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Hardy–Copson type inequalities for nabla time scale calculus

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Abstract: This paper is devoted to the nabla unification of the discrete and continuous Hardy–Copson type inequalities. Some of the obtained inequalities are nabla counterparts of their delta versions while the others are new even for the discrete, continuous, and delta cases. Moreover, these dynamic inequalities not only generalize and unify the related ones in the literature but also improve them in the special cases.

Key words: Nabla time scale calculus, Hardy's inequality, Copson's inequality

1. Introduction

Hardy provided the inspiration for mathematicians to study integral inequalities and to highlight their importance in analysis of qualitative nature of solutions of initial and boundary value problems for any kind of differential equations, such as ordinary, partial, dynamic, and fractional. The discrete inequality

$$\sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}b(j)\right)^{\zeta} \leq \left(\frac{\zeta}{\zeta-1}\right)^{\zeta}\sum_{m=1}^{\infty}b^{\zeta}(m), \quad b(m)\geq 0,\ \zeta>1 \tag{1.1}$$

was established by Hardy [17] in 1920 to find a simpler proof for the Hilbert's inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \leq \pi\left(\sum_{m=1}^{\infty}a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2},$$

where $a_m, b_n \geq 0$ and $\sum_{m=1}^{\infty} a_m^2$ and $\sum_{n=1}^{\infty} b_n^2$ are convergent, see [21].

Then the continuous counterpart of inequality (1.1) was proved by Hardy [18] in 1925 by using the calculus of variations. Continuous Hardy inequality asserts that for nonnegative function φ, if $\int_0^t \varphi(s)ds < \infty$ and $\int_0^{\infty} \varphi^{\zeta}(s)ds < \infty$, then inequality

$$\int_0^{\infty}\left(\frac{1}{t}\int_0^t \varphi(s)ds\right)^{\zeta} dt \leq \left(\frac{\zeta}{\zeta-1}\right)^{\zeta}\int_0^{\infty}\varphi^{\zeta}(t)dt, \quad \zeta>1 \tag{1.2}$$

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holds. The constant $\left(\frac{\zeta}{\zeta-1}\right)^{\zeta}$ that appears in the above inequalities also has been found as the best possible one, since if it is replaced by a smaller constant then inequalities (1.1) and (1.2) are no longer satisfied for the involved sequences and functions, respectively. Later in [19, Theroem 330], Hardy et al. generalized inequality (1.2) to

$$\int_0^\infty \frac{\Phi^\zeta(t)}{t^\theta}dt \leq \left|\frac{\zeta}{\theta-1}\right|^\zeta \int_0^\infty \frac{\varphi^\zeta(t)}{t^{\theta-\zeta}}dt, \quad \zeta>1, \tag{1.3}$$

where φ is nonnegative function and $\Phi(t) = \begin{cases} \displaystyle\int_0^t \varphi(s)ds, & \text{if } \theta>1, \\ \displaystyle\int_t^\infty \varphi(s)ds, & \text{if } \theta<1. \end{cases}$

The discrete and continuous Hardy inequalities have been improved and generalized in many directions and used in many applications, see [7, 8, 13, 20, 22, 23, 26, 27, 29] and references therein.

Copson [14, Theorem 1.1, Theorem 2.1] improved inequality (1.1) by replacing the arithmetic mean of a sequence by a weighted arithmetic mean in two following manners: Let $b(m)\geq 0, w(m)\geq 0$ for all m.

If $\zeta>1, \theta>1$, then

$$\sum_{m=1}^\infty \frac{w(m)}{\left[\overline{A}(m)\right]^\theta}\left(\sum_{j=1}^m b(j)w(j)\right)^\zeta \leq \left(\frac{\zeta}{\theta-1}\right)^\zeta \sum_{m=1}^\infty w(m)\left[\overline{A}(m)\right]^{\zeta-\theta} b^\zeta(m), \tag{1.4}$$

and if $0\leq\theta<1<\zeta$, then

$$\sum_{m=1}^\infty \frac{w(m)}{\left[\overline{A}(m)\right]^\theta}\left(\sum_{j=m}^\infty b(j)w(j)\right)^\zeta \leq \left(\frac{\zeta}{1-\theta}\right)^\zeta \sum_{m=1}^\infty w(m)\left[\overline{A}(m)\right]^{\zeta-\theta} b^\zeta(m), \tag{1.5}$$

where $\overline{A}(m)=\sum_{j=1}^m w(j)$.

Other refinements of discrete Hardy–Copson inequalities have been developed by Bennett [9, Corollary 3-Corollary 6] (see also Leindler [25, Proposition 1-Proposition 4]) as follows: Let $b(m)\geq 0, w(m)\geq 0$ for all m and $\sum_{j=1}^\infty w(j)<\infty$.

If $0\leq\theta<1<\zeta$, then

$$\sum_{m=1}^\infty \frac{w(m)}{[A(m)]^\theta}\left(\sum_{j=1}^m b(j)w(j)\right)^\zeta \leq \left(\frac{\zeta}{1-\theta}\right)^\zeta \sum_{m=1}^\infty w(m)A^{\zeta-\theta}(m)b^\zeta(m), \tag{1.6}$$

and

$$\sum_{m=1}^\infty \frac{w(m)}{\left[\overline{A}(m)\right]^\theta}\left(\sum_{j=m}^\infty b(j)w(j)\right)^\zeta \leq \left(\frac{\zeta}{1-\theta}\right)^\zeta \sum_{m=1}^\infty w(m)\overline{A}^{\zeta-\theta}(m)b^\zeta(m), \tag{1.7}$$

where $A(m)=\sum_{j=m}^{\infty} w(j)$ and $\overline{A}(m)=\sum_{j=1}^{m} w(j)$.

If $1<\theta\leq\zeta$, then

$$\sum_{m=1}^{\infty} \frac{w(m)}{[A(m)]^{\theta}} \left(\sum_{j=m}^{\infty} b(j)w(j)\right)^{\zeta} \leq \left(\frac{\zeta}{\theta-1}\right)^{\zeta} \sum_{m=1}^{\infty} w(m)A^{\zeta-\theta}(m)b^{\zeta}(m), \tag{1.8}$$

and

$$\sum_{m=1}^{\infty} \frac{w(m)}{[\overline{A}(m)]^{\theta}} \left(\sum_{j=1}^{m} b(j)w(j)\right)^{\zeta} \leq \left(\frac{\zeta}{\theta-1}\right)^{\zeta} \sum_{m=1}^{\infty} w(m)\overline{A}^{\zeta-\theta}(m)b^{\zeta}(m). \tag{1.9}$$

Similar to the discrete Hardy inequality (1.1), the continuous versions (1.2) or (1.3) have attracted many mathematicians' interests and expansions of these continuous inequalities have appeared in the literature. Copson [15, Theorem 1, Theorem 3] improved inequality (1.3) in the two following manners and obtained the continuous versions of discrete inequalities (1.4) (or (1.9)) and (1.5) (or (1.7)), respectively: Let $w(t)$ and $f(t)$ be nonnegative functions and $\overline{A}=\int_0^t w(s)ds,\ B(t)=\int_0^t w(s)f(s)ds,\ \overline{B}(t)=\int_t^{\infty} w(s)f(s)ds.$

If $1<\theta,\ 1\leq\zeta,\ 0<b\leq\infty$, then

$$\int_0^b \frac{w(t)}{[\overline{A}(t)]^{\theta}}[B(t)]^{\zeta}dt \leq \left(\frac{\zeta}{\theta-1}\right)^{\zeta} \int_0^b w(t)[\overline{A}(t)]^{\zeta-\theta} f^{\zeta}(t)dt \tag{1.10}$$

and if $\theta<1\leq\zeta,\ a>0$, then

$$\int_a^{\infty} \frac{w(t)}{[\overline{A}(t)]^{\theta}}[\overline{B}(t)]^{\zeta}dt \leq \left(\frac{\zeta}{1-\theta}\right)^{\zeta} \int_a^{\infty} w(t)[\overline{A}(t)]^{\zeta-\theta} f^{\zeta}(t)dt. \tag{1.11}$$

Various generalizations and numerous variants of continuous Hardy–Copson inequalities (1.10) and (1.11) can be found in Pachpatte [31] and the references therein.

Other refinements of continuous Hardy–Copson inequalities, which are generalizations of (1.10) and (1.11), respectively, have been introduced by Pečarić and Hanjš [32] as follows: Let $w(t)$ and $f(t)$ be nonnegative functions and $\overline{A}(t)=\int_0^t w(s)ds,\ B(t)=\int_0^t w(s)f(s)ds,\ \overline{B}(t)=\int_t^{\infty} w(s)f(s)ds.$

If $\eta\geq 0,\ \zeta>1, \eta+\theta>1$, then

$$\int_0^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta}dt}{[\overline{A}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} \int_a^{\infty} \frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[\overline{A}(t)]^{\eta+\theta-\zeta}}dt, \tag{1.12}$$

and if $\eta\geq 0,\ \zeta>1, \eta+\theta<1$, then

$$\int_0^{\infty} \frac{w(t)[\overline{B}(t)]^{\eta+\zeta}dt}{[\overline{A}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{w(t)f^{\zeta}(t)[\overline{B}(t)]^{\eta}}{[\overline{A}(t)]^{\eta+\theta-\zeta}}dt. \tag{1.13}$$

Following the development of the time scale concept, the analysis of dynamic inequalities have become a popular research area and most classical inequalities have been extended to an arbitrary time scale. We can refer to the surveys [1, 33] and the monograph [2] for exhibition of these results. The aforementioned Hardy–Copson inequalities have been unified to an arbitrary time scale in the book [3] and in the articles [4, 34–40] by using delta time scale calculus.

The time scale delta unifications of the foregoing inequalities in an arbirtary time scale are given by the following theorems. In the next four theorems, we assume that $a \in [0,\infty)_{\mathbb{T}}$, where $\mathbb{T}$ is time scale, $w(t)$ and $f(t)$ are nonnegative, rd-continuous, Δ-differentiable, and locally delta integrable functions.

The following theorem establishes the delta unification of the discrete Bennett's inequality (1.6).

Theorem 1.1 *[39] For the functions w and f, let us define the functions $A(t) = \int_t^\infty w(s)\Delta s$ and $B(t) = \int_a^t w(s)f(s)\Delta s$. If $\eta \geq 0,\ \zeta > 1,\ \eta + \theta < 1,$ then*

$$\int_a^\infty \frac{w(t)[B^\sigma(t)]^{\eta+\zeta}\Delta t}{[A(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[B^\sigma(t)]^\eta}{[A(t)]^{\eta+\theta-\zeta}}\Delta t. \tag{1.14}$$

The delta unification of the discrete Bennett's inequality (1.8) can be established as follows.

Theorem 1.2 *[39] For the functions w and f, let us define the functions $A(t) = \int_t^\infty w(s)\Delta s$ and $\overline{B}(t) = \int_t^\infty w(s)f(s)\Delta s$. For $\frac{A^\sigma(t)}{A(t)} \geq \frac{1}{K} > 0$, if $\eta \geq 0,\ \zeta > 1,\ \eta + \theta > 1,$ then*

$$\int_a^\infty \frac{w(t)[\overline{B}(t)]^{\eta+\zeta}\Delta t}{[A(t)]^{\eta+\theta}} \leq \left[K^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[\overline{B}(t)]^\eta}{[A(t)]^{\eta+\theta-\zeta}}\Delta t. \tag{1.15}$$

The following theorem given in [39] establishes the delta unification of the Copson's discrete inequality (1.5) (or the Bennett's discrete inequality (1.7)) and its continuous counterpart (1.11) as well as its continuous generalization (1.13).

Theorem 1.3 *[39] For the functions w and f, let us define the functions $\overline{A}(t) = \int_a^t w(s)\Delta s$ and $\overline{B}(t) = \int_t^\infty w(s)f(s)\Delta s$. If $\eta \geq 0,\ \zeta > 1,\ \eta + \theta < 1,$ then*

$$\int_a^\infty \frac{w(t)[\overline{B}(t)]^{\eta+\zeta}\Delta t}{[\overline{A}^\sigma(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[\overline{B}(t)]^\eta}{[\overline{A}^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t. \tag{1.16}$$

Although the authors obtained the results of Theorems 1.1–1.3 in [39] for delta time scale calculus, they did not include the following theorem. For the completeness of the paper, we give the next theorem, which is the delta unification of the Copson's discrete inequality (1.4) (or the Bennett's discrete inequality (1.9)) and its continuous counterpart (1.10) as well as the continuous generalization (1.12).

Theorem 1.4 *For the functions w and f, we set $\overline{A}(t) = \int_a^t w(s)\Delta s$ and $B(t) = \int_a^t w(s)f(s)\Delta s$. For $\frac{\overline{A}(t)}{\overline{A}^\sigma(t)} \geq \frac{1}{J} > 0$, if $\eta \geq 0$, $\zeta > 1$, $\eta + \theta > 1$, then*

$$\int_a^\infty \frac{w(t)[B^\sigma(t)]^{\eta+\zeta}\Delta t}{[\overline{A}^\sigma(t)]^{\eta+\theta}} \leq \left[J^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[B^\sigma(t)]^\eta}{[\overline{A}^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t. \tag{1.17}$$

Since two types of calculus, the nabla and the delta on time scales, are the "dual" of each other [12], the nabla counterparts of some of the the well known discrete, continuous, and delta inequalities have been obtained by using the nabla calculus and the nabla fractional calculus, see [5].

Although time scale unification of Hardy–Copson type inequalities obtained by using delta derivative and integral have taken place in the literature, the nabla unification established by nabla derivative and integral have not appeared in the literature so far. Therefore, the main purpose of this article is to expand Hardy–Copson type inequalities to nabla time scale calculus and to merge them for an arbitrary time scale. In that regard, this is a major contribution to the limited academic literature on nabla Hardy–Copson type inequalities. The inequalities in [39] covering the results of delta Hardy–Copson type inequalities are the main sources of inspiration for their nabla counterparts. Just like the delta case, the nabla Hardy–Copson type inequalities generalize and unify all of the known results mentioned above and obtained for arbitrary time scales $\mathbb{T}$, e.g., the set of real numbers and/or the set of integers. Besides, some of our results are new not only for the nabla case but also for the delta, continuous, and discrete cases.

This article is planned in the following manner: the nabla time scale calculus is introduced and some of its properties are given in Section 2. The nabla calculus versions of Theorems 1.1–1.4, which provide the unifications of Hardy–Copson type inequalities (1.1)-(1.13) for nabla time scale calculus, are obtained in Section 3. The conclusion and ideas for future work are presented in Section 4.

2. Preliminaries

This section is devoted to present the main definitions and theorems of nabla time scale calculus. We refer the reader to [6, 10] for the concept of time scale calculus in delta and nabla senses.

If $\mathbb{T} \neq \emptyset$ is a closed subset of $\mathbb{R}$, then $\mathbb{T}$ is called a time scale. The backward jump operator ρ is defined as $\rho(t) := \sup(-\infty, t)_\mathbb{T}$, for $t \in \mathbb{T}$, provided $\sup\emptyset = \inf\mathbb{T}$. The backward graininess function $\nu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\nu(t) := t - \rho(t)$, for $t \in \mathbb{T}$.

The ∇-derivative of $\psi : \mathbb{T} \to \mathbb{R}$ at the point $t \in \mathbb{T}_\kappa = \mathbb{T}/[\inf\mathbb{T}, \sigma(\inf\mathbb{T}))$ denoted by $\psi^\nabla(t)$ is the number that has the property that for all $\epsilon > 0$, there exists a neighborhood $V \subset \mathbb{T}$ of $t \in \mathbb{T}_\kappa$ such that

$$|\psi(s) - \psi(\rho(t)) - \psi^\nabla(t)(s - \rho(t))| \leq \epsilon|s - \rho(t)|$$

for all $s \in V$.

A function $\psi : \mathbb{T} \to \mathbb{R}$ is ld-continuous if it is continuous at each left-dense points in $\mathbb{T}$ and $\lim_{s\to t^+}\psi(s)$ exists as a finite number for all right-dense points in $\mathbb{T}$. The set $C_{ld}(\mathbb{T}, \mathbb{R})$ denotes the class of real, ld-continuous functions defined on a time scale $\mathbb{T}$. If $\psi \in C_{ld}(\mathbb{T}, \mathbb{R})$, then there exists a function $\Psi(t)$ such that $\Psi^\nabla(t) = \psi(t)$ and the nabla integral of ψ is defined by $\int_a^b \psi(s)\nabla s = \Psi(b) - \Psi(a)$.

Some important features of the nabla derivative can be listed below.

Theorem 2.1 *[6, 10] Suppose that $\varphi : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$.*

1. *If φ is nabla differentiable at t, then φ is continuous at t.*
2. *If φ is continuous at a left scattered point t, then φ is nabla differentiable at t with $\varphi^\nabla(t) = \dfrac{\varphi(t) - \varphi(\rho(t))}{\nu(t)}$.*
3. *φ is nabla differentiable at a left dense point t if and only if the limit $\varphi^\nabla(t) = \lim_{s \to t} \dfrac{\varphi(t) - \varphi(s)}{t - s}$ exists as a finite number.*
4. *If φ is nabla differentiable at t, then $\varphi^\rho(t) = \varphi(t) - \nu(t)\varphi^\nabla(t)$.*

Some of the properties of the nabla integral are gathered next.

Theorem 2.2 *[6, 10] Let $t_1, t_2, t_3 \in \mathbb{T}$ with $t_1 < t_3 < t_2$ and $a \in \mathbb{R}$. If $\varphi, \psi : \mathbb{T} \to \mathbb{R}$ are ld-continuous, then*

1) $\int_{t_1}^{t_2} [\varphi(s) + \psi(s)]\nabla s = \int_{t_1}^{t_2} \varphi(s)\nabla(s) + \int_{t_1}^{t_2} \psi(s)\nabla s.$

2) $\int_{t_1}^{t_2} a\varphi(s)\nabla s = a\int_{t_1}^{t_2} \varphi(s)\nabla s.$

3) $\int_{t_1}^{t_2} \varphi(s)\nabla s = -\int_{t_2}^{t_1} \varphi(s)\nabla s.$

4) $\int_{t_1}^{t_2} \varphi(s)\nabla s = \int_{t_1}^{t_3} \varphi(s)\nabla s + \int_{t_3}^{t_2} \varphi(s)\nabla s.$

5) $\int_{t_1}^{t_1} \varphi(s)\nabla(s) = 0.$

6) *integration by parts formula-I holds:*

$$\int_{t_1}^{t_2} \varphi(s)\psi^\nabla(s)\nabla s = \varphi(t_2)\psi(t_2) - \varphi(t_1)\psi(t_1) - \int_{t_1}^{t_2} \varphi^\nabla(s)\psi(\rho(s))\nabla s.$$

7) *integration by parts formula-II holds:*

$$\int_{t_1}^{t_2} \varphi(\rho(s))\psi^\nabla(s)\nabla s = \varphi(t_2)\psi(t_2) - \varphi(t_1)\psi(t_1) - \int_{t_1}^{t_2} \varphi^\nabla(s)\psi(s)\nabla s.$$

Theorem 2.3 *[6, 10] Let $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$ and $\varphi : \mathbb{T} \to \mathbb{R}$ be ld-continuous function.*

(i) If $\mathbb{T}=\mathbb{R}$, then the nabla integral of φ becomes classical Riemann integral, namely

$$\int_{t_1}^{t_2} \varphi(s)\nabla s = \int_{t_1}^{t_2} \varphi(s)ds.$$

(ii) If $\mathbb{T}=\mathbb{Z}$, then $\displaystyle\int_{t_1}^{t_2} \varphi(s)\nabla s = \sum_{s=t_1+1}^{t_2} \varphi(s).$

Lemma 2.4 (Hölder's inequality) *[30] Let $t_1, t_2 \in \mathbb{T}$. For $\varphi, \psi \in C_{ld}([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ and for the conjugate numbers $\eta, \zeta > 1$ satisfying $\frac{1}{\eta}+\frac{1}{\zeta}=1$, Hölder's inequality*

$$\int_{t_1}^{t_2} \varphi(s)\psi(s)\nabla s \leq \left[\int_{t_1}^{t_2} \varphi^{\eta}(s)\nabla s\right]^{1/\eta} \left[\int_{t_1}^{t_2} \psi^{\zeta}(s)\nabla s\right]^{1/\zeta} \tag{2.1}$$

holds.

Lemma 2.5 (Chain rule for the nabla derivative) *[16] If $\varphi:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $\psi:\mathbb{T}\to\mathbb{R}$ is continuous and nabla differentiable on $\mathbb{T}_\kappa$, then $\varphi\circ\psi$ is nabla differentiable and there exists d in the real interval $[\rho(t), t]$ with*

$$(\varphi\circ\psi)^{\nabla}(s) = \varphi'(\psi(d))\psi^{\nabla}(s). \tag{2.2}$$

Lemma 2.6 (Chain rule for the nabla derivative) *[16] If $\varphi:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $\psi:\mathbb{T}\to\mathbb{R}$ is nabla differentiable, then $\varphi\circ\psi$ is nabla differentiable and*

$$(\varphi\circ\psi)^{\nabla}(s) = \psi^{\nabla}(s)\left[\int_0^1 \varphi'(\psi(\rho(s)) + h\nu(s)\psi^{\nabla}(s))dh\right]. \tag{2.3}$$

3. Hardy–Copson type inequalities

The main interest of this section is to obtain corresponding results of Theorems 1.1–1.4 for nonnegative, ld-continuous, ∇-differentiable and locally nabla integrable functions $w(t)$ and $f(t)$. We also assume that $a\in[0,\infty)_{\mathbb{T}}$, where $\mathbb{T}$ is a time scale.

The next theorem states the nabla unification of the discrete inequalities (1.1), (1.4), and (1.9) and the continuous inequalities (1.2), (1.3), (1.10), and (1.12) as well as a nabla analogue of the delta inequality (1.17) for nonnegative, ld-continuous, ∇-differentiable, and locally nabla integrable functions $w(t)$ and $f(t)$. Moreover, this theorem provides novel results for all of the above three time scales.

Theorem 3.1 *For the functions w and f satisfying the properties stated above, we set $\overline{A}(t)=\int_a^t w(s)\nabla s$ and $B(t)=\int_a^t w(s)f(s)\nabla s$. Assume that $B(\infty)<\infty$ and $\int_a^\infty \frac{w(t)\nabla t}{[\overline{A}(t)]^{\eta+\theta}}<\infty$. Suppose that there exists $L>0$ such that $\frac{\overline{A}(t)}{\overline{A}^{\rho}(t)}\leq L$ for $t\in(a,\infty)_{\mathbb{T}}$. If $\eta\geq 0$, $\zeta>1$ and $\eta+\theta>1$ are real constants, then*

(i)

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[\overline{A}^\rho(t)]^{\eta+\theta-1}}\nabla t. \tag{3.1}$$

(ii)

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[B(t)]^\eta[\overline{A}(t)]^{(\eta+\theta)(\zeta-1)}}{[\overline{A}^\rho(t)]^{(\eta+\theta-1)\zeta}}\nabla t. \tag{3.2}$$

(iii)

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta L^{(\eta+\theta)(\zeta-1)}\int_a^\infty \frac{w(t)f^\zeta(t)[B(t)]^\eta}{[\overline{A}^\rho(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.3}$$

(iv)

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} \leq \left[L^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[B(t)]^\eta}{[\overline{A}(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.4}$$

(v)

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}^\rho(t)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{\eta+\theta-1}L^{\eta+\theta}\int_a^\infty \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[\overline{A}^\rho(t)]^{\eta+\theta-1}}\nabla t. \tag{3.5}$$

(vi)

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}^\rho(t)]^{\eta+\theta}} \leq \left(\frac{\eta+\zeta}{\eta+\theta-1}\right)^\zeta L^{(\eta+\theta)\zeta}\int_a^\infty \frac{w(t)f^\zeta(t)[B(t)]^\eta}{[\overline{A}^\rho(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.6}$$

Proof

(i) After employing the integration by parts formula-I to the left hand side of inequality (3.1) for the functions $u^\nabla(t) = \dfrac{w(t)}{[\overline{A}(t)]^{\eta+\theta}}$ and $v(t) = [B(t)]^{\eta+\zeta}$, it becomes

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} = u(t)B^{\eta+\zeta}(t)|_a^\infty + \int_a^\infty -u^\rho(t)\left[B^{\eta+\zeta}(t)\right]^\nabla \nabla t.$$

In view of how u^∇ is given above, we deduce that $u(t) = -\displaystyle\int_t^\infty \frac{w(s)}{[\overline{A}(s)]^{\eta+\theta}}\nabla s$. By using $B(a)=0$ and $u(\infty)=0$, one can have

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} = \int_a^\infty -u^\rho(t)\left[B^{\eta+\zeta}(t)\right]^\nabla \nabla t. \tag{3.7}$$

Applying (2.2) to $\left[B^{\eta+\zeta}(t)\right]^\nabla$ and using $B^\nabla(t) = f(t)w(t) \geq 0$, we obtain

$$\left[B^{\eta+\zeta}(t)\right]^\nabla = (\eta+\zeta)B^{\eta+\zeta-1}(d)B^\nabla(t) = (\eta+\zeta)w(t)f(t)B^{\eta+\zeta-1}(d).$$

For $\eta+\zeta>1$ and $t\geq d$, one can observe that $[B(t)]^{\eta+\zeta-1}\geq[B(d)]^{\eta+\zeta-1}$ and

$$\left[B^{\eta+\zeta}(t)\right]^{\nabla}\leq(\eta+\zeta)w(t)f(t)B^{\eta+\zeta-1}(t). \tag{3.8}$$

Since $\overline{A}^{\nabla}(t)=w(t)\geq 0$ and $\eta+\theta>1$, by (2.3), we get

$$\begin{aligned}\left[\overline{A}^{1-\eta-\theta}(t)\right]^{\nabla}&=\int_0^1\frac{(1-\eta-\theta)\overline{A}^{\nabla}(t)dh}{[h\overline{A}(t)+(1-h)\overline{A}^{\rho}(t)]^{\eta+\theta}}=\int_0^1\frac{-(\eta+\theta-1)w(t)dh}{[h\overline{A}(t)+(1-h)\overline{A}^{\rho}(t)]^{\eta+\theta}}\\&\leq-(\eta+\theta-1)\int_0^1\frac{w(t)}{[h\overline{A}(t)+(1-h)\overline{A}(t)]^{\eta+\theta}}dh=-(\eta+\theta-1)\frac{w(t)}{[\overline{A}(t)]^{\eta+\theta}}\end{aligned} \tag{3.9}$$

providing $\dfrac{w(t)}{[\overline{A}(t)]^{\eta+\theta}}\leq-\dfrac{\left[\overline{A}^{1-\eta-\theta}(t)\right]^{\nabla}}{\eta+\theta-1}$.

Then we can infer that

$$-u^{\rho}(t)=\int_{\rho(t)}^{\infty}\frac{w(s)\nabla s}{[\overline{A}(s)]^{\eta+\theta}}\leq\int_{\rho(t)}^{\infty}-\frac{\left[\overline{A}^{1-\eta-\theta}(s)\right]^{\nabla}\nabla s}{\eta+\theta-1}=\frac{-\overline{A}^{1-\eta-\theta}(\infty)+[\overline{A}^{\rho}(t)]^{1-\eta-\theta}}{\eta+\theta-1}\leq\frac{[\overline{A}^{\rho}(t)]^{1-\eta-\theta}}{\eta+\theta-1}. \tag{3.10}$$

Substituting (3.8) and (3.10) into (3.7) leads to

$$\int_a^{\infty}\frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}}\leq\frac{\eta+\zeta}{\eta+\theta-1}\int_a^{\infty}\frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[\overline{A}^{\rho}(t)]^{\eta+\theta-1}}\nabla t,$$

which is the desired result (3.1).

(ii) In order to obtain inequality (3.2), we use inequality (3.1). Let us define the functions $C(t)=\dfrac{w^{\frac{1}{\zeta}}(t)f(t)[B(t)]^{\frac{\eta}{\zeta}}[\overline{A}(t)]^{\frac{(\eta+\theta)(\zeta-1)}{\zeta}}}{[\overline{A}^{\rho}(t)]^{\eta+\theta-1}}$ and $D(t)=\left[\dfrac{w(t)[B(t)]^{\eta+\zeta}}{[\overline{A}(t)]^{\eta+\theta}}\right]^{\frac{\zeta-1}{\zeta}}$. Then inequality (3.1) becomes

$$\int_a^{\infty}\frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}}\leq\frac{\eta+\zeta}{\eta+\theta-1}\int_a^{\infty}C(t)D(t)\nabla t. \tag{3.11}$$

For the indices $\zeta>1$ and $\frac{\zeta}{\zeta-1}>1$ and for the functions C and D, employing Hölder's inequality (2.1) leads to an estimation on the last term of inequality (3.11) as

$$\int_a^{\infty}\frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}}\leq\frac{\eta+\zeta}{\eta+\theta-1}\left[\int_a^{\infty}\frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}[\overline{A}(t)]^{(\eta+\theta)(\zeta-1)}}{[\overline{A}^{\rho}(t)]^{(\eta+\theta-1)\zeta}}\nabla t\right]^{\frac{1}{\zeta}}\left[\int_a^{\infty}\frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}}\right]^{\frac{\zeta-1}{\zeta}},$$

which implies

$$\int_a^{\infty}\frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}}\leq\left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta}\int_a^{\infty}\frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}[\overline{A}(t)]^{(\eta+\theta)(\zeta-1)}}{[\overline{A}^{\rho}(t)]^{(\eta+\theta-1)\zeta}}\nabla t,$$

which is the desired result (3.2).

(iii) In order to obtain inequality (3.3), we use

$$\frac{[\overline{A}(t)]^{(\eta+\theta)(\zeta-1)}}{[\overline{A}^{\rho}(t)]^{(\eta+\theta-1)\zeta}} = \left(\frac{\overline{A}(t)}{\overline{A}^{\rho}(t)}\right)^{(\eta+\theta)(\zeta-1)} \frac{[\overline{A}^{\rho}(t)]^{-\eta-\theta}}{[\overline{A}^{\rho}(t)]^{-\zeta}} \leq \frac{L^{(\eta+\theta)(\zeta-1)}}{[\overline{A}^{\rho}(t)]^{\eta+\theta-\zeta}}$$

in inequality (3.2). Then inequality (3.2) becomes

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} L^{(\eta+\theta)(\zeta-1)} \int_a^\infty \frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[\overline{A}^{\rho}(t)]^{\eta+\theta-\zeta}}\nabla t,$$

which is the desired inequality (3.3).

(iv) In order to obtain inequality (3.4), we use inequality (3.1) and the constant L as follows.

$$\begin{aligned}\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} &\leq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[\overline{A}^{\rho}(t)]^{\eta+\theta-1}}\nabla t\\ &\leq L^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[\overline{A}(t)]^{\eta+\theta-1}}\nabla t.\end{aligned}$$

Now, if we use Hölder's inequality for the indices $\zeta>1$ and $\frac{\zeta}{\zeta-1}$ with the functions $C(t)=\dfrac{w^{\frac{1}{\zeta}}(t)f(t)[B(t)]^{\frac{\eta}{\zeta}}}{[\overline{A}(t)]^{\frac{\eta+\theta-\zeta}{\zeta}}}$ and $D(t)=\left[\dfrac{w(t)[B(t)]^{\eta+\zeta}}{[\overline{A}(t)]^{\eta+\theta}}\right]^{\frac{\zeta-1}{\zeta}}$, then we obtain

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} \leq \left[L^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta}\int_a^\infty \frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[\overline{A}(t)]^{\eta+\theta-\zeta}}\nabla t,$$

which is the desired inequality (3.4).

(v) After employing the integration by parts formula-I to the left hand side of inequality (3.5) for the functions $u^{\nabla}(t)=\dfrac{w(t)}{[\overline{A}^{\rho}(t)]^{\eta+\theta}}$ and $v(t)=[B(t)]^{\eta+\zeta}$, it becomes

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}^{\rho}(t)]^{\eta+\theta}} = u(t)B^{\eta+\zeta}(t)|_a^\infty + \int_a^\infty -u^{\rho}(t)\left[B^{\eta+\zeta}(t)\right]^{\nabla}\nabla t.$$

In view of how u^{∇} is given above, we deduce that $u(t)=-\displaystyle\int_t^\infty \frac{w(s)}{[\overline{A}^{\rho}(s)]^{\eta+\theta}}\nabla s$. By using $B(a)=0$ and $u(\infty)=0$, one can have

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[\overline{A}^{\rho}(t)]^{\eta+\theta}} = \int_a^\infty -u^{\rho}(t)\left[B^{\eta+\zeta}(t)\right]^{\nabla}\nabla t. \tag{3.12}$$

By following the procedure in the proof of (i), we arrive inequality (3.9) as

$$\left[\overline{A}^{1-\eta-\theta}(t)\right]^{\nabla} \leq -(\eta+\theta-1)\frac{w(t)}{[\overline{A}(t)]^{\eta+\theta}} \leq -(\eta+\theta-1)\frac{w(t)}{L^{\eta+\theta}[\overline{A}^{\rho}(t)]^{\eta+\theta}} \tag{3.13}$$

which implies $\dfrac{w(t)}{[\overline{A}^{\rho}(t)]^{\eta+\theta}} \leq -L^{\eta+\theta} \dfrac{\left[\overline{A}^{1-\eta-\theta}(t)\right]^{\nabla}}{\eta+\theta-1}$.

Then we can infer that

$$\begin{aligned}
-u^{\rho}(t) &= \int_{\rho(t)}^{\infty} \frac{w(s)\nabla s}{[\overline{A}^{\rho}(s)]^{\eta+\theta}} \leq \int_{\rho(t)}^{\infty} -L^{\eta+\theta} \frac{\left[\overline{A}^{1-\eta-\theta}(s)\right]^{\nabla} \nabla s}{\eta+\theta-1} = L^{\eta+\theta} \frac{-\overline{A}^{1-\eta-\theta}(\infty) + [\overline{A}^{\rho}(t)]^{1-\eta-\theta}}{\eta+\theta-1} \\
&\leq L^{\eta+\theta} \frac{[\overline{A}^{\rho}(t)]^{1-\eta-\theta}}{\eta+\theta-1}.
\end{aligned} \tag{3.14}$$

Substituting (3.8) and (3.14) into (3.12) leads to

$$\int_a^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta} \nabla t}{[\overline{A}^{\rho}(t)]^{\eta+\theta}} \leq L^{\eta+\theta} \frac{\eta+\zeta}{\eta+\theta-1} \int_a^{\infty} \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[\overline{A}^{\rho}(t)]^{\eta+\theta-1}} \nabla t,$$

which is the desired result (3.5).

(vi) In order to obtain inequality (3.6), we use inequality (3.5). Let us define the functions $C(t) = \dfrac{w^{\frac{1}{\zeta}}(t)f(t)[B(t)]^{\frac{\eta}{\zeta}}}{[\overline{A}^{\rho}(t)]^{\frac{\eta+\theta-\zeta}{\zeta}}}$ and $D(t) = \left[\dfrac{w(t)[B(t)]^{\eta+\zeta}}{[\overline{A}^{\rho}(t)]^{\eta+\theta}}\right]^{\frac{\zeta-1}{\zeta}}$. Then inequality (3.5) becomes

$$\int_a^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta} \nabla t}{[\overline{A}^{\rho}(t)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{\eta+\theta-1} \int_a^{\infty} C(t)D(t)\nabla t. \tag{3.15}$$

For the indices $\zeta > 1$ and $\frac{\zeta}{\zeta-1} > 1$ and for the functions C and D, employing Hölder's inequality (2.1) leads to an estimation on the last term of inequality (3.15) as

$$\int_a^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta} \nabla t}{[\overline{A}^{\rho}(t)]^{\eta+\theta}} \leq L^{\eta+\theta} \frac{\eta+\zeta}{\eta+\theta-1} \left[\int_a^{\infty} \frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[\overline{A}^{\rho}(t)]^{\eta+\theta-\zeta}} \nabla t\right]^{\frac{1}{\zeta}} \left[\int_a^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta} \nabla t}{[\overline{A}^{\rho}(t)]^{\eta+\theta}}\right]^{\frac{\zeta-1}{\zeta}}$$

providing

$$\int_a^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta} \nabla t}{[\overline{A}^{\rho}(t)]^{\eta+\theta}} \leq \left[L^{\eta+\theta} \frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} \int_a^{\infty} \frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[\overline{A}^{\rho}(t)]^{\eta+\theta-\zeta}} \nabla t,$$

which is the desired result (3.6).

□

Remark 3.2 *In the article [39], since the authors did not obtain Hardy–Copson type inequalities for the functions* $\overline{A}(t) = \int_a^t w(s)\Delta s$ *and* $B(t) = \int_a^t w(s)f(s)\Delta s$ *via delta time scale calculus, for the further purposes of this work, we need to present delta counterparts of the inequalities (3.1)–(3.6) in the following, respectively.*

Assume that $B(\infty) < \infty$ *and* $\int_a^{\infty} \dfrac{w(t)\Delta t}{[\overline{A}(t)]^{\eta+\theta}} < \infty$*. Suppose that there exists* $J > 0$ *such that* $\dfrac{\overline{A}^{\sigma}(t)}{\overline{A}(t)} \leq J$ *for* $t \in (a,\infty)_{\mathbb{T}}$*. If* $\eta \geq 0$*,* $\zeta > 1$ *and* $\eta+\theta > 1$ *are real constants, then*

(i)

$$\int_a^\infty \frac{w(t)[B^\sigma(t)]^{\eta+\zeta}\Delta t}{[\overline{A}^\sigma(t)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[B^\sigma(t)]^{\eta+\zeta-1}}{[\overline{A}(t)]^{\eta+\theta-1}}\Delta t.$$

(ii)

$$\int_a^\infty \frac{w(t)[B^\sigma(t)]^{\eta+\zeta}\Delta t}{[\overline{A}^\sigma(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[B^\sigma(t)]^{\eta}[\overline{A}^\sigma(t)]^{(\eta+\theta)(\zeta-1)}}{[\overline{A}(t)]^{(\eta+\theta-1)\zeta}}\Delta t.$$

(iii)

$$\int_a^\infty \frac{w(t)[B^\sigma(t)]^{\eta+\zeta}\Delta t}{[\overline{A}^\sigma(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta J^{(\eta+\theta)(\zeta-1)} \int_a^\infty \frac{w(t)f^\zeta(t)[B^\sigma(t)]^{\eta}}{[\overline{A}(t)]^{\eta+\theta-\zeta}}\Delta t.$$

(iv)

$$\int_a^\infty \frac{w(t)[B^\sigma(t)]^{\eta+\zeta}\Delta t}{[\overline{A}^\sigma(t)]^{\eta+\theta}} \leq \left[J^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[B^\sigma(t)]^{\eta}}{[\overline{A}^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t.$$

(v)

$$\int_a^\infty \frac{w(t)[B^\sigma(t)]^{\eta+\zeta}\Delta t}{[\overline{A}(t)]^{\eta+\theta}} \leq J^{\eta+\theta}\frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[B^\sigma(t)]^{\eta+\zeta-1}}{[\overline{A}(t)]^{\eta+\theta-1}}\Delta t.$$

(vi)

$$\int_a^\infty \frac{w(t)[B^\sigma(t)]^{\eta+\zeta}\Delta t}{[\overline{A}(t)]^{\eta+\theta}} \leq \left(J^{\eta+\theta}\frac{\eta+\zeta}{\eta+\theta-1}\right)^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[B^\sigma(t)]^{\eta}}{[\overline{A}(t)]^{\eta+\theta-\zeta}}\Delta t.$$

The proofs of (i)–(vi) are similar to that of (i)–(vi) of Theorem 3.1, respectively.

Since $[B(t)]^{\eta+\zeta} \leq [B^\sigma(t)]^{\eta+\zeta}$, the term $[B^\sigma(t)]^{\eta+\zeta}$ on the left hand sides of the above inequalities can be replaced by $[B(t)]^{\eta+\zeta}$.

Remark 3.3 *Although the inequalities in (i)–(iv) of Remark 3.2 can be obtained from [39], inequalities in (v) and (vi) are new even for the delta case. They are the delta counterparts of inequalities* (3.5) *and* (3.6) *and have not been obtained so far. Therefore, by Theorem 3.1, we derive novel results not only for the nabla case but also for the delta case.*

Remark 3.4 *Since $[B^\rho(t)]^{\eta+\zeta} \leq [B(t)]^{\eta+\zeta}$, the term $[B(t)]^{\eta+\zeta}$ on the left hand sides of the inequalities* (3.1)*–*(3.6) *can be replaced by $[B^\rho(t)]^{\eta+\zeta}$.*

Remark 3.5 *Let $\eta = 0$. Then inequality* (3.1) *and* (3.2) *are nabla counterparts of inequalties (2.7) and (2.8) of [35, Theorem 2.1], respectively and inequality* (3.3) *is a nabla analogue of inequality (2.18) of [35, Corollary 2.2]. Inequality* (3.4) *follows from inequality* (3.1) *while inequalities* (3.5) *and* (3.6) *are new results obtained for an arbitrary calculus of time scale even for the case $\eta = 0$.*

Remark 3.6 *If the time scale is set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L = 1$ in (3.1)–(3.6). This choice forces inequalities (3.1) and (3.5) to be the same, and inequalities (3.2)–(3.4) and (3.6) being coincident with the outcomes as*

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}dt}{[\overline{A}(t)]^{\eta+\theta}} \le \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[\overline{A}(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}dt}{[\overline{A}(t)]^{\eta+\theta}} \le \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[B(t)]^\eta}{[\overline{A}(t)]^{\eta+\theta-\zeta}}dt, \tag{3.16}$$

respectively, where $\eta \ge 0,\ \zeta > 1,\ \eta+\theta > 1,\ \overline{A}(t) = \int_a^t w(s)ds$ and $B(t) = \int_a^t w(s)f(s)ds$.

For $a = 0$, inequality (3.16) provides exactly the same result as inequality (6) in [31, Theorem 1] and inequality (3) in [32, Theorem 1] if $H(x) = x$.

Inequality (3.16) reduces to Copson's inequality in [15, Theorem 1] if $\eta = 0 = a$.

When $\eta = 0 = a$ and $w(t) = 1$ in inequality (3.16), we obtain Hardy's inequality (1.3), and if in addition $\zeta = \theta$, Hardy's inequality (1.2) is established.

If we write inequality (1.2) in the following form

$$\int_0^\infty \frac{\Phi^\zeta(t)}{t^\zeta}dt \le \left(\frac{\zeta}{\zeta-1}\right)^\zeta \int_0^\infty [\Phi'(t)]^\zeta dt,$$

then we have (note that $\Phi(0) = 0$) a generalization of Wirtinger's inequality [19, Theorem 257]. Observe that if $\zeta = \theta = 2$ in (1.3), we get the well-known inequality

$$\int_0^\infty \frac{1}{t^2}\left(\int_0^t f(s)ds\right)^2 dt \le 4\int_0^\infty f^2(t)dt,$$

due to Hardy [22, Remark 5] with the best constant 4.

Remark 3.7 *If the time scale is set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator becomes $\rho(t) = t-1$.*

Using $\int_a^{\rho(t)} w(s)\nabla s = \sum_{k=a+1}^{\rho(t)} w(k)$, we have $\overline{A}^\rho(t) = \overline{A}(t-1) = \sum_{k=a+1}^{t-1} w(k)$, where $\overline{A}(t) = \sum_{k=a+1}^{t} w(k)$ and $B(t) = \sum_{k=a+1}^{t} w(k)f(k)$. Let us assume that there exists $L > 0$ such that $\frac{\overline{A}(t)}{\overline{A}(t-1)} \le L$. In this case for $a = 0,\ \eta \ge 0,\ \zeta > 1$, and $\eta+\theta > 1$, inequalities (3.1)–(3.6) become

$$\sum_{t=1}^\infty \frac{w(t)[B(t)]^{\eta+\zeta}}{[\overline{A}(t)]^{\eta+\theta}} \le \frac{\eta+\zeta}{\eta+\theta-1}\sum_{t=1}^\infty \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[\overline{A}(t-1)]^{\eta+\theta-1}},$$

$$\sum_{t=1}^\infty \frac{w(t)[B(t)]^{\eta+\zeta}}{[\overline{A}(t)]^{\eta+\theta}} \le \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \sum_{t=1}^\infty \frac{w(t)f^\zeta(t)[B(t)]^\eta[\overline{A}(t)]^{(\eta+\theta)(\zeta-1)}}{[\overline{A}(t-1)]^{(\eta+\theta-1)\zeta}},$$

$$\sum_{t=1}^{\infty}\frac{w(t)[B(t)]^{\eta+\zeta}}{[\overline{A}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} L^{(\eta+\theta)(\zeta-1)} \sum_{t=1}^{\infty}\frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[\overline{A}(t-1)]^{\eta+\theta-\zeta}},$$

$$\sum_{t=1}^{\infty}\frac{w(t)[B(t)]^{\eta+\zeta}}{[\overline{A}(t)]^{\eta+\theta}} \leq \left[L^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} \sum_{t=1}^{\infty}\frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[\overline{A}(t)]^{\eta+\theta-\zeta}},$$

$$\sum_{t=1}^{\infty}\frac{w(t)[B(t)]^{\eta+\zeta}}{[\overline{A}(t-1)]^{\eta+\theta}} \leq L^{\eta+\theta}\frac{\eta+\zeta}{\eta+\theta-1} \sum_{t=1}^{\infty}\frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[\overline{A}(t-1)]^{\eta+\theta-1}},$$

and

$$\sum_{t=1}^{\infty}\frac{w(t)[B(t)]^{\eta+\zeta}}{[\overline{A}(t-1)]^{\eta+\theta}} \leq \left(L^{\eta+\theta}\frac{\eta+\zeta}{\eta+\theta-1}\right)^{\zeta} \sum_{t=1}^{\infty}\frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[\overline{A}(t-1)]^{\eta+\theta-\zeta}},$$

respectively. Let $\eta = 0$. If $\theta-\zeta > 0$, since inequality (1.4) implies the third and the fourth inequalities of the above estimates, inequality (1.4) is sharper than these inequalities. However, the other inequalities not only generalize the results of Hardy (1.1), Copson in [14, Theorem 1.1], and of Bennett in [9, Corollary 3] (or Leindler in [25, Proposition 1]) for $\eta \geq 0$, they are also new for the discrete case even for $\eta = 0$.

The next theorem states the nabla unification of the discrete inequality (1.8) and a nabla analogue of the delta inequality (1.15) for nonnegative, ld-continuous, ∇-differentiable, and locally nabla integrable functions $w(t)$ and $f(t)$. Moreover, this theorem provides new results for the continuous case.

Theorem 3.8 *For the functions w and f, we set $A(t) = \int_t^{\infty} w(s)\nabla s$ and $\overline{B}(t) = \int_t^{\infty} w(s)f(s)\nabla s$. Assume that $\overline{B}(a) < \infty$ and $\int_a^{\infty}\frac{w(t)\nabla t}{[A^{\rho}(t)]^{\theta+\eta}} < \infty$. Suppose that there exists $M > 0$ such that $\frac{A^{\rho}(t)}{A(t)} \leq M$ for $t \in (a,\infty)_{\mathbb{T}}$. If $\eta \geq 0$, $\zeta > 1$ and $\eta+\theta > 1$ are real constants, then*

(i)

$$\int_a^{\infty}\frac{w(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta}\nabla t}{[A^{\rho}(t)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^{\infty}\frac{w(t)f(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta-1}}{[A(t)]^{\eta+\theta-1}}\nabla t. \tag{3.17}$$

(ii)

$$\int_a^{\infty}\frac{w(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta}\nabla t}{[A^{\rho}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta}\int_a^{\infty}\frac{w(t)f^{\zeta}(t)[\overline{B}^{\rho}(t)]^{\eta}[A^{\rho}(t)]^{(\eta+\theta)(\zeta-1)}}{[A(t)]^{(\eta+\theta-1)\zeta}}\nabla t. \tag{3.18}$$

(iii)

$$\int_a^{\infty}\frac{w(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta}\nabla t}{[A^{\rho}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} M^{(\eta+\theta)(\zeta-1)}\int_a^{\infty}\frac{w(t)f^{\zeta}(t)[\overline{B}^{\rho}(t)]^{\eta}}{[A(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.19}$$

(iv)

$$\int_a^{\infty}\frac{w(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta}\nabla t}{[A^{\rho}(t)]^{\eta+\theta}} \leq \left[M^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta}\int_a^{\infty}\frac{w(t)f^{\zeta}(t)[\overline{B}^{\rho}(t)]^{\eta}}{[A^{\rho}(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.20}$$

(v)

$$\int_a^\infty \frac{w(t)[\overline{B}^\rho(t)]^{\eta+\zeta}\nabla t}{[A(t)]^{\eta+\theta}} \leq M^{\eta+\theta}\frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[\overline{B}^\rho(t)]^{\eta+\zeta-1}}{[A(t)]^{\eta+\theta-1}}\nabla t. \tag{3.21}$$

(vi)

$$\int_a^\infty \frac{w(t)[\overline{B}^\rho(t)]^{\eta+\zeta}\nabla t}{[A(t)]^{\eta+\theta}} \leq \left(M^{\eta+\theta}\frac{\eta+\zeta}{\eta+\theta-1}\right)^\zeta\int_a^\infty \frac{w(t)f^\zeta(t)[\overline{B}^\rho(t)]^{\eta}}{[A(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.22}$$

Proof

(i) After employing the integration by parts formula-II to the left hand side of inequality (3.17) for the functions $u^\nabla(t) = \dfrac{w(t)}{[A^\rho(t)]^{\eta+\theta}}$ and $v(t) = [\overline{B}(t)]^{\eta+\zeta}$, it becomes

$$\int_a^\infty \frac{w(t)[\overline{B}^\rho(t)]^{\eta+\zeta}\nabla t}{[A^\rho(t)]^{\eta+\theta}} = u(t)\overline{B}^{\eta+\zeta}(t)|_a^\infty + \int_a^\infty -u(t)\left[\overline{B}^{\eta+\zeta}(t)\right]^\nabla \nabla t.$$

In view of how u^∇ is given above, one can find that $u(t) = \displaystyle\int_a^t \frac{w(s)}{[A^\rho(s)]^{\eta+\theta}}\nabla s$. Using $\overline{B}(\infty) = 0$ and $u(a) = 0$ imply that

$$\int_a^\infty \frac{w(t)[\overline{B}^\rho(t)]^{\eta+\zeta}\nabla t}{[A^\rho(t)]^{\eta+\theta}} = \int_a^\infty -u(t)\left[\overline{B}^{\eta+\zeta}(t)\right]^\nabla \nabla t. \tag{3.23}$$

Applying (2.2) to $\left[\overline{B}^{\eta+\zeta}(t)\right]^\nabla$ and using $\overline{B}^\nabla(t) = -f(t)w(t) \leq 0$, we obtain

$$-\left[\overline{B}^{\eta+\zeta}(t)\right]^\nabla = -(\eta+\zeta)\overline{B}^{\eta+\zeta-1}(d)\overline{B}^\nabla(t) = (\eta+\zeta)w(t)f(t)\overline{B}^{\eta+\zeta-1}(d).$$

For $\eta+\zeta > 1$ and $t \geq d$, one can observe that $[\overline{B}^\rho(t)]^{\eta+\zeta-1} \geq [\overline{B}(d)]^{\eta+\zeta-1}$ and

$$-\left[\overline{B}^{\eta+\zeta}(t)\right]^\nabla \leq (\eta+\zeta)w(t)f(t)[\overline{B}^\rho(t)]^{\eta+\zeta-1}. \tag{3.24}$$

Since $A^\nabla(t) = -w(t) \leq 0$ and $\eta+\theta > 1$, by (2.3), we get

$$\begin{aligned}\left[A^{1-\eta-\theta}(t)\right]^\nabla &= \int_0^1 \frac{(1-\eta-\theta)A^\nabla(t)dh}{[hA(t)+(1-h)A^\rho(t)]^{\eta+\theta}} = \int_0^1 \frac{(\eta+\theta-1)w(t)dh}{[hA(t)+(1-h)A^\rho(t)]^{\eta+\theta}} \\ &\geq (\eta+\theta-1)\int_0^1 \frac{w(t)}{[hA^\rho(t)+(1-h)A^\rho(t)]^{\eta+\theta}}dh = (\eta+\theta-1)\frac{w(t)}{[A^\rho(t)]^{\eta+\theta}},\end{aligned} \tag{3.25}$$

which implies $\dfrac{w(t)}{[A^\rho(t)]^{\eta+\theta}} \leq \dfrac{\left[A^{1-\eta-\theta}(t)\right]^\nabla}{\eta+\theta-1}$.

Then we can infer that

$$u(t) = \int_a^t \frac{w(s)\nabla s}{[A^\rho(s)]^{\eta+\theta}} \leq \int_a^t \frac{\left[A^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{\eta+\theta-1} = \frac{-A^{1-\eta-\theta}(a)+[A(t)]^{1-\eta-\theta}}{\eta+\theta-1} \leq \frac{[A(t)]^{1-\eta-\theta}}{\eta+\theta-1}. \tag{3.26}$$

Substituting (3.24) and (3.26) into (3.23) leads to

$$\int_a^\infty \frac{w(t)[\overline{B}^\rho(t)]^{\eta+\zeta}\nabla t}{[A^\rho(t)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[\overline{B}^\rho(t)]^{\eta+\zeta-1}}{[A(t)]^{\eta+\theta-1}}\nabla t, \tag{3.27}$$

which is the desired result (3.17).

(ii) In order to obtain inequality (3.18), we use inequality (3.17). Employing the same procedure of the proof of (ii) of Theorem 3.1, we obtain the desired result (3.18).

(iii) In order to obtain inequality (3.19), we use

$$\frac{[A^\rho(t)]^{(\eta+\theta)(\zeta-1)}}{[A(t)]^{(\eta+\theta-1)\zeta}} = \left(\frac{A^\rho(t)}{A(t)}\right)^{(\eta+\theta)(\zeta-1)} \frac{[A(t)]^{-\eta-\theta}}{[A(t)]^{-\zeta}} \leq \frac{M^{(\eta+\theta)(\zeta-1)}}{[A(t)]^{\eta+\theta-\zeta}}$$

in inequality (3.18). Then the desired inequality (3.19) can be proved directly.

(iv) In order to obtain inequality (3.20), we use inequality (3.17). Employing the same procedure of the proof of (iv) of Theorem 3.1, we obtain the desired inequality (3.20).

(v) After employing the integration by parts formula-II to the left hand side of inequality (3.20) for the functions $u^\nabla(t) = \dfrac{w(t)}{[A(t)]^{\eta+\theta}}$ and $v(t) = [\overline{B}(t)]^{\eta+\zeta}$, it becomes

$$\int_a^\infty \frac{w(t)[\overline{B}(t)]^{\eta+\zeta}\nabla t}{[A(t)]^{\eta+\theta}} = u(t)\overline{B}^{\eta+\zeta}(t)|_a^\infty + \int_a^\infty -u(t)\left[\overline{B}^{\eta+\zeta}(t)\right]^\nabla \nabla t.$$

In view of how u^∇ is given above, we deduce that $u(t) = \displaystyle\int_a^t \frac{w(s)}{[A(s)]^{\eta+\theta}}\nabla s$. By using $\overline{B}(\infty) = 0$ and $u(a) = 0$, one can have

$$\int_a^\infty \frac{w(t)[\overline{B}(t)]^{\eta+\zeta}\nabla t}{[A(t)]^{\eta+\theta}} = \int_a^\infty -u(t)\left[\overline{B}^{\eta+\zeta}(t)\right]^\nabla \nabla t. \tag{3.28}$$

By following the procedure in the proof of (i), we obtain inequality (3.25) as

$$\left[A^{1-\eta-\theta}(t)\right]^\nabla \geq (\eta+\theta-1)\frac{w(t)}{[A^\rho(t)]^{\eta+\theta}} \geq \frac{\eta+\theta-1}{M^{\eta+\theta}}\frac{w(t)}{[A(t)]^{\eta+\theta}},$$

which implies $\dfrac{w(t)}{[A(t)]^{\eta+\theta}} \leq M^{\eta+\theta}\dfrac{\left[A^{1-\eta-\theta}(t)\right]^\nabla}{\eta+\theta-1}$.

Then, we can infer that

$$\begin{aligned} u(t) = \int_a^t \frac{w(s)\nabla s}{[A(s)]^{\eta+\theta}} &\leq \int_a^t M^{\eta+\theta}\frac{\left[A^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{\eta+\theta-1} = M^{\eta+\theta}\frac{-A^{1-\eta-\theta}(a)+[A(t)]^{1-\eta-\theta}}{\eta+\theta-1} \\ &\leq M^{\eta+\theta}\frac{[A(t)]^{1-\eta-\theta}}{\eta+\theta-1}. \end{aligned} \tag{3.29}$$

Substituting (3.24) and (3.29) into (3.28) leads to

$$\int_a^\infty \frac{w(t)[\overline{B}(t)]^{\eta+\zeta}\nabla t}{[A(t)]^{\eta+\theta}} \leq M^{\eta+\theta}\frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[\overline{B}^\rho(t)]^{\eta+\zeta-1}}{[A(t)]^{\eta+\theta-1}}\nabla t,$$

which is the desired result (3.21).

(vi) In order to obtain inequality (3.22), we use inequality (3.21). Employing the same procedure of the proof of (vi) of Theorem 3.1, we obtain the desired inequality (3.22).

□

Remark 3.9 *Since* $[\overline{B}(t)]^{\eta+\zeta} \leq [\overline{B}^\rho(t)]^{\eta+\zeta}$*, the term* $[B^\rho(t)]^{\eta+\zeta}$ *on the left hand sides of the inequalities* (3.17)–(3.22) *can be replaced by* $[B(t)]^{\eta+\zeta}$*.*

Remark 3.10 *Let* $\eta = 0$*. Then inequalities* (3.17) *and* (3.18) *are nabla counterparts of inequalties (2.48) and (2.49) of [35, Theorem 2.10], respectively. If* $\eta \geq 0$*, then inequality* (3.20) *is a nabla analogue of (2.17) of [39, Theorem 2.3]. Inequality* (3.19) *follows from inequality* (3.18) *while inequalities* (3.21) *and* (3.22) *are new results obtained for an arbitrary calculus of time scale even for the case* $\eta = 0$*.*

Remark 3.11 *If the time scale is set of real numbers, then for all* $t \in \mathbb{R}$*, the backward jump operator results in* $\rho(t) = t$ *and* $M = 1$ *in* (3.17)-(3.22)*. This choice forces inequalities* (3.17) *and inequality* (3.21) *to be the same, and inequalities* (3.18)–(3.20) *and* (3.22) *being coincident with the outcomes as*

$$\int_a^\infty \frac{w(t)[\overline{B}(t)]^{\eta+\zeta}dt}{[A(t)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{w(t)f(t)[\overline{B}(t)]^{\eta+\zeta-1}}{[A(t)]^{\eta+\theta-1}}dt, \tag{3.30}$$

and

$$\int_a^\infty \frac{w(t)[\overline{B}(t)]^{\eta+\zeta}dt}{[A(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[\overline{B}(t)]^{\eta}}{[A(t)]^{\eta+\theta-\zeta}}dt, \tag{3.31}$$

respectively, where $\eta \geq 0,\ \zeta > 1$ *and* $\eta+\theta > 1,\ A(t) = \int_t^\infty w(s)ds$ *and* $\overline{B}(t) = \int_t^\infty w(s)f(s)ds$*.*

Since these inequalities have not been obtained so far, they are new for the continuous case. Moreover, if $\eta = 0$*, inequality* (3.31) *reduces to a continuous analogue of Bennett's inequality in [9, Corollary 6] (or Leindler's inequality in [25, Proposition 4]).*

Remark 3.12 *If the time scale is set of natural numbers, then for all* $t \in \mathbb{N}$*, the backward jump operator the backward jump operator results in* $\rho(t) = t-1$ *in* (3.17)–(3.22)*.*

Using $\int_{\rho(t)}^\infty w(s)\nabla s = \sum_{k=\rho(t)+1}^\infty w(k)$*, we have* $A^\rho(t) = A(t-1) = \sum_{k=t}^\infty w(k)$*, where* $A(t) = \sum_{k=t+1}^\infty w(k)$ *and* $\overline{B}(t) = \sum_{k=t+1}^\infty w(k)f(k)$*. Let us assume that there exist* $M > 0$ *such that* $\frac{A(t)}{A(t-1)} \geq \frac{1}{M}$*. In this case for* $a = 0$*,* $\eta \geq 0,\ \zeta > 1,$ *and* $\eta+\theta > 1,$ *inequalities* (3.17)–(3.22) *become*

$$\sum_{t=1}^\infty \frac{w(t)[\overline{B}(t-1)]^{\eta+\zeta}}{[A(t-1)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{\eta+\theta-1}\sum_{t=1}^\infty \frac{w(t)f(t)[\overline{B}(t-1)]^{\eta+\zeta-1}}{[A(t)]^{\eta+\theta-1}},$$

$$\sum_{t=1}^{\infty} \frac{w(t)[\overline{B}(t-1)]^{\eta+\zeta}}{[A(t-1)]^{\eta+\theta}} \le \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} \sum_{t=1}^{\infty} \frac{w(t)f^{\zeta}(t)[\overline{B}(t-1)]^{\eta}[A(t-1)]^{(\eta+\theta)(\zeta-1)}}{[A(t)]^{(\eta+\theta-1)\zeta}},$$

$$\sum_{t=1}^{\infty} \frac{w(t)[\overline{B}(t-1)]^{\eta+\zeta}}{[A(t-1)]^{\eta+\theta}} \le \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} M^{(\eta+\theta)(\zeta-1)} \sum_{t=1}^{\infty} \frac{w(t)f^{\zeta}(t)[\overline{B}(t-1)]^{\eta}}{[A(t)]^{\eta+\theta-\zeta}},$$

$$\sum_{t=1}^{\infty} \frac{w(t)[\overline{B}(t-1)]^{\eta+\zeta}}{[A(t-1)]^{\eta+\theta}} \le \left[M^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} \sum_{t=1}^{\infty} \frac{w(t)f^{\zeta}(t)[\overline{B}(t-1)]^{\eta}}{[A(t-1)]^{\eta+\theta-\zeta}},$$

$$\sum_{t=1}^{\infty} \frac{w(t)[\overline{B}(t-1)]^{\eta+\zeta}}{[A(t)]^{\eta+\theta}} \le M^{\eta+\theta}\frac{\eta+\zeta}{\eta+\theta-1} \sum_{t=1}^{\infty} \frac{w(t)f(t)[\overline{B}(t-1)]^{\eta+\zeta-1}}{[A(t)]^{\eta+\theta-1}},$$

and

$$\sum_{t=1}^{\infty} \frac{w(t)[\overline{B}(t-1)]^{\eta+\zeta}}{[A(t)]^{\eta+\theta}} \le \left(M^{\eta+\theta}\frac{\eta+\zeta}{\eta+\theta-1}\right)^{\zeta} \sum_{t=1}^{\infty} \frac{w(t)f^{\zeta}(t)[\overline{B}(t-1)]^{\eta}}{[A(t)]^{\eta+\theta-\zeta}},$$

respectively. Let $\eta = 0$. If $\theta-\zeta<0$, since inequality (1.8) implies the third and the fourth inequalities of the above estimates, inequality (1.8) is sharper than these inequalities. However, the other inequalities not only generalize the results of Bennett in [9, Corollary 6] (or Leindler in [25, Proposition 4]) for $\eta \ge 0$, they are also new for the discrete case even for $\eta = 0$.

The next theorem states the nabla unification of the discrete inequalities (1.5) and (1.7) and their continuous versions (1.3), (1.11), and (1.13) as well as a nabla analogue of the delta inequality (1.16) for nonnegative, ld-continuous, ∇-differentiable, and locally nabla integrable functions $w(t)$ and $f(t)$. Moreover, this theorem provides novel results for discrete and continuous cases.

Theorem 3.13 *For the functions w and f, we set $\overline{A}(t) = \int_a^t w(s)\nabla s$ and $\overline{B}(t) = \int_t^{\infty} w(s)f(s)\nabla s$. Assume that $\overline{B}(a) < \infty$ and $\int_a^{\infty} \frac{w(t)\nabla t}{[\overline{A}^{\rho}(t)]^{\theta+\eta}} < \infty$. If $\eta \ge 0$, $\zeta > 1$ and $\eta+\theta<1$ are real constants, then*

(i)

$$\int_a^{\infty} \frac{w(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} \le \frac{\eta+\zeta}{1-\eta-\theta} \int_a^{\infty} \frac{w(t)f(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta-1}}{[\overline{A}(t)]^{\eta+\theta-1}}\nabla t. \tag{3.32}$$

(ii)

$$\int_a^{\infty} \frac{w(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} \le \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{w(t)f^{\zeta}(t)[\overline{B}^{\rho}(t)]^{\eta}}{[\overline{A}(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.33}$$

Proof

(i) After employing the integration by parts formula-II to the left hand side of inequality (3.32) for the functions $u^{\nabla}(t) = \frac{w(t)}{[\overline{A}(t)]^{\eta+\theta}}$ and $v(t) = [\overline{B}(t)]^{\eta+\zeta}$, it becomes

$$\int_a^{\infty} \frac{w(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta}\nabla t}{[\overline{A}(t)]^{\eta+\theta}} = u(t)\overline{B}^{\eta+\zeta}(t)|_a^{\infty} + \int_a^{\infty} -u(t)\left[\overline{B}^{\eta+\zeta}(t)\right]^{\nabla}\nabla t.$$

In view of how u^{∇} is given above, we deduce that $u(t) = \int_a^t \frac{w(s)}{[\overline{A}(s)]^{\eta+\theta}} \nabla s$. By using $\overline{B}(\infty) = 0$ and $u(a) = 0$, one can have

$$\int_a^{\infty} \frac{w(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta} \nabla t}{[\overline{A}(t)]^{\eta+\theta}} = \int_a^{\infty} -u(t) \left[\overline{B}^{\eta+\zeta}(t)\right]^{\nabla} \nabla t. \tag{3.34}$$

Applying (2.2) to $\left[\overline{B}^{\eta+\zeta}(t)\right]^{\nabla}$ and using $\overline{B}^{\nabla}(t) = -f(t)w(t) \leq 0$, we obtain

$$-\left[\overline{B}^{\eta+\zeta}(t)\right]^{\nabla} = -(\eta+\zeta)\overline{B}^{\eta+\zeta-1}(d)\overline{B}^{\nabla}(t) = (\eta+\zeta)w(t)f(t)\overline{B}^{\eta+\zeta-1}(d).$$

For $\eta+\zeta > 1$ and $t \geq d$, one can observe that $[\overline{B}^{\rho}(t)]^{\eta+\zeta-1} \geq [\overline{B}(d)]^{\eta+\zeta-1}$ and

$$-\left[\overline{B}^{\eta+\zeta}(t)\right]^{\nabla} \leq (\eta+\zeta)w(t)f(t)[\overline{B}^{\rho}(t)]^{\eta+\zeta-1}. \tag{3.35}$$

Since $\overline{A}^{\nabla}(t) = w(t) \geq 0$ and $0 \leq \eta+\theta < 1$, by (2.3), we get

$$\begin{aligned}\left[\overline{A}^{1-\eta-\theta}(t)\right]^{\nabla} &= \int_0^1 \frac{(1-\eta-\theta)\overline{A}^{\nabla}(t)dh}{[h\overline{A}(t)+(1-h)\overline{A}^{\rho}(t)]^{\eta+\theta}} = \int_0^1 \frac{(1-\eta-\theta)w(t)dh}{[h\overline{A}(t)+(1-h)\overline{A}^{\rho}(t)]^{\eta+\theta}} \\ &\geq (1-\eta-\theta)\int_0^1 \frac{w(t)}{[h\overline{A}(t)+(1-h)\overline{A}(t)]^{\eta+\theta}}dh = (1-\eta-\theta)\frac{w(t)}{[\overline{A}(t)]^{\eta+\theta}},\end{aligned}$$

which implies $\frac{w(t)}{[\overline{A}(t)]^{\eta+\theta}} \leq \frac{\left[\overline{A}^{1-\eta-\theta}(t)\right]^{\nabla}}{1-\eta-\theta}$.

Then we can infer that

$$u(t) = \int_a^t \frac{w(s)\nabla s}{[\overline{A}(s)]^{\eta+\theta}} \leq \int_a^t \frac{\left[\overline{A}^{1-\eta-\theta}(s)\right]^{\nabla} \nabla s}{1-\eta-\theta} = \frac{-\overline{A}^{1-\eta-\theta}(a) + [\overline{A}(t)]^{1-\eta-\theta}}{1-\eta-\theta} \leq \frac{[\overline{A}(t)]^{1-\eta-\theta}}{1-\eta-\theta}. \tag{3.36}$$

Substituting (3.35) and (3.36) into (3.34) leads to the desired result (3.32).

(ii) In order to obtain inequality (3.33), we will use inequality (3.32). Employing the same procedure of the proof of (ii) of Theorem 3.1, we obtain the desired result (3.33).

□

Remark 3.14 *Since $[\overline{B}(t)]^{\eta+\zeta} \leq [\overline{B}^{\rho}(t)]^{\eta+\zeta}$, the term $[B^{\rho}(t)]^{\eta+\zeta}$ on the left hand sides of the inequalities (3.32) and (3.33) can be replaced by $[B(t)]^{\eta+\zeta}$.*

Remark 3.15 *Let $\eta = 0$. Then inequalities (3.32) and (3.33) are nabla counterparts of inequalties (2.21) and (2.22) of [35, Theorem 2.5], respectively. If $\eta \geq 0$, then inequality (3.33) is a nabla analogue of (2.4) of [39, Theorem 2.1].*

Remark 3.16 *If we choose $\eta = 0$ and $w(t) = 1$ in inequality (3.33), then for $0 \le \theta < 1$, we have*

$$\int_a^\infty \frac{1}{(t-a)^\theta} \left(\int_t^\infty f(s) \nabla s \right)^\zeta \nabla t \le \left[\frac{\zeta}{1-\theta} \right]^\zeta \int_a^\infty \frac{f^\zeta(t)}{(t-a)^{\theta-\zeta}} \nabla t. \tag{3.37}$$

Remark 3.17 *If the time scale is set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ in (3.32) and (3.33). Hence, inequalities (3.32) and (3.33) become*

$$\int_a^\infty \frac{w(t)[\overline{B}(t)]^{\eta+\zeta} dt}{[\overline{A}(t)]^{\eta+\theta}} \le \frac{\eta+\zeta}{1-\eta-\theta} \int_a^\infty \frac{w(t)f(t)[\overline{B}(t)]^{\eta+\zeta-1}}{[\overline{A}(t)]^{\eta+\theta-1}} dt,$$

and

$$\int_a^\infty \frac{w(t)[\overline{B}(t)]^{\eta+\zeta} dt}{[\overline{A}(t)]^{\eta+\theta}} \le \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[\overline{B}(t)]^{\eta}}{[\overline{A}(t)]^{\eta+\theta-\zeta}} dt, \tag{3.38}$$

respectively, where $\eta \ge 0$, $\zeta > 1$, and $\eta + \theta < 1$, $\overline{A}(t) = \int_a^t w(s) ds$ and $\overline{B}(t) = \int_t^\infty w(s) f(s) ds$.

For $a = 0$, inequality (3.38) provides exactly the same result as inequality (6) in [31, Theorem 1] and inequality (9) in [32, Theorem 3] if $H(x) = x$.

Inequality (3.38) reduces to Copson's inequality in [15, Theorem 3] if $\eta = 0$.

When $\eta = 0 = a$ and $w(t) = 1$ in inequality (3.38), we have Hardy inequality (1.3) and if in addition $\zeta = \theta$, Hardy's inequality (1.2) is established.

Remark 3.18 *If the time scale is set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (3.32)-(3.33).*

Using $\int_a^{\rho(t)} w(s) \nabla s = \sum_{k=a+1}^{\rho(t)} w(k)$, we have $\overline{A}^\rho(t) = \overline{A}(t-1) = \sum_{k=a+1}^{t-1} w(k)$, where $\overline{A}(t) = \sum_{k=a+1}^{t} w(k)$ and $\overline{B}(t) = \sum_{k=t+1}^{\infty} w(k) f(k)$. In this case for $a = 0$, $\eta \ge 0$, $\zeta > 1$ and $\eta + \theta < 1$, inequalities (3.32) and (3.33) become

$$\sum_{t=1}^\infty \frac{w(t)[\overline{B}(t-1)]^{\eta+\zeta}}{[\overline{A}(t)]^{\eta+\theta}} \le \frac{\eta+\zeta}{\eta+\theta-1} \sum_{t=1}^\infty \frac{w(t)f(t)[\overline{B}(t-1)]^{\eta+\zeta-1}}{[\overline{A}(t)]^{\eta+\theta-1}},$$

and

$$\sum_{t=1}^\infty \frac{w(t)[\overline{B}(t-1)]^{\eta+\zeta}}{[\overline{A}(t)]^{\eta+\theta}} \le \left[\frac{\eta+\zeta}{\eta+\theta-1} \right]^\zeta \sum_{t=1}^\infty \frac{w(t)f^\zeta(t)[\overline{B}(t-1)]^{\eta}}{[\overline{A}(t)]^{\eta+\theta-\zeta}}.$$

respectively. These inequalities are sharper than the results of Copson in [14, Theorem 2.1] and of Bennett in [9, Corollary 4] (or Leindler in [25, Proposition 2]) in the special cases and generalize them for $\eta \ge 0$.

The next theorem states the nabla unification of the discrete inequality (1.6) and a nabla analogue of the delta inequality (1.14) for nonnegative, ld-continuous, ∇-differentiable, and locally nabla integrable functions $w(t)$ and $f(t)$. Moreover, this theorem provides new results for the continuous case.

Theorem 3.19 *For the functions w and f, we set $A(t)=\int_t^\infty w(s)\nabla s$ and $B(t)=\int_a^t w(s)f(s)\nabla s$. Assume that $B(\infty)<\infty$ and $\int_a^\infty \frac{w(t)\nabla t}{[A^\rho(t)]^{\theta+\eta}}<\infty$. If $\eta\geq 0,\ \zeta>1$ and $\eta+\theta<1$ are real constants, then*

(i)

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[A^\rho(t)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^\infty \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[A^\rho(t)]^{\eta+\theta-1}}\nabla t. \tag{3.39}$$

(ii)

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[A^\rho(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{w(t)f^\zeta(t)[B(t)]^{\eta}}{[A^\rho(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.40}$$

Proof

(i) After employing the integration by parts formula-I to the left hand side of inequality (3.39) for the functions $u^\nabla(t)=\frac{w(t)}{[A^\rho(t)]^{\eta+\theta}}$ and $v(t)=[B(t)]^{\eta+\zeta}$, it becomes

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[A^\rho(t)]^{\eta+\theta}} = u(t)B^{\eta+\zeta}(t)|_a^\infty + \int_a^\infty -u^\rho(t)\left[B^{\eta+\zeta}(t)\right]^\nabla \nabla t.$$

In view of how u^∇ is given above, we deduce that $u(t)=-\int_t^\infty \frac{w(s)}{[A^\rho(s)]^{\eta+\theta}}\nabla s$. By using $B(a)=0$ and $u(\infty)=0$, one can have

$$\int_a^\infty \frac{w(t)[B(t)]^{\eta+\zeta}\nabla t}{[A^\rho(t)]^{\eta+\theta}} = \int_a^\infty -u^\rho(t)\left[B^{\eta+\zeta}(t)\right]^\nabla \nabla t. \tag{3.41}$$

Applying (2.2) to $\left[B^{\eta+\zeta}(t)\right]^\nabla$ and using $B^\nabla(t)=f(t)w(t)\geq 0$, we obtain

$$\left[B^{\eta+\zeta}(t)\right]^\nabla = (\eta+\zeta)B^{\eta+\zeta-1}(d)B^\nabla(t) = (\eta+\zeta)w(t)f(t)B^{\eta+\zeta-1}(d).$$

For $\eta+\zeta>1$ and $t\geq d$, one can observe that $[B(t)]^{\eta+\zeta-1}\geq [B(d)]^{\eta+\zeta-1}$ and

$$\left[B^{\eta+\zeta}(t)\right]^\nabla \leq (\eta+\zeta)w(t)f(t)[B(t)]^{\eta+\zeta-1}. \tag{3.42}$$

Since $A^\nabla(t)=-w(t)\leq 0$ and $0\leq\eta+\theta<1$, by the chain rule for nabla derivative (2.3), we get

$$\begin{aligned}\left[A^{1-\eta-\theta}(t)\right]^\nabla &= \int_0^1 \frac{(1-\eta-\theta)A^\nabla(t)dh}{[hA(t)+(1-h)A^\rho(t)]^{\eta+\theta}} = \int_0^1 \frac{-(1-\eta-\theta)w(t)dh}{[hA(t)+(1-h)A^\rho(t)]^{\eta+\theta}}\\ &\leq (1-\eta-\theta)\int_0^1 \frac{w(t)}{[hA^\rho(t)+(1-h)A^\rho(t)]^{\eta+\theta}}dh = -(1-\eta-\theta)\frac{w(t)}{[A^\rho(t)]^{\eta+\theta}},\end{aligned}$$

which implies that $\frac{w(t)}{[A^\rho(t)]^{\eta+\theta}} \leq \frac{-\left[A^{1-\eta-\theta}(t)\right]^\nabla}{1-\eta-\theta}$.

Then we can infer that

$$\begin{aligned}-u^{\rho}(t) &= \int_{\rho(t)}^{\infty} \frac{w(s)\nabla s}{[A^{\rho}(s)]^{\eta+\theta}} \leq \int_{\rho(t)}^{\infty} \frac{-\left[A^{1-\eta-\theta}(s)\right]^{\nabla} \nabla s}{1-\eta-\theta} = \frac{-A^{1-\eta-\theta}(\infty)+[A^{\rho}(t)]^{1-\eta-\theta}}{1-\eta-\theta} \\ &\leq \frac{[A^{\rho}(t)]^{1-\eta-\theta}}{1-\eta-\theta}.\end{aligned} \tag{3.43}$$

Substituting (3.42) and (3.43) into (3.41) leads to the desired result (3.39).

(ii) In order to obtain inequality (3.40), we will use inequality (3.39) and we will apply the same arguments used in the proof of (ii) of Theorem 3.1.

□

Remark 3.20 *Since $[B^{\rho}(t)]^{\eta+\zeta} \leq [B(t)]^{\eta+\zeta}$, the term $[B(t)]^{\eta+\zeta}$ on the left hand sides of the inequalities (3.39) and (3.40) can be replaced by $[B^{\rho}(t)]^{\eta+\zeta}$.*

Remark 3.21 *Let $\eta = 0$. Then inequalities (3.39) and (3.40) are nabla counterparts of the inequalties given in [35, Theorem 2.9], respectively. If $\eta \geq 0$, then inequality (3.40) is a nabla analogue of inequality (2.11) of [39, Theorem 2.2].*

Remark 3.22 *If the time scale is set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ in (3.39) and (3.40). Hence, inequalities (3.39) and (3.40) become*

$$\int_{a}^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta} dt}{[A(t)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{1-\eta-\theta} \int_{a}^{\infty} \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[A(t)]^{\eta+\theta-1}} dt,$$

and

$$\int_{a}^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta} dt}{[A(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_{a}^{\infty} \frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[A(t)]^{\eta+\theta-\zeta}} dt, \tag{3.44}$$

respectively, where $\eta \geq 0$, $\zeta > 1$ and $\eta+\theta < 1$ $A(t) = \int_{t}^{\infty} w(s)ds$ and $B(t) = \int_{a}^{t} w(s)f(s)ds$.

Since these inequalities have not been obtained so far, they are new for the continuous case. Moreover, if $\eta = 0$, inequality (3.44) reduces to a continuous analogue of Bennett's inequality in [9, Corollary 5] (or Leindler's inequality in [25, Proposition 3]).

Remark 3.23 *If the time scale is set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator becomes $\rho(t) = t-1$ in (3.39) and (3.40).*

Using $\int_{\rho(t)}^{\infty} w(s)\nabla s = \sum_{k=\rho(t)+1}^{\infty} w(k)$, we have $A^{\rho}(t) = A(t-1) = \sum_{k=t}^{\infty} w(k)$, where $A(t) = \sum_{k=t+1}^{\infty} w(k)$. Moreover, $B(t) = \sum_{k=a+1}^{t} w(k)f(k)$. In this case, for $a = 0$, $\eta \geq 0$, $\zeta > 1$, and $\eta+\theta < 1$, inequalities (3.39) and (3.40) become

$$\sum_{t=1}^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta}}{[A(t-1)]^{\eta+\theta}} \leq \frac{\eta+\zeta}{1-\eta-\theta} \sum_{t=1}^{\infty} \frac{w(t)f(t)[B(t)]^{\eta+\zeta-1}}{[A(t-1)]^{\eta+\theta-1}},$$

and

$$\sum_{t=1}^{\infty} \frac{w(t)[B(t)]^{\eta+\zeta}}{[A(t-1)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \sum_{t=1}^{\infty} \frac{w(t)f^{\zeta}(t)[B(t)]^{\eta}}{[A(t-1)]^{\eta+\theta-\zeta}},$$

respectively. The second inequality is sharper than the result of Bennett [9, Corollary 5] (or Leindler [25, Proposition 3]) in the special case $\eta = 0$ and generalizes it for $\eta \geq 0$ while the first one is new for the discrete case and for all $\eta \geq 0$.

4. Conclusion

Since time scale calculus enables us to avoid the separate discussion of the two cases, which are continuous and discrete cases, the unification of these cases by nabla calculus has gained importance in recent years. In this paper, nabla Hardy–Copson type inequalities, which are analagous to the results of the delta time scale calculus and are unifications of the related continuous and discrete results, are established. Some of these inequalities are obtained similarly from the results on the delta calculus while the others appear here for the first time even for the special cases. Beside their generalization properties, these inequalities could serve as starting points for the new results in diamond alpha calculus, which is a linear combination of delta and nabla time scale calculi.

References

[1] Agarwal RP, Bohner M, Peterson A. Inequalities on time scales: A survey. Mathematical Inequalities & Applications 2001; 4 (4): 535-557.

[2] Agarwal RP, O'Regan D, Saker S. Dynamic Inequalities On Time Scales. Switzerland: Springer International Publishing, 2014.

[3] Agarwal RP, O'Regan D, Saker S. Hardy Type Inequalities on Time Scales. Switzerland: Springer International Publishing, 2016.

[4] Agarwal RP, Mahmoud RR, Saker S, Tunç C. New generalizations of Németh-Mohapatra type inequalities on time scales. Acta Mathematica Hungarica 2017; 152 (2): 383-403.

[5] Anastassiou GA. Foundations of nabla fractional calculus on time scales and inequalities. Computers and Mathematics with Applications 2010; 59: 3750-3762.

[6] Atici FM, Guseinov GSh. On Green's functions and positive solutions for boundary value problems on time scales. Journal of Computational and Applied Mathematics 2002; 141: 75-99.

[7] Balinsky AA, Evans WD, Lewis RT. The Analysis and Geometry of Hardy's Inequality. Switzerland: Springer International Publishing, 2015.

[8] Beesack PR. Hardy's inequality and its extensions. Pacific Journal of Mathematics 1961; 11: 39-61.

[9] Bennett G. Some elementary inequalities. Quarterly Journal of Mathematics 1987; 38 (4): 401-425.

[10] Bohner M, Peterson A. Dynamic Equations on Times Scales: An Introduction with Applications. Boston, MA, USA: Springer Science+Business Media New York, 2001.

[11] Bohner M, Mahmoud R, Saker SH. Discrete, continuous, delta, nabla, and diamond-alpha Opial inequalities. Mathematical Inequalities & Applications 2015; 18 (3): 923-940.

[12] Caputo MC. Time scales: From nabla calculus to delta calculus and vice versa via duality. International Journal of Difference Equations 2010; 5 (1): 25–40.

[13] Chu YM, Xu Q, Zhang XM. A note on Hardy's inequality. Journal of Inequalities and Applications 2014; 2014 (271): 1-10.

[14] Copson ET. Note on series of positive terms. Journal of the London Mathematical Society 1928; 3: 49-51.

[15] Copson ET. Some integral inequalities. Proceedings of the Royal Society of Edinburgh Section A 1976; 75 (2): 157-164.

[16] Güvenilir, AF, Kaymakçalan B, Pelen NN. Constantin's inequality for nabla and diamond-alpha derivative. Journal of Inequalities and Applications 2015; 2015 (167): 1-17.

[17] Hardy GH. Notes on a theorem of Hilbert. Mathematische Zeitschrift 1920; 6 (3-4): 314-317.

[18] Hardy GH. Notes on some points in the integral calculus, LX. An inequality between integrals. Messenger of Mathematics 1925; 54: 150-156.

[19] Hardy GH, Littlewood JE, Pólya G. Inequalities. London, UK: Cambridge University Press, 1934.

[20] Iddrisu MM, Okpoti AC, Gbolagade AK. Some proofs of the classical integral Hardy inequality. The Korean Journal of Mathematics 2014; 22: 407-417.

[21] Kufner A, Maligranda L, Persson LE. The Prehistory of the Hardy Inequality. The American Mathematical Monthly 2006; 113 (8): 715-732.

[22] Kufner A, Maligranda L, Persson LE. The Hardy Inequality. About Its History and Some Related Results. Pilsen: Vydavatelský Servis, 2007.

[23] Kufner A, Persson LE, Samko N. Weighted Inequalities of Hardy Type. New Jersey, USA: World Scientific, 2003.

[24] Landau E, Schur I, Hardy GH. A note on a theorem concerning series of positive terms: extract from a letter. Journal of the London Mathematical Society 1926; 1: 38-39.

[25] Leindler L. Some inequalities pertaining to Bennett's results. Acta Scientiarum Mathematicarum 1993; 58: 261-279.

[26] Liao ZW. Discrete Hardy-type inequalities. Advanced Nonlinear Studies 2015; 15: 805-834.

[27] Masmoudi N. About the Hardy inequality. In: Schleicher D, Lackmann M (editor). An Invitation to Mathematics. From Competitions to Research. Heidelberg, Germany: Springer, 2011, pp. 165-180.

[28] Mitrinovic DS, Dragoslav S, Pecaric J, Fink AM. Inequalities Involving Functions and Their Integrals and Derivatives. The Netherlands: Springer, 1991.

[29] Nikolidakis EN. A sharp integral Hardy type inequality and applications to Muckenhoupt weights on $\mathbb{R}$. Annales Academiae Scientiarum Fennicae Mathematica 2014; 39: 887-896.

[30] Özkan UM, Sarikaya MZ, Yildirim H. Extensions of certain integral inequalities on time scales. Applied Mathematics Letters 2008; 21: 993-1000.

[31] Pachpatte BG. On some generalizations of Hardy's integral inequality. Journal of Mathematical Analysis and Applications 1999; 234: 15-30.

[32] Pečarić J, Hanjš Ž. On some generalizations of inequalities given by B. G. Pachpatte. Annals of the Alexandru Cuza University. Mathematics 1999; 45 (1): 103-114.

[33] Saker SH. Dynamic inequalities on time scales: A survey. Fractional Calculus and Applied Analysis 2012; 3(S) (2): 1-36.

[34] Saker SH, O'Regan D, Agarwal RP. Dynamic inequalities of Hardy and Copson type on time scales. Analysis 2014; 34 (4): 391-402.

[35] Saker SH, O'Regan D, Agarwal RP. Generalized Hardy, Copson, Leindler and Bennett inequalities on time scales. Mathematische Nachrichten 2014; 287 (5-6): 686-698.

[36] Saker SH, O'Regan D, Agarwal RP. Converses of Copson's inequalities on time scales. Mathematical Inequalities & Applications 2015; 18 (1): 241-254.

[37] Saker SH, Mahmoud RR, Peterson A. Some Bennett-Copson type inequalities on time scales. Journal of Inequalities and Applications 2016; 10 (2): 471-489.

[38] Saker SH, Mahmoud RR, Osman MM, Agarwal RP. Some new generalized forms of Hardy's type inequality on time scales. Mathematical Inequalities & Applications 2017; 20 (2): 459-481.

[39] Saker SH, Osman MM, O'Regan D, Agarwal RP. Inequalities of Hardy type and generalizations on time scales. Analysis 2018; 38 (1): 47-62.

[40] Saker SH, Mahmoud RR, Peterson A. A unified approach to Copson and Ceesack type inequalities on time scales. Mathematical Inequalities & Applications 2018; 21 (4): 985-1002.