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Gauss–Bonnet theorems and the Lorentzian Heisenberg group

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Abstract: In this paper, we compute sub-Riemannian limits of Gaussian curvature for a C^2-smooth surface in the Lorentzian Heisenberg group for the second Lorentzian metric and the third Lorentzian metric and signed geodesic curvature for C^2-smooth curves on surfaces. We get Gauss–Bonnet theorems in the Lorentzian Heisenberg group for the second Lorentzian metric and the third Lorentzian metric.

Key words: Lorentzian Heisenberg group, the second Lorentzian metric, the third Lorentzian metric, Gauss–Bonnet theorem, sub-Riemannian limit

1. Introduction

In [6], Diniz and Veloso prove a version of the Gauss–Bonnet theorem in sub-Riemannian Heisenberg space $\mathbb{H}^1$ and define a Gaussian curvature for nonhorizontal surfaces in sub-Riemannian Heisenberg space $\mathbb{H}^1$. The definition was analogous to Gauss curvature of surfaces in $\mathbb{R}^3$ with particular normal to surface and Hausdorff measure of area. The image of Gauss map was in the cylinder of radius one. In [2], Balogh et al. propose a suitable candidate for the notion of intrinsic Gaussian curvature for Euclidean C^2-smooth surface in the first Heisenberg group $\mathbb{H}^1$. These results were then used to prove a Heisenberg version of the Gauss–Bonnet theorem. In [10], Veloso verified that Gausssian curvature of surfaces and normal curvature of curves in surfaces introduced by [6] and by [2] to prove Gauss–Bonnet theorems in Heisenberg space $\mathbb{H}^1$ were unequal and he applied the same formalism of [6] to get the curvatures of [2]. With the obtained formulas, it is possible to prove the Gauss–Bonnet theorem as a straightforward application of Stokes theorem. In [8] Gilkey and Park use analytic continuation to derive the Chern–Gauss–Bonnet theorem for pseudo-Riemannian manifolds with boundary directly from the corresponding result in the Riemannian setting. In [9], Rahmani and Rahmani proved that there are three different metrics of the sense of isometry in the Lorentzian groups. In [11], Wang and Wei gave sub-Riemannian limits of Gaussian curvature for a C^2-smooth surface in the Lorentzian Heisenberg group for the first Lorentzian metric and the Lorentzian group of rigid motions of the Minkowski plane away from characteristic points and signed geodesic curvature for C^2-smooth curves on surfaces. And they got Gauss–Bonnet theorems in the Lorentzian Heisenberg group for the first Lorentzian metric and the Lorentzian group of rigid motions of the Minkowski plane. In this paper, we will use similar methods to show that Gauss–Bonnet theorem holding in the Lorentzian Heisenberg group for the second Lorentzian metric and for the third Lorentzian metric.

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The Riemannian approximation scheme used in [2], can in general depend upon the choice of the complement to the horizontal distribution. In the context of $\mathbb{H}^1$ the choice which they have adopted is rather natural. The existence of the limit defining the intrinsic curvature of a surface depends crucially on the cancellation of certain divergent quantities in the limit. Such cancellation stems from the specific choice of the adapted frame bundle on the surface, and on symmetries of the underlying left-invariant group structure on the Heisenberg group. In [2], they proposed an interesting question which is solved in this paper to understand to what extent similar phenomena hold in other sub-Riemannian geometric structures. The main results of this paper are Gauss–Bonnet type theorems for Lorentzian surfaces in Lorentzian Heisenberg group for the second Lorentzian metric and the Lorentzian Heisenberg group for the third Lorentzian metric.

In Section 2, we compute the sub-Riemannian limit of curvature of curves in the Lorentzian Heisenberg group for the second Lorentzian metric. In Section 3, we compute sub-Riemannian limits of geodesic curvature of curves on Lorentzian surfaces and the Riemannian Gaussian curvature of surfaces in the Lorentzian Heisenberg group for the second Lorentzian metric. We prove the Gauss–Bonnet theorem in the Lorentzian Heisenberg group for the second Lorentzian metric. In Section 4, we compute the sub-Riemannian limit of curvature of curves in the Lorentzian Heisenberg group for the third Lorentzian metric. In Section 5, we compute sub-Riemannian limits of geodesic curvature of curves on Lorentzian surfaces and the Riemannian Gaussian curvature of surfaces in the Lorentzian Heisenberg group for the third Lorentzian metric. We prove the Gauss–Bonnet theorem in the Lorentzian Heisenberg group for the third Lorentzian metric.

2. The sub-Riemannian limit of curvature of curves in the Lorentzian Heisenberg group for the second Lorentzian metric

In this section we prepare some basic notions in the Lorentzian Heisenberg group. Let $\mathbb{H}$ be the Heisenberg group where the noncommutative group law is given by

$$(\overline{x},\overline{y},\overline{z})\star(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\overline{x}y+x\overline{y}).$$

Let

$$X_1=\partial_{x_1},\quad X_2=\partial_{x_2}-x_1\partial_{x_3},\quad X_3=\partial_{x_3},\tag{2.1}$$

then

$$\partial_{x_1}=X_1,\quad \partial_{x_2}=X_2+x_1X_3,\quad \partial_{x_3}=X_3,\tag{2.2}$$

and $\mathrm{span}\{X_1,X_2,X_3\}=T\mathbb{H}$. Let $H=\mathrm{span}\{X_1,X_2\}$ be the horizontal distribution on $\mathbb{H}$. Let $\omega_1=dx_1$, $\omega_2=dx_2$, $\omega=x_1dx_2+dx_3$. Then $H=\mathrm{Ker}\omega$. For the constant $L>0$, let $g_L=\omega_1\otimes\omega_1+\omega_2\otimes\omega_2-L\omega\otimes\omega$, $g=g_1$ be a Lorentzian metric on $\mathbb{H}$(see [9]). We call $(\mathbb{H},g_L)$ a Lorentzian Heisenberg group and wirte $\mathbb{H}^2_L$ instead of $(\mathbb{H},g_L)$. Then $X_1,X_2,\widetilde{X_3}:=L^{-\frac{1}{2}}X_3$ are orthonormal basis on $T\mathbb{H}^2_L$ with respect to g_L. We have

$$[X_1,X_2]=-X_3,\quad [X_2,X_3]=0,\quad [X_1,X_3]=0.\tag{2.3}$$

We say that a nonzero vector $\mathbf{x}\in\mathbb{H}^2_L$ is *spacelike*, *null* or *timelike* if $\langle\mathbf{x},\mathbf{x}\rangle>0$, $\langle\mathbf{x},\mathbf{x}\rangle=0$ or $\langle\mathbf{x},\mathbf{x}\rangle<0$ respectively. The norm of the vector $\mathbf{x}\in\mathbb{H}^2_L$ is defined by $\|\mathbf{x}\|=\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|}$. Let $\gamma:I\to\mathbb{H}^2_L$ be a regular curve, where I is an open interval in $\mathbb{R}$. The regular curve γ is called a spacelike curve, timelike curve or null curve if $\gamma'(t)$ is a spacelike vector, timelike vector or null vector at any $t\in I$ respectively.

Let ∇^L be the Levi–Civita connection on $\mathbb{H}^2_L$ with respect to g_L. By the Koszul formula, we have

$$2\langle\nabla^L_{X_i}X_j,X_k\rangle_L=\langle[X_i,X_j],X_k\rangle_L-\langle[X_j,X_k],X_i\rangle_L+\langle[X_k,X_i],X_j\rangle_L,\tag{2.4}$$

where $i, j, k = 1, 2, 3$. By (2.3) and (2.4), we have

Lemma 2.1

$$\nabla^L_{X_j} X_j = 0, \quad 1 \le j \le 3, \quad \nabla^L_{X_1} X_2 = -\frac{1}{2} X_3, \quad \nabla^L_{X_2} X_1 = \frac{1}{2} X_3, \tag{2.5}$$
$$\nabla^L_{X_1} X_3 = \nabla^L_{X_3} X_1 = -\frac{L}{2} X_2,, \quad \nabla^L_{X_2} X_3 = \nabla^L_{X_3} X_2 = \frac{L}{2} X_1.$$

Definition 2.2 *Let $\gamma : I \to \mathbb{H}^2_L$ be a C^1-smooth curve. We say that γ is regular if $\dot{\gamma} \neq 0$ for every $t \in I$. Moreover we say that $\gamma(t)$ is a horizontal point of γ if*

$$\omega(\dot{\gamma}(t)) = \gamma_1(t)\dot{\gamma}_2(t) + \dot{\gamma}_3(t) = 0.$$

We recall that in Riemannian geometry the standard definition of curvature for a curve γ parametrized by arc length is $k^L_\gamma := \|\nabla^L_{\dot{\gamma}} \dot{\gamma}\|$. For curves with an arbitrary parametrization, we give the definitions as follows:

Definition 2.3 *Let $\gamma : I \to \mathbb{H}^2_L$ be a C^2-smooth regular curve.*
(1)If $\nabla^L_{\dot{\gamma}} \dot{\gamma}$ is a spacelike vector, the curvature k^L_γ of γ at $\gamma(t)$ is defined as

$$k^L_\gamma := \sqrt{\frac{||\nabla^L_{\dot{\gamma}} \dot{\gamma}||^2_L}{||\dot{\gamma}||^4_L} - \frac{\langle \nabla^L_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle^2_L}{\langle \dot{\gamma}, \dot{\gamma} \rangle^3_L}}. \tag{2.6}$$

(2)If $\nabla^L_{\dot{\gamma}} \dot{\gamma}$ is a timelike vector, the curvature k^L_γ of γ at $\gamma(t)$ is defined as

$$k^L_\gamma := \sqrt{\frac{||\nabla^L_{\dot{\gamma}} \dot{\gamma}||^2_L}{||\dot{\gamma}||^4_L} + \frac{\langle \nabla^L_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle^2_L}{\langle \dot{\gamma}, \dot{\gamma} \rangle^3_L}}. \tag{2.7}$$

Lemma 2.4 *Let $\gamma : I \to \mathbb{H}^2_L$ be a C^2-smooth regular curve.*
(1)If $\nabla^L_{\dot{\gamma}} \dot{\gamma}$ is a spacelike vector, then

$$k^L_\gamma = \Bigg\{ \Bigg\{ [\ddot{\gamma}_1 + L\dot{\gamma}_2 \omega(\dot{\gamma}(t))]^2 + [\ddot{\gamma}_2 - L\dot{\gamma}_1 \omega(\dot{\gamma}(t))]^2 - L \left[\frac{d}{dt} (\omega(\dot{\gamma}(t))) \right]^2 \Bigg\} \tag{2.8}$$
$$\cdot \left[\dot{\gamma}_1^2 + \dot{\gamma}_2^2 - L(\omega(\dot{\gamma}(t)))^2 \right]^{-2} - [\dot{\gamma}_1(\ddot{\gamma}_1 + L\dot{\gamma}_2 \omega(\dot{\gamma}(t)))$$
$$+ \dot{\gamma}_2(\ddot{\gamma}_2 - L\dot{\gamma}_1 \omega(\dot{\gamma}(t))) - L\omega(\dot{\gamma}(t)) \frac{d}{dt} (\omega(\dot{\gamma}(t))) \Big]^2 \cdot \left[\dot{\gamma}_1^2 + \dot{\gamma}_2^2 - L(\omega(\dot{\gamma}(t)))^2 \right]^{-3} \Bigg\}^{\frac{1}{2}}.$$

In particular, if $\gamma(t)$ is a horizontal point of γ, then

$$k^L_\gamma = \Bigg\{ \Bigg\{ \ddot{\gamma}_1^2 + \ddot{\gamma}_2^2 - L \left[\frac{d}{dt} (\omega(\dot{\gamma}(t))) \right]^2 \Bigg\} \cdot \left[\dot{\gamma}_1^2 + \dot{\gamma}_2^2 \right]^{-2} \tag{2.9}$$
$$- \left\{ \left[\dot{\gamma}_1 \ddot{\gamma}_1 + \dot{\gamma}_2 \ddot{\gamma}_2 \right]^2 \cdot \left[\dot{\gamma}_1^2 + \dot{\gamma}_2^2 \right]^{-3} \right\} \Bigg\}^{\frac{1}{2}}.$$

(2)If $\nabla^L_{\dot\gamma}\dot\gamma$ is a timelike vector, then

$$k^L_\gamma = \Bigg\{ -\Bigg\{ [\ddot\gamma_1 + L\dot\gamma_2\omega(\dot\gamma(t))]^2 + [\ddot\gamma_2 - L\dot\gamma_1\omega(\dot\gamma(t))]^2 - L\left[\frac{d}{dt}(\omega(\dot\gamma(t)))\right]^2 \Bigg\} \tag{2.10}$$
$$\cdot \left[\dot\gamma_1^2 + \dot\gamma_2^2 - L(\omega(\dot\gamma(t)))^2\right]^{-2} + [\dot\gamma_1(\ddot\gamma_1 + L\dot\gamma_2\omega(\dot\gamma(t)))$$
$$+\dot\gamma_2(\ddot\gamma_2 - L\dot\gamma_1\omega(\dot\gamma(t))) - L\omega(\dot\gamma(t))\frac{d}{dt}(\omega(\dot\gamma(t)))\Big]^2 \cdot \left[\dot\gamma_1^2 + \dot\gamma_2^2 - L(\omega(\dot\gamma(t)))^2\right]^{-3} \Bigg\}^{\frac{1}{2}}.$$

In particular, if $\gamma(t)$ is a horizontal point of γ, then

$$k^L_\gamma = \Bigg\{ -\Bigg\{ \ddot\gamma_1^2 + \ddot\gamma_2^2 - L\left[\frac{d}{dt}(\omega(\dot\gamma(t)))\right]^2 \Bigg\} \cdot \left[\dot\gamma_1^2 + \dot\gamma_2^2\right]^{-2} \tag{2.11}$$
$$+ \left[\dot\gamma_1\ddot\gamma_1 + \dot\gamma_2\ddot\gamma_2\right]^2 \cdot \left[\dot\gamma_1^2 + \dot\gamma_2^2\right]^{-3} \Bigg\}^{\frac{1}{2}}.$$

Proof By (2.2), we have

$$\dot\gamma(t) = \dot\gamma_1 X_1 + \dot\gamma_2 X_2 + \omega(\dot\gamma(t))X_3. \tag{2.12}$$

By Lemma 2.1 and (2.12), we have

$$\nabla^L_{\dot\gamma}X_1 = -\frac{L}{2}\left[\gamma_1(t)\dot\gamma_2(t) + \dot\gamma_3(t)\right]X_2 + \frac{\dot\gamma_2(t)}{2}X_3, \tag{2.13}$$
$$\nabla^L_{\dot\gamma}X_2 = \frac{L}{2}\left[\gamma_1(t)\dot\gamma_2(t) + \dot\gamma_3(t)\right]X_1 - \frac{\dot\gamma_1(t)}{2}X_3,$$
$$\nabla^L_{\dot\gamma}X_3 = \frac{L}{2}\dot\gamma_2(t)X_1 - \frac{L}{2}\dot\gamma_1(t)X_2.$$

By (2.12) and (2.13), we have

$$\nabla^L_{\dot\gamma}\dot\gamma = \left[\ddot\gamma_1 + L\dot\gamma_2\omega(\dot\gamma(t))\right]X_1 + \left[\ddot\gamma_2 - L\dot\gamma_1\omega(\dot\gamma(t))\right]X_2 + \left[\frac{d}{dt}(\omega(\dot\gamma(t)))\right]X_3. \tag{2.14}$$

By Definition 2.3, (2.12) and (2.14), we get Lemma 2.4. □

Definition 2.5 *Let $\gamma : I \to \mathbb{H}^2_L$ be a C^2-smooth regular curve. We define the intrinsic curvature k^∞_γ of γ at $\gamma(t)$ to be*

$$k^\infty_\gamma := \lim_{L\to+\infty} k^L_\gamma,$$

if the limit exists.

We introduce the following notation: for continuous functions $f_1, f_2 : (0, +\infty) \to \mathbb{R}$,

$$f_1(L) \sim f_2(L), \quad as \quad L \to +\infty \Leftrightarrow \lim_{L\to+\infty}\frac{f_1(L)}{f_2(L)} = 1. \tag{2.15}$$

Lemma 2.6 *Let $\gamma : I \to \mathbb{H}_L^2$ be a C^2-smooth regular curve. Then*
(1)If $\nabla^L_{\dot\gamma}\dot\gamma$ is a spacelike vector, then

$$k_\gamma^\infty = \frac{\sqrt{\dot\gamma_1^2 + \dot\gamma_2^2}}{|\omega(\dot\gamma(t))|}, \quad if \quad \omega(\dot\gamma(t)) \neq 0; \tag{2.16}$$

$$k_\gamma^\infty = \left\{ \left(\ddot\gamma_1^2 + \ddot\gamma_2^2\right) \cdot \left(\dot\gamma_1^2 + \dot\gamma_2^2\right)^{-2} - \left(\dot\gamma_1\ddot\gamma_1 + \dot\gamma_2\ddot\gamma_2\right)^2 \cdot \left(\dot\gamma_1^2 + \dot\gamma_2^2\right)^{-3} \right\}^{\frac{1}{2}}, \tag{2.17}$$
$$if \quad \omega(\dot\gamma(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot\gamma(t))) = 0;$$

$$\lim_{L\to+\infty} \frac{k_\gamma^L}{\sqrt{L}} = \frac{\sqrt{-\frac{d}{dt}(\omega(\dot\gamma(t)))^2}}{|\dot\gamma_1^2 + \dot\gamma_2^2|}, \quad if \quad \omega(\dot\gamma(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0. \tag{2.18}$$

Therefore, this situation does not exist.
(2)If $\nabla^L_{\dot\gamma}\dot\gamma$ is a timelike vector, then

$$k_\gamma^\infty = \frac{\sqrt{-\dot\gamma_1^2 - \dot\gamma_2^2}}{|\omega(\dot\gamma(t))|}, \quad if \quad \omega(\dot\gamma(t)) \neq 0; \tag{2.19}$$

Therefore, this situation does not exist.

$$k_\gamma^\infty = \left\{ -\left(\ddot\gamma_1^2 + \ddot\gamma_2^2\right) \cdot \left(\dot\gamma_1^2 + \dot\gamma_2^2\right)^{-2} + \left(\dot\gamma_1\ddot\gamma_1 + \dot\gamma_2\ddot\gamma_2\right)^2 \cdot \left(\dot\gamma_1^2 + \dot\gamma_2^2\right)^{-3} \right\}^{\frac{1}{2}}, \tag{2.20}$$
$$if \quad \omega(\dot\gamma(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot\gamma(t))) = 0.$$

$$\lim_{L\to+\infty} \frac{k_\gamma^L}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot\gamma(t)))|}{|\dot\gamma_1^2 + \dot\gamma_2^2|}, \quad if \quad \omega(\dot\gamma(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0. \tag{2.21}$$

Proof Using the notation introduced in (2.15), when $\omega(\dot\gamma(t)) \neq 0$, we have

$$\langle \nabla^L_{\dot\gamma}\dot\gamma, \nabla^L_{\dot\gamma}\dot\gamma \rangle_L \sim [\omega(\dot\gamma(t))]^2(\dot\gamma_1^2 + \dot\gamma_2^2)L^2, \quad as \quad L \to +\infty,$$

$$\langle \dot\gamma, \dot\gamma \rangle_L \sim -L\omega(\dot\gamma(t))^2, \quad as \quad L \to +\infty,$$

$$\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma \rangle_L^2 \sim O(L^2) \quad as \quad L \to +\infty.$$

Therefore

$$\frac{\langle \nabla^L_{\dot\gamma}\dot\gamma, \nabla^L_{\dot\gamma}\dot\gamma \rangle_L}{||\dot\gamma||_L^4} \to \frac{\dot\gamma_1^2 + \dot\gamma_2^2}{[\omega(\dot\gamma(t))]^2}, \quad as \quad L \to +\infty,$$

$$\frac{\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma \rangle_L^2}{\langle \dot\gamma, \dot\gamma \rangle_L^3} \to 0, \quad as \quad L \to +\infty.$$

So by Definition 2.3, we have (2.16) and (2.19). (2.17) comes from (2.9) and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$. (2.20) comes from (2.11) and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$. When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$, we have

$$\langle \nabla^L_{\dot{\gamma}}\dot{\gamma}, \nabla^L_{\dot{\gamma}}\dot{\gamma}\rangle_L \sim -L[\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2, \quad as \quad L \to +\infty,$$

$$\langle \dot{\gamma}, \dot{\gamma}\rangle_L = \dot{\gamma}_1^2 + \dot{\gamma}_2^2,$$

$$\langle \nabla^L_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}\rangle_L^2 = O(1) \quad as \quad L \to +\infty.$$

By (2.6), we get (2.18). □

3. Lorentzian surfaces and a Gauss–Bonnet theorem in the Lorentzian Heisenberg group for the second Lorentzian metric

We will say that a surface $\Sigma \subset \mathbb{H}^2_L$ is regular if Σ is a C^2-smooth compact and oriented surface. In particular we will assume that there exists a C^2-smooth function $u : \mathbb{H}^2_L \to \mathbb{R}$ such that

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H}^2_L : u(x_1, x_2, x_3) = 0\}$$

and $\nabla_{\mathbb{H}^2_L} u = u_{x_1}\partial_{x_1} + u_{x_2}\partial_{x_2} + u_{x_3}\partial_{x_3} \neq 0$. A point $x \in \Sigma$ is called *characteristic* if $\nabla_H u(x) = (0, 0)$. Our computations will be local and away from characteristic points of Σ.

Let us define first

$$p := X_1 u, \quad q := X_2 u, \quad \text{and} \quad r := \widetilde{X_3} u.$$

Because $p^2 + q^2 > 0$, we say that $\Sigma \subset \mathbb{H}^2_L$ is Horizontal spacelike surface. When $L \to +\infty$, then $p^2 + q^2 - r^2 > 0$. And we define

$$l := \sqrt{p^2 + q^2}, \quad l_L := \sqrt{p^2 + q^2 - r^2}, \quad \overline{p} := \frac{p}{l}, \tag{3.1}$$
$$\overline{q} := \frac{q}{l}, \quad \overline{p_L} := \frac{p}{l_L}, \quad \overline{q_L} := \frac{q}{l_L}, \quad \overline{r_L} := \frac{r}{l_L}.$$

In particular, $\overline{p}^2 + \overline{q}^2 = 1$. These functions are well defined at every non-characteristic point. Let

$$v_L = \overline{p_L} X_1 + \overline{q_L} X_2 - \overline{r_L}\widetilde{X_3}, \quad e_1 = \overline{q} X_1 - \overline{p} X_2, \quad e_2 = \overline{r_L}\ \overline{p} X_1 + \overline{r_L}\ \overline{q} X_2 - \frac{l}{l_L}\widetilde{X_3}, \tag{3.2}$$

then v_L is the unit spacelike normal vector to Σ and e_1 is the unit spacelike vector, e_2 is the unit timelike vector. $\{e_1, e_2\}$ are the orthonormal basis of Σ. We call Σ a Lorentzian surface in tne Lorentzian Hersenberg group.

Let $\dot{\gamma} = \lambda_1 e_1 + \lambda_2 e_2$. If $\gamma : I \to \mathbb{H}^2_L$ be a C^2-smooth timelike curve, then we define $J_L(\dot{\gamma}) := -\lambda_1 e_2 - \lambda_2 e_1$. If $\gamma : I \to \mathbb{H}^2_L$ be a C^2-smooth spacelike curve, then $J_L(\dot{\gamma}) := \lambda_1 e_2 + \lambda_2 e_1$. Then $\langle \dot{\gamma}, J_L(\dot{\gamma})\rangle = 0$ and $(\dot{\gamma}, J_L(\dot{\gamma}))$ has the same orientation with $\{e_1, e_2\}$.

For every $U, V \in T\Sigma$, we define $\nabla^{\Sigma,L}_U V = \pi \nabla^L_U V$ where $\pi : T\mathbb{H}^2_L \to T\Sigma$ is the projection. Then $\nabla^{\Sigma,L}$ is the Levi–Civita connection on Σ with respect to the metric g_L. By (2.14), (3.2) and

$$\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma} = \langle \nabla^L_{\dot{\gamma}}\dot{\gamma}, e_1\rangle_L e_1 - \langle \nabla^L_{\dot{\gamma}}\dot{\gamma}, e_2\rangle_L e_2, \tag{3.3}$$

we have

$$\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma = \{\overline{q}\,[\ddot\gamma_1 + L\dot\gamma_2\omega(\dot\gamma(t))] - \overline{p}\,[\ddot\gamma_2 - L\dot\gamma_1\omega(\dot\gamma(t))]\}\, e_1 \\ - \{\overline{r_L}\ \ \overline{p}\,[\ddot\gamma_1 + L\dot\gamma_2\omega(\dot\gamma(t))] + \overline{r_L}\ \ \overline{q}\,[\ddot\gamma_2 - L\dot\gamma_1\omega(\dot\gamma(t))] \\ + \frac{l}{l_L}L^{\frac{1}{2}}\left[\frac{d}{dt}(\omega(\dot\gamma(t)))\right]\right\} e_2. \tag{3.4}$$

Moreover if $\omega(\dot\gamma(t)) = 0$, then

$$\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma = [\overline{q}\ddot\gamma_1 - \overline{p}\ddot\gamma_2]e_1 - \left\{\overline{r_L}\ \ \overline{p}\ddot\gamma_1 + \overline{r_L}\ \ \overline{q}\ddot\gamma_2 + \frac{l}{l_L}L^{\frac{1}{2}}\left[\frac{d}{dt}(\omega(\dot\gamma(t)))\right]\right\} e_2 \tag{3.5}$$

Definition 3.1 *Let $\Sigma \subset \mathbb{H}^2_L$ be a regular Lorentzian surface. Let $\gamma : I \to \Sigma$ be a C^2-smooth regular curve. (1)If $\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma$ is spacelike vectors, the geodesic curvature $k^L_{\gamma,\Sigma}$ of γ at $\gamma(t)$ is defined as*

$$k^L_{\gamma,\Sigma} := \sqrt{\frac{||\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma||^2_{\Sigma,L}}{||\dot\gamma||^4_{\Sigma,L}} - \frac{\langle\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma, \dot\gamma\rangle^2_{\Sigma,L}}{\langle\dot\gamma,\dot\gamma\rangle^3_{\Sigma,L}}}. \tag{3.6}$$

(2)If $\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma$ is timelike vectors, the curvature k^L_γ of γ at $\gamma(t)$ is defined as

$$k^L_{\gamma,\Sigma} := \sqrt{\frac{||\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma||^2_{\Sigma,L}}{||\dot\gamma||^4_{\Sigma,L}} + \frac{\langle\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma, \dot\gamma\rangle^2_{\Sigma,L}}{\langle\dot\gamma,\dot\gamma\rangle^3_{\Sigma,L}}}. \tag{3.7}$$

Definition 3.2 *Let $\Sigma \subset \mathbb{H}^2_L$ be a regular Lorentzian surface. Let $\gamma : I \to \Sigma$ be a C^2-smooth regular curve. We define the intrinsic geodesic curvature $k^\infty_{\gamma,\Sigma}$ of γ at $\gamma(t)$ to be*

$$k^\infty_{\gamma,\Sigma} := \lim_{L\to+\infty} k^L_{\gamma,\Sigma},$$

if the limit exists.

Lemma 3.3 *Let $\Sigma \subset \mathbb{H}^2_L$ be a regular Lorentzian surface. Let $\gamma : I \to \Sigma$ be a C^2-smooth regular curve. (1)If $\nabla^{\Sigma,L}_{\dot\gamma}\dot\gamma$ is a spacelike vector, then*

$$k^\infty_{\gamma,\Sigma} = \frac{|\overline{p}\dot\gamma_1 + \overline{q}\dot\gamma_2|}{|\omega(\dot\gamma(t))|}, \quad if \quad \omega(\dot\gamma(t)) \neq 0; \tag{3.8}$$

$$k^\infty_{\gamma,\Sigma} = 0, \quad if \quad \omega(\dot\gamma(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot\gamma(t))) = 0; \tag{3.9}$$

$$\lim_{L\to+\infty}\frac{k^L_{\gamma,\Sigma}}{\sqrt{L}} = \frac{\sqrt{-\frac{d}{dt}(\omega(\dot\gamma(t)))^2}}{(\overline{q}\dot\gamma_1 - \overline{p}\dot\gamma_2)^2}, \quad if \quad \omega(\dot\gamma(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0. \tag{3.10}$$

Therefore, this situation does not exist.
(2)If $\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}$ *is a timelike vector, then*

$$k^{\infty}_{\gamma,\Sigma} = \frac{\sqrt{-(\overline{p}\dot{\gamma}_1+\overline{q}\dot{\gamma}_2)^2}}{|\omega(\dot{\gamma}(t))|}, \quad if \quad \omega(\dot{\gamma}(t)) \neq 0; \tag{3.11}$$

Therefore, this situation does not exist.

$$k^{\infty}_{\gamma,\Sigma} = 0, \quad if \quad \omega(\dot{\gamma}(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0; \tag{3.12}$$

$$\lim_{L\to+\infty}\frac{k^{L}_{\gamma,\Sigma}}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot{\gamma}(t)))|}{(\overline{q}\dot{\gamma}_1-\overline{p}\dot{\gamma}_2)^2}, \quad if \quad \omega(\dot{\gamma}(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \tag{3.13}$$

Proof By (2.12) and $\dot{\gamma} \in T\Sigma$, we have

$$\dot{\gamma} = (\overline{q}\dot{\gamma}_1-\overline{p}\dot{\gamma}_2)e_1 - \frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot{\gamma}(t))e_2. \tag{3.14}$$

By (3.4), we have

$$\begin{aligned}\langle\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma},\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}\rangle_{L,\Sigma} &= \{\overline{q}\,[\ddot{\gamma}_1+L\dot{\gamma}_2\omega(\dot{\gamma}(t))]-\overline{p}\,[\ddot{\gamma}_2-L\dot{\gamma}_1\omega(\dot{\gamma}(t))]\}^2 \\ &\quad -\{\overline{r_L}\ \ \overline{p}\,[\ddot{\gamma}_1-L\dot{\gamma}_2\omega(\dot{\gamma}(t))]+\overline{r_L}\ \ \overline{q}\,[\ddot{\gamma}_2-L\dot{\gamma}_1\omega(\dot{\gamma}(t))] \\ &\quad +\frac{l}{l_L}L^{\frac{1}{2}}\left[\frac{d}{dt}(\omega(\dot{\gamma}(t)))\right]\Big\}^2.\end{aligned} \tag{3.15}$$

Similarly, we have that when $\omega(\dot{\gamma}(t)) \neq 0$,

$$\langle\dot{\gamma},\dot{\gamma}\rangle_{\Sigma,L} = (\overline{q}\dot{\gamma}_1-\overline{p}\dot{\gamma}_2)^2 - \left[\frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot{\gamma}(t))\right]^2 \sim -L[\omega(\dot{\gamma}(t))]^2, \quad \text{as} \quad L\to+\infty. \tag{3.16}$$

By (3.4) and (3.14), we have

$$\begin{aligned}\langle\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma},\dot{\gamma}\rangle_{\Sigma,L} &= (\overline{q}\dot{\gamma}_1-\overline{p}\dot{\gamma}_2)\cdot\{\overline{q}\,[\ddot{\gamma}_1+L\dot{\gamma}_2\omega(\dot{\gamma}(t))] \\ &\quad -\overline{p}\,[\ddot{\gamma}_2-L\dot{\gamma}_1\omega(\dot{\gamma}(t))]\} - \left[\frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot{\gamma}(t))\right]\cdot\{\overline{r_L}\ \ \overline{p}\,[\ddot{\gamma}_1+L\gamma_2\omega(\dot{\gamma}(t))] \\ &\quad +\overline{r_L}\ \ \overline{q}\,[\ddot{\gamma}_2-L\gamma_1\omega(\dot{\gamma}(t))]+\frac{l}{l_L}L^{\frac{1}{2}}\left[\frac{d}{dt}(\omega(\dot{\gamma}(t)))\right]\Big\} \sim M_0L,\end{aligned} \tag{3.17}$$

where M_0 does not depend on L. By Definition 3.1, (3.15)–(3.17), we get (3.8).
When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$, we have

$$\begin{aligned}\langle\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma},\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}\rangle_{L,\Sigma} &= (\overline{q}\ddot{\gamma}_1-\overline{p}\ddot{\gamma}_2)^2+(\overline{r_L}\ \ \overline{p}\ddot{\gamma}_1+\overline{r_L}\ \ \overline{q}\ddot{\gamma}_2)^2 \\ &\sim (\overline{q}\ddot{\gamma}_1-\overline{p}\ddot{\gamma}_2)^2, \quad \text{as} \quad L\to+\infty,\end{aligned} \tag{3.18}$$

$$\langle \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma,L} = (\overline{q}\dot{\gamma}_1 - \overline{p}\dot{\gamma}_2)^2, \tag{3.19}$$

$$\langle \nabla^{\Sigma,L}_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma,L} = (\overline{q}\dot{\gamma}_1 - \overline{p}\dot{\gamma}_2) \cdot (\overline{q}\ddot{\gamma}_1 - \overline{p}\ddot{\gamma}_2) \tag{3.20}$$

By (3.18)–(3.20) and Definition 3.1, we get $k^{\infty}_{\gamma,\Sigma} = 0$.

When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$, we have

$$\langle \nabla^{\Sigma,L}_{\dot{\gamma}} \dot{\gamma}, \nabla^{\Sigma,L}_{\dot{\gamma}} \dot{\gamma} \rangle_{L,\Sigma} \sim -L[\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2,$$

$$\langle \nabla^{\Sigma,L}_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma,L} = O(1),$$

so we get (3.10). □

Definition 3.4 *Let $\Sigma \subset \mathbb{H}^2_L$ be a regular surface. Let $\gamma : I \to \Sigma$ be a C^2-smooth regular curve. The signed geodesic curvature $k^{L,s}_{\gamma,\Sigma}$ of γ at $\gamma(t)$ is defined as*

$$k^{L,s}_{\gamma,\Sigma} := \frac{\langle \nabla^{\Sigma,L}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{\Sigma,L}}{||\dot{\gamma}||^3_{\Sigma,L}}. \tag{3.21}$$

Definition 3.5 *Let $\Sigma \subset \mathbb{H}^2_L$ be a regular surface. Let $\gamma : [a,b] \to \Sigma$ be a C^2-smooth regular curve. We define the intrinsic geodesic curvature $k^{\infty}_{\gamma,\Sigma}$ of γ at the noncharacteristic point $\gamma(t)$ to be*

$$k^{\infty,s}_{\gamma,\Sigma} := \lim_{L\to+\infty} k^{L,s}_{\gamma,\Sigma},$$

if the limit exists.

Lemma 3.6 *Let $\Sigma \subset \mathbb{H}^2_L$ be a regular Lorentzian surface.*
(1)If $\gamma : I \to \Sigma$ be a spacelike C^2-smooth regular curve, then

$$k^{\infty,s}_{\gamma,\Sigma} = -\frac{\overline{p}\dot{\gamma}_1 + \overline{q}\dot{\gamma}_2}{\sqrt{-\omega(\dot{\gamma}(t))^2}}, \quad if \omega(\dot{\gamma}(t)) \neq 0. \tag{3.22}$$

Therefore, the situation does not exist.

$$k^{\infty}_{\gamma,\Sigma} = 0, \quad if \omega(\dot{\gamma}(t)) = 0, \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0, \tag{3.23}$$

$$\lim_{L\to+\infty} \frac{k^{L,s}_{\gamma,\Sigma}}{\sqrt{L}} = \frac{(\overline{q}\dot{\gamma}_1 - \overline{p}\dot{\gamma}_2)\,(\frac{d}{dt}(\omega(\dot{\gamma}(t))))}{(\overline{q}\dot{\gamma}_1 + \overline{p}\dot{\gamma}_2)^3}, \quad if \quad \omega(\dot{\gamma}(t)) = 0, \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \tag{3.24}$$

(2)If $\gamma : I \to \Sigma$ be a timelike C^2-smooth regular curve , then

$$k^{\infty,s}_{\gamma,\Sigma} = -\frac{\overline{p}\dot{\gamma}_1 + \overline{q}\dot{\gamma}_2}{|\omega(\dot{\gamma}(t))|}, \quad if \omega(\dot{\gamma}(t)) \neq 0. \tag{3.25}$$

$$\|\dot{\gamma}\|_{L,\Sigma} = \sqrt{-\left(\overline{q}\dot{\gamma}_1 + \overline{p}\dot{\gamma}_2\right)^2}, \quad if \omega(\dot{\gamma}(t)) = 0, \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0, \tag{3.26}$$

Therefore, the situation does not exist.

$$\|\dot{\gamma}\|_{L,\Sigma} = \sqrt{-\left(\overline{q}\dot{\gamma}_1 + \overline{p}\dot{\gamma}_2\right)^2}, \quad if \quad \omega(\dot{\gamma}(t)) = 0, \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \tag{3.27}$$

Therefore, the situation does not exist.

Proof For (1), by (3.14), we have

$$J_L(\dot{\gamma}) = -\frac{l_L}{l} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) e_1 + (\overline{q}\dot{\gamma}_1 - \overline{p}\dot{\gamma}_2) e_2. \tag{3.28}$$

By (3.4) and (3.28), we have

$$\begin{aligned} \langle \nabla^{\Sigma,L}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L,\Sigma} = & -\frac{l_L}{l} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) \left\{ \overline{q} \left[\ddot{\gamma}_1 + L\dot{\gamma}_2 \omega(\dot{\gamma}(t)) \right] - \overline{p} \left[\ddot{\gamma}_2 - L\dot{\gamma}_1 \omega(\dot{\gamma}(t)) \right] \right\} \\ & + (\overline{q}\dot{\gamma}_1 - \overline{p}\dot{\gamma}_2) \cdot \{ \overline{r_L} \quad \overline{p} \left[\ddot{\gamma}_1 + L\dot{\gamma}_2 \omega(\dot{\gamma}(t)) \right] + \overline{r_L} \quad \overline{q} \left[\ddot{\gamma}_2 - L\dot{\gamma}_1 \omega(\dot{\gamma}(t)) \right] \\ & + \frac{l}{l_L} L^{\frac{1}{2}} \left[\frac{d}{dt}(\omega(\dot{\gamma}(t))) \right] \Big\} \sim -L^{\frac{3}{2}} \omega(\dot{\gamma}(t))^2 (\overline{p}\dot{\gamma}_1 + \overline{q}\dot{\gamma}_2), \quad \text{as} \quad L \to +\infty. \end{aligned} \tag{3.29}$$

When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$, we get

$$\langle \nabla^{\Sigma,L}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L,\Sigma} = (\overline{q}\dot{\gamma}_1 - \overline{p}\dot{\gamma}_2) \cdot \{ \overline{r_L} \quad \overline{p}\ddot{\gamma}_1 + \overline{r_L} \quad \overline{q}\ddot{\gamma}_2] \sim O(L^{-\frac{1}{2}}) \quad \text{as} \quad L \to +\infty. \tag{3.30}$$

So $k^{\infty,s}_{\gamma,\Sigma} = 0$. When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$, we have

$$\langle \nabla^{\Sigma,L}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L,\Sigma} \sim L^{\frac{1}{2}} (\overline{q}\dot{\gamma}_1 - \overline{p}\dot{\gamma}_2) \frac{d}{dt}(\omega(\dot{\gamma}(t))) \quad \text{as} \quad L \to +\infty. \tag{3.31}$$

So we get (3.24). Similarly, we have (2). □

Next, we compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in the Lorentzian Heisenberg space. We define the second fundamental form II^L of the embedding of Σ into $\mathbb{H}^2_L$:

$$II^L = \begin{pmatrix} \langle \nabla^L_{e_1} v_L, e_1 \rangle \rangle_L, & \langle \nabla^L_{e_1} v_L, e_2 \rangle \rangle_L \\ \langle \nabla^L_{e_2} v_L, e_1 \rangle \rangle_L, & \langle \nabla^L_{e_2} v_L, e_2 \rangle \rangle_L \end{pmatrix}. \tag{3.32}$$

Similarly to Theorem 4.3 in [5], we have

Theorem 3.7 *The second fundamental form II^L of the embedding of Σ into $\mathbb{H}^2_L$ is given by*

$$II^L = \begin{pmatrix} \frac{l}{l_L}[X_1(\overline{p}) + X_2(\overline{q})], & -\frac{l_L}{l}\langle e_1, \nabla_H(\overline{r_L}) \rangle_L - \frac{\sqrt{L}}{2} \\ \frac{-l_L}{l}\langle e_1, \nabla_H(\overline{r_L}) \rangle_L - \frac{\sqrt{L}}{2}, & -\frac{l^2}{l_L^2}\langle e_2, \nabla_H(\frac{r}{l}) \rangle_L + \widetilde{X_3}(\overline{r_L}) \end{pmatrix}. \tag{3.33}$$

The mean curvature $\mathcal{H}_L$ of Σ is defined by

$$\mathcal{H}_L := \mathrm{tr}(II^L).$$

Define the curvature of a connection ∇ by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{3.34}$$

Let

$$\mathcal{K}^{\Sigma,L}(e_1,e_2) = -\langle R^{\Sigma,L}(e_1,e_2)e_1, e_2\rangle_{\Sigma,L}, \quad \mathcal{K}^L(e_1,e_2) = -\langle R^L(e_1,e_2)e_1,e_2\rangle_L. \tag{3.35}$$

By the Gauss equation, we have

$$\mathcal{K}^{\Sigma,L}(e_1,e_2) = \mathcal{K}^L(e_1,e_2) + \det(II^L). \tag{3.36}$$

Proposition 3.8 *Away from characteristic points, the horizontal mean curvature $\mathcal{H}_\infty$ of $\Sigma \subset \mathbb{H}$ is given by*

$$\mathcal{H}_\infty = \lim_{L\to+\infty}\mathcal{H}_L = X_1(\overline{p}) + X_2(\overline{q}). \tag{3.37}$$

Proof By

$$\frac{l^2}{l_L^2}\langle e_2, \nabla_H(\frac{r}{l})\rangle_L = \frac{\overline{p}r}{l}X_1(\overline{r_L}) + \frac{\overline{q}r}{l}X_2(\overline{r_L}) \to O(L^{-1})$$

$$\frac{l}{l_L}[X_1(\overline{p}) + X_2(\overline{q})] \to X_1(\overline{p}) + X_2(\overline{q}), \quad \widetilde{X_3}(\overline{r_L}) \to 0, \quad \overline{p_L} \to \overline{p},$$

we get (3.37). □

By Lemma 2.1 and (3.35), we have

Lemma 3.9 *Let $\mathbb{H}_L^2$ be the Lorentzian Heisenberg space, then*

$$\begin{aligned} &R^L(X_1,X_2)X_1 = -\frac{3}{4}LX_2, \quad R^L(X_1,X_2)X_2 = \frac{3}{4}LX_1, \quad R^L(X_1,X_2)X_3 = 0, \\ &R^L(X_1,X_3)X_1 = \frac{1}{4}LX_3, \quad R^L(X_1,X_3)X_2 = 0, \quad R^L(X_1,X_3)X_3 = \frac{L^2}{4}X_1, \\ &R^L(X_2,X_3)X_1 = 0, \quad R^L(X_2,X_3)X_2 = \frac{L}{4}X_3, \quad R^L(X_2,X_3)X_3 = \frac{L^2}{4}X_2. \end{aligned} \tag{3.38}$$

Proposition 3.10 *Away from characteristic points, we have*

$$\mathcal{K}^{\Sigma,L}(e_1,e_2) \to A_0 + O(L^{-1}), \quad \text{as} \quad L \to +\infty, \tag{3.39}$$

where

$$A_0 := -\langle e_1, \nabla_H(\frac{X_3u}{|\nabla_H u|})\rangle + \frac{(X_3u)^2}{l^2}. \tag{3.40}$$

Proof By (3.2), we have

$$
\begin{aligned}
&\langle R^L(e_1,e_2)e_1,e_2\rangle_L \qquad (3.41)\\
&=\overline{r_L}^2\langle R^L(X_1,X_2)X_1,X_2\rangle_L-2\frac{l}{l_L}\overline{q}L^{-\frac{1}{2}}\overline{r_L}\langle R^L(X_1,X_2)X_1,X_3\rangle_L\\
&+2\frac{l}{l_L}\overline{p}L^{-\frac{1}{2}}\overline{r_L}\langle R^L(X_1,X_2)X_2,X_3\rangle_L+(\frac{l}{l_L}\overline{q})^2L^{-1}\langle R^L(X_1,X_3)X_1,X_3\rangle_L\\
&-2(\frac{l}{l_L})^2\overline{pq}L^{-1}\langle R^L(X_1,X_3)X_2,X_3\rangle_L+(\overline{p}\frac{l}{l_L})^2L^{-1}\langle R^L(X_2,X_3)X_2,X_3\rangle_L.
\end{aligned}
$$

By Lemma 3.9, we have

$$\mathcal{K}^L(e_1,e_2)=\frac{L}{4}(\frac{l}{l_L})^2+\frac{3}{4}L\overline{r_L}^2. \tag{3.42}$$

By (3.33) and

$$\nabla_H(\overline{r_L})=L^{-\frac{1}{2}}\nabla_H(\frac{X_3u}{|\nabla_Hu|})+O(L^{-1}) \quad \text{as} \quad L\to+\infty$$

we get

$$\det(II^L)=-\frac{L}{4}-\langle e_1,\nabla_H(\frac{X_3u}{|\nabla_Hu|})\rangle+O(L^{-1}) \quad \text{as} \quad L\to+\infty. \tag{3.43}$$

By (3.33),(3.42),(3.43) we get (3.39). □

Let us first consider the case of a timelike curve $\gamma:I\to\mathbb{H}^2_L$. We define the Riemannian length measure $ds_L=||\dot{\gamma}||_Ldt$.

Lemma 3.11 *Let $\gamma:I\to\mathbb{H}^2_L$ be a C^2-smooth timelike curve. Let*

$$ds:=|\omega(\dot{\gamma}(t))|dt, \qquad d\overline{s}:=\frac{1}{2}\frac{1}{|\omega(\dot{\gamma}(t))|}\left(-\dot{\gamma}_1^2-\dot{\gamma}_2^2\right)dt. \tag{3.44}$$

Then

$$\lim_{L\to+\infty}\frac{1}{\sqrt{L}}\int_\gamma ds_L=\int_a^b ds. \tag{3.45}$$

When $\omega(\dot{\gamma}(t))\neq 0$, we have

$$\frac{1}{\sqrt{L}}ds_L=ds+d\overline{s}L^{-1}+O(L^{-2}) \quad \text{as} \quad L\to+\infty. \tag{3.46}$$

When $\omega(\dot{\gamma}(t))=0$, we have

$$\frac{1}{\sqrt{L}}ds_L=\frac{1}{\sqrt{L}}\sqrt{-\dot{\gamma}_1^2-\dot{\gamma}_2^2}. \tag{3.47}$$

Therefore, the situation does not exist.

Proof We know that $||\dot{\gamma}(t)||_L = \sqrt{-\dot{\gamma}_1^2 - \dot{\gamma}_2^2 + L\omega(\dot{\gamma}(t))^2}$, similar to the proof of Lemma 6.1 in [11], we can prove (3.45). When $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\frac{1}{\sqrt{L}} ds_L = \sqrt{L^{-1}\left(-\dot{\gamma}_1^2 - \dot{\gamma}_2^2\right) + \omega(\dot{\gamma}(t))^2} dt.$$

Using the Taylor expansion, we can prove (3.45). From the denifition of ds_L, and $\omega(\dot{\gamma}(t)) = 0$, we get (3.47). □

Proposition 3.12 *Let $\Sigma \subset \mathbb{H}_L^2$ be a regular Lorentzian C^2-smooth surface. Let $d\sigma_L$ denote the surface measure on Σ with respect to the Riemannian metric g_L. Let*

$$d\sigma_\Sigma := (\overline{p}\omega_2 - \overline{q}\omega_1) \wedge \omega, \qquad d\overline{\sigma_\Sigma} := \frac{X_3 u}{l}\omega_1 \wedge \omega_2 + \frac{(X_3 u)^2}{2l^2}(\overline{p}\omega_2 - \overline{q}\omega_1) \wedge \omega. \tag{3.48}$$

Then

$$\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = d\sigma_\Sigma + d\overline{\sigma_\Sigma} L^{-1} + O(L^{-2}), \quad \text{as} \quad L \to +\infty. \tag{3.49}$$

If $\Sigma = f(D)$ with

$$f = f(u_1, u_2) = (f_1, f_2, f_3) : D \subset \mathbb{R}^2 \to \mathbb{H}_L^2,$$

then

$$\lim_{L\to+\infty} \frac{1}{\sqrt{L}} \int_\Sigma d\sigma_{\Sigma,L} = \int_D \Big\{ \left[(f_2)_{u_1}(f_3)_{u_2} - (f_3)_{u_1}(f_2)_{u_2}\right]^2 \tag{3.50}$$
$$+ \left[((f_1)_{u_2}(f_2)_{u_1} - (f_2)_{u_2}(f_1)_{u_1})f_1 + (f_3)_{u_1}(f_1)_{u_2} - (f_1)_{u_1}(f_3)_{u_2}\right]^2 \Big\}^{\frac{1}{2}} du_1 du_2.$$

Proof We know that

$$g_L(X_1, \cdot) = \omega_1, \quad g_L(X_2, \cdot) = \omega_2, \quad g_L(X_3, \cdot) = -L\omega,$$

so

$$e_1^\star = g_L(e_1, \cdot) = \overline{q}\omega_1 - \overline{p}\omega_2, \quad e_2^\star = g_L(e_2, \cdot) = \overline{r_L}\ \overline{p}\omega_1 + \overline{r_L}\ \overline{q}\omega_2 - \frac{l}{l_L} L^{\frac{1}{2}}\omega. \tag{3.51}$$

Then

$$\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = \frac{1}{\sqrt{L}} e_1^\star \wedge e_2^\star = \frac{l}{l_L}(\overline{p}\omega_2 - \overline{q}\omega_1) \wedge \omega + \frac{1}{\sqrt{L}}\overline{r_L}\omega_1 \wedge \omega_2. \tag{3.52}$$

By

$$\overline{r_L} = \frac{(X_3 u)L^{-\frac{1}{2}}}{\sqrt{p^2 + q^2 - L^{-1}(X_3 u)^2}}$$

and the Taylor expansion

$$\frac{1}{l_L} = \frac{1}{l} - \frac{1}{2l^3}(X_3 u)^2 L^{-1} + O(L^{-2}) \quad \text{as} \quad L \to +\infty$$

we get (3.55). By (2.2), we have

$$\begin{aligned} f_{u_1} &= (f_1)_{u_1}\partial_{x_1} + (f_2)_{u_1}\partial_{x_2} + (f_3)_{u_1}\partial_{x_3} \\ &= (f_1)_{u_1} X_1 + (f_2)_{u_1} X_2 + \sqrt{L}\left[f_1(f_2)_{u_1} + (f_3)_{u_1}\right] \widetilde{X_3}, \end{aligned} \tag{3.53}$$

and

$$f_{u_2} = (f_1)_{u_2} X_1 + (f_2)_{u_2} X_2 + \sqrt{L}\,[f_1(f_2)_{u_2} + (f_3)_{u_2}]\,\widetilde{X_3}. \tag{3.54}$$

Let

$$\overline{v_L} = \begin{vmatrix} X_1, & X_2, & -\widetilde{X_3} \\ (f_1)_{u_1}, & (f_2)_{u_1}, & \sqrt{L}\,[f_1(f_2)_{u_1} + (f_3)_{u_1}] \\ (f_1)_{u_2}, & (f_2)_{u_2}, & \sqrt{L}\,[f_1(f_2)_{u_2} + (f_3)_{u_2}] \end{vmatrix}. \tag{3.55}$$

We know that

$$d\sigma_{\Sigma,L} = \sqrt{-\det(g_{ij})}du_1 du_2, \quad g_{ij} = g_L(f_{u_i}, f_{u_j}), \quad \det(g_{ij}) = ||\overline{v_L}||_L^2 = -\langle \overline{v_L}, \overline{v_L} \rangle,$$

so by the dominated convergence theorem, we get (3.51). □

Theorem 3.13 *Let $\Sigma \subset \mathbb{H}_L^2$ be a regular Lorentzian surface with finitely many boundary components $(\partial\Sigma)_i$, $i \in \{1, \cdots, n\}$, given by C^2-smooth regular and closed timelike curves $\gamma_i : [0, 2\pi] \to (\partial\Sigma)_i$. Suppose that the characteristic set $C(\Sigma)$ satisfies $\mathcal{H}^1(C(\Sigma)) = 0$ and that $||\nabla_H u||_H^{-1}$ is locally summable with respect to the 2-dimensional Hausdorff measure near the characteristic set $C(\Sigma)$, Let A_0 be defined by (3.40) and $d\sigma_\Sigma$ be defined by (3.48) and $k_{\gamma_i,\Sigma}^{\infty,s}$ be the sub-Riemannian signed geodesic curvature of γ_i relative to Σ. Then we have*

$$\int_\Sigma A_0 d\sigma_\Sigma + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i,\Sigma}^{\infty,s} ds = 0. \tag{3.56}$$

Proof Using the discussions in [2], we may assume that all points satisfy $\omega(\dot{\gamma}_i(t)) \neq 0$ on γ_i. Then by Lemma 3.6, we have

$$k_{\gamma_i,\Sigma}^{L,s} = k_{\gamma_i,\Sigma}^{\infty,s} + O(L^{-1}). \tag{3.57}$$

By the Gauss-Bonnet theorem (see [8] page 90 Theorem 1.4), we have

$$\int_\Sigma \mathcal{K}^{\Sigma,L} \frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i,\Sigma}^{L,s} \frac{1}{\sqrt{L}} ds_L = 0. \tag{3.58}$$

So by (3.57),(3.58),(3.40),(3.48).(3.49), we get

$$\left(\int_\Sigma A_0 d\sigma_\Sigma + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i,\Sigma}^{\infty,s} ds \right) + O(L^{-\frac{1}{2}}) = 0. \tag{3.59}$$

Let L go to the infinity and using the dominated convergence theorem, then we get (3.56). □

Example 3.14 *In $\mathbb{H}_L^2$, let $u = x_1^2 + x_2^2 + x_3^2 - 1$ and $\sum = S^2$. $\sum$ is a regular surface. By (2.1), we get*

$$X_1(u) = 2x_1; \quad X_2(u) = 2x_2 - 2x_1 x_3. \tag{3.60}$$

Solve the equations $X_1(u) = X_2(u) = 0$, then we get

$$C(\Sigma) = \{(0, 0, 1), (0, 0, -1)\}$$

and $\mathcal{H}^1(C(\Sigma)) = 0$.
A parametrization of Σ *is*

$$x_1 = cos(\varphi)cos(\theta), \quad x_2 = cos(\varphi)sin(\theta), \qquad (3.61)$$
$$x_3 = sin(\varphi), \quad for \quad \varphi \in (-\frac{\pi}{2}, \quad \frac{\pi}{2}), \quad \theta \in [0, 2\pi).$$

Then

$$\begin{aligned}\|\nabla_H u\|_H^2 &= X_1(u)^2 + X_2(u)^2 \\ &= 4x_1^2 + 4(x_2 - x_1x_3)^2 \\ &= 4cos(\varphi)^2 + 4cos(\varphi)^2 sin(\varphi)^2 cos(\theta)^2 - 8cos(\varphi)^2 cos(\theta)sin(\varphi)sin(\theta).\end{aligned} \qquad (3.62)$$

By the definitions of w_j *for* $1 \leq j \leq 3$ *and* (3.48)*, we have*

$$\begin{aligned}d\sigma_\Sigma &= \frac{1}{\|\nabla_H u\|_H}[(X_1(u))dx_2 - (X_2(u))dx_1] \wedge (x_1dx_2 + dx_3) \\ &= \frac{1}{\|\nabla_H u\|_H} 2cos(\varphi)^3 \left[1 - 2sin(\theta)cos(\theta)sin(\varphi) + cos(\theta)^2 sin(\varphi)^2\right] d\theta \wedge d\varphi.\end{aligned} \qquad (3.63)$$

By (3.62) *and* (3.63)*, we have* $\|\nabla_H u\|_H^{-1}$ *is locally summable around the isolated characteristic points with respect to the measure* $d\sigma_\Sigma$.

4. The sub-Riemannian limit of curvature of curves in the Lorentzian Heisenberg group for the third Lorentzian metric

We consider the Lorentzian Heisenberg group for the third lorentzian metric, let $\mathbb{H}$ be the Heisenberg group where the noncommutative group law is given by

$$(\overline{x}, \overline{y}, \overline{z}) \star (x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \overline{x}y + x\overline{y}).$$

Let

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + (1 - x_1)\partial_{x_3}, \quad X_3 = \partial_{x_2} - x_1\partial_{x_3}. \qquad (4.1)$$

Then

$$\partial_{x_1} = X_1, \quad \partial_{x_2} = x_1X_2 + (1 - x_1)X_3, \quad \partial_{x_3} = X_2 - X_3, \qquad (4.2)$$

and $\text{span}\{X_1, X_2, X_3\} = T\mathbb{H}$. Let $H = \text{span}\{X_1, X_2\}$ be the horizontal distribution on $\mathbb{H}$. Let $\omega_1 = dx_1$, $\omega_2 = x_1dx_2 + dx_3$, $\omega = (1 - x_1)dx_2 - dx_3$. Then $H = \text{Ker}\omega$. For the constant $L > 0$, let $g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 - L\omega \otimes \omega$, $g = g_1$ be the third Lorentzian metric on $\mathbb{H}$ (see [8]). We call $(\mathbb{H}, g_L)$ a Lorentzian Heisenberg group for the Lorentzian third metric and write $\mathbb{H}_L^3$ instead of $(\mathbb{H}, g_L)$. Then $X_1, X_2, \widetilde{X_3} := L^{-\frac{1}{2}}X_3$ are orthonormal basis on $T\mathbb{H}_L^3$ with respect to g_L. We have

$$[X_1, X_2] = X_3 - X_2, \quad [X_2, X_3] = 0, \quad [X_1, X_3] = X_3 - X_2. \qquad (4.3)$$

We say that a nonzero vector $\mathbf{x} \in \mathbb{H}_L^3$ is *spacelike*, *null* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. The *norm* of the vector $\mathbf{x} \in \mathbb{H}_L^3$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$.

Let $\gamma : I \to \mathbb{H}^3_L$ be a regular curve, where I is an open interval in $\mathbb{R}$. The regular curve γ is called a spacelike curve, timelike curve or null curve if $\gamma'(t)$ is a spacelike vector, timelike vector or null vector at any $t \in I$ respectively.

Let ∇^L be the Levi–Civita connection on $\mathbb{H}^3_L$ with respect to g_L. By the Koszul formula and (4.3), we have

Lemma 4.1 *Let $\mathbb{H}^3_L$ be the Lorentzian Heisenberg group for the third metric, then*

$$\nabla^L_{X_1}X_1 = 0, \quad \nabla^L_{X_1}X_3 = \frac{L-1}{2}X_2, \nabla^L_{X_2}X_2 = -X_1, \tag{4.4}$$
$$\nabla^L_{X_1}X_2 = \frac{L-1}{2L}X_3, \quad \nabla^L_{X_2}X_1 = \frac{-L-1}{2L}X_3 + X_2, \qquad \nabla^L_{X_3}X_3 = -LX_1,$$
$$\nabla^L_{X_3}X_1 = \frac{1+L}{2}X_2 - X_3, \nabla^L_{X_2}X_3 = \nabla^L_{X_3}X_2 = -\frac{1+L}{2}X_1.$$

Definition 4.2 *Let $\gamma : I \to \mathbb{H}^3_L$ be a C^1-smooth curve. We say that $\gamma(t)$ is a horizontal point of γ if*

$$\omega(\dot\gamma(t)) = (1-\gamma_1(t))\dot\gamma_2(t) - \dot\gamma_3(t) = 0.$$

Similar to the Definition 2.3 and Definition 2.5, we can define k^L_γ and k^∞_γ for the Lorentzian Heisenberg group for the third Lorentzian metric, we have

Lemma 4.3 *Let $\gamma : I \to \mathbb{H}^3_L$ be a C^2-smooth regular curve.*
(1)If $\nabla^L_{\dot\gamma}\dot\gamma$ is a spacelike vector, then

$$k^\infty_\gamma = \frac{\sqrt{\dot\gamma_1^2+\dot\gamma_2^2}}{|\omega(\dot\gamma(t))|}, \quad if \quad \omega(\dot\gamma(t)) \neq 0, \tag{4.5}$$

$$k^\infty_\gamma = \left\{ \frac{\left[\ddot\gamma_1 - \gamma_1^2\dot\gamma_2^2 - \dot\gamma_3^2 - 2\gamma_1\dot\gamma_2\dot\gamma_3\right]^2 + \left[\gamma_1\dot\gamma_1\dot\gamma_2 + \dot\gamma_1\dot\gamma_3 + \gamma_1\ddot\gamma_2 + \dot\gamma_1\dot\gamma_2 + \ddot\gamma_3\right]^2}{\left\{\dot\gamma_1^2 + (\gamma_1\dot\gamma_2+\dot\gamma_3)^2\right\}^2} \right. \tag{4.6}$$
$$\left. - \frac{\left\{\dot\gamma_1\left[\ddot\gamma_1 - \gamma_1^2\dot\gamma_2^2 - \dot\gamma_3^2 - 2\gamma_1\dot\gamma_2\dot\gamma_3\right] + (\gamma_1\dot\gamma_2+\dot\gamma_3)\left[\gamma_1\dot\gamma_1\dot\gamma_2 + \dot\gamma_1\dot\gamma_3 + \gamma_1\ddot\gamma_2 + \dot\gamma_1\dot\gamma_2 + \ddot\gamma_3\right]\right\}^2}{\left\{\dot\gamma_1^2 + (\gamma_1\dot\gamma_2+\dot\gamma_3)^2\right\}^3} \right\}^{\frac{1}{2}}$$
$$if \quad \omega(\dot\gamma(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot\gamma(t))) = 0,$$

$$\lim_{L\to+\infty}\frac{k^L_\gamma}{\sqrt{L}} = \frac{\sqrt{-\frac{d}{dt}(\omega(\dot\gamma(t)))^2}}{\left|\dot\gamma_1^2 + (\gamma_1\dot\gamma_2+\dot\gamma_3)^2\right|}, \quad if \quad \omega(\dot\gamma(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0. \tag{4.7}$$

Therefore, the situation does not exist.
(2)If $\nabla^L_{\dot\gamma}\dot\gamma$ is a timelike vector, then

$$k^\infty_\gamma = \frac{\sqrt{-\dot\gamma_1^2-\dot\gamma_2^2}}{|\omega(\dot\gamma(t))|}, \quad if \quad \omega(\dot\gamma(t)) \neq 0, \tag{4.8}$$

$$\lim_{L\to+\infty}\frac{k^L_{\gamma,\Sigma_1}}{\sqrt{L}}=\frac{|\frac{d}{dt}(\omega(\dot\gamma(t)))|}{\left[\overline{q}\dot\gamma_1-\overline{p}\left(\dot\gamma_2\gamma_1+\dot\gamma_3\right)\right]^2},\quad if\quad \omega(\dot\gamma(t))=0\quad and\quad \frac{d}{dt}(\omega(\dot\gamma(t)))\neq 0. \tag{4.9}$$

Therefore, the situation does not exist.

$$k^{\infty}_{\gamma}=\Bigg\{-\frac{\left[\ddot\gamma_1-\gamma_1^2\dot\gamma_2^2-\dot\gamma_3^2-2\gamma_1\dot\gamma_2\dot\gamma_3\right]^2+\left[\gamma_1\dot\gamma_1\dot\gamma_2+\dot\gamma_1\dot\gamma_3+\gamma_1\ddot\gamma_2+\dot\gamma_1\dot\gamma_2+\ddot\gamma_3\right]^2}{\left\{\dot\gamma_1^2+(\gamma_1\dot\gamma_2+\dot\gamma_3)^2\right\}^2} \tag{4.10}$$

$$+\frac{\left\{\dot\gamma_1\left[\ddot\gamma_1-\gamma_1^2\dot\gamma_2^2-\dot\gamma_3^2-2\gamma_1\dot\gamma_2\dot\gamma_3\right]+(\gamma_1\dot\gamma_2+\dot\gamma_3)\left[\gamma_1\dot\gamma_1\dot\gamma_2+\dot\gamma_1\dot\gamma_3+\gamma_1\ddot\gamma_2+\dot\gamma_1\dot\gamma_2+\ddot\gamma_3\right]\right\}^2}{\left\{\dot\gamma_1^2+(\gamma_1\dot\gamma_2+\dot\gamma_3)^2\right\}^3}\Bigg\}^{\frac{1}{2}}$$

$$if\quad \omega(\dot\gamma(t))=0\quad and\quad \frac{d}{dt}(\omega(\dot\gamma(t)))=0,$$

$$\lim_{L\to+\infty}\frac{k^L_{\gamma}}{\sqrt{L}}=\frac{|\frac{d}{dt}(\omega(\dot\gamma(t)))|}{|\dot\gamma_1^2+\left(\gamma_1\dot\gamma_2+\dot\gamma_3\right)^2|},\quad if\quad \omega(\dot\gamma(t))=0\quad and\quad \frac{d}{dt}(\omega(\dot\gamma(t)))\neq 0. \tag{4.11}$$

Proof By (4.2), we have

$$\dot\gamma(t)=\dot\gamma_1X_1+(\gamma_1\dot\gamma_2+\dot\gamma_3)X_2+\omega(\dot\gamma(t))X_3. \tag{4.12}$$

By Lemma 4.1 and (4.12), we have

$$\nabla^L_{\dot\gamma}X_1=\left[\gamma_1\dot\gamma_2+\dot\gamma_3+\frac{L+1}{2}\omega(\dot\gamma(t))\right]X_2-\left[\frac{L+1}{2L}\left(\gamma_1\dot\gamma_2+\dot\gamma_3\right)+\omega(\dot\gamma(t))\right]X_3, \tag{4.13}$$

$$\nabla^L_{\dot\gamma}X_2=-\left[\gamma_1\dot\gamma_2+\dot\gamma_3+\frac{L+1}{2}\omega(\dot\gamma(t))\right]X_1+\frac{L-1}{2L}\dot\gamma_1X_3,$$

$$\nabla^L_{\dot\gamma}X_3=\left[\frac{L-1}{2}\dot\gamma_3+\frac{L-1}{2}\gamma_1\dot\gamma_2-L\dot\gamma_2\right]X_1+\frac{L-1}{2}\dot\gamma_1X_2.$$

By (4.12) and (4.13), we have

$$\nabla^L_{\dot\gamma}\dot\gamma=\left[\ddot\gamma_1-\gamma_1^2\dot\gamma_2^2-\dot\gamma_3^2-2\gamma_1\dot\gamma_2\dot\gamma_3+(-L\dot\gamma_2-\gamma_1\dot\gamma_2-\dot\gamma_3)\omega(\dot\gamma(t))\right]X_1 \tag{4.14}$$

$$+\left[\gamma_1\dot\gamma_2\dot\gamma_1+\dot\gamma_1\dot\gamma_3+\gamma_1\ddot\gamma_2+\dot\gamma_2\dot\gamma_1+\ddot\gamma_3+L\dot\gamma_1\omega(\dot\gamma(t))\right]X_2$$

$$+\left[-\frac{1}{L}\gamma_1\dot\gamma_2\dot\gamma_1-\frac{1}{L}\dot\gamma_3\dot\gamma_1-\dot\gamma_1\omega(\dot\gamma(t))+\frac{d}{dt}(\omega(\dot\gamma(t)))\right]X_3$$

By (4.12) and (4.14), when $\omega(\dot\gamma(t))\neq 0$, we have

$$\langle\nabla^L_{\dot\gamma}\dot\gamma,\nabla^L_{\dot\gamma}\dot\gamma\rangle_L\sim\left[\dot\gamma_1^2+\dot\gamma_2^2\right]\omega(\dot\gamma(t))^2L^2,\quad as\quad L\to+\infty,$$

$$\langle\dot\gamma,\dot\gamma\rangle_L\sim -L\omega(\dot\gamma(t))^2,\quad as\quad L\to+\infty,$$

$$\langle\nabla^L_{\dot\gamma}\dot\gamma,\dot\gamma\rangle_L^2\sim O(L^2)\quad as\quad L\to+\infty.$$

Therefore

$$\frac{\langle\nabla^L_{\dot\gamma}\dot\gamma,\nabla^L_{\dot\gamma}\dot\gamma\rangle_L}{||\dot\gamma||_L^4}\to\frac{\left[\dot\gamma_1^2+\dot\gamma_2^2\right]}{\omega(\dot\gamma(t))^2},\quad as\quad L\to+\infty,$$

$$\frac{\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma\rangle_L^2}{||\dot\gamma||_L^6} \to 0, \quad as \quad L \to +\infty.$$

So by (2.6), we have (4.5) and (4.8). (4.6) and (4.9) come from (4.12), (4.14), (2.6) and $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) = 0$. When $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0$, we have

$$\langle \nabla^L_{\dot\gamma}\dot\gamma, \nabla^L_{\dot\gamma}\dot\gamma\rangle_L \sim -L[\frac{d}{dt}(\omega(\dot\gamma(t)))]^2, \quad as \quad L \to +\infty,$$

$$\langle \dot\gamma, \dot\gamma\rangle_L = \left\{\dot\gamma_1^2 + (\gamma_1\dot\gamma_2 + \dot\gamma_3)^2\right\},$$

$$\langle \nabla^L_{\dot\gamma}\dot\gamma, \dot\gamma\rangle_L^2 = O(1) \quad as \quad L \to +\infty.$$

By (2.6), we get (4.7). □

5. Lorentzian surfaces and a Gauss–Bonnet theorem in the Lorentzian Heisenberg group for the second Lorentzian metric

We will consider a regular surface $\Sigma_1 \subset \mathbb{H}^3_L$ and regular curve $\gamma \subset \Sigma_1$. We will assume that there exists a C^2-smooth function $u : \mathbb{H}^3_L \to \mathbb{R}$ such that

$$\Sigma_1 = \{(x_1, x_2, x_3) \in \mathbb{H}^3_L : u(x_1, x_2, x_3) = 0\}.$$

Similar to Section 3, we define $p, q, r, l, l_L, \overline{p}, \overline{q}, \overline{p_L}, \overline{q_L}, \overline{r_L}, v_L, e_1, e_2, J_L, k^L_{\gamma,\Sigma_1}, k^\infty_{\gamma,\Sigma_1}, k^{L,s}_{\gamma,\Sigma_1}, k^{\infty,s}_{\gamma,\Sigma_1}$. We call Σ a Lorentzian surface in the Lorentzian Heisenberg group for the third lorentzian metric. By (3.3) and (5.12), we have

$$\begin{aligned}
\nabla^{\Sigma_1,L}_{\dot\gamma}\dot\gamma = &\left\{\overline{q}\left[\ddot\gamma_1 - \gamma_1^2\dot\gamma_2^2 - \dot\gamma_3^2 - 2\gamma_1\dot\gamma_2\dot\gamma_3 + (-L\gamma_2 - \dot\gamma_2\gamma_1 - \dot\gamma_3)\,\omega(\dot\gamma(t))\right]\right.\\
&\left.-\overline{p}\left[\dot\gamma_1\gamma_1\dot\gamma_2 + \dot\gamma_1\dot\gamma_3 + \gamma_1\ddot\gamma_2 + \dot\gamma_1\dot\gamma_2 + \ddot\gamma_3 + L\dot\gamma_1\omega(\dot\gamma(t))\right]\right\} e_1\\
&-\left\{\overline{r_L}\ \overline{p}\left[\ddot\gamma_1 - \gamma_1^2\dot\gamma_2^2 - \dot\gamma_3^2 - 2\gamma_1\dot\gamma_2\dot\gamma_3 + (-L\gamma_2 - \dot\gamma_2\gamma_1 - \dot\gamma_3)\,\omega(\dot\gamma(t))\right]\right.\\
&+\overline{r_L}\ \overline{q}\left[\dot\gamma_1\gamma_1\dot\gamma_2 + \dot\gamma_1\dot\gamma_3 + \gamma_1\ddot\gamma_2 + \dot\gamma_1\dot\gamma_2 + \ddot\gamma_3 + L\dot\gamma_1\omega(\dot\gamma(t))\right]\\
&\left.+\frac{l}{l_L}L^{\frac{1}{2}}\left[\frac{d}{dt}(\omega(\dot\gamma(t))) - \frac{1}{L}\dot\gamma_1\gamma_1\dot\gamma_2 - \frac{1}{L}\dot\gamma_1\dot\gamma_3 - \dot\gamma_1\omega(\dot\gamma(t))\right]\right\} e_2\\
&= N_1e_1 + N_2e_2.
\end{aligned} \tag{5.1}$$

By (4.12) and $\dot\gamma(t) \in T\Sigma_1$, we have

$$\dot\gamma(t) = \left[\overline{q}\dot\gamma_1 - \overline{p}\left(\gamma_1\dot\gamma_2 + \dot\gamma_3\right)\right] e_1 - \frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot\gamma(t))e_2. \tag{5.2}$$

We have

Lemma 5.1 *Let $\Sigma_1 \subset \mathbb{H}^3_L$ be a regular Lorentzian surface. Let $\gamma : I \to \Sigma_1$ be a C^2-smooth regular curve.*
(1)If $\nabla^{\Sigma_1,L}_{\dot\gamma}\dot\gamma$ is a spacelike vector, then

$$k^\infty_{\gamma,\Sigma_1} = \frac{|\overline{p}\dot\gamma_1 + \overline{q}\dot\gamma_2|}{|\omega(\dot\gamma(t))|}, \quad if \quad \omega(\dot\gamma(t)) \neq 0, \tag{5.3}$$

$$k^{\infty}_{\gamma,\Sigma_1}=0,\quad if\quad \omega(\dot{\gamma}(t))=0,\quad and\quad \frac{d}{dt}(\omega(\dot{\gamma}(t)))=0,$$

$$\lim_{L\to+\infty}\frac{k^{L}_{\gamma,\Sigma_1}}{\sqrt{L}}=\frac{\sqrt{-\frac{d}{dt}(\omega(\dot{\gamma}(t)))^2}}{\left[\overline{q}\dot{\gamma}_1-\overline{p}\left(\dot{\gamma}_2\gamma_1+\dot{\gamma}_3\right)\right]^2},\quad if\quad \omega(\dot{\gamma}(t))=0\quad and\quad \frac{d}{dt}(\omega(\dot{\gamma}(t)))\neq 0. \tag{5.4}$$

Therefore, the situation does not exist.
(2)If $\nabla^{\Sigma_1,L}_{\dot{\gamma}}\dot{\gamma}$ *is a timelike vector, then*

$$k^{\infty}_{\gamma,\Sigma_1}=\frac{-(\overline{p}\dot{\gamma}_1+\overline{q}\dot{\gamma}_2)^2}{|\omega(\dot{\gamma}(t))|},\quad if\quad \omega(\dot{\gamma}(t))\neq 0, \tag{5.5}$$

Therefore, the situation does not exist.

$$k^{\infty}_{\gamma,\Sigma_1}=0,\quad if\quad \omega(\dot{\gamma}(t))=0,\quad and\quad \frac{d}{dt}(\omega(\dot{\gamma}(t)))=0,$$

$$\lim_{L\to+\infty}\frac{k^{L}_{\gamma,\Sigma_1}}{\sqrt{L}}=\frac{|\frac{d}{dt}(\omega(\dot{\gamma}(t)))|}{\left[\overline{q}\dot{\gamma}_1-\overline{p}\left(\dot{\gamma}_2\gamma_1+\dot{\gamma}_3\right)\right]^2},\quad if\quad \omega(\dot{\gamma}(t))=0\quad and\quad \frac{d}{dt}(\omega(\dot{\gamma}(t)))\neq 0. \tag{5.6}$$

Proof By (5.1), we have

$$\langle\nabla^{\Sigma_1,L}_{\dot{\gamma}}\dot{\gamma},\nabla^{\Sigma_1,L}_{\dot{\gamma}}\dot{\gamma}\rangle_{L,\Sigma_1}\sim L^2\omega(\dot{\gamma}(t))^2\left[\overline{q}\dot{\gamma}_2+\overline{p}\dot{\gamma}_1\right]^2, \tag{5.7}$$
$$\text{as}\quad L\to+\infty.$$

By (5.2), we have that when $\omega(\dot{\gamma}(t))\neq 0$,

$$\langle\dot{\gamma},\dot{\gamma}\rangle_{\Sigma_1,L}\sim -L[\omega(\dot{\gamma}(t))]^2,\quad \text{as}\quad L\to+\infty. \tag{5.8}$$

By (5.1) and (5.2), we have

$$\langle\nabla^{\Sigma_1,L}_{\dot{\gamma}}\dot{\gamma},\dot{\gamma}\rangle_{\Sigma_1,L}\sim M_0L, \tag{5.9}$$

where M_0 does not depend on L. By Definition 3.1, (5.7) – (5.9), we get (5.3). When $\omega(\dot{\gamma}(t))=0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t)))=0$, we have

$$\langle\nabla^{\Sigma_1,L}_{\dot{\gamma}}\dot{\gamma},\nabla^{\Sigma_1,L}_{\dot{\gamma}}\dot{\gamma}\rangle_{L,\Sigma_1}\sim N_1^2 \tag{5.10}$$
$$\text{as}\quad L\to+\infty,$$

$$\langle\dot{\gamma},\dot{\gamma}\rangle_{\Sigma_1,L}=\left[\overline{q}\dot{\gamma}_1-\overline{p}\left(\gamma_1\dot{\gamma}_2+\dot{\gamma}_3\right)\right]^2, \tag{5.11}$$

$$\langle\nabla^{\Sigma_1,L}_{\dot{\gamma}}\dot{\gamma},\dot{\gamma}\rangle_{\Sigma_1,L}=\left[\overline{q}\dot{\gamma}_1-\overline{p}\left(\gamma_1\dot{\gamma}_2+\dot{\gamma}_3\right)\right]N_1. \tag{5.12}$$

By (5.10) – (5.12) and Definition 3.1, we get $k^{\infty}_{\gamma,\Sigma_1}=0$. When $\omega(\dot{\gamma}(t))=0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t)))\neq 0$, we have

$$\langle\nabla^{\Sigma_1,L}_{\dot{\gamma}}\dot{\gamma},\nabla^{\Sigma_1,L}_{\dot{\gamma}}\dot{\gamma}\rangle_{L,\Sigma_1}\sim -L[\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2,$$

$$\langle \nabla^{\Sigma_1,L}_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma_1,L} = O(1),$$

so we get (5.6). □

Lemma 5.2 *Let $\Sigma_1 \subset \mathbb{H}^3_L$ be a regular Lorentzian surface.*
(1)If $\gamma : I \to \Sigma_1$ be a spacelike C^2-smooth curve, then

$$k^{\infty,s}_{\gamma,\Sigma_1} = -\frac{\overline{q}\dot{\gamma}_2 + \overline{p}\dot{\gamma}_1}{\sqrt{-\omega(\dot{\gamma}(t))^2}}, \qquad if \quad \omega(\dot{\gamma}(t)) \neq 0. \tag{5.13}$$

Therefore, the situation does not exist.

$$k^{\infty,s}_{\gamma,\Sigma_1} = 0 \quad if \quad \omega(\dot{\gamma}(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0; \tag{5.14}$$

$$\lim_{L\to+\infty} \frac{k^{L,s}_{\gamma,\Sigma_1}}{\sqrt{L}} = \frac{\left|\frac{d}{dt}(\omega(\dot{\gamma}(t)))\right|}{\left|\overline{q}\dot{\gamma}_1 - \overline{p}\left(\gamma_1\dot{\gamma}_2 + \dot{\gamma}_3\right)\right|^2}, \quad if \quad \omega(\dot{\gamma}(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \tag{5.15}$$

(2)If $\gamma : [a,b] \to \Sigma_1$ be a timelike C^2-smooth curve, then

$$k^{\infty,s}_{\gamma,\Sigma_1} = -\frac{\overline{q}\dot{\gamma}_2 + \overline{p}\dot{\gamma}_1}{|\omega(\dot{\gamma}(t))|}, \qquad if \quad \omega(\dot{\gamma}(t)) \neq 0. \tag{5.16}$$

$$\|\dot{\gamma}\|_{L,\Sigma} = \sqrt{-\left[\overline{q}\dot{\gamma}_1 - \overline{p}\left(\dot{\gamma}_2\gamma_1 + \dot{\gamma}_3\right)\right]^2}, \quad if \quad \omega(\dot{\gamma}(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0; \tag{5.17}$$

Therefore, the situation does not exist.

$$\|\dot{\gamma}\|_{L,\Sigma} = \sqrt{-\left[\overline{q}\dot{\gamma}_1 - \overline{p}\left(\dot{\gamma}_2\gamma_1 + \dot{\gamma}_3\right)\right]^2}, \quad if \quad \omega(\dot{\gamma}(t)) = 0 \quad and \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \tag{5.18}$$

Therefore, the situation does not exist.

Proof For (1), by (3.3) and (5.2), we have

$$J_L(\dot{\gamma}) = -\frac{l_L}{l} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) e_1 + \left[\overline{q}\dot{\gamma}_1 - \overline{p}\left(\gamma_1\dot{\gamma}_2 + \dot{\gamma}_3\right)\right] e_2. \tag{5.19}$$

By (5.1) and (5.19), we have

$$\langle \nabla^{\Sigma_1,L}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L,\Sigma_1} \sim L^{\frac{3}{2}} \omega(\dot{\gamma}(t))^2 \left[\overline{q}\dot{\gamma}_2 + \overline{p}\dot{\gamma}_1\right], \quad \text{as} \quad L \to +\infty. \tag{5.20}$$

When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$, we get

$$\langle \nabla^{\Sigma_1,L}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L,\Sigma_1} \sim M_0 L^{-\frac{1}{2}} \quad \text{as} \quad L \to +\infty. \tag{5.21}$$

So $k^{\infty,s}_{\gamma,\Sigma_1}=0$. When $\omega(\dot\gamma(t))=0$ and $\frac{d}{dt}(\omega(\dot\gamma(t)))\neq 0$, we have

$$\langle\nabla^{\Sigma_1,L}_{\dot\gamma}\dot\gamma, J_L(\dot\gamma)\rangle_{L,\Sigma_1}\sim L^{\frac{1}{2}}\left[\overline{p}\left(\gamma_1\dot\gamma_2+\dot\gamma_3\right)-\overline{q}\dot\gamma_1\right]\frac{d}{dt}(\omega(\dot\gamma(t))),\quad \text{as}\quad L\to+\infty. \tag{5.22}$$

So we get (5.15). □

Next, we compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in the Lorentzian Heisenberg group for the third Lorentzian metric. Similarly to Theorem 4.3 in [5], we have

Theorem 5.3 *The second fundamental form II^L_1 of the embedding of Σ_1 into $\mathbb{H}^3_L$ is given by*

$$II^L_1=\begin{pmatrix} h_{11}, & h_{12}\\ h_{21}, & h_{22}\end{pmatrix}, \tag{5.23}$$

where

$$\begin{aligned}
h_{11}&=\frac{l}{l_L}[X_1(\overline{p})+X_2(\overline{q})]+\overline{p_L}-\overline{pqr_L}L^{-\frac{1}{2}},\\
h_{12}&=h_{21}=-\frac{l_L}{l}\langle e_1,\nabla_H(\overline{r_L})\rangle_L+\overline{p_L}^2L^{-\frac{1}{2}}\frac{L+1}{2}-\overline{p}^2\overline{r_L}^2\frac{L+1}{2}L^{-\frac{1}{2}}\\
&+\overline{q_L}^2L^{-\frac{1}{2}}\frac{L-1}{2}-\overline{q}^2\overline{r_L}^2\frac{L-1}{2}L^{-\frac{1}{2}},\\
h_{22}&=-\frac{l^2}{l_L^2}\langle e_2,\nabla_H(\frac{r}{l})\rangle_L+\widetilde{X_3}(\overline{r_L})-\overline{p_Lqr_L}\frac{l}{l_L}L^{-\frac{1}{2}}+\overline{p_L}\frac{l^2}{l_L^2}\\
&+\overline{r_L}^3\overline{qp}L^{-\frac{1}{2}}-\overline{r_L}^2\frac{l}{l_L}\overline{p}.
\end{aligned}$$

Similar to Proposition 3.8, we have

Proposition 5.4 *Away from characteristic points, the horizontal mean curvature $\mathcal{H}^1_\infty$ of $\Sigma_1\subset\mathbb{H}^3_L$ is given by*

$$\mathcal{H}^1_\infty=X_1(\overline{p})+X_2(\overline{q})+2\overline{p}. \tag{5.24}$$

By Lemma 4.1, we have

Lemma 5.5 *Let $\mathbb{H}^3_L$ be the Lorentzian Heisenberg group for the third metric, then*

$$\begin{aligned}
&R^L(X_1,X_2)X_1=\frac{-3L^2+2L+1}{4L}X_2+\frac{L-1}{L}X_3,\quad R^L(X_1,X_2)X_2=\frac{3L^2-2L-1}{4L}X_1, \qquad (5.25)\\
&R^L(X_1,X_2)X_3=(L-1)X_1,\quad R^L(X_1,X_3)X_1=(1-L)X_2+\frac{L^2+2L-3}{4L}X_3,\\
&R^L(X_1,X_3)X_2=(L-1)X_1,\quad R^L(X_1,X_3)X_3=\frac{L^2+2L-3}{4L}X_1,\\
&R^L(X_2,X_3)X_1=0,\quad R^L(X_2,X_3)X_2=\frac{L^2-2L+1}{4L}X_3,\quad R^L(X_2,X_3)X_3=\frac{L^2-2L+1}{4L}X_3.
\end{aligned}$$

Proposition 5.6 *Away from characteristic points, we have*

$$\mathcal{K}^{\Sigma_1,\infty}(e_1,e_2)=\overline{A}+O(L^{-1}). \tag{5.26}$$

where

$$\overline{A}=\frac{(X_3u)^2}{l^2}-2\overline{q}\frac{X_3u}{|\nabla_H u|}+\frac{\overline{q}^2+\overline{p}^2}{2}+\langle e_1,\nabla_H(\frac{X_3u}{|\nabla_H u|})\rangle+\overline{p}\left[X_1(\overline{p})+X_2(\overline{q})\right]. \tag{5.27}$$

Proof By (3.41) and Lemma 5.5, we have

$$\mathcal{K}^{\mathbb{H}^3_L}(e_1,e_2)=\overline{r_L}^2\frac{3L^2-2L-1}{4L}-2\overline{r_Lq_L}L^{-\frac{1}{2}}(L-1)+\overline{q_L}^2\frac{L^2+2L-3}{4L}+\overline{P_L}^2\frac{L^2-2L+1}{4L}.$$

Similar to (3.43), we have

$$\det(II^L_1)=-\frac{L}{4}+\langle e_1,\nabla_H(\frac{X_3u}{|\nabla_H u|})\rangle+\overline{p}\left[X_1(\overline{p})+X_2(\overline{q})+\overline{p}\right]+O(L^{-1}) \text{ as } L\to+\infty. \tag{5.28}$$

By (5.27) and (5.28), we have (5.26). □

Similar to (3.46) and (3.49), for the Lorentzian Heisenberg group for the third Lorentzian metric. Let $\gamma:I\to\mathbb{H}^3_L$ be a timelike C^2-smooth curve, we have

$$\lim_{L\to+\infty}\frac{1}{\sqrt{L}}ds_L=ds,\quad \lim_{L\to+\infty}\frac{1}{\sqrt{L}}d\sigma_{\Sigma_1,L}=d\sigma_{\Sigma_1}. \tag{5.29}$$

By (5.26), (5.29) and Lemma 5.2, similar to the proof of Theorem 3.13, we have

Theorem 5.7 *Let $\Sigma_1\subset\mathbb{H}^3_L$ be a regular Lorentzian surface with finitely many boundary components $(\partial\Sigma_1)_i$, $i\in\{1,\cdots,n\}$, given by C^2-smooth regular and closed timelike curves $\gamma_i:[0,2\pi]\to(\partial\Sigma_1)_i$. Suppose that the characteristic set $C(\Sigma_1)$ satisfies $\mathcal{H}^1(C(\Sigma_1))=0$ and that $||\nabla_H u||_H^{-1}$ is locally summable with respect to the 2-dimensional Hausdorff measure near the characteristic set $C(\Sigma_1)$, then*

$$\int_{\Sigma_1}\mathcal{K}^{\Sigma_1,\infty}d\sigma_{\Sigma_1}+\sum_{i=1}^n\int_{\gamma_i}k^{\infty,s}_{\gamma_i,\Sigma_1}ds=0. \tag{5.30}$$

Example 5.8 *In $\mathbb{H}^3_L$, let $u=x_1^2+x_2^2+x_3^2-1$ and $\sum_1=S^2$. $\sum_1$ is a regular surface. By (4.1), we get*

$$X_1(u)=2x_1;\quad X_2(u)=2x_2+2(1-x_1)x_3. \tag{5.31}$$

Solve the equations $X_1(u)=X_2(u)=0$,
then we get

$$C(\Sigma_1)=\{(0,\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}),(0,-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})\}$$

and $\mathcal{H}^1(C(\Sigma_1))=0$.
A parametrization of Σ_1 is

$$x_1=cos(\varphi)cos(\theta),\quad x_2=cos(\varphi)sin(\theta), \tag{5.32}$$

$$x_3=sin(\varphi),\quad for\quad \varphi\in(-\frac{\pi}{2},\ \frac{\pi}{2}),\quad \theta\in[0,2\pi).$$

Then

$$\|\nabla_H u\|_H^2 = X_1(u)^2 + X_2(u)^2 \tag{5.33}$$
$$= 4x_1^2 + 4\left[x_2 + (1-x_1)x_3\right]^2$$
$$= 4 + 4cos(\varphi)^2 cos(\theta)^2 sin(\varphi)^2 - 8cos(\varphi)cos(\theta)sin(\varphi)^2$$
$$- 8cos(\varphi)^2 sin(\varphi)cos(\theta)sin(\theta) + 8cos(\varphi)sin(\varphi)sin(\theta).$$

By the definitions of w_j *for* $1 \leq j \leq 3$ *and* (5.29), *we have*

$$d\sigma_{\Sigma_1} = \frac{1}{\|\nabla_H u\|_H}[(X_1(u))(x_1dx_2 + dx_3) - (X_2(u))dx_1] \wedge [(1-x_1)dx_2 - dx_3] \tag{5.34}$$
$$= -\frac{1}{\|\nabla_H u\|_H} 2cos(\varphi)\lambda_0 d\theta \wedge d\varphi.$$

where

$$\lambda_0 = cos(\varphi)^2 - 2cos(\varphi)^2 cos(\theta)sin(\varphi)sin(\theta) - 2cos(\varphi)cos(\theta)sin(\varphi)^2 \tag{5.35}$$
$$+ 2cos(\varphi)sin(\theta)sin(\varphi) + sin(\varphi)^2 + cos(\varphi)^2 sin(\varphi)^2 cos(\theta)^2.$$

And λ_0 *is a bounded smooth function on* Σ_1*. By* (5.33) *and* (5.34), *we have* $\|\nabla_H u\|_H^{-1}$ *is locally summable around the isolated characteristic points with respect to the measure* $d\sigma_{\Sigma_1}$*.*

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