$. If $e = 0$, then $x = r + u0 = r \in Reg(R)$. If $e = 1$, then $x = r + u1 = r + u \Rightarrow r = x - u \in Reg(R)$, as desired.
(2) $\Rightarrow$ (1) First, we show that R is an indecomposable ring. Now, assume that $1 \neq e \in Id(R)$. Then, $e(e-1) = 0$. By the hypothesis, by taking $u = 1$, e or $e-1$ is regular, we get $e = 0$. Thus R is indecomposable.
Now, we show that R is almost quasi-clean. If x is a regular element in R, then, $x = x + 0 = x + u.0$ where $u \in UC(R)$ and $0 \in Id(R)$. If $x - v$ is a regular for $v \in UC(R)$, then there exists an $r \in Reg(R)$ such that $x - v = r$. So $x = r + v = r + v.1$ where $1 \in Id(R)$. Hence R is almost quasi-clean.
(3) $\Rightarrow$ (2) Suppose that I and J are ideals consisting of zero divisor and $I + J = R$. So, $i + j = u$ where $i \in I$, $j \in J$ and $u \in UC(R)$. Then, i and $-j = i - u$ are both zero divisor. Hence both i and $i - u$ are not regular in R. It is a contradiction.
(2) $\Rightarrow$ (3) Let $x \in R$ and suppose that, for a $u \in UC(R)$, x and $x - u$ are both not regular elements. Then $(x) + (x - u) \neq R$, a contradiction. □

Let R be a ring and M be an R-bimodule. The set pairs (r, m) with $r \in R$ and $m \in M$ under coordinate wise addition and multiplication defined by

$$(r, m)(r', m') = (rr', rm' + r'm)$$

for all $r, r' \in R$ and $m, m' \in M$. Then $T(R, M)$ is called the trivial extension of R by M. We have the followings:

$$Z(M) = \{r \in R \mid \exists 0 \neq m \in M \text{ such that } rm = 0\}$$

$$C(M) = \{r \in R \mid m \in M \text{ such that } rm = mr\}$$

$$U(M) = \{r \in R \mid \exists m \in M \text{ such that } rm = m\}.$$

Recall that a Morita context is a 4-tuple $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where A and B are rings, ${}_AM_B$ and ${}_BN_A$ are bimodules, and there exist context products $M \times N \to A$ and $N \times M \to B$ written multiplicatively as $(w, z) = wz$ and $(z, w) = zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operations. A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called trivial if the context products are trivial, i.e. $MN = 0$ and $NM = 0$. We have

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a trivial Morita context by [8]. Furthermore, if $q = ue$, where $u \in UC(R)$ and $e \in Id(R)$, then $\{(u,0) \mid u \in UC(R) \cap UC(M)\} \subseteq UC(T(R,M))$. For $(x,m) \in T(R,M)$,

$$(u,0)(x,m) = (ux, um + 0m) = (x,m) = (xu, x0 + um) = (x,m)(u,0),$$

so $\{(u,0) \mid u \in UC(R) \cap UC(M)\} \subseteq UC(T(R,M))$. As $Id(T(R,M)) = \{(e,0) \in R(M) \mid e \in Id(R)\}$, we get

$$QId(T(R,M)) = \{(q,0) \in T(R,M) \mid q \in QId(R)\}.$$

Theorem 3.3 *The trivial extension $T(R,M)$ is almost quasi-clean if and only if each $x \in R$ can be written in the form $x = r + q$ where $r \in R - (Z(R) \cup Z(M))$ and $q \in QId(R)$.*

Proof (:⇒) Let $x \in R$. Then $(x,0) = (r,0) + (q,0)$, where $(r,0) \in Reg(T(R,M))$ and $(e,0) \in Id(T(R,M))$. Let $(q,0) = (u,0)(e,0)$, where $(u,0) \in UC(T(R,M))$ and $(e,0) \in Id(T(R,M))$. By [1, Theorem 2.11], if $(r,0) \in Reg(T(R,M))$ then $r \in R - (Z(R) \cup Z(M))$. Thus we get $x = r + ue$, as desired.
(⇐:) Let $x \in R$ and $m \in M$. Write, $x = r + q$ where $r \in R - (Z(R) \cup Z(M))$ and $q \in QId(R)$. Let $q = ue$, where $u \in UC(T(R,M))$ and $e \in Id(R)$. Then, $x = r + ue$, and hence, $(x,m) = (r,m) + (u,0)(e,0)$ as desired. □

Theorem 3.4 *If R is a ring with $Z(R)$ closed under addition then R is almost quasi-clean ring.*

Proof If $x \notin Z(R)$ then $x \in Reg(R)$. So, $x = x + 0$ where $0 \in QId(R)$. For $u \in UC(R)$, suppose that $u \in Z(R)$. Since $u \in Z(R)$, there exist non-zero element a in R such that $ua = 0$.

Also, since $u \in UC(R)$ then there exists $v \in R$ such that $uv = 1 = vu$. Then we get, $v(ua) = 0 \Rightarrow (vu)a = 0 \Rightarrow a = 0$, a contradiction. Hence $u \notin Z(R)$.

Let $x \in Z(R)$ then, for $u \in UC(R)$, $x - u \notin Z(R)$. So, $x = (x-u) + u$, where $x - u \in Reg(R)$ and $u \in QId(R)$. Hence R is almost quasi-clean. □

Theorem 3.5 *Let P_1 and P_2 be prime ideals of a ring R such that $Z(R) = P_1 \cup P_2$ and $nil(R) = P_1 \cap P_2$. Then R is almost quasi-clean ring.*

Proof If P_1 and P_2 are comparable, the result follows from Theorem 3.4. Now we show that $x \in (P_1 \cup P_2) \setminus (P_1 \cap P_2)$ is almost quasi-clean. Since, $P_1 \cup P_2 = (P_1 \setminus P_2) \cup (P_2 \setminus P_1)$ and $(P_1 \setminus P_2) \cap (P_2 \setminus P_1) = \emptyset$, without lost of generality we may assume that $x \in P_1 \setminus P_2$. First of all assume that R is not indecomposable. To take $q \in QId(R)$, $q = ue$ where $u \in UC(R)$ and $e \in Id(R)$ which is non-trivial idempotent in $P_2 \setminus P_1$. Since $e \in P_1 \setminus P_2$, $q \in P_1 \setminus P_2$. Then $x = (x-q) + q$ where $x - q \in Z(R)$ (i.e. $x - q \in Reg(R)$) and $q \in QId(R)$. Now assume that R is indecomposable. Then, $R \setminus (P_1 \cap P_2)$ is indecomposable, so $P_1 + P_2 \neq R$. Now, for some $u \in UC(R)$, $x - u \in P_1$. If $(x-u) \in P_2$, then $u = x - (x-u) \in P_1 + P_2$, so $1 \in P_1 + P_2$ is a contradiction. Thus, $(x-u) \notin P_1 \cup P_2 = Z(R)$. Hence, $x = (x-u) + u$ where $(x-u) \in Reg(R)$ and $u \in QId(R)$. □

Theorem 3.6 *Let R be an integral domain and M be an $R-$module.*

(1) If $Z(M) = P$ a prime ideal then $T(R,M)$ is almost quasi-clean.

(2) $T(\mathbb{Z}, M)$ *is almost quasi-clean if and only if* $Z(M) = (p)$ *where* p *is prime, possibly* 0*.*

Proof
(1) If $Z(M) = P$ a prime ideal, then $Z(T(R, M)) = T(P, M)$ is prime ideal. By Theorem 3.4, $T(R, M)$ is almost quasi-clean.
(2) ($\Leftarrow$:) This follows from (1).
(:$\Rightarrow$) Suppose that $(p) \cup (q) = Z(M)$ where p and q are different with each other (nonzero) primes. Choose $x \in \mathbb{Z}$ with $x \equiv 0 \pmod p$ and $x \equiv 1 \pmod q$ by Chinese Remainder Theorem. Then, $x - 0, x - 1 \in Z(M)$, so we cannot write $x = r + q$ where $r \in Reg(R)$ and $q \in \{0, 1\} \subseteq QId(\mathbb{Z}) = \{-1, 0, 1\}$. □

Corollary 3.7 *The idealization* $T(\mathbb{Z}, \mathbb{Z}_n)$ *is almost quasi-clean if and only if* $n = p^\alpha$ *where* p *is prime integer and* $\alpha \geq 1$

Note that let R be a ring and $\mathbb{U}_n(R)$ (resp. $\mathbb{L}_n(R)$) the ring of $n \times n$ upper (resp. lower) triangular matrices over R.

Proposition 3.8 *For* $A = [a_{ij}] \in \mathbb{U}_n(R)$ *(resp.* $A \in \mathbb{L}_n(R)$*), if* $A \in QId(\mathbb{U}_n(R))$ *(resp.* $A \in QId(\mathbb{L}_n(R))$*) then, for* $i = 1, ...n$*,* $a_{ii} \in QId(R)$*. Also, if for* $i = 1, ...n$*,* $a_{ii} \in QId(R)$ *then* $diag(a_{11}, \ldots, a_{nn}) \in QId(\mathbb{U}_n(R))$ *(resp.* $diag(a_{11}, \ldots, a_{nn}) \in QId(\mathbb{L}_n(R))$*).*

Proof Without losing generality, we do upper triangular case. Let $A = [a_{ij}] \in QId(\mathbb{U}_n(R))$. Note that $A \in UC(\mathbb{U}_n(R))$ if and only $a_{ii} \in UC(R)$. Also, if $A \in Id(\mathbb{U}_n(R))$ then $a_{ii} \in Id(R)$ and if $a_{ii} \in Id(R)$ then $diag(a_{11}, \ldots, a_{nn}) \in Id(\mathbb{U}_n(R))$. If $A \in QId(\mathbb{U}_n(R))$ then $A = UE$ where $U = [u_{ij}] \in UC(\mathbb{U}_n(R))$ and $E = [e_{ij}] \in Id(\mathbb{U}_n(R))$. Since $U = [u_{ij}] \in UC(\mathbb{U}_n(R))$ and $E = [e_{ij}] \in Id(\mathbb{U}_n(R))$, $u_{ii} \in UC(R)$ and $e_{ii} \in Id(R)$. Because of $a_{ii} = u_{ii}e_{ii}$ where $u_{ii} \in UC(R)$ and $e_{ii} \in Id(R)$, for all $1 \leq i \leq n$, $a_{ii} \in QId(R)$. Also, let $a_{ii} \in QId(R)$. Then there exist $u_i \in UC(R)$ and $e_i \in Id(R)$ such that $a_{ii} = u_i e_i$ for all $1 \leq i \leq n$. So,

$$\begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} u_1e_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_ne_n \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_n \end{bmatrix} \begin{bmatrix} e_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_n \end{bmatrix}$$

where $U = diag(u_1, \cdots, u_n) \in UC(\mathbb{U}_n(R))$ and $E = diag(e_1, \cdots, e_n) \in Id(\mathbb{U}_n(R))$. Hence $diag(a_{11}, \cdots, a_{nn}) \in QId(\mathbb{U}_n(R))$ □

Theorem 3.9 R *is almost quasi-clean ring if and only if* $\mathbb{U}_n(R)$ *(resp.* $\mathbb{L}_n(R)$*) is almost quasi-clean.*

Proof Without lose generality we do upper triangular case. Note that $A \in \mathbb{U}_n(R))$ is a regular element if and only if for $i = 1, ...n$, $a_{ii} \in Reg(R)$.
(:$\Rightarrow$) Suppose that R is almost quasi-clean ring. Let $A = [a_{ij}] \in \mathbb{U}_n(R)$. Since R is almost quasi-clean, we can write $a_{ii} = r_i + q_i$ where $r_i \in Reg(R)$ and $q_i \in QId(R)$. Put,

$$R = \begin{bmatrix} r_1 & a_{12} & \cdots & a_{1n} \\ 0 & r_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix} \text{ and } Q = \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_n \end{bmatrix}.$$

Then $A = R + Q$ where $R \in Reg(\mathbb{U}_n(R))$ and $Q \in QId(\mathbb{U}_n(R))$ by proposition 3.8.

($\Leftarrow$:) Let $\mathbb{U}_n(R)$ be almost quasi-clean and $a \in R$. Then $\begin{bmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in U_n(R)$ can be written $\begin{bmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} =$ $R + Q$ where $R = [r_{ij}] \in Reg(\mathbb{U}_n(R))$ and $Q = [q_{ij}] \in OId(\mathbb{U}_n(R))$. Then, $a = r_{11} + q_{11}$ where $r_{11} \in Reg(R)$ and $q_{11} \in QId(R)$. Hence R is almost quasi-clean. □

Theorem 3.10 *Let R be an elementary divisor ring, which is every matrix over R is a diagonal reduction, and $\mathbb{M}_n(R)$ the ring of $n \times n$ matrices over R. If R is almost quasi-clean, then $\mathbb{M}_n(R)$ is almost quasi-clean for all $n \in \mathbb{N}$.*

Proof Let $M \in \mathbb{M}_n(R)$. Then we have some invertible $P, Q \in \mathbb{M}_n(R)$ such that $PMQ = diag(d_1, \cdots, d_n)$. Because the R is almost quasi-clean, for each i, $d_i = r_i + q_i$ where $r_i \in Reg(R)$ and $q_i \in QId(R)$. Hence,

$$PMQ = diag(r_1 + q_1, \cdots r_n + q_n) = diag(r_1, \cdots, r_n) + diag(q_1, \cdots, q_n).$$

By Proposition 3.8, $diag(r_1, \cdots, r_n) \in Reg(\mathbb{M}_n(R))$ and $diag(q_1, \cdots, q_n) \in QId(\mathbb{M}_n(R))$. Therefore, $\mathbb{M}_n(R)$ is almost quasi-clean. □

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