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Unitals in projective planes of order 16

Stoicho Dimitrov STOICHEV[1], **Mustafa GEZEK**[2,*]
[1]Department of Computer Systems, Technical University of Sofia, Sofia, Bulgaria
[2]Department of Mathematics, Namık Kemal University, Tekirdağ, Turkey

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Abstract: In this study, we perform computer searches for unitals in planes of order 16. The number of known nonisomorphic unitals in these planes is improved to be 261. Some data related to 2-$(65,5,1)$ designs associated with unitals are given. New lower bounds on the number of unital designs in projective planes of order 16 and 2-$(65,5,1)$ designs are established. The computations show that thirty-nine unitals can be embedded in two or more nonisomorphic projective planes of order 16. Fifteen new connections between planes of order 16 (based on unitals) are found. All unitals found by the algorithms used in this study are explicitly listed.

Key words: Unital, projective plane, Steiner design, graph isomorphism, orbits, automorphism group

1. Introduction

We assume familiarity with the basic facts from combinatorial design theory and finite geometries [5, 9, 16].

A t-(v,k,λ) design (t-design) is a pair $D=\{X,B\}$ of a set X of cardinality v, called points, and a collection B of k-subsets of X, called blocks, such that every t points appear together in exactly λ blocks. A 2-design with $\lambda=1$ is called a Steiner design.

The incidence matrix of a 2-(v,k,λ) design D is a matrix $M=(m_{ij})$ with rows labeled by the blocks of D, columns labeled by the points of D, where $m_{i,j}=1$ if the ith block contains the jth point and $m_{i,j}=0$ otherwise. For a prime p, the rank of the incidence matrix of design D over a finite field of characteristic p is called the p-rank of D.

Two designs D and D' are called isomorphic if there is a bijection between their point sets that maps every block of D to a block of D'. An isomorphism of a design D with itself is called an automorphism of D.

The dual design of a design D, denoted by $D^{\perp}$, has point set as the block set of D, and the block set as the point set of D. If the number of points of D is equal to the number of blocks of D, then D is called a symmetric design. A symmetric design D is self-dual if D and $D^{\perp}$ are isomorphic.

A parallel clas of a design D is a collection of blocks that partitions the point set of D. A partial parallel class of D is a set of blocks that contain no point of D more than once. A resolution of D is a partition of the collection of blocks of D into disjoint parallel classes. A design D is resolvable if it has at least one resolution.

Point sets in projective planes with two line intersections have been studied in finite geometry a lot. Maximal arcs [11–13, 15, 26] and unitals [1–4, 6, 7, 10, 14, 17–19, 26] are examples of such sets.

*Correspondence: mgezek@nku.edu.tr
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A unital in a projective plane Π of order q^2 is a set U of q^3+1 points that meets every line of the plane in either one or $q+1$ points. If U is a unital in Π, the lines that meet U in one point form a unital $U^\perp$ in $\Pi^\perp$. The point set of a unital with the line intersections of size $q+1$ form a $2-(q^3+1, q+1, 1)$ design, called unital design associated with U.

A classical example of a unital is the Hermitian unital $H(q)$ defined by the absolute points of a unitary polarity in the Desarguesian plane of order q^2. Since $PG(2,q^2)$ admits unitary polarities for each prime power q, unitals exists in these planes. Unitals also exist for $q=6$ [1, 20]. For more on unitals we refer to the book [4].

In this study, the results of computer searches for unitals in planes of order 16 are reported. The point sets of all known unitals in planes of order 16 (including new ones found with our algorithms) are listed explicitly (previously only 77 of them were published [18, 19, 27]).

In Section 2, we provide a brief history of unitals and unitals in planes of order 16 are discussed in details. We provide the details of the algorithms we used. The number of known unitals in these planes with their automorphism group orders are listed.

In Section 3, details of 2-$(65,5,1)$ designs associated with new sets of unitals are provided: the orders of the automorphism group, the 5-rank and the number of parallel classes of designs associated with unitals and dual unitals and the resolvability of the unital designs are computed. The lower bounds on (1) the number of nonisomorphic unital designs embedded in planes of order 16, (2) the number of resolvable 2-$(65,5,1)$ designs and (3) the number of known 2-$(65,5,1)$ designs are improved.

In Section 4, we show that some sets of unitals are unitals in more than one plane. This observation provides new links between planes of order 16.

In the Appendix, we list explicitly point sets of all unitals found with our algorithms.

2. Unitals in planes of order 16

Thirteen projective planes of order 16 are known to exist, four of which are self-dual, and the rest have nonisomorphic duals. The names of the planes are in accordance with [25]: BBH1, BBH2, BBS4, DEMP, DSFP, HALL, JOHN, JOWK, LMRH, MATH, SEMI2, SEMI4 and PG(2,16).

The only projective plane of order 16 for which all inequivalent unitals have been completely classified is the plane PG(2,16) [3]. There are exactly two inequivalent unitals in PG(2,16), with automorphism groups of order 768 and 249,600. Unitals were found in all known planes of order 16 [26, 27] and they have not been classified completely in any of the remaining twenty one planes.

Stoichev and Tonchev [27] found 38 unitals in the known planes of order 16, which are explicitly listed in their paper. In 2011, Krčadinac and Smoljak [19] claimed that they found three more unitals, one in LMRH plane with group order 16 and two in HALL plane with group orders 16 and 80, respectively. Isomorphism checks were performed between these unitals with the ones we found. It is observed that isomorphic copies of these unitals are the unital 2 in LMRH plane, the unitals 5 and 6 in HALL planes, respectively.

The authors developed two different computer programs independently. The algorithms used in this paper found all previously known unitals in the known projective planes of order 16 [2, 8, 10, 18, 19, 22, 27] as well as some new unitals.

Details of the first author's algorithm, it is referred to as Algorithm 1 through the paper, can be found in [26]. We provide the details of the second author's algorithm, it is referred to as Algorithm 2 through the paper. The search was restricted to unitals that are invariant under the subgroups of order 12, 16, and 20 of the collineation groups of the underlying planes. This algorithm is based on the following steps:

Step 1: Find the automorphism group G of plane P.

Step 2: Find all conjugacy classes of subgroups of G of order $t \in \{12, 16, 20\}$.

Step 3: For each class, find orbits.

Step 4: For each set of orbits, find all possible union of orbits having length 65.

Step 5: Save the 65-length sets meeting every line of the plane P in one or five points.

We give a small example to show how to find a unital in BBH2 plane with Algorithm 2 described above:

Example 2.1 *Let P be BBH2 plane and G be its automorphism group. Our goal is to find the unital 1 of P. Let H be a subgroup of G of order 20. A set of generating permutations of H and its orbits on the set of points of P are given in Tables 1 and 2, respectively. One can easily check that the unital 1 of P comes from the union of orbit numbers 6, 17, 19, 22, and 25 given in Table 2.*

Algorithm 2 assumes the presence of a nontrivial group of automorphisms. For each automorphism groups of planes of order 16, the subgroups of order 16 and 20 can be handled computationally completely (by Algorithm 2). We classify all solutions for these subgroups (up to isomorphism). We would like to make this assumed group of automorphisms small but, obviously, computational complexity makes it difficult to go below a certain value. For example, subgroups of order 12 also were checked, but the classification of the solutions was not computationally possible at the time of writing this paper. Future improvements could look into smaller assumed automorphism groups. Algorithm 2 was able to find all known unitals of group order ≥ 12 (including the new ones given in this study).

Algorithm 1 is heuristic (approximate) and it is faster and more powerful than Algorithm 2 (Algorithm 1 finds the unitals in the MATH plane on a single laptop* around 106,655 s, while Algorithm 2 finds the same unitals on a single laptop† around 1,183,750 s). Subgroups of order at least 4 were considered (subgroups of this order could not be handled by Algorithm 2), and the search was not exhaustive. Algorithm 1 was able to find all known unitals (including the new ones given in this paper).

Some of the unitals found with our algorithms have been considered earlier in [26], but they are listed here for the first time. There are also some mistakes in [26] that are corrected here: In BBH2 plane, [26] states that the number of unitals with group orders 80, 32 and 16 are at least 2, 4 and 2, respectively, but Algorithm 2 finds that these numbers are actually 1, 2, and 1, respectively (and this is confirmed by Algorithm 1). Since the classification is complete for these subgroups (by Algorithm 2), we conclude that the numbers given in [26] for BBH2 plane for these orders are not correct. [26] also states that BBH2 plane has at least 7 unitals with group order 4, but Algorithm 1 finds only 5 unitals with this group order.

The specific line sets of the projective planes of order 16 that we are using in this paper are from [12]‡

Table 3 gives the number of unitals we found and the ones already known in planes of order 16. Column 1 lists the names of the planes, Columns 2–17 give the number of known unitals with the specific automorphism group orders, and the last column shows the total number of known unitals in each plane.

*Dell, Intel(R) Core(TM)i7-682HQ CPU@2.7GHz, Memory 32 GB, OS:64 bit Microsoft Windows10 Professional.
†MacBook Pro, Intel(R) Core i7 CPU@2.9GHz, Memory 16 GB, macOS High Sierra.
‡Available online at https://www.combinatorics.org/ojs/index.php/eljc/article/view/v27i1p62/8052

Table 1. Generating permutations of H.

g_1	(1 74 117 215 258)(2 115 76 271 220)(3 214 119 261 66)(4 78 259 125 219)(5 257 16 6 10)(7 213 270 72 126)(8 77 209 267 123)(9 118 269 224 65)(11 260 218 121 70)(12 221 73 114 272)(13 273 122 69 212)(14 128 210 71 268)(15 266 75 223 116)(17 83 25 32 23)(18 53 103 97 64)(19 246 39 38 254)(20 242 44 43 247)(21 164 87 82 170)(22 182 166 161 189)(24 54 110 107 58)(26 132 204 202 144)(27 134 194 193 133)(29 90 188 179 96)(30 155 234 240 150)(31 147 229 238 146)(33 231 156 173 245)(34 253 93 61 99)(35 241 162 152 230)(36 56 163 252 112)(37 207 243 91 142)(40 130 89 250 208)(41 102 244 176 63)(42 100 51 84 248)(45 249 186 143 206)(46 160 177 251 239)(47 227 255 190 145)(48 199 138 185 256)(49 86 154 98 237)(50 101 200 139 167)(52 183 157 235 104)(55 196 109 135 191)(57 225 105 148 95)(59 180 141 106 201)(60 108 228 153 184)(62 165 140 198 111)(67 79 80 262 68)(81 151 236 203 136)(85 129 195 233 159)(113 120 263 124 127)(131 232 197 149 169)(137 171 158 205 226)(211 265 222 217 216)
g_2	(1 214)(3 74)(4 218)(5 263)(6 127)(7 72)(8 223)(9 14)(10 124)(11 259)(12 73)(13 273)(15 267)(16 113)(17 23)(18 144)(19 146)(20 150)(21 170)(22 189)(24 133)(26 64)(27 58)(29 96)(30 247)(31 254)(32 83)(34 151)(35 152)(36 131)(37 98)(38 147)(39 229)(40 57)(41 205)(42 195)(43 155)(44 234)(45 157)(46 251)(48 108)(49 91)(50 200)(51 85)(52 186)(53 202)(54 193)(55 109)(56 169)(59 180)(60 199)(61 203)(62 165)(63 226)(65 128)(66 117)(67 217)(68 216)(70 125)(71 269)(75 77)(76 271)(78 260)(79 222)(80 265)(81 253)(82 164)(84 159)(86 243)(88 187)(89 148)(90 179)(92 181)(93 136)(95 130)(97 132)(99 236)(100 129)(102 158)(103 204)(104 143)(105 250)(107 134)(110 194)(111 140)(112 232)(114 272)(115 220)(116 123)(118 268)(119 258)(120 257)(121 219)(122 212)(135 191)(137 176)(138 184)(139 167)(141 201)(142 237)(145 227)(149 163)(153 185)(154 207)(156 173)(160 177)(161 182)(162 241)(168 172)(171 244)(183 249)(190 255)(192 264)(197 252)(206 235)(208 225)(209 266)(210 224)(211 262)(213 270)(215 261)(228 256)(231 245)(233 248)(238 246)(240 242)
g_3	(1 125)(2 126)(3 121)(4 117)(5 120)(6 127)(7 115)(8 128)(9 116)(10 113)(11 119)(12 122)(13 114)(14 123)(15 118)(16 124)(18 26)(19 31)(20 30)(24 27)(33 230)(34 233)(35 231)(36 226)(37 225)(38 238)(39 229)(40 237)(41 232)(42 236)(43 240)(44 234)(45 228)(46 227)(47 239)(48 235)(49 130)(50 140)(51 136)(52 138)(53 132)(54 134)(55 141)(56 137)(57 142)(58 133)(59 135)(60 143)(61 129)(62 139)(63 131)(64 144)(65 223)(66 218)(67 217)(68 222)(69 221)(70 214)(71 209)(72 220)(73 212)(74 219)(75 224)(76 213)(77 210)(78 215)(79 216)(80 211)(81 84)(85 93)(86 89)(88 181)(91 95)(92 187)(94 178)(97 202)(98 208)(99 195)(100 203)(101 198)(102 197)(103 204)(104 199)(105 207)(106 196)(107 193)(108 206)(109 201)(110 194)(111 200)(112 205)(145 251)(146 254)(147 246)(148 243)(149 244)(150 247)(151 248)(152 245)(153 249)(154 250)(155 242)(156 241)(157 256)(158 252)(159 253)(160 255)(162 173)(163 171)(165 167)(168 264)(169 176)(172 192)(174 175)(177 190)(180 191)(183 185)(184 186)(257 263)(258 259)(260 261)(262 265)(266 269)(267 268)(270 271)(272 273)

3. Unital designs

Table 4 contains some data related to unital designs found with our algorithms. The numbering of unitals are as continuation of the ones given in [27], that is, if a plane's unital number starts with n, the details of the $n-1$ unitals in this plane are given in [27]. Column 3 lists the order of the automorphism group of each unital, Columns 4 and 5 give the 3-rank and 5-rank of each design, respectively. Columns 6 and 7 provide the number of parallel classes and the number of partial parallel classes of each design of unitals and designs of the dual

Table 2. Orbits of H.

Orbit no.	Elements
1	28
2	94 178
3	174 175
4	88 187 181 92
5	168 172 264 192
6	17 83 23 25 32
7	21 164 170 87 82
8	22 182 189 166 161
9	29 90 96 188 179
10	2 115 126 76 220 7 271 213 72 270
11	5 257 263 120 16 124 6 113 127 10
12	12 221 73 122 69 114 212 272 13 273
13	18 53 144 26 103 202 132 64 97 204
14	19 246 146 31 39 238 147 254 38 229
15	20 242 150 30 44 240 155 247 43 234
16	24 54 133 27 110 193 134 58 107 194
17	33 231 230 156 245 35 173 241 152 162
18	46 160 251 227 177 255 239 145 190 47
19	50 101 200 140 198 139 111 167 62 165
20	55 196 109 141 106 135 201 191 59 180
21	67 79 217 80 222 216 262 265 211 68
22	1 74 214 125 117 3 219 119 70 215 66 4 121 261 258 11 78 218 260 259
23	8 77 223 128 209 75 210 116 65 267 266 71 224 15 123 9 268 269 118 14
24	34 253 151 233 93 81 159 236 248 61 136 85 84 203 99 42 129 51 100 195
25	36 56 131 226 163 169 137 232 63 252 149 171 176 197 112 41 158 244 102 205
26	37 207 98 225 243 154 105 237 208 91 86 148 250 49 142 40 95 89 130 57
27	45 249 157 228 186 183 153 235 256 143 52 184 185 104 206 48 60 138 199 108

unitals, respectively, and Column 8 gives if the unital design is isomorphic to any other known unital designs, including designs of dual unitals. The last column contains information about the resolvability of the design.

The total number of designs given in Table 4 is 118, and they are pairwise nonisomorphic. Considering the designs of dual unitals, we see that the number of unital designs embedded in planes of order 16 is improved by 184 with the results given in Table 4 (excluding unitals in HALL and LMRH planes, since they were already known before [19]). Since previously known number of unital designs embedded in planes of order 16 was 77 [18, 19, 27], we see that the number of nonisomorphic unital designs embedded in projective planes of order 16 is at least 261.

Table 3. Number of unitals in projective planes of order 16.

Plane \\$|Aut(U)|$	4	8	10	12	16	20	24	32	48	64	80	100	128	192	1200	768	249600	TOTAL
BBH1	3	7			4			2										16
BBH2	5	8	1		8	1		2			1							26
BBS4	5	5		2			1											13
DEMP				1	1		1		1									4
DSFP				1			1											2
HALL					1			1	1		1	1			1			6
LMRH					1			1										2
JOHN	6	15			4		1	2	1									29
JOWK	1	1		1	1		1	1	1									7
MATH					6			1		3			6					16
SEMI2					1			2	3	12				3				21
SEMI4		4		2					1	1			3	1				12
PG(2,16)																1	1	2

Table 4. Designs associated with unitals in planes of order 16.

Plane	Unital no.	$\|Aut(U)\|$	3−rank	5−rank	Par. clas. of $D(U)/D(U^{\perp})$	Part. par. clas. of $D(U)/D(U^{\perp})$	Isomorphic to	Resolvable
BBH1	4	4	65	62	12/12	379/379	BBH1.4$^{\perp}$	No
	5	4	65	63	34/34	686/688	BBH1.6$^{\perp}$	No
	6	4	65	63	34/34	688/686	BBH1.5$^{\perp}$	No
	7	8	65	63	34/34	680/696	BBH1.3$^{\perp}$	No
	8	8	65	63	50/54	904/980	BBH1.10$^{\perp}$	No
	9	8	65	63	46/46	840/840	BBH1.9$^{\perp}$	No
	10	8	65	63	54/50	980/904	BBH1.8$^{\perp}$	No
	11	8	65	63	34/30	680/668	BBH1.12$^{\perp}$	No
	12	8	65	63	30/34	668/680	BBH1.11$^{\perp}$	No
	13	16	65	63	34/34	684/684	BBH1.15$^{\perp}$	No
	14	16	65	62	90/138	1428/2124	BBH1.2$^{\perp}$	Yes
	15	16	65	63	34/34	684/684	BBH1.13$^{\perp}$	No
	16	32	65	64	24/24	568/568	BBH1.16$^{\perp}$	Yes
BBH2	7	4	65	62	12/11	383/373		No
	8	4	65	63	34/34	698/706		No
	9	4	65	63	34/34	690/694		No
	10	4	65	63	46/50	856/908		No
	11	8	65	63	34/36	684/712		No
	12	8	65	63	30/34	628/676		No
	13	8	65	63	38/34	738/696		No
	14	8	65	63	34/38	692/756		No

Table 4. (Continued).

Plane	Unital no.	$\lvert Aut(U)\rvert$	3−rank	5−rank	Par. clas. of $D(U)/D(U^{\perp})$	Part. par. clas. of $D(U)/D(U^{\perp})$	Isomorphic to	Resolvable
	15	8	65	63	38/42	724/784		No
	16	8	65	63	38/42	746/784		No
	17	8	65	63	30/34	624/688		No
	18	10	65	62	14/14	426/421		No
	19	16	65	64	28/32	676/672		Yes
	20	16	65	64	36/24	824/610		Yes
	21	16	65	64	32/36	696/730		Yes
	22	16	65	64	36/36	840/750		Yes
	23	16	65	64	28/32	620/666		Yes
	24	16	65	63	38/34	768/708		No
	25	16	65	63	50/62	896/1064		No
	26	32	65	64	48/40	880/788		Yes
DEMP	3	12	65	64	4/4	263/263		No
	4	48	65	64	10/10	404/404		No
HALL	5	16	65	64	20/16	500/452		Yes
	6	80	65	60	224/184	3556/2796		Yes
JOWK	5	12	65	64	4/4	299/263		No
	6	16	65	63	38/46	748/844		No
	7	32	65	64	16/16	480/448		Yes
LMRH	2	16	65	64	16/16	456/456		Yes
MATH	5	16	65	64	32/28	688/608		Yes
	6	16	65	64	0/0	208/212		No
	7	16	65	64	0/0	212/208		No
	8	16	65	64	0/0	212/212		No
	9	16	65	64	22/22	540/532		No
	10	64	65	64	16/16	416/448		Yes
	11	64	65	64	16/16	416/416		Yes
	12	128	65	64	16/16	448/448		Yes
	13	128	65	64	80/80	1296/1328		Yes
	14	128	65	64	16/16	528/496		Yes
	15	128	65	64	16/16	480/464		Yes
	16	128	65	64	16/16	480/464		Yes
SEMI2	4	16	65	64	4/4.	260/260	SEMI2.$4^{\perp}$	No
	5	32	65	64	16/16	424/424	SEMI2.$6^{\perp}$	Yes
	6	32	65	64	16/16	424/424	SEMI2.$5^{\perp}$	Yes
	7	48	65	64	4/4	260/260	SEMI2.$8^{\perp}$	No

Table 4. (Continued).

Plane	Unital no.	$\lvert Aut(U)\rvert$	3−rank	5−rank	Par. clas. of $D(U)/D(U^{\perp})$	Part. par. clas. of $D(U)/D(U^{\perp})$	Isomorphic to	Resolvable
	8	48	65	64	4/4	260/260	SEMI2.7$^{\perp}$	No
	9	64	65	64	16/16	416/416	SEMI2.9$^{\perp}$	Yes
	10	64	65	64	32/32	688/656	SEMI2.2$^{\perp}$	Yes
	11	64	65	64	16/16	416/416	SEMI2.11$^{\perp}$	Yes
	12	64	65	64	16/16	416/416	SEMI2.13$^{\perp}$	Yes
	13	64	65	64	16/16	416/416	SEMI2.12$^{\perp}$	Yes
	14	64	65	64	16/16	416/416	SEMI2.18$^{\perp}$	Yes
	15	64	65	64	16/16	416/416	SEMI2.19$^{\perp}$	Yes
	16	64	65	64	16/16	480/480	SEMI2.16$^{\perp}$	Yes
	17	64	65	64	16/16	416/416	SEMI2.17$^{\perp}$	Yes
	18	64	65	64	16/16	416/416	SEMI2.14$^{\perp}$	Yes
	19	64	65	64	16/16	416/416	SEMI2.15$^{\perp}$	Yes
	20	192	65	64	16/16	416/416	SEMI2.21$^{\perp}$	Yes
	21	192	65	64	16/16	416/416	SEMI2.20$^{\perp}$	Yes
SEMI4	2	8	65	64	0/0	224/224	SEMI4.2$^{\perp}$	No
	3	8	65	64	14/14	424/424	SEMI4.3$^{\perp}$	No
	4	8	65	64	24/20	554/502	SEMI4.1$^{\perp}$	Yes
	5	12	65	64	16/16	440/440	SEMI4.6$^{\perp}$	Yes
	6	12	65	64	16/16	440/440	SEMI4.5$^{\perp}$	Yes
	7	48	65	64	4/4	260/260	SEMI4.7$^{\perp}$	No
	8	64	65	64	16/16	448/448	SEMI4.8$^{\perp}$	Yes
	9	128	65	64	16/16	608/608	SEMI4.11$^{\perp}$	Yes
	10	128	65	64	16/16	512/512	SEMI4.10$^{\perp}$	Yes
	11	128	65	64	16/16	608/608	SEMI4.9$^{\perp}$	Yes
	12	192	65	64	16/16	416/416	SEMI4.12$^{\perp}$	Yes
BBS4	2	4	65	63	34/34	688/690		No
	3	4	65	63	34/34	696/700		No
	4	4	65	63	38/38	742/740		No
	5	4	65	63	34/34	688/692		No
	6	4	65	63	38/38	734/738		No
	7	8	65	63	40/42	784/796		No
	8	8	65	63	38/42	772/792		No
	9	8	65	63	34/38	736/740		No
	10	8	65	63	30/34	672/708		No
	11	12	65	64	4/4	266/260		No
	12	12	65	63	34/34	668/668		No

Table 4. (Continued).

Plane	Unital no.	$\lvert Aut(U)\rvert$	3−rank	5−rank	Par. clas. of $D(U)/D(U^{\perp})$	Part. par. clas. of $D(U)/D(U^{\perp})$	Isomorphic to	Resolvable
	13	24	65	63	38/34	760/692		No
JOHN	6	4	65	63	34/34	694/690		No
	7	4	65	63	34/34	686/686		No
	8	4	65	63	34/34	684/684		No
	9	4	65	63	34/34	680/694		No
	10	4	65	63	38/38	742/736		No
	11	4	65	63	38/38	744/740		No
	12	8	65	63	38/42	734/772		No
	13	8	65	63	38/38	754/728		No
	14	8	65	63	30/34	664/692		No
	15	8	65	63	38/38	740/736		No
	16	8	65	63	42/38	800/744		No
	17	8	65	63	30/34	674/704		No
	18	8	65	63	38/42	760/784		No
	19	8	65	63	34/34	744/752		No
	20	8	65	63	30/34	672/708		No
	21	8	65	63	38/38	752/724		No
	22	8	65	64	38/34	764/732		No
	23	8	65	63	38/38	744/744		No
	24	8	65	64	34/34	700/700		No
	25	8	65	64	30/30	648/648		No
	26	16	65	64	28/24	664/552		Yes
	27	16	65	63	34/42	680/776		No
	28	16	65	63	46/42	864/784		No
	29	32	65	64	16/16	592/496		Yes

In [18], a combined method (using the Kramer–Mesner method and the method of tactical decomposition) of construction of 2-$(65, 5, 1)$ designs was introduced. The total number of such designs given in [18] is 1777, including the unital designs given in [27]. Isomorphisms between the designs we have in Table 4 and 1777 designs in [18] was checked, we have the following results:

- The design of unital 11 of SEMI2 plane is isomorphic to the design 1774 given in [18].
- The designs of unital 12, 13, 14, 15, and 16 of MATH plane are isomorphic to the designs 1603, 1688, 1648, 1744, and 1698 given in [18], respectively.
- The designs of the dual unitals of unital 12, 13, 14, 15, and 16 of MATH plane are isomorphic to the designs 1604, 1745, 1763, 1709, and 1686 given in [18], respectively.

- The designs of unital 9, 10, and 11 of SEMI4 plane are isomorphic to the designs 1656, 1773, and 1662 given in [18], respectively.

These observations show that fourteen of the 2-$(65,5,1)$ designs found by the method given in [18] has also embedded in projective planes of order 16, and many of the designs we provide in this paper are new. This shows that the number of nonisomorphic 2-$(65,5,1)$ designs is at least 1947.

The number of resolvable 2-(65,5,1) designs embedded in projective planes of order 16 in [27] was 25. From the designs presented in Table 4, we have 62 more resolvable 2-(65,5,1) designs (including the resolvable designs from the dual unitals), so in total, there are 87 resolvable 2-(65,5,1) designs embedded in projective planes of order 16. 216 of the 1777 designs given in [18, design 1467 and designs 1563-1777] are also resolvable, of which 14 are already counted in Table 4. This shows that the number of resolvable 2-(65,5,1) designs is at least 289.

We can summarize the results obtained in this section by the following theorem:

Theorem 3.1
- *The number of nonisomorphic unital designs embedded in projective planes of order 16 is* $\geq$ *261.*
- *The number of resolvable 2-(65,5,1) designs is* $\geq$ *289.*
- *The number of nonisomorphic 2-*$(65,5,1)$ *designs is* $\geq$ *1947.*

4. New links between planes of order 16

In [7], 2-$(28,4,1)$ designs embedded in the projective planes of order 9 were studied. Later, Penttila and Royle enumerated all unitals in these planes and they showed that there are exactly 17 unitals (up to isomorphism) [24]. Some other studies of 2-$(28,4,1)$ designs can be found in [6, 17, 21, 23].

Previously, it was known that some unital 2-(28,4,1) designs can be embedded in a plane of order 9 and its dual, and that plane was not self-dual [7]. In [14], it was shown that for every prime power q, a unital of order q can be constructed such that it can be embedded both in the HALL plane and the dual of the HALL plane of order q^2.

In [18, 19], it was reported that the unital 1 in SEMI4 plane and the unital 4 in HALL plane represent unitals in both planes, and in [19], dual unital of unital 2 in SEMI2 plane represents a unital in both SEMI2$^\perp$ (which is expected, since SEMI2 is a self-dual plane) and LMRH$^\perp$ planes.

Our computations show that there are more sets of unitals in planes of order 16 which are unitals in more than one plane, given in Table 5. Column 1 lists the planes where there are unitals in several planes, Column 2 presents the unital numbers for such sets. The last column contains which planes share the unital given in Column 2.

In Column 2 of Table 5, a unital number with $(')$ and/or $('')$ implies that there are isomorphic copies of this unital which are unitals in different sets of planes. For example, three isomorphic copies of the unital 5 of SEMI2, labeled by 5, $5'$ and $5''$ in Table 5, are unital in different sets of planes: the unital 5 is a unital in SEMI2 and JOWK$^\perp$, the unital $5'$ is a unital in SEMI2 and SEMI4, and the unital $5''$ is a unital in SEMI2, LMRH, and LMRH$^\perp$. The point sets of these isomorphic copies are also given in the Appendix with the same labeling. Some of the isomorphic copies of the dual unitals are also unitals in more than one plane, which are labeled with $(^{\perp})$ in Table 5, and the point sets of these isomorphic copies are given in the Appendix as well. This observation shows that there are more unitals in planes of order 16 which can be embedded in two or more different planes.

Table 5. Unitals in more than one plane.

Plane	Unital no.	Also a unital in
DEMP	$(1^{\perp})'$	HALL
	$(2^{\perp})'$	JOWK
	4	HALL$^{\perp}$
DSFP	1	JOWK
HALL	$(1^{\perp})'$, $(3^{\perp})'$	DEMP
	3	DEMP$^{\perp}$
	4	SEMI4, JOWK, HALL$^{\perp}$,JOWK$^{\perp}$
	$4^{\perp}$	HALL, JOWK, JOWK$^{\perp}$, SEMI4
	$(4^{\perp})'$	LMRH, LMRH$^{\perp}$
	5,6	SEMI4
LMRH	1	LMRH$^{\perp}$, HALL$^{\perp}$, JOWK$^{\perp}$
	$1'$,2	LMRH$^{\perp}$, SEMI2, SEMI4
	$1^{\perp}$	LMRH, HALL$^{\perp}$, JOWK$^{\perp}$
	$(1^{\perp})'$, $(2^{\perp})'$	LMRH, SEMI2, SEMI4
	$2'$	LMRH$^{\perp}$, HALL$^{\perp}$
	$(2^{\perp})''$	LMRH, HALL$^{\perp}$
JOWK	2	DSFP
	6	DEMP$^{\perp}$
	$(6^{\perp})'$	HALL$^{\perp}$
	7	SEMI4, HALL, HALL$^{\perp}$, JOWK$^{\perp}$
	$7^{\perp}$	JOWK, HALL, HALL$^{\perp}$
	$(7^{\perp})'$	LMRH, LMRH$^{\perp}$
	$(7^{\perp})''$	SEMI2
SEMI2	2,9,10,13-19	SEMI4
	5	JOWK$^{\perp}$
	$5'$	SEMI4
	$5''$,12	LMRH, LMRH$^{\perp}$
	6	SEMI4, LMRH, LMRH$^{\perp}$
SEMI4	1	HALL, JOWK
	5,11	HALL
	8	SEMI2, LMRH, LMRH$^{\perp}$
	$8'$	JOWK$^{\perp}$
	10	SEMI2
	$10'$	LMRH, LMRH$^{\perp}$
	12	HALL, HALL$^{\perp}$, JOWK, JOWK$^{\perp}$

Table 5 shows that the follwing sets of planes share a unital: DEMP plane with HALL$^{\perp}$ plane, DEMP$^{\perp}$ plane with HALL and JOWK planes, DSFP plane with JOWK plane, HALL plane with DEMP$^{\perp}$, SEMI4,

$\mathrm{HALL}^{\perp}$, JOWK, and $\mathrm{JOWK}^{\perp}$ planes, $\mathrm{HALL}^{\perp}$ plane with DEMP, HALL, JOWK, $\mathrm{JOWK}^{\perp}$, SEMI4, LMRH, and $\mathrm{LMRH}^{\perp}$ planes, LMRH plane with $\mathrm{LMRH}^{\perp}$, SEMI2, SEMI4, $\mathrm{HALL}^{\perp}$, and $\mathrm{JOWK}^{\perp}$ planes, $\mathrm{LMRH}^{\perp}$ plane with LMRH, $\mathrm{HALL}^{\perp}$, $\mathrm{JOWK}^{\perp}$, SEMI2, and SEMI4 planes, JOWK plane with $\mathrm{DEMP}^{\perp}$, SEMI4, HALL, $\mathrm{HALL}^{\perp}$, and DSFP planes, $\mathrm{JOWK}^{\perp}$ plane with JOWK, HALL, $\mathrm{HALL}^{\perp}$, LMRH, $\mathrm{LMRH}^{\perp}$, and SEMI2 planes, SEMI2 plane with SEMI4, LMRH, $\mathrm{LMRH}^{\perp}$, and $\mathrm{JOWK}^{\perp}$ planes, SEMI4 plane with SEMI2, LMRH, $\mathrm{LMRH}^{\perp}$, HALL, $\mathrm{HALL}^{\perp}$, JOWK, and $\mathrm{JOWK}^{\perp}$ planes. Table 5 shows that there are at least thirty-nine unitals in projective planes of order 16 which can be embedded in two or more different planes.

All previously known connections between planes of order 16 were reported in [12]. Table 6 lists all previously known connections [12] and some new connections given in this section. Rows (and columns) are labeled by the known projective planes of order 16.

Entries 1 and 2 indicate that the corresponding planes are connected by derivation [15] and superderivation, respectively. Entries 3–5 indicate that the planes share a semibiplane, a 2-$(52,4,1)$ design associated with a maximal $(52,4)$-arc [13], and a binary linear code of length 52 generated by nonisomorphic designs associated with maximal arcs in the corresponding planes [12], respectively. Entry 6 indicates a new connection between a pair of nonisomorphic planes that share a unital. All connections given in Table 6 are symmetric.

Table 6. Links between planes of order 16.

	PG(2,16)	DEMP	SEMI4	SEMI2	LMRH	MATH	HALL	BBH1	JOWK	JOHN	DSFP	BBH2	BBS4
PG(2,16)							1						
DEMP				1,5	5	5	5,**6**		3,**6**	5	2,5		
SEMI4			2	3,5,**6**	1,**6**		**6**		1,**6**	1	1		1
SEMI2		1,5	3,5,**6**		5,**6**	5			**6**	5			
LMRH		5	1,**6**	5,**6**	4,5,**6**	5	**6**		2,**6**	5	3		
MATH		5		5	5	5	5		2,5	5			
HALL	1	5,**6**	**6**		**6**	5	**6**	1	**6**	1		1	
BBH1							1			4,5			
JOWK		3,**6**	1,**6**	**6**	2,**6**	2,5	**6**		**6**	4,5	**6**		
JOHN		5	1	5	5	5	1	4,5	4,5				
DSFP		2,5	1		3				**6**				
BBH2							1						
BBS4			1										

Table 6 shows that the fifteen new links found in this section provides new connections between seven pairs of the planes where previously no other connections were known to exist.

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A. Point sets of unitals

We list point sets of all unitals (since some isomorphic copies of unitals given in [27] are also unital in different planes, we include isomorphic copies of these sets as well) found by our algorithms in Tables A1 and A2.

Table A1. Unitals in planes of order 16.

Plane	Unital no.	Unital
BBH1	1	5 11 13 15 18 20 26 30 33 39 42 48 49 56 59 64 66 68 72 75 83 88 92 94 97 102 105 109 122 123 124 125 129 134 140 143 148 149 150 160 162 165 167 173 179 182 188 189 194 195 198 206 209 211 213 215 226 227 237 240 245 246 248 256 257
	2	3 19 24 26 28 30 36 37 39 41 43 49 53 55 59 62 70 84 88 90 92 96 102 103 105 107 112 122 129 134 136 139 144 146 151 152 155 159 161 166 168 171 175 177 180 184 186 192 193 198 199 206 208 213 228 229 231 235 237 245 247 250 251 255 267
	3	4 7 9 10 18 23 25 26 33 39 41 47 52 54 63 64 70 73 77 79 91 92 94 95 99 106 108 112 122 123 124 125 132 139 140 144 150 154 157 159 168 169 172 176 180 187 191 192 195 196 205 206 229 230 232 239 247 248 251 254 257 270 271 272 273
	4	3 6 12 13 16 25 38 51 53 58 59 63 65 66 68 77 79 82 83 85 92 93 101 103 105 108 112 114 115 116 120 125 141 153 167 168 171 173 174 185 195 197 202 204 206 211 212 215 217 223 226 230 232 237 239 242 249 251 254 256 258 259 267 269 271
	5	2 7 15 16 17 23 25 32 34 36 39 44 53 54 58 59 70 72 74 78 81 84 85 93 98 101 104 109 113 114 125 127 129 136 141 144 149 152 154 156 179 182 188 189 193 195 196 207 210 211 223 224 231 232 235 240 243 249 252 256 257 262 263 264 265
	6	11 20 22 24 27 30 33 34 38 39 47 49 53 56 57 58 70 83 87 89 90 94 97 98 105 107 112 121 136 138 140 141 143 145 147 149 155 160 162 167 169 171 175 178 179 181 186 189 198 199 201 203 206 218 227 229 232 239 240 245 246 253 254 255 260
	7	1 2 3 6 8 17 18 20 23 24 33 35 36 45 46 57 71 72 76 78 80 83 85 89 92 95 100 102 107 108 109 114 118 119 120 123 141 158 162 163 172 174 176 177 181 185 191 192 193 195 202 203 207 217 219 221 222 223 228 229 231 237 240 242 262
	8	4 7 9 10 18 23 27 28 36 42 45 47 50 52 56 57 82 92 93 95 106 107 109 110 114 118 119 120 134 137 139 141 149 151 158 159 161 172 173 176 179 183 191 192 196 198 203 207 217 219 221 223 231 235 239 240 242 244 254 256 257 262 263 264 265
	9	4 7 9 10 18 20 21 22 34 36 45 46 50 56 61 64 66 68 72 75 87 89 95 96 102 107 109 111 114 118 119 120 134 137 143 144 149 150 151 152 163 164 170 176 177 182 188 191 199 202 203 207 217 219 221 223 231 237 238 240 244 246 247 256 257
	10	12 17 18 23 26 28 33 36 39 41 43 49 50 53 54 63 72 82 83 91 92 95 99 105 108 109 112 122 130 131 132 134 140 145 148 151 154 157 161 169 172 174 176 179 180 181 191 192 195 201 205 206 207 220 226 229 230 232 240 243 248 250 251 256 263
	11	18 20 28 29 33 35 36 41 52 59 62 64 66 68 72 75 82 89 90 95 97 98 101 107 114 118 119 120 132 140 141 143 152 154 157 158 165 167 168 174 181 184 186 187 193 195 205 207 209 211 213 215 225 227 231 236 247 253 254 255 257 266 267 268 269

Table A1. (Continued)

Plane	Unital no.	Unital
	12	6 7 8 10 16 19 24 26 27 31 36 49 53 55 58 59 65 67 70 72 77 81 84 85 91 93 109 113 115 121 125 126 132 136 137 141 144 149 150 154 156 158 161 164 166 169 170 179 180 182 188 189 207 210 214 219 221 224 240 243 249 250 252 255 265
	13	1 2 4 15 18 20 26 30 33 37 38 43 49 55 58 60 67 68 76 77 82 87 90 96 99 102 104 106 113 120 122 128 129 131 139 144 147 153 154 160 183 184 185 186 194 200 204 207 212 215 219 224 227 232 233 235 242 244 245 253 257 262 263 264 265
	14	8 10 11 14 18 20 26 30 40 42 44 45 53 56 61 62 72 73 74 80 82 87 90 96 101 107 109 112 114 117 124 126 133 135 142 143 150 152 158 159 183 184 185 186 202 203 206 208 209 210 217 218 226 229 231 239 243 246 249 251 257 262 263 264 265
	15	8 19 21 24 26 30 37 40 41 46 48 49 52 57 59 63 77 85 88 90 92 96 98 102 104 105 106 123 135 136 139 140 143 145 151 152 153 156 161 163 166 168 175 177 180 184 186 187 196 198 200 203 206 209 227 228 230 231 236 243 244 250 251 256 267
	16	2 3 6 8 17 23 25 32 37 41 44 47 51 52 58 62 70 73 77 79 82 87 90 96 97 102 105 109 114 118 119 120 129 134 140 143 146 153 156 159 161 164 166 170 178 181 187 192 197 201 202 207 218 220 222 224 225 232 234 239 245 246 248 256 257
BBH2	1	1 3 4 11 17 23 25 32 33 35 36 41 50 56 62 63 66 70 74 78 83 101 102 111 112 117 119 121 125 131 137 139 140 149 152 156 158 162 163 165 167 169 171 173 176 197 198 200 205 214 215 218 219 226 230 231 232 241 244 245 252 258 259 260 261
	2	2 3 5 6 7 8 9 12 14 15 18 21 28 34 41 52 71 72 73 75 77 80 83 87 89 90 104 106 113 114 117 119 121 122 123 126 127 128 139 159 161 164 169 182 189 190 193 208 213 214 215 216 220 224 231 234 254 260 261 262 265 267 268 272 273
	3	10 22 26 27 29 31 34 35 37 39 44 50 51 56 59 62 66 84 89 91 101 105 106 109 110 126 132 134 138 142 143 148 153 158 159 160 164 165 166 172 173 174 177 178 187 188 190 192 196 199 203 207 208 211 226 227 228 230 236 242 243 245 246 252 272
	4	8 12 15 17 19 22 23 25 26 32 33 35 43 51 53 55 56 57 60 63 68 74 80 83 84 87 89 97 108 111 119 123 126 129 130 138 147 156 160 170 171 173 182 185 191 195 200 202 213 217 218 225 229 231 232 234 239 240 245 247 254 259 265 266 272
	5	5 7 8 11 13 15 16 17 22 23 24 25 26 32 36 37 39 51 56 63 68 74 79 83 85 95 96 100 101 104 107 109 111 112 116 124 126 131 138 144 148 156 158 162 163 166 179 185 190 195 196 207 214 217 222 234 235 240 252 254 255 259 263 267 273
	6	6 8 10 13 15 24 26 28 29 31 48 49 50 55 61 64 65 70 72 73 75 85 87 90 93 97 101 109 110 111 114 121 122 124 127 129 130 137 141 142 146 170 174 186 198 202 205 206 207 209 214 215 216 222 237 242 245 247 250 256 258 261 269 270 271
	7	2 4 8 10 33 37 38 43 52 57 62 63 71 72 75 76 84 87 91 92 98 100 103 104 113 114 116 120 122 124 127 128 134 136 138 140 148 152 154 156 168 169 172 174 176 177 181 191 193 196 203 204 209 214 217 222 242 246 252 256 258 259 263 270 272

Table A1. (Continued)

Plane	Unital no.	Unital
	8	6 10 12 15 17 18 20 23 25 26 30 32 34 40 47 48 56 61 62 63 66 70 74 80 82 83 87 90 96 101 102 103 107 116 119 124 128 132 136 140 144 147 148 153 154 194 200 201 208 212 213 221 224 228 232 236 240 251 254 255 256 266 267 268 269
	9	6 10 11 12 17 21 22 23 25 28 29 32 33 34 44 46 51 54 56 63 66 70 74 76 83 97 106 107 108 116 118 124 128 132 139 143 144 146 148 151 154 163 169 171 176 195 200 205 207 211 212 221 224 225 226 228 229 245 249 251 255 258 259 260 261
	10	4 7 9 10 17 19 23 24 25 27 31 32 33 34 35 38 39 42 45 48 49 51 52 56 58 59 62 64 71 76 78 80 83 121 126 127 128 145 147 151 152 154 155 157 158 161 163 164 166 169 170 171 176 196 197 201 202 204 205 207 208 218 220 222 224
	11	3 9 16 17 18 22 23 24 25 32 39 44 46 51 59 63 65 68 70 83 85 91 96 100 101 102 104 106 107 111 114 115 118 119 120 124 127 130 131 138 150 156 158 162 166 171 177 179 185 195 202 205 216 221 222 225 235 237 244 252 254 260 265 269 271
	12	4 10 11 13 18 22 28 30 32 44 47 60 61 71 73 77 80 86 90 95 96 98 99 101 108 110 112 114 115 116 119 129 143 146 151 153 156 157 159 163 164 166 169 177 179 180 182 205 208 209 211 214 216 218 220 222 224 234 239 248 256 257 260 266 273
	13	6 8 26 28 29 30 32 33 35 38 39 42 48 49 54 59 60 69 74 86 87 91 96 107 111 121 122 123 124 125 126 129 130 133 135 136 138 139 140 147 148 157 160 163 164 170 176 177 179 189 191 199 203 217 223 228 233 243 245 248 255 259 262 268 271
	14	1 11 13 16 21 22 24 25 28 29 34 35 39 40 42 62 63 72 75 76 78 81 99 102 104 107 110 118 120 122 123 132 136 153 154 163 169 171 173 176 177 179 180 182 186 188 189 190 191 198 200 201 203 206 209 211 218 222 228 230 235 238 240 251 254
	15	7 9 21 22 28 29 31 32 39 42 44 52 53 54 55 56 62 73 77 82 84 86 87 89 90 91 95 96 99 106 110 113 117 131 133 136 142 143 144 145 148 151 153 155 156 167 177 180 186 190 191 198 205 206 214 216 225 230 235 245 250 252 253 254 255
	16	1 9 10 11 14 15 17 21 22 28 29 35 38 51 53 62 63 68 70 72 73 76 80 100 104 113 116 118 119 123 125 131 132 136 139 145 146 153 154 177 180 183 184 185 186 190 191 199 203 211 212 213 214 220 222 231 238 249 250 251 254 266 267 268 269
	17	1 9 10 11 15 16 19 32 34 35 47 55 56 58 61 62 63 70 72 73 75 76 80 81 103 104 107 113 116 118 119 122 123 132 136 138 140 142 144 147 148 153 154 155 159 165 184 200 203 208 211 212 213 214 218 222 228 232 238 241 251 252 254 255 256
	18	2 4 6 11 12 13 16 19 21 29 35 43 45 60 61 62 66 67 70 71 72 77 78 87 94 101 106 108 115 116 118 129 134 144 153 154 156 162 168 173 175 181 182 183 184 188 190 194 198 200 221 222 223 228 233 235 246 249 253 263 264 265 269 270 273
	19	5 7 9 15 18 34 41 50 52 53 55 60 71 73 77 80 83 89 97 102 104 105 106 109 113 114 117 119 132 139 140 141 143 145 148 149 150 152 155 157 159 160 169 190 193 197 201 202 207 208 213 214 215 216 231 234 242 243 244 245 247 251 254 255 256

Table A1. (Continued)

Plane	Unital no.	Unital
	20	1 14 19 26 30 31 37 42 44 48 53 63 66 75 81 83 84 86 91 97 98 101 102 103 105 109 110 123 125 131 132 148 149 150 155 158 160 162 163 173 176 177 183 185 191 194 196 198 205 217 218 219 221 223 224 232 235 236 239 242 252 257 259 269 270
	21	4 5 7 13 18 20 28 29 41 47 50 61 71 77 78 79 82 83 89 90 95 97 102 105 106 109 112 114 115 117 120 129 140 148 149 150 151 152 160 164 169 170 171 180 182 188 190 204 208 209 210 215 216 217 219 221 223 225 234 245 248 257 259 269 270
	22	19 21 22 31 35 36 38 40 42 43 46 48 51 54 60 62 81 83 84 87 96 101 107 110 111 121 126 127 128 130 133 134 135 137 138 143 144 146 147 153 157 161 162 166 173 179 183 185 189 194 198 199 203 228 233 235 236 241 246 253 256 260 262 267 272
	23	9 10 11 12 15 16 34 45 52 55 57 58 68 70 72 73 76 80 83 97 100 102 104 105 109 113 116 118 119 122 124 136 139 141 142 145 148 149 150 154 156 159 160 193 200 211 212 213 214 217 219 220 221 222 223 231 238 249 250 251 254 257 259 269 270
	24	1 6 7 13 17 23 25 32 35 37 43 48 53 58 59 60 66 67 73 76 81 83 84 85 86 89 91 93 95 97 98 108 111 114 115 125 127 132 133 135 143 146 155 157 160 200 202 206 208 212 213 217 218 225 231 235 240 242 254 255 256 270 271 272 273
	25	3 5 16 17 18 22 23 25 31 32 35 42 43 59 61 62 65 71 72 81 83 86 87 97 98 103 106 110 111 112 114 125 127 131 132 133 141 142 143 144 149 152 159 162 163 170 180 182 183 198 200 202 215 221 224 231 235 240 243 246 249 260 263 268 270
	26	1 16 18 19 22 24 29 33 34 41 48 52 66 68 82 83 84 89 93 96 97 101 102 104 105 106 108 109 124 125 139 148 149 150 159 160 166 167 169 170 173 179 184 185 188 190 193 194 195 197 201 202 207 208 218 222 229 231 234 236 243 244 247 254 255
BBS4	1	7 10 21 28 33 48 51 60 62 66 69 76 78 79 82 85 91 92 95 99 101 104 105 110 115 116 120 121 126 129 130 138 139 143 147 148 153 157 159 161 164 168 169 170 178 182 184 187 188 196 197 202 210 212 220 225 235 238 243 249 251 257 261 272 273
	2	8 10 12 20 27 28 34 40 43 50 53 57 60 61 63 64 69 73 74 75 85 89 93 96 98 105 106 112 113 115 119 124 133 138 139 142 152 157 159 160 161 169 173 175 177 178 184 189 209 210 213 214 219 220 223 224 242 245 252 255 259 262 266 269 271
	3	2 8 10 26 27 28 34 43 46 50 53 54 57 60 61 63 68 73 76 80 84 94 95 96 98 102 104 109 119 121 127 128 129 132 133 136 149 150 157 158 165 167 169 171 177 183 188 190 209 210 213 214 219 220 223 224 228 231 234 237 258 259 264 265 267
	4	14 15 17 29 44 46 50 53 55 60 63 64 72 74 80 84 87 89 90 93 95 96 97 103 108 114 116 120 141 142 144 145 146 148 151 152 154 157 164 171 172 183 184 188 193 194 206 210 211 213 218 220 223 224 228 238 239 242 246 253 259 261 263 265 266
	5	2 3 20 32 35 37 49 50 53 58 60 63 65 66 69 72 76 77 79 88 92 96 102 106 108 115 116 127 129 131 141 146 148 151 153 154 157 160 165 171 173 181 184 186 193 204 206 210 211 212 213 220 223 224 228 232 239 242 253 256 259 260 267 269 273

Table A1. (Continued)

Plane	Unital no.	Unital
	6	12 14 17 26 41 44 50 53 55 59 60 63 66 69 71 73 76 79 80 81 82 89 101 103 112 114 121 125 129 131 135 148 151 153 154 156 157 160 161 165 173 181 186 190 194 206 208 210 213 218 220 222 223 224 237 238 239 246 253 255 259 260 267 269 273
	7	2 3 12 14 17 20 26 32 37 40 44 46 54 55 58 63 64 66 74 79 85 87 90 99 110 112 115 118 123 129 136 139 143 150 151 152 158 163 165 167 173 178 183 187 188 209 210 213 214 219 220 223 224 226 229 236 239 241 243 246 248 249 251 254 256
	8	3 5 12 14 17 20 29 32 37 40 41 44 49 55 58 64 73 75 78 79 80 87 89 91 94 96 98 99 101 106 109 116 119 123 124 127 139 152 173 178 194 210 212 213 220 223 227 228 231 234 237 241 243 248 249 254 257 258 259 261 264 265 267 272 273
	9	2 3 12 14 17 20 26 32 37 40 44 46 54 55 58 63 64 65 67 68 73 80 83 86 94 95 96 98 100 102 108 109 114 119 121 125 127 133 136 150 157 165 169 177 183 210 212 213 215 218 220 221 223 243 248 249 254 258 262 264 265 266 267 269 271
	10	5 8 14 17 27 29 37 41 47 49 50 52 58 66 67 72 73 75 78 79 84 89 93 98 99 101 106 108 110 111 113 124 128 132 135 138 141 179 184 185 190 194 196 198 202 203 210 211 220 221 222 225 226 228 229 232 235 236 237 239 243 245 249 251 252
	11	2 5 8 17 18 21 41 46 47 50 53 57 58 60 62 63 66 68 74 76 78 80 86 88 97 99 101 103 105 107 125 127 130 134 137 140 145 149 158 159 162 163 169 173 181 184 186 190 195 198 202 204 205 207 211 214 225 227 230 232 236 237 252 253 259
	12	5 8 10 17 20 31 33 46 47 50 53 56 58 59 60 63 69 71 72 80 81 82 84 89 103 110 111 112 114 121 122 123 129 133 135 137 146 148 152 160 162 170 174 176 183 185 187 191 194 203 205 206 215 219 220 222 228 229 230 233 243 244 245 256 259
	13	4 5 9 10 12 14 20 21 27 29 31 32 33 34 35 37 43 46 50 52 54 55 56 59 62 66 68 71 82 85 87 105 107 112 121 126 128 133 137 143 148 154 160 162 168 174 177 183 187 200 201 202 209 223 224 225 226 239 247 248 250 263 264 271 273
DEMP	1	16 17 20 22 23 25 34 38 40 44 47 50 51 57 58 59 80 85 87 88 91 94 100 101 106 109 111 114 115 121 122 123 133 135 136 139 142 145 147 156 157 158 162 166 168 172 175 192 196 197 202 205 207 224 225 227 236 237 238 241 244 246 247 249 257
	2	1 2 4 10 21 24 31 32 35 43 44 46 51 54 55 60 72 75 78 80 86 87 89 93 99 105 108 109 120 123 126 128 131 133 140 143 153 155 157 158 165 166 167 175 182 183 187 190 195 197 204 207 213 214 215 223 232 233 237 240 248 249 253 256 273
	3	1 2 8 10 13 17 18 24 26 29 46 56 57 58 63 64 69 70 71 75 76 94 110 117 121 124 127 128 129 130 131 132 141 152 153 154 159 160 161 162 163 164 173 190 195 196 198 199 203 211 212 214 215 219 229 233 236 239 240 245 246 247 251 252 257
	4	11 27 33 36 40 46 47 59 66 67 68 70 77 81 84 88 94 95 98 101 106 110 112 113 119 122 124 125 131 135 136 137 144 155 163 167 168 169 176 178 181 186 190 192 197 198 201 204 207 213 214 217 220 223 225 231 234 236 237 242 243 244 246 253 257

Table A1. (Continued)

Plane	Unital no.	Unital
DSFP	1	1 2 8 9 14 17 18 25 30 32 34 39 43 44 45 52 57 58 62 63 71 84 87 91 92 93 97 99 102 107 109 124 131 134 135 140 141 148 149 153 154 158 161 162 164 169 170 179 182 186 187 189 195 198 199 203 204 209 210 212 218 222 227 246 266
	2	25 27 29 30 34 41 44 47 53 56 63 64 70 72 74 75 82 85 87 94 98 102 104 109 117 119 121 122 131 132 136 139 147 150 151 156 164 165 169 172 178 179 187 192 196 199 206 207 212 214 221 224 227 234 237 240 250 252 254 255 257 258 260 266 273
HALL	1	1 7 9 15 18 19 21 25 39 40 42 46 52 54 56 57 68 76 77 79 81 85 92 94 99 100 110 112 114 119 125 128 153 154 156 160 161 162 164 170 178 182 190 191 195 198 199 204 213 216 223 224 229 230 234 237 241 243 248 253 257 258 260 266 273
	2	3 8 12 14 15 17 19 21 22 31 37 38 40 43 46 54 65 74 77 78 79 83 85 87 93 94 102 103 104 109 111 117 119 122 124 127 130 131 138 140 141 152 161 186 193 195 199 202 208 220 225 228 230 231 238 245 248 249 252 253 261 262 263 265 272
	3	21 24 31 32 34 44 46 48 49 50 52 58 68 70 77 80 81 91 92 95 104 106 108 109 114 117 118 123 130 135 136 137 153 155 157 158 164 167 174 175 180 181 185 188 199 202 203 208 209 213 215 221 230 233 234 239 241 246 248 254 257 258 260 266 273
	4	5 8 15 16 25 27 29 30 37 40 47 48 53 56 63 64 69 72 79 80 85 88 95 96 99 102 103 108 121 123 125 126 137 139 141 142 153 155 157 158 163 166 167 172 179 182 183 188 201 203 205 206 211 214 215 220 225 226 228 234 243 246 247 252 273
	5	6 7 9 14 16 17 21 25 29 32 35 51 53 54 56 62 67 69 75 76 77 85 87 88 89 92 100 104 107 110 111 115 118 125 127 128 135 139 140 143 144 151 166 179 185 188 190 191 200 201 202 205 207 214 215 216 219 221 236 242 245 251 254 256 264
	6	2 6 9 10 11 20 21 25 29 32 38 40 45 46 48 63 65 66 70 71 72 83 84 87 88 95 102 103 107 109 111 114 115 120 122 128 130 132 135 137 140 146 148 155 159 160 164 168 170 171 173 187 195 211 215 217 218 221 234 243 246 249 255 256 263
PG(2,16)	1	1 2 5 10 15 21 22 24 28 38 46 47 49 50 55 60 65 68 69 71 75 84 88 96 97 99 103 111 112 115 116 118 122 137 138 140 144 147 149 153 161 167 168 170 174 178 180 185 190 194 198 203 208 211 219 220 222 226 229 234 239 248 249 251 255
	2	1 19 27 28 32 41 44 46 49 56 58 64 65 68 71 74 80 82 84 89 92 98 100 109 110 114 119 125 129 130 131 136 141 151 152 157 158 162 167 169 172 184 186 190 192 195 201 207 208 212 216 221 225 227 230 234 236 243 245 249 250 260 263 270 273
LMRH	1	1 2 4 10 17 18 20 26 33 34 36 42 57 59 61 62 67 70 71 76 83 86 87 92 97 98 100 106 121 123 125 126 133 136 143 144 153 155 157 158 163 166 167 172 179 182 183 188 193 194 196 202 217 219 221 222 227 230 231 236 249 251 253 254 273
	1’	4 7 10 12 20 23 26 28 40 45 46 47 56 61 62 63 65 66 67 70 88 93 94 95 101 105 107 112 117 121 123 128 132 135 138 140 149 153 155 160 168 173 174 175 184 189 190 191 197 201 203 208 212 215 218 220 228 231 234 236 245 249 251 256 273

Table A1. (Continued)

Plane	Unital no.	Unital
	2	4 7 10 12 20 23 26 28 33 34 35 38 56 61 62 63 72 77 78 79 81 82 83 86 104 109 110 111 116 119 122 124 136 141 142 143 148 151 154 156 161 162 163 166 180 183 186 188 200 205 206 207 213 217 219 224 225 226 227 230 241 242 243 246 273
	2’	3 6 7 12 21 24 31 32 37 40 47 48 49 50 52 58 65 66 68 74 85 88 95 96 99 102 103 108 117 120 127 128 131 134 135 140 153 155 157 158 161 162 164 170 179 182 183 188 193 194 196 202 211 214 215 220 225 226 228 234 245 248 255 256 273
JOWK	1	1 2 6 8 11 17 18 20 22 24 36 41 42 43 47 51 57 59 63 64 66 81 82 86 93 94 109 115 116 121 122 123 131 132 133 138 143 147 153 155 156 159 163 164 167 170 175 177 198 200 202 205 206 214 216 217 221 222 225 226 232 237 238 254 269
	2	4 8 9 10 16 20 21 26 29 31 33 34 35 43 46 52 53 58 59 63 65 73 75 77 78 81 82 89 92 93 97 98 102 105 109 114 121 123 125 126 136 148 152 154 158 160 176 177 178 183 187 190 197 223 228 229 232 239 240 245 248 250 255 256 266
	3	2 4 8 16 21 22 24 28 34 35 41 44 51 52 54 58 69 71 77 78 91 92 94 95 102 105 107 111 114 115 118 126 132 133 135 139 147 151 159 160 184 186 190 192 194 202 203 205 212 220 221 223 229 233 234 240 247 248 249 253 257 258 259 262 273
	4	4 5 7 8 16 30 39 51 52 54 60 63 67 70 73 78 79 87 88 89 94 96 112 117 119 121 126 128 132 133 135 136 144 149 150 152 153 156 164 165 166 168 172 179 185 188 190 191 193 194 202 203 205 211 212 214 220 223 239 243 258 261 265 267 272
	5	3 5 8 12 13 21 22 27 28 32 37 41 43 46 47 51 57 60 63 64 67 71 76 77 78 81 82 85 88 93 101 107 109 110 111 115 118 121 123 124 135 147 151 152 158 159 167 180 185 186 191 192 198 214 231 232 233 238 240 246 248 251 253 256 260
	6	3 6 7 12 25 27 29 30 35 39 43 45 53 56 63 64 73 78 79 80 86 91 92 93 99 103 105 110 123 125 127 128 133 134 136 140 145 146 148 154 163 165 167 168 182 185 188 190 198 204 207 208 211 215 223 224 229 232 233 238 245 248 251 253 273
	7	5 8 15 16 25 27 29 30 33 34 36 42 53 56 63 64 67 70 71 76 81 82 84 90 105 107 109 110 113 114 116 122 133 136 143 144 153 155 157 158 161 162 164 170 185 187 189 190 197 200 207 208 209 210 212 218 233 235 237 238 245 248 255 256 273
MATH	1	5 9 10 13 16 29 36 40 41 42 43 50 53 55 57 63 72 83 84 85 88 96 100 102 107 110 112 114 115 116 121 125 129 135 137 139 144 149 151 152 154 158 163 178 179 183 184 187 195 199 205 206 208 210 212 213 220 222 226 234 235 237 238 250 268
	2	1 5 6 10 16 18 21 23 24 26 47 50 51 54 58 60 67 75 77 78 80 83 85 92 95 96 97 99 106 107 111 118 134 136 140 141 144 145 168 169 170 171 176 178 184 187 188 191 204 209 213 216 221 223 225 226 230 235 237 242 243 244 245 253 268
	3	2 3 5 9 13 17 18 27 30 32 33 37 39 40 48 49 72 73 77 79 80 89 97 101 106 107 109 115 118 120 121 122 130 134 138 141 144 145 147 152 157 158 163 170 171 172 174 190 194 195 196 198 203 214 226 229 232 234 238 245 246 249 251 256 268

Table A1. (Continued)

Plane	Unital no.	Unital
	4	5 6 9 10 16 17 26 27 30 32 36 40 41 43 48 56 65 68 72 73 78 87 88 90 92 93 97 100 102 103 104 122 129 131 132 134 135 157 163 166 167 169 174 179 180 186 189 191 195 199 203 206 208 211 225 226 235 237 238 246 249 251 253 256 268
	5	1 3 4 6 9 17 21 24 28 31 33 35 39 47 48 64 66 67 68 69 74 82 86 87 91 96 98 100 104 111 112 127 131 133 135 139 140 156 162 165 167 168 173 177 180 181 187 192 196 198 200 203 204 219 225 230 231 232 238 242 243 246 252 255 268
	6	4 8 10 12 16 21 22 26 27 32 34 36 37 38 41 50 51 58 60 63 66 67 69 79 80 81 83 85 89 93 97 99 102 106 107 116 121 123 127 128 130 150 161 163 164 174 176 177 185 187 188 191 207 219 225 226 228 230 236 245 247 249 250 252 268
	7	2 5 10 15 16 20 21 22 28 31 45 55 65 73 74 78 79 83 84 85 88 91 99 105 107 110 111 113 115 117 120 121 135 157 161 164 168 170 171 177 180 186 187 190 199 200 203 205 207 209 213 215 221 222 227 231 233 234 238 243 244 248 249 253 268
	8	3 5 9 12 15 20 22 23 28 32 39 40 41 46 48 53 66 67 71 76 77 94 98 99 100 105 107 114 120 123 127 128 133 135 136 142 144 147 149 155 156 160 164 165 168 170 175 185 195 196 201 203 206 210 226 232 236 238 239 241 244 247 251 255 268
	9	15 20 26 28 31 32 34 36 39 42 45 49 53 57 62 63 65 66 68 70 77 81 83 84 90 94 101 117 118 121 126 127 130 145 146 153 156 158 161 165 166 167 170 182 183 184 185 189 196 197 199 202 205 210 214 217 220 222 236 247 251 252 253 255 268
	10	2 9 11 12 16 21 23 25 26 31 33 37 42 45 48 53 65 74 75 76 79 86 88 89 90 96 98 102 105 110 111 118 129 133 137 140 142 145 166 172 173 175 176 177 179 187 189 190 194 198 202 203 205 210 229 235 238 239 240 242 244 252 253 254 268
	11	1 2 6 9 16 26 35 37 46 47 48 50 58 59 62 64 65 66 69 74 79 89 100 102 109 111 112 113 121 124 125 127 129 135 138 139 140 150 154 156 158 159 162 165 166 172 173 190 194 200 201 203 204 213 217 219 221 224 225 229 230 235 238 253 268
	12	1 7 8 9 13 17 21 22 24 26 33 35 41 44 46 55 67 76 77 79 80 84 88 92 93 94 110 113 116 119 122 124 137 148 150 151 157 159 168 170 171 172 175 179 183 185 186 191 198 200 201 206 207 212 227 228 230 234 238 241 242 243 246 253 268
	13	1 4 5 9 11 32 33 38 39 40 41 50 53 54 57 63 69 82 84 89 92 96 100 102 107 111 112 116 124 125 126 127 129 135 139 140 144 151 153 154 156 159 162 181 183 187 190 191 193 195 196 198 206 209 210 213 220 222 226 230 231 238 240 251 268
	14	5 7 11 12 14 21 33 38 39 41 45 49 50 52 53 60 80 81 82 87 94 96 97 105 107 108 112 116 120 124 126 127 134 139 142 143 144 147 151 153 156 159 171 180 181 182 185 187 193 196 198 202 206 210 213 214 215 223 226 228 233 239 240 242 268
	15	3 11 14 15 16 19 20 26 27 29 35 38 41 45 47 56 68 76 77 79 80 83 84 89 92 94 100 101 106 110 112 119 130 135 137 139 141 148 167 170 171 172 175 183 184 185 190 191 193 200 202 204 206 211 232 233 235 236 240 247 248 250 253 256 268

Table A1. (Continued)

Plane	Unital no.	Unital
	16	3 5 14 15 16 18 26 27 30 32 33 34 38 41 48 58 68 70 77 79 80 81 89 92 93 95 97 98 101 106 111 121 130 133 134 140 141 158 161 167 170 171 172 182 186 188 190 191 193 197 198 203 206 221 226 232 233 235 236 245 249 251 253 256 268
SEMI2	1	3 10 13 16 17 24 25 28 35 42 45 48 50 55 59 63 65 72 73 76 84 85 86 94 98 103 107 111 114 119 123 127 131 138 141 144 147 154 157 160 162 167 171 175 178 183 187 191 193 200 201 204 211 218 221 224 225 232 233 236 241 248 249 252 273
	2	1 9 10 13 18 20 27 30 34 36 43 46 51 56 60 64 67 72 76 80 83 88 92 96 97 105 106 109 113 121 122 125 133 134 135 143 146 148 155 158 162 164 171 174 178 180 187 190 193 201 202 205 209 217 218 221 227 232 236 240 243 248 252 256 273
	3	1 6 8 14 23 26 27 32 34 38 46 47 49 56 57 60 70 74 78 80 81 83 88 93 97 103 104 107 114 118 126 127 130 135 139 143 145 152 153 156 162 167 171 175 178 185 188 191 199 202 203 208 212 213 217 220 233 234 236 240 246 250 254 256 273
	4	3 11 12 14 17 21 22 25 35 42 43 47 50 51 59 64 70 71 73 77 81 82 85 96 97 98 101 112 116 120 124 126 131 134 137 139 147 155 156 158 162 167 173 176 178 183 189 192 199 204 205 206 209 213 220 222 230 231 233 237 241 245 246 249 273
	5	4 9 10 11 20 25 26 27 35 38 40 47 51 54 56 63 65 66 77 78 85 87 92 96 99 102 104 111 116 121 122 123 132 137 138 139 148 153 154 155 163 166 168 175 177 178 189 190 193 194 205 206 209 210 221 222 225 226 237 238 243 246 248 255 273
	5’	5 7 12 16 17 18 29 30 36 41 42 43 49 50 61 62 65 66 77 78 84 89 90 91 97 98 109 110 115 118 120 127 129 130 141 142 148 153 154 155 164 169 170 171 179 182 184 191 196 201 202 203 211 214 216 223 227 230 232 239 243 246 248 255 273
	5”	3 6 8 15 19 22 24 31 35 38 40 47 53 55 60 64 67 70 72 79 84 89 90 91 100 105 106 107 115 118 120 127 133 135 140 144 148 153 154 155 164 169 170 171 177 178 189 190 197 199 204 208 213 215 220 224 228 233 234 235 245 247 252 256 273
	6	1 2 3 6 20 23 26 28 37 41 43 48 52 55 58 60 72 77 78 79 88 93 94 95 101 105 107 112 117 121 123 128 132 135 138 140 149 153 155 160 168 173 174 175 180 183 186 188 200 205 206 207 216 221 222 223 229 233 235 240 244 247 250 252 273
	7	1 6 8 14 20 21 23 27 35 42 45 48 50 57 60 63 66 73 76 79 81 87 88 91 100 101 103 107 116 117 121 124 129 134 136 142 145 152 153 156 164 165 167 171 177 182 184 190 196 197 198 206 210 217 220 223 226 230 238 239 242 247 251 255 273
	8	4 5 9 12 25 26 28 32 38 42 46 48 49 51 56 61 68 69 73 76 84 85 86 94 99 106 109 112 114 118 126 127 131 138 141 144 151 154 155 160 162 169 172 175 178 185 188 191 196 197 198 206 211 212 213 221 226 227 237 239 242 246 254 255 273
	9	5 6 7 15 18 20 27 30 37 38 39 47 49 57 58 61 69 70 71 79 83 88 92 96 99 104 108 112 113 121 122 125 129 137 138 141 149 150 151 159 163 168 172 176 179 184 188 192 197 198 199 207 209 217 218 221 225 233 234 237 243 248 252 256 273

Table A1. (Continued)

Plane	Unital no.	Unital
	10	1 4 5 8 22 25 28 30 33 36 37 40 54 57 60 62 66 74 79 80 86 89 92 94 98 106 111 112 115 119 123 125 129 132 133 136 146 154 159 160 161 164 165 168 182 185 188 190 194 202 207 208 209 212 213 216 230 233 236 238 242 250 255 256 273
	11	2 4 8 16 18 20 24 32 35 43 44 46 50 52 56 64 67 75 76 78 81 85 90 95 98 100 104 112 113 117 122 127 131 139 140 142 147 155 156 158 163 171 172 174 177 181 186 191 198 199 201 205 209 213 218 223 226 228 232 240 241 245 250 255 273
	12	8 13 14 15 24 29 30 31 36 39 42 44 53 57 59 64 68 71 74 76 84 87 90 92 100 103 106 108 117 121 123 128 133 137 139 144 145 146 147 150 164 167 170 172 184 189 190 191 200 205 206 207 213 217 219 224 232 237 238 239 245 249 251 256 273
	13	5 7 12 16 21 23 28 32 35 38 40 47 53 55 60 64 69 71 76 80 81 82 93 94 101 103 108 112 113 114 125 126 132 137 138 139 147 150 152 159 161 162 173 174 177 178 189 190 195 198 200 207 211 214 216 223 227 230 232 239 241 242 253 254 273
	14	2 4 11 14 18 20 27 30 34 36 43 46 51 56 60 64 67 72 76 80 81 89 90 93 98 100 107 110 114 116 123 126 129 137 138 141 145 153 154 157 161 169 170 173 179 184 188 192 193 201 202 205 211 216 220 224 227 232 236 240 245 246 247 255 273
	15	5 6 7 15 19 24 28 32 33 41 42 45 53 54 55 63 66 68 75 78 82 84 91 94 101 102 103 111 114 116 123 126 129 137 138 141 145 153 154 157 161 169 170 173 177 185 186 189 197 198 199 207 210 212 219 222 229 230 231 239 242 244 251 254 273
	16	3 8 12 16 18 20 27 30 33 41 42 45 50 52 59 62 66 68 75 78 83 88 92 96 98 100 107 110 115 120 124 128 133 134 135 143 149 150 151 159 163 168 172 176 179 184 188 192 197 198 199 207 213 214 215 223 229 230 231 239 242 244 251 254 273
	17	2 4 11 14 19 24 28 32 34 36 43 46 51 56 60 64 65 73 74 77 83 88 92 96 97 105 106 109 117 118 119 127 130 132 139 142 145 153 154 157 162 164 171 174 179 184 188 192 193 201 202 205 210 212 219 222 227 232 236 240 241 249 250 253 273
	18	3 8 12 16 21 22 23 31 34 36 43 46 49 57 58 61 66 68 75 78 85 86 87 95 99 104 108 112 117 118 119 127 130 132 139 142 147 152 156 160 162 164 171 174 181 182 183 191 195 200 204 208 210 212 219 222 229 230 231 239 243 248 252 256 273
	19	3 8 12 16 17 25 26 29 33 41 42 45 53 54 55 63 67 72 76 80 81 89 90 93 97 105 106 109 117 118 119 127 130 132 139 142 147 152 156 160 165 166 167 175 181 182 183 191 197 198 199 207 211 216 220 224 225 233 234 237 243 248 252 256 273
	20	2 7 11 15 17 24 25 28 36 37 38 46 52 53 54 62 68 69 70 78 82 87 91 95 98 103 107 111 116 117 118 126 129 136 137 140 148 149 150 158 161 168 169 172 179 186 189 192 193 200 201 204 210 215 219 223 226 231 235 239 241 248 249 252 273
	21	3 10 13 16 20 21 22 30 35 42 45 48 49 56 57 60 66 71 75 79 82 87 91 95 97 104 105 108 115 122 125 128 129 136 137 140 147 154 157 160 163 170 173 176 177 184 185 188 194 199 203 207 209 216 217 220 226 231 235 239 242 247 251 255 273

Table A1. (Continued)

Plane	Unital no.	Unital
SEMI4	1	3 4 7 8 15 19 20 23 24 31 37 42 43 46 48 51 52 55 56 63 65 70 73 76 77 85 90 91 94 96 101 106 107 110 112 113 118 121 124 125 130 147 148 151 152 159 162 181 186 187 190 192 194 210 225 230 233 236 237 241 246 249 252 253 257
	2	3 11 12 14 18 21 24 26 33 37 42 47 50 54 56 61 65 69 74 79 82 85 88 90 97 99 110 111 118 123 124 125 133 135 137 138 146 150 152 157 163 171 172 174 177 179 190 191 198 203 204 205 212 219 220 224 226 227 232 238 241 246 253 255 273
	3	2 3 13 15 20 23 25 32 36 38 40 41 49 50 61 62 69 70 72 76 82 85 92 93 97 106 107 110 113 119 126 128 129 130 141 142 149 151 156 160 165 167 172 176 178 180 185 189 193 198 200 206 212 215 217 224 228 230 232 233 245 246 248 252 273
	4	18 21 28 29 37 40 47 48 52 56 60 62 65 67 72 77 84 86 93 96 97 107 108 111 113 114 116 122 129 135 142 144 147 149 154 158 166 168 170 171 178 182 190 191 195 198 199 204 212 213 215 219 226 227 235 240 247 250 253 255 257 263 270 272 273
	5	6 17 18 25 28 29 35 37 40 43 46 51 53 56 59 62 68 71 74 79 80 84 87 90 95 96 102 113 114 121 124 125 134 148 151 154 159 160 161 162 169 172 173 177 178 185 188 189 196 199 202 207 208 211 213 216 219 222 230 243 245 248 251 254 257
	6	3 5 10 14 37 40 47 48 50 51 59 64 67 70 71 76 82 83 91 96 97 101 102 105 116 117 119 123 131 134 135 140 146 149 156 157 162 166 174 175 178 182 190 191 193 203 204 207 209 219 220 223 225 231 238 240 241 247 254 256 259 261 266 270 273
	7	9 10 12 16 17 24 25 28 41 42 44 48 49 55 56 59 70 74 78 80 84 85 86 94 102 106 110 112 116 117 119 123 129 134 136 142 148 149 153 156 163 164 165 173 183 186 187 192 196 197 199 203 210 215 219 223 225 232 233 236 241 246 248 254 273
	8	4 7 10 12 20 23 26 28 40 45 46 47 52 55 58 60 72 77 78 79 88 93 94 95 97 98 99 102 120 125 126 127 133 137 139 144 145 146 147 150 161 162 163 166 180 183 186 188 200 205 206 207 209 210 211 214 225 226 227 230 244 247 250 252 273
	8’	3 6 8 15 20 25 26 27 37 39 44 48 53 55 60 64 68 73 74 75 85 87 92 96 101 103 108 112 115 118 120 127 132 137 138 139 147 150 152 159 161 162 173 174 180 185 186 187 195 198 200 207 212 217 218 219 227 230 232 239 245 247 252 256 273
	9	3 5 10 14 18 23 24 25 36 38 45 48 50 55 56 57 68 70 77 80 82 87 88 89 99 101 106 110 113 123 124 127 132 134 141 144 146 151 152 153 164 166 173 176 180 182 189 192 195 197 202 206 210 215 216 217 227 229 234 238 243 245 250 254 273
	10	3 8 12 16 19 24 28 32 37 38 39 47 53 54 55 63 65 73 74 77 83 88 92 96 97 105 106 109 114 116 123 126 131 136 140 144 147 152 156 160 161 169 170 173 177 185 186 189 197 198 199 207 209 217 218 221 229 230 231 239 245 246 247 255 273
	10’	5 7 12 16 17 18 29 30 33 34 45 46 51 54 56 63 65 66 77 78 81 82 93 94 100 105 106 107 115 118 120 127 133 135 140 144 149 151 156 160 163 166 168 175 181 183 188 192 193 194 205 206 213 215 220 224 227 230 232 239 243 246 248 255 273

Table A1. (Continued)

Plane	Unital no.	Unital
	11	2 7 13 16 18 23 29 32 34 39 45 48 49 53 60 62 68 70 72 73 84 86 88 89 98 103 109 112 113 117 124 126 132 134 136 137 145 149 156 158 163 170 171 175 180 182 184 185 193 197 204 206 210 215 221 224 228 230 232 233 241 245 252 254 273
	12	5 8 15 16 25 27 29 30 33 34 36 42 53 56 63 64 67 70 71 76 81 82 84 90 105 107 109 110 113 114 116 122 133 136 143 144 153 155 157 158 161 162 164 170 185 187 189 190 197 200 207 208 209 210 212 218 233 235 237 238 245 248 255 256 273
JOHN	1	3 8 9 14 18 20 21 23 26 28 29 31 60 81 84 86 87 90 91 93 96 98 99 101 104 105 108 110 111 131 132 135 136 137 138 141 142 161 162 165 166 171 172 175 176 196 199 202 205 210 213 220 223 227 232 233 238 241 246 251 256 258 264 265 267
	2	5 23 40 54 63 67 73 75 81 88 91 99 101 104 105 109 110 111 113 114 118 119 123 125 128 131 133 135 141 142 145 149 157 159 160 162 166 169 175 176 177 180 184 189 190 194 197 199 202 204 207 208 210 217 223 227 230 231 232 233 235 238 242 243 254
	3	50 65 70 75 80 97 98 101 102 107 108 111 112 114 116 117 119 122 124 125 127 146 148 149 151 154 156 157 159 164 167 170 173 178 179 181 184 185 188 190 191 193 194 197 198 203 204 207 208 228 231 234 237 241 244 246 247 250 251 253 256 257 266 267 268
	4	5 8 12 14 22 23 26 32 37 40 41 47 54 55 59 60 61 75 77 83 95 103 112 126 127 133 140 142 145 146 148 149 156 157 159 168 169 172 177 179 183 184 185 190 192 195 201 204 209 219 221 225 227 229 230 235 239 240 243 246 247 248 249 253 254
	5	2 4 9 10 14 21 26 27 31 32 33 34 37 43 46 50 52 54 55 56 62 65 68 70 71 75 80 92 95 97 102 114 117 121 124 126 127 133 135 137 139 143 148 153 154 156 160 161 162 163 169 170 182 183 188 190 192 196 199 202 205 210 213 220 223
	6	17 22 27 32 35 40 41 46 50 53 60 63 67 70 74 79 82 88 90 96 101 102 103 104 113 117 122 126 135 136 139 140 150 151 158 159 164 165 171 174 177 179 181 183 199 200 207 208 209 211 221 223 234 235 238 239 243 245 250 256 257 259 261 272 273
	7	34 37 44 47 49 50 53 54 59 60 63 64 68 69 70 78 81 84 88 92 103 108 110 112 117 119 120 123 129 135 136 140 150 151 152 159 165 170 171 174 177 179 183 191 196 198 206 207 217 219 220 221 228 229 238 240 244 248 255 256 257 259 261 272 273
	8	17 22 27 32 35 40 41 46 50 53 60 63 68 69 70 78 81 84 88 92 103 108 110 112 117 119 120 123 131 133 135 139 145 154 158 159 162 163 164 171 179 180 181 192 193 195 196 197 214 215 220 222 231 232 236 240 246 248 252 253 257 259 261 272 273
	9	10 14 15 22 28 29 34 43 46 50 53 57 58 60 63 64 65 74 79 81 82 84 86 91 94 96 102 103 105 118 124 126 138 143 144 151 153 159 161 162 163 165 167 172 175 178 184 187 199 201 204 210 215 219 236 238 240 241 249 250 259 261 263 265 266

Table A1. (Continued)

Plane	Unital no.	Unital
	10	4 8 12 23 27 28 34 46 48 49 50 53 56 60 61 63 68 73 75 85 86 89 106 107 111 116 124 126 130 137 138 145 151 159 161 164 165 167 169 170 173 184 189 192 194 197 198 199 204 206 207 213 222 224 230 234 239 242 248 253 258 259 266 270 273
	11	4 8 15 20 27 28 34 41 48 50 53 54 56 60 61 63 65 68 72 83 85 91 98 100 107 114 125 126 129 134 138 139 142 143 144 150 154 159 161 162 168 177 184 186 198 205 206 212 215 218 221 222 223 224 234 239 240 242 253 254 257 259 260 265 271
	12	2 26 46 50 54 65 66 69 70 75 80 83 88 89 90 93 94 99 104 123 128 133 135 137 139 140 151 153 154 156 160 161 162 163 170 174 177 183 188 190 192 195 197 200 204 209 210 213 214 215 218 220 223 226 229 232 233 243 244 246 247 248 249 251 254
	13	2 3 9 17 26 27 34 44 46 50 52 54 58 70 71 74 80 82 83 89 95 99 102 103 104 105 107 109 110 113 114 115 118 123 124 126 128 135 137 139 153 156 160 161 162 170 183 188 190 193 208 211 222 231 234 245 252 257 260 262 263 265 267 271 272
	14	7 8 10 14 15 17 18 23 27 31 33 35 37 47 48 50 52 56 57 58 59 66 68 69 74 85 90 93 95 99 102 104 112 115 121 123 128 129 132 137 147 159 160 162 164 166 178 183 185 195 197 200 204 209 214 215 218 226 229 232 233 244 246 247 251
	15	3 5 13 22 28 29 38 41 44 49 50 51 53 60 61 63 65 67 69 70 71 75 80 90 91 92 97 99 111 119 126 128 140 142 144 151 155 158 165 173 174 177 180 191 200 203 207 210 211 213 214 220 221 223 229 231 238 247 255 256 259 262 265 268 273
	16	2 4 8 17 21 26 43 46 47 50 53 54 58 60 62 63 65 67 69 70 74 75 80 85 90 91 99 111 112 115 119 128 129 142 143 147 148 155 164 165 169 178 180 182 195 198 207 209 210 213 220 221 222 223 226 234 238 242 244 256 259 262 265 268 273
	17	3 7 8 9 14 18 20 23 26 29 48 49 52 53 54 55 57 58 59 61 64 66 72 73 79 86 87 90 91 98 104 105 111 113 116 125 128 129 134 135 143 147 148 152 159 169 170 171 173 179 182 188 191 194 203 208 211 212 216 227 234 237 241 242 245
	18	7 8 9 10 21 22 27 28 33 34 47 48 51 52 53 61 62 66 71 74 79 84 85 92 93 98 99 101 102 107 108 110 111 113 120 121 128 131 136 137 142 194 195 206 207 214 215 218 219 228 229 231 232 233 234 236 237 241 244 253 256 258 268 271 272
	19	20 23 26 29 33 38 43 48 50 52 53 55 58 60 61 63 65 67 70 72 73 75 78 80 113 115 118 120 121 123 126 128 163 164 167 168 169 170 173 174 179 180 183 184 185 186 189 190 209 212 214 215 218 219 221 224 242 243 245 248 249 252 254 255 259
	20	3 10 12 13 14 22 27 28 29 31 33 34 35 38 47 51 54 55 56 58 67 81 84 87 90 93 98 113 116 118 123 128 132 135 156 159 164 167 170 171 173 176 178 179 181 184 188 191 199 200 202 204 210 212 219 223 225 226 232 240 248 249 253 256 259
	21	3 7 13 17 21 31 33 34 43 51 52 53 57 67 68 71 73 80 82 85 86 89 96 97 101 102 103 111 114 115 116 120 122 134 153 164 191 194 195 197 210 212 213 215 219 220 223 225 230 233 235 238 239 240 247 251 256 260 262 264 265 266 268 272 273

Table A1. (Continued)

Plane	Unital no.	Unital
	22	3 6 7 10 11 19 21 25 27 28 38 43 47 53 55 56 57 69 79 80 83 84 90 99 105 106 117 118 128 130 143 144 151 153 154 163 167 174 178 182 187 199 201 205 210 219 220 230 236 240 242 243 245 249 250 252 255 257 261 262 263 264 265 266 270
	23	3 10 15 27 28 29 43 46 47 50 53 55 57 60 63 64 65 67 68 72 73 76 78 87 92 94 103 110 112 113 120 124 129 131 135 136 137 142 143 146 151 152 163 170 176 185 187 191 197 201 202 218 219 223 225 227 236 248 250 256 259 261 263 265 266
	24	2 3 6 9 10 11 13 17 19 20 25 27 28 31 35 37 43 47 48 49 53 55 57 61 62 69 77 85 90 97 98 99 101 108 111 116 119 121 122 125 128 140 143 148 151 163 168 187 192 195 199 201 209 210 219 226 228 231 234 236 237 240 244 249 250
	25	13 16 30 31 36 42 48 53 62 65 66 70 76 87 89 93 94 100 103 104 110 113 123 124 127 129 134 135 143 145 146 147 150 151 152 155 160 162 163 165 170 172 173 175 176 185 187 188 191 194 199 205 210 211 216 217 220 221 222 230 232 240 243 249 251
	26	7 10 15 21 28 29 33 46 48 51 62 63 64 65 67 73 81 91 94 101 103 111 119 124 125 129 134 135 139 141 143 144 149 151 159 163 166 176 178 181 184 187 188 190 191 195 198 206 211 214 219 229 236 237 244 252 253 257 260 262 263 265 267 271 272
	27	3 8 9 14 33 38 43 48 51 52 53 55 56 57 58 61 62 68 71 74 77 97 99 102 104 105 107 110 112 113 116 118 119 122 123 125 128 145 150 155 160 194 195 197 200 201 204 206 207 227 228 231 232 233 234 237 238 244 247 250 253 261 263 265 266
	28	3 17 34 52 53 66 72 73 75 79 86 87 90 91 94 98 100 104 105 111 113 116 117 125 128 131 137 144 146 147 149 150 156 159 160 165 170 175 179 180 181 184 185 186 190 197 206 207 212 214 218 225 226 230 232 235 238 240 241 243 248 249 250 251 254
	29	4 5 8 13 14 17 18 23 27 31 37 38 40 43 47 52 54 56 57 58 63 66 68 69 74 75 76 79 84 85 87 88 90 93 95 102 109 112 114 115 121 133 157 169 177 200 201 208 209 212 215 217 218 221 224 226 231 239 241 242 246 247 250 251 256

Table A2. Some dual unitals in planes of order 16.

Plane	Unital no.	Unital
DEMP	$(1^{\perp})'$	19 22 23 28 35 36 40 43 57 59 61 62 67 69 74 78 82 87 88 89 104 106 108 109 114 117 118 123 129 139 140 143 145 146 148 154 164 167 174 175 177 182 184 190 194 195 205 207 214 217 218 223 225 229 231 237 244 245 249 252 257 258 260 266 273
	$(2^{\perp})'$	9 11 13 14 19 22 23 28 33 36 37 40 49 50 52 58 73 78 79 80 82 85 88 90 98 101 104 106 123 125 127 128 129 132 137 142 149 152 159 160 161 164 171 173 177 180 181 184 194 201 202 206 210 218 219 221 235 237 239 240 249 254 255 256 273
HALL	$(1^{\perp})'$	13 29 33 39 42 43 44 61 67 69 71 78 79 82 83 84 86 91 97 103 106 107 108 115 117 119 126 127 129 132 133 137 144 157 162 168 169 170 175 178 179 180 182 187 194 200 201 202 207 209 212 213 217 224 230 232 236 238 240 246 248 252 254 256 257
	$(3^{\perp})'$	5 21 35 36 40 42 44 53 66 74 75 78 80 81 82 86 87 88 99 100 104 106 108 114 122 123 126 128 134 137 139 140 143 149 163 167 173 174 175 177 178 182 183 184 195 199 205 206 207 214 217 219 220 223 225 228 233 237 240 241 244 249 253 256 257
	$(4^{\perp})'$	9 11 13 14 25 27 29 30 33 34 36 42 49 50 52 58 73 75 77 78 83 86 87 92 97 98 100 106 115 118 119 124 137 139 141 142 145 146 148 154 161 162 164 170 179 182 183 188 195 198 199 204 213 216 223 224 233 235 237 238 243 246 247 252 273
LMRH	$(1^{\perp})'$	8 13 14 15 24 29 30 31 36 39 42 44 53 57 59 64 72 77 78 79 84 87 90 92 104 109 110 111 117 121 123 128 133 137 139 144 152 157 158 159 161 162 163 166 181 185 187 192 197 201 203 208 212 215 218 220 228 231 234 236 244 247 250 252 273
	$(2^{\perp})'$	4 7 10 12 17 18 19 22 40 45 46 47 56 61 62 63 72 77 78 79 85 89 91 96 97 98 99 102 116 119 122 124 132 135 138 140 145 146 147 150 168 173 174 175 184 189 190 191 193 194 195 198 209 210 211 214 228 231 234 236 244 247 250 252 273
	$(2^{\perp})''$	9 11 13 14 19 22 23 28 41 43 45 46 51 54 55 60 65 66 68 74 89 91 93 94 97 98 100 106 113 114 116 122 129 130 132 138 153 155 157 158 169 171 173 174 177 178 180 186 197 200 207 208 211 214 215 220 227 230 231 236 243 246 247 252 273
JOWK	$(6^{\perp})'$	3 6 7 12 39 40 42 46 51 54 55 60 68 76 77 79 81 83 88 93 99 106 107 111 113 119 121 127 132 140 141 143 145 146 148 154 165 166 170 173 177 183 185 191 196 198 200 201 217 218 220 224 225 227 232 237 244 246 248 249 261 264 271 272 273
	$(7^{\perp})'$	3 6 7 12 21 24 31 32 35 38 39 44 49 50 52 58 65 66 68 74 89 91 93 94 105 107 109 110 113 114 116 122 129 130 132 138 153 155 157 158 161 162 164 170 179 182 183 188 201 203 205 206 217 219 221 222 227 230 231 236 243 246 247 252 273
	$(7^{\perp})''$	5 7 12 16 21 23 28 32 37 39 44 48 51 54 56 63 69 71 76 80 83 86 88 95 100 105 106 107 113 114 125 126 132 137 138 139 148 153 154 155 163 166 168 175 181 183 188 192 195 198 200 207 211 214 216 223 228 233 234 235 244 249 250 251 273