

A new implicit-explicit local differential method for boundary value problems

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Abstract: In this study, an effective numerical method based on Taylor expansions is presented for boundary value problems. This method is arbitrary directional and called as implicit-explicit local differential transform method (IELDTM). With the completion of this study, a reliable numerical method is derived by optimizing the required degrees of freedom. It is shown that the order refinement procedure of the IELDTM does not affect the degrees of freedom. A priori error analysis of the current method is constructed and order conditions are presented in a detailed analysis. The theoretical order expectations are verified for nonlinear BVPs. Stability of the IELDTM is investigated by following the analysis of approximation matrices. To illustrate efficiency of the method, qualitative and quantitative results are presented for various challenging BVPs. It is tested that the current method is reliable and accurate for a broad range of problems even for strongly nonlinear and singularly perturbed BVPs. The produced results have revealed that the IELDTM is more accurate than the existing ones in literature.

Key words: Implicit-explicit methods, boundary value problem, Taylor series, numerical algorithms, stiff problems

1. Introduction

Boundary value problems (BVPs) of differential equations arise in many disciplines such as physics, chemistry, engineering, finance, mathematical biology and so on. Analytical solutions are often not available for most of those problems. While some series-based techniques are capable of producing semianalytical solutions for BVPs, convergence of those methods is largely dependent on the global smoothness of exact solutions, as observed from examples discussed in the literature [38, 42]. In addition, some BVPs involving significant local behaviors such as sharp discontinuities or boundary layers are areas where various notable difficulties are encountered [36, 37]. In such cases, any analytical or numerical approach to such problems should be well defined.

Numerical methods for boundary value problems are mainly divided into two categories: direct methods and shooting methods [1]. The shooting strategies are important for using IVP algorithms to solve BVPs, but the computational mechanisms involve additional costs to find out high order initial conditions [2–4]. A well-known direct method for BVPs is the finite difference method widely used in the literature [5–10]. Even finite difference methods produce acceptable results for many BVPs, their local order refinement (p-refinement) is not easy task due to the direct disconnection of the higher order formulations. Another kind of direct method is the collocation methods which are based on direct substitutions of approximate solutions and getting algebraic equations at collocation points [11–16]. These types of methods are suitable for local order refinement and can also be used in either local form or global form. Besides these widely used techniques, local discontinuous

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Galerkin method [17], spline interpolation based numerical methods [18–20], Adomian decomposition methods [21, 22] and reproducing kernel methods [22–24] have been applied to solve various kinds of BVPs.

Classical structure of the differential transform method (DTM) is semianalytic and leads to nonlocal series approximation [39–41]. The BVPs can be solved by the differential transform method (DTM) even the original structure of the method is designed for IVPs. Thus, DTM based methods for BVPs can be considered to be shooting methods [25–32]. Various kinds of BVPs were solved by DTMs in literature [25–32]. Generally, those semianalytic approaches give good agreement with exact solution only about one of the boundaries due to the structure of the Taylor series expansion. The idea of discretization of finite domain and then applying DTM procedure numerically is known as local DTM (LDTM) or multistep DTM [34, 35, 43]. Although the LDTM is better than the DTM in terms of convergence, as will be shown later, the LDTM does not lead to stable results as a shooting method [43] for especially stiff problems. Thus, the development of a higher order and stability preserved method that can be used in solving boundary value problems will fill the gap in this area.

In this study, we have proposed a new implicit-explicit local differential transformation method (IELDTM) to solve various BVP types. In this sense, a Taylor series based high-order numerical method has been created here, which is flexible in terms of accuracy, stability and local adaptability. In the way of developing this idea, the presented method here eliminates the well-known disadvantages of the DTMs such as divergence and numerical instability. The IELDTM has advantage over the finite difference method as it provides p-refinement convenience, and the finite element method as it minimizes the degrees of freedom. The present technique is seen to be suitable for stiff and nonstiff BVPs as well as for strongly nonlinear BVPs as will be shown in the following sections. A priori error analysis of the present method has been done, and order conditions have been produced. Stability results are discussed together with the selection of the direction parameter that determines whether the method is explicit or implicit. In numerical experiments, various types of BVPs, particularly high-order BVPs, singular BVPs, singularly perturbed BVPs, and strongly nonlinear BVPs have been studied to test the derived method.

2. Local differential transformation

The following propositions and definitions are needed to construct a numerical algorithm, and are the localized forms of the semianalytical forms presented in literature [33].

Definition 2.1 *When $y(x)$ is analytic in domain S , the function $\varphi(x, k)$ can be defined as follows:*

$$\frac{d^k y(x)}{dx^k} = \varphi(x, k) \quad \text{for all } x \in S, \quad (2.1)$$

where k is a nonnegative integer.

By considering Definition 2.1, the differential transform of function $y(x)$ at any position $x = x_i$ in the domain is locally defined as follows:

$$Y_i(k) = \frac{\varphi(x, k)}{k!} = \frac{1}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=x_i}. \quad (2.2)$$

Definition 2.2 When $y(x)$ is analytic in domain S , then $y(x)$ can be denoted by Taylor series at $x = x_i$ as follows:

$$y_i(x) = \sum_{k=0}^{\infty} Y_i(k)(x - x_i)^k = D^{-1}Y_i(k), \quad x_i - \rho \leq x \leq x_i + \rho \tag{2.3}$$

where D^{-1} denotes the inverse differential transform operator and ρ is the radius of convergence.

By truncating series 2.3, $y(x)$ can be represented as follows:

$$y_i(x) = \sum_{k=0}^K Y_i(k)(x - x_i)^k + O((x - x_i)^{K+1}), \quad x_i - \rho \leq x \leq x_i + \rho. \tag{2.4}$$

Proposition 2.3 Let $y(x)$ and $w(x)$ be analytic in domain S and $x_i \in S$. If $D(y_i(x), k) = Y_i(k)$, $D(w_i(x), k) = W_i(k)$ and $z_i(x) = y_i(x)w_i(x)$, then the following equality holds

$$Z_i(k) = D(z_i(x), k) = \sum_{n=0}^k Y_i(n)W_i(k - n). \tag{2.5}$$

Proposition 2.4 Let $y(x)$ be analytic in domain S and $x_i \in S$. If $D(y_i(x), k) = Y_i(k)$ and $z_i(x) = y_i^m(x)$ for $m \in N$, then the following equality holds

$$Z_i(k) = D(z_i(x), k) = \sum_{n_1=0}^k \sum_{n_2=0}^{k-n_1} \dots \sum_{n_{m-1}=0}^{k-\sum_{i=1}^{m-2} n_i} Y_i(n_1) \dots Y_i(n_{m-1}) Y_i(k - \sum_{i=1}^{m-1} n_i). \tag{2.6}$$

Proposition 2.5 Let $y(x)$ be analytic in domain S and $x_i \in S$. If $D(y_i(x), k) = Y_i(k)$ and $z_i(x) = \frac{d^m y_i(x)}{dx^m}$ for $m \in N$, then the following equality holds

$$Z_i(k) = D(z_i(x), k) = (k + 1)(k + 2) \dots (k + m) Y_i(k + m). \tag{2.7}$$

Definition 2.6 Let $y(x)$ be analytic in domain S and $x_i \in S$ and $m, n, p \in N$. If $D(y_i(x), k) = Y_i(k)$ and $z_i(x) = (y_i^m(x)) \left(\frac{d^p y_i(x)}{dx^p} \right)^n$, we define the following transformation notation

$$Z_i(k) = D(Z_i(x), k) = \bar{Y}_i(m; p, n; k) \tag{2.8}$$

Note that we assume $p = 0$ and $n = 0$ in case of $z_i(x) = y_i^m(x)$. Throughout the rest of the paper we use this notation given in Definition 2.6. Before giving the propositions about the general values of $m, n, p \in N$, we need to consider the following special cases;

- $\bar{Y}_i(m; 0, 0; k) = \sum_{n_1=0}^k \sum_{n_2=0}^{k-n_1} \sum_{n_3=0}^{k-n_1-n_2} \dots \sum_{n_{m-1}=0}^{k-\sum_{i=1}^{m-2} n_i} Y_i(n_1) \dots Y_i(n_{m-1}) Y_i(k - \sum_{i=1}^{m-1} n_i)$
- $\bar{Y}_i(0; p, 1; k) = (k + 1)(k + 2) \dots (k + p) Y_i(k + p)$

$$\cdot \bar{Y}_i(0; p, n; k) = \sum_{n_1=0}^k \dots \sum_{n_{m-1}=0}^{k-\sum_{i=1}^{m-2} n_i} \bar{Y}_i(0; p, 1; n_1) \dots \bar{Y}_i(0; p, 1; n_{m-1}) \bar{Y}_i(0; p, 1; k - \sum_{i=1}^{m-1} n_i).$$

Proposition 2.7 *Let $y(x)$ be analytic in domain S and $x_i \in S$ and $m, n, p \in N$. If $D(y_i(x), k) = Y_i(k)$, then the following equality holds*

$$\bar{Y}_i(m; p, n; k) = \sum_{n=0}^k \bar{Y}_i(m; 0, 0; k) \bar{Y}_i(0; p, n; k - n). \tag{2.9}$$

Proposition 2.8 *Assume that the power series*

$$\sum_{k=0}^{\infty} Y_i(k) (x - x_i)^k \tag{2.10}$$

has the radius of convergence ρ . For any point x_j satisfying $|x_i - x_j| < \rho$, then the power series

$$\sum_{k=0}^{\infty} Y_j(k) (x - x_j)^k \tag{2.11}$$

has the radius of convergence at least $\tau = \rho - |x_i - x_j|$ and on the interval $(x_j - \tau, x_j + \tau)$ it converges to $y(x)$.

In the light of Proposition 2.8, we present the foundations of the IELDTM in the following section.

3. The implicit-explicit local differential transform method

This section is devoted to numerical foundations of the IELDTM for general second order BVPs. A general description of the IELDTM is explained and some comments on this method are provided. Numerical implementation of the present method to the general second order linear BVP is presented. Priors error analysis of the IELDTM is done and order conditions are determined. Stability properties of the IELDTM for the considered BVP is explained and some illustrations are presented.

Let us consider the following general second order two-point boundary value problem,

$$y''(x) = G(y'(x), y(x), x), \quad x \in S \tag{3.1}$$

with the mixed boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = \delta_1 \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = \delta_2 \tag{3.2}$$

where $S = [a, b] \subset R$, $y(x) \in R$, $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2 \in R$ and $G : R \times R \times S \rightarrow R$.

We assume that the BVP (3.1-3.2) is well-posed and the exact solution $y(x)$ is analytic on the considered domain S . Let us divide the interval $[a, b]$ into at most N spatial elements with $dx_i = x_{i+1} - x_i$ and the partition of the interval is given by $\omega = \{a = x_0 < x_1 < \dots < x_{N^*} = b\}$ where $N^* \leq N$. Let us consider the convergent local Taylor series representation of the function $y(x)$ about $x_i \in S$ as follows:

$$y_i(x) = \sum_{k=0}^K Y_i(k) (x - x_i)^k + O\left((x - x_i)^{K+1}\right), \quad x_i - \rho^i \leq x \leq x_i + \rho^i \tag{3.3}$$

where $i = 0, 1, \dots, N^*$. The function $y_i(x)$ is assumed to be analytic for all i and has the radius of convergence $\rho^i > dx_i$. This assumption leads us to search about the relations between neighbour solutions $y_i(x)$ and $y_{i+1}(x)$ in terms of their local behaviours. With Proposition 2.8, since the condition $|x_{i+1} - x_i| = dx_i < \rho_i$ holds, the function $y(x)$ has also a local convergent Taylor series representation about $x = x_{i+1}$ with the radius of convergence at least $\rho^{i+1} = \rho^i - dx_i$ and the interval of convergence is $(x_{i+1} - \rho^{i+1}, x_{i+1} + \rho^{i+1})$. Then, the function $y(x)$ can be locally approximated as

$$y_{i+1}(x) = \sum_{k=0}^K Y_{i+1}(k) (x - x_{i+1})^k + O\left((x - x_{i+1})^{K+1}\right), \tag{3.4}$$

where $x_{i+1} - \rho^{i+1} \leq x \leq x_{i+1} + \rho^{i+1}$. Let us define the intervals $P^i = [x_i - \rho^i, x_i + \rho^i]$, $P^{i+1} = [x_{i+1} - \rho^{i+1}, x_{i+1} + \rho^{i+1}]$ and $P = [x_i, x_{i+1}]$. Assume that the condition $P \subset (P^i \cap P^{i+1})$ holds. Then we state the following conclusions:

- . Due to the convergence assumptions, the two convergent representations (3.3) and (3.4) should give the same numerical result at any point in the interval $[x_i, x_{i+1}]$.
- . If necessary, C^m continuity condition i.e. continuous slopes, continuous curvatures etc. can be applied in the interval $[x_i, x_{i+1}]$.
- . Any point in the interval $[x_i, x_{i+1}]$ can be written as $x^* = x_i + (1 - \theta)dx_i$, where $0 \leq \theta \leq 1$.
- . We produce the IELDTM with the use of continuity conditions of the solutions at such interior points.

Convergent solutions $y_i(x)$ and $y_{i+1}(x)$ at the interior points $x^* = x_i + (1 - \theta)dx_i$ need to satisfy the following C^0 and C^1 continuity relations

$$y_{i+1}(x_i + (1 - \theta)dx_i) = y_i(x_i + (1 - \theta)dx_i), \tag{3.5}$$

$$y'_{i+1}(x_i + (1 - \theta)dx_i) = y'_i(x_i + (1 - \theta)dx_i). \tag{3.6}$$

Substituting interior node value $x^* = x_i + (1 - \theta)dx_i$ to approximate solutions (3.3-3.4) and using continuity relations (3.5-3.6) yield the following equations,

$$\sum_{k=0}^K Y_{i+1}(k) (-\theta dx_i)^k = \sum_{k=0}^K Y_i(k) ((1 - \theta)dx_i)^k + O\left((dx_i)^{K+1}\right) \tag{3.7}$$

$$\sum_{k=1}^K Y_{i+1}(k) k (-\theta dx_i)^{k-1} = \sum_{k=1}^K Y_i(k) k ((1 - \theta)dx_i)^{k-1} + O\left((dx_i)^K\right). \tag{3.8}$$

The next issue is to determine the algebraic relations between $Y_i(k)$ and $Y_{i+1}(k)$ for all $k = 0, 1, \dots, K$. Taking differential transform of equation (3.1) with the use of Proposition 2.5 given in Section 2 leads to the following relation

$$Y_i(k + 2) = \frac{1}{(k + 1)(k + 2)} F(Y_i(k + 1), Y_i(k), \dots, Y_i(1), Y_i(0), H(k)) \tag{3.9}$$

where $i = 0, 1, \dots, N^*$, $k = 0, 1, \dots, K - 2$, F is the transformed form of the function arbitrary function $G(y'(x), y(x), x)$ and $H(k)$ is the transformed form of the nonautonomous part of the function $G(y'(x), y(x), x)$. It is obvious that $Y_i(k + 2)$ can be written in terms of $Y_i(k + 1)$ and $Y_i(k)$ for each i and k . Thus, to completely describe approximation functions (3.3-3.4), one should evaluate the initial position $Y_i(0)$ and the initial slope $Y_i(1)$ for all $i = 0, 1, \dots, N^*$. For each spatial position, $Y_i(0)$ and $Y_i(1)$ are assumed to be unknown values to determine highly accurate local approximations (3.3-3.4). With the use of algebraic relations in (3.9), substitution of all algebraic coefficients $Y_i(k)$ into continuity equations (3.7-3.8) leads to the following explicit-implicit local equations

$$g_1(Y_{i+1}(0), Y_{i+1}(1), \theta, dx_i) = h_1(Y_i(0), Y_i(1), \theta, dx_i) \tag{3.10}$$

$$g_2(Y_{i+1}(0), Y_{i+1}(1), \theta, dx_i) = h_2(Y_i(0), Y_i(1), \theta, dx_i) \tag{3.11}$$

where $i = 0, 1, \dots, N^*$ and the functions g_1, g_2, h_1 and h_2 are obtained from the right- and left-hand sides of equations (3.7-3.8). If $\theta = 0$ then the equations (3.10-3.11) are explicit equations of $Y_{i+1}(0)$ and $Y_{i+1}(1)$ irrespective of the linearity-nonlinearity of the differential equation. On the other hand, if $\theta \neq 0$ and the differential equation is nonlinear then equations (3.10-3.11) are implicit equations of $Y_{i+1}(0)$ and $Y_{i+1}(1)$. The boundary conditions lead to the following equations:

$$\alpha_1 Y_0(0) + \alpha_2 Y_0(1) = \delta_1 \tag{3.12}$$

$$\beta_1 Y_{N^*}(0) + \beta_2 Y_{N^*}(1) = \delta_2. \tag{3.13}$$

By eliminating $Y_0(0)$ and $Y_{N^*}(0)$ from the equations (3.12-3.13) and substituting into system of algebraic equations (3.10-3.11), equations (3.10-3.11) can be rewritten as follows:

$$\varphi(Y_0(1), Y_1(0), Y_1(1), Y_2(0), Y_2(1), \dots, Y_{N^*-1}(0), Y_{N^*-1}(1), Y_{N^*}(1)) = \mathbf{Z} \tag{3.14}$$

where the system is $(2N^*) \times (2N^*)$ including all local positions and slopes. Depending on the function $G(y'(x), y(x), x)$, $\varphi(\mathbf{Y}) = \mathbf{Z}$ is either a linear or nonlinear system of algebraic equations with

$$\mathbf{Y} = (Y_0(1), Y_1(0), Y_1(1), Y_2(0), Y_2(1), \dots, Y_{N^*-1}(0), Y_{N^*-1}(1), Y_{N^*}(1)).$$

MATLAB built-in function *fsolve* has been used when equation system (3.14) is nonlinear. The function φ also depends on the selection of the direction parameter θ . As will be demonstrated later on, this parameter is crucial in terms of accuracy, stability and computational efficiency. Therefore, the following notes are important for the rest of this study

- . $N^* = 1$ and $\theta = 0$ yields the classical semianalytic DTM [25-30].
- . $\theta = 0$ yields the explicit forward scheme and known as local differential transform method which is used in the literature [34, 35].
- . The rest of the selections of the parameter θ , i.e. $\theta \neq 0$, leads to the implicit schemes. To the best of the authors knowledge, this generalization for boundary value problems is carried out for the first time.

- $\theta = 0.5$ leads to implicit central scheme which will give us stability and order preserved method for linear and nonlinear cases.
- $\theta = 1$ leads to the implicit backward scheme which is also a stable and order preserved method for linear and nonlinear cases.

3.1. Numerical implementation and error analysis

Consider BVP (3.1-3.2) as

$$G(y'(x), y(x), x) = -(py'(x) + qy) \tag{3.15}$$

where $p, q \in R$. Thus, problem (3.1-3.2) becomes a linear two-point BVP as a test problem to explain numerical implementation of the current IELDTM. With the guide of Section 2, taking the transform of equation (3.1) yields,

$$Y_i(k+2) = -\frac{1}{(k+1)(k+2)} [p(k+1)Y_i(k+1) + qY_i(k)] \tag{3.16}$$

where $k = 0, 1, 2, \dots, K-2$ and $i = 0, 1, 2, \dots, N$. The following coefficients can be evaluated

$$Y_i(2) = -\frac{q}{2}Y_i(0) - \frac{p}{2}Y_i(1), \tag{3.17}$$

$$Y_i(3) = \frac{pq}{6}Y_i(0) + \frac{(p^2 - q)}{6}Y_i(1), \tag{3.18}$$

$$Y_i(4) = \frac{(q^2 - p^2q)}{24}Y_i(0) + \frac{(2pq - p^3)}{24}Y_i(1), \tag{3.19}$$

$$Y_i(5) = \frac{(7pq^2 - 3p^2q)}{480}Y_i(0) + \frac{(10p^2q - 4q^2 - 3p^4)}{480}Y_i(1). \tag{3.20}$$

This evaluation can be increased up to desired approximation order. If we apply the IELDTM to the considered BVP, the following equations are found for each spatial element $[x_i, x_{i+1}]$

$$\sum_{k=0}^K Y_{i+1}(k) (-\theta dx)^k = \sum_{k=0}^K Y_i(k) ((1-\theta)dx)^k, \tag{3.21}$$

$$\sum_{k=1}^K Y_{i+1}(k) k(-\theta dx)^{k-1} = \sum_{k=1}^K Y_i(k) k((1-\theta)dx)^{k-1}, \tag{3.22}$$

where $i = 0, 1, \dots, N - 1$ and the fixed spatial step size $dx = \frac{(b-a)}{N}$ is assumed. With the use of coefficients (3.17-3.20), equations (3.21-3.22) become

$$\begin{aligned}
 & Y_{i+1}(0) + Y_{i+1}(1)(-\theta dx) + \left[-\frac{q}{2}Y_{i+1}(0) - \frac{p}{2}Y_{i+1}(1)\right](-\theta dx)^2 + \left[\frac{pq}{6}Y_{i+1}(0) + \frac{(p^2 - q)}{6}Y_{i+1}(1)\right](-\theta dx)^3 + \\
 & \left[\frac{(q^2 - p^2q)}{24}Y_{i+1}(0) + \frac{(2pq - p^3)}{24}Y_{i+1}(1)\right](-\theta dx)^4 + \dots \\
 & = Y_i(0) + Y_i(1)((1 - \theta)dx) + \left[-\frac{q}{2}Y_i(0) - \frac{p}{2}Y_i(1)\right] \\
 & ((1 - \theta)dx)^2 + \left[\frac{pq}{6}Y_i(0) + \frac{(p^2 - q)}{6}Y_i(1)\right]((1 - \theta)dx)^3 + \left[\frac{(q^2 - p^2q)}{24}Y_i(0) + \frac{(2pq - p^3)}{24}Y_i(1)\right]((1 - \theta)dx)^4 + \dots
 \end{aligned} \tag{3.23}$$

$$\begin{aligned}
 & Y_{i+1}(1) + 2\left[-\frac{q}{2}Y_{i+1}(0) - \frac{p}{2}Y_{i+1}(1)\right](-\theta dx) + 3\left[\frac{pq}{6}Y_{i+1}(0) + \frac{(p^2 - q)}{6}Y_{i+1}(1)\right](-\theta dx)^2 + \\
 & 4\left[\frac{(q^2 - p^2q)}{24}Y_{i+1}(0) + \frac{(2pq - p^3)}{24}Y_{i+1}(1)\right](-\theta dx)^3 + \dots \\
 & = Y_i(1) + 2\left[-\frac{q}{2}Y_i(0) - \frac{p}{2}Y_i(1)\right]((1 - \theta)dx) + \\
 & 3\left[\frac{pq}{6}Y_i(0) + \frac{(p^2 - q)}{6}Y_i(1)\right]((1 - \theta)dx)^2 + 4\left[\frac{(q^2 - p^2q)}{24}Y_i(0) + \frac{(2pq - p^3)}{24}Y_i(1)\right]((1 - \theta)dx)^3 + \dots
 \end{aligned} \tag{3.24}$$

where $i = 0, 1, \dots, N - 1$. If equations (3.23-3.24) is rearranged in terms of $Y_{i+1}(0)$, $Y_{i+1}(1)$, $Y_i(0)$ and $Y_i(1)$, equations (3.23-3.24) turn into the following equation

$$\begin{aligned}
 & \left[1 - \frac{q}{2}(-\theta dx)^2 + \frac{pq}{6}(-\theta dx)^3 + \frac{(q^2 - p^2q)}{24}(-\theta dx)^4 + \dots\right]Y_{i+1}(0) + \\
 & \left[-\theta dx - \frac{p}{2}(-\theta dx)^2 + \frac{(p^2 - q)}{6}(-\theta dx)^3 + \frac{(2pq - p^3)}{24}(-\theta dx)^4 + \dots\right]Y_{i+1}(1) = \\
 & \left[1 - \frac{q}{2}((1 - \theta)dx)^2 + \frac{pq}{6}((1 - \theta)dx)^3 + \frac{(q^2 - p^2q)}{24}((1 - \theta)dx)^4 + \dots\right]Y_i(0) + \\
 & \left[(1 - \theta)dx - \frac{p}{2}((1 - \theta)dx)^2 + \frac{(p^2 - q)}{6}((1 - \theta)dx)^3 + \frac{(2pq - p^3)}{24}((1 - \theta)dx)^4 + \dots\right]Y_i(1),
 \end{aligned} \tag{3.25}$$

$$\begin{aligned}
 & \left[-q(-\theta dx) + \frac{pq}{2}(-\theta dx)^2 + \frac{(q^2 - p^2q)}{6}(-\theta dx)^3 + \dots \right] Y_{i+1}(0) + \\
 & \left[1 - p(-\theta dx) + \frac{(p^2 - q)}{2}(-\theta dx)^2 + \frac{(2pq - p^3)}{6}(-\theta dx)^3 + \dots \right] Y_{i+1}(1) = \\
 & \left[-q((1 - \theta)dx) + \frac{pq}{2}((1 - \theta)dx)^2 + \frac{(q^2 - p^2q)}{6}((1 - \theta)dx)^3 + \dots \right] Y_i(0) + \\
 & \left[1 - p((1 - \theta)dx) + \frac{(p^2 - q)}{2}((1 - \theta)dx)^2 + \frac{(2pq - p^3)}{6}((1 - \theta)dx)^3 + \dots \right] Y_i(1)
 \end{aligned} \tag{3.26}$$

where $i = 0, 1, \dots, N - 1$. Equations (3.25–3.26) include $2N$ equations with $2N + 2$ unknowns for N spatial elements. Thus, imposing mixed boundary conditions defined in (3.2) we can eliminate $Y_0(0)$ and $Y_N(0)$. Finally, we reach the following linear system

$$A\hat{Y} = \hat{F}, \tag{3.27}$$

where $\hat{Y} = [Y_0(1), Y_1(0), Y_1(1), \dots, Y_{N-1}(0), Y_{N-1}(1), Y_N(1)]^T$ including all unknown positions and slopes, A is a $((2N) \times (2N))$ known matrix and \hat{F} is the residual vector coming from boundary conditions. With the use of any suitable linear solver for Equation (3.27), we find the desired solutions at discrete positions. Now, it is time to analyse the global error of the numerical approximation. Complete form of continuity Equations (3.21–3.22) can be written as follows:

$$\sum_{k=0}^K Y_{i+1}(k) (-\theta dx)^k = \sum_{k=0}^K Y_i(k) ((1 - \theta)dx)^k + dx\rho_i^1, \tag{3.28}$$

$$\sum_{k=1}^K Y_{i+1}(k) k(-\theta dx)^{k-1} = \sum_{k=1}^K Y_i(k) k((1 - \theta)dx)^{k-1} + dx\rho_i^2, \tag{3.29}$$

where ρ_i^1 and ρ_i^2 are the local truncation errors defined as

$$\rho_i^1 = \left[Y_{i_*}(K + 1) (1 - \theta)^{K+1} - Y_{(i+1)_*}(K + 1) (-\theta)^{K+1} \right] dx^K \tag{3.30}$$

$$\rho_i^2 = \left[Y_{i_*}(K + 1) (K + 1) (1 - \theta)^K - Y_{(i+1)_*}(K + 1) (K + 1) (-\theta)^K \right] dx^{K-1} \tag{3.31}$$

where $i = 0, 1, \dots, N - 1$ and Y_{i_*} defines the local transform at the point $x_{i_*} \in [x_{i-1}, x_{i+1}]$. Then exact equation system corresponding to Equation (3.27) can be stated as

$$AY = \hat{F} + dxG(\theta, dx), \tag{3.32}$$

where Y is a $((2N) \times 1)$ column vector corresponds to exact form of \hat{Y} and $G(\theta, dx) = G_1(\theta, dx) + G_2(\theta, dx)$ is a $((2N) \times 1)$ column vector corresponds to the truncation errors as follows:

$$G_1(\theta, dx) = \begin{bmatrix} \left[(1-\theta)^{K+1} Y_{0_*}(K+1) - (-\theta)^{K+1} Y_{1_*}(K+1) \right] \\ 0 \\ \left[(1-\theta)^{K+1} Y_{1_*}(K+1) - (-\theta)^{K+1} Y_{2_*}(K+1) \right] \\ 0 \\ \vdots \\ \vdots \\ \left[(1-\theta)^{K+1} Y_{(N-1)_*}(K+1) - (-\theta)^{K+1} Y_{N_*}(K+1) \right] \\ 0 \end{bmatrix} dx^K \tag{3.33}$$

$$G_2(\theta, dx) = \begin{bmatrix} 0 \\ \left[(K+1)(1-\theta)^K Y_{0_*}(K+1) - (K+1)(-\theta)^K Y_{1_*}(K+1) \right] \\ 0 \\ \left[(K+1)(1-\theta)^K Y_{1_*}(K+1) - (K+1)(-\theta)^K Y_{2_*}(K+1) \right] \\ \vdots \\ \vdots \\ 0 \\ \left[(K+1)(1-\theta)^K Y_{(N-1)_*}(K+1) - (K+1)(-\theta)^K Y_{N_*}(K+1) \right] \end{bmatrix} dx^{K-1} \tag{3.34}$$

Now, defining the error column vector as $\varepsilon = Y - \hat{Y}$ and subtracting Equation (3.32) from Equation (3.27) yield

$$A\varepsilon = dxG(\theta, dx) \tag{3.35}$$

and defining new matrix $A^* = dxA^{-1}$ leads to

$$\|\varepsilon\| \leq \|A^*\| \|G(\theta, dx)\|, \tag{3.36}$$

where $\|\cdot\|$ is an arbitrary norm. The explicit derivation for the bound of $\|G(\theta, dx)\|$ can be found in Appendix. Assume that the following inequalities are satisfied

$$\|G(\theta, dx)\| \leq L_2 dx^S, \tag{3.37}$$

$$\|A^*\| \leq L_1, \tag{3.38}$$

where L_2 can be seen in the Appendix and $S = K - 1, K$. Thus, we obtain the following error estimates

$$\|\varepsilon\| \leq \begin{cases} L_1 L_2 dx^K, & \theta = 0.5 \text{ and } K \text{ is even} \\ L_1 L_2 dx^{K-1}, & \text{otherwise} \end{cases} \tag{3.39}$$

Note that, the norm $\|A^*\|$ is almost independent of the spatial increment dx . Similar analysis can be executed for different order boundary value problems. If the order of the differential equation is $2p$, then one

needs $2p$ continuity equations to get consistent system of algebraic equations. Since one needs at least C^{2p-1} continuity, the truncation error terms will start with dx^{K-2p+1} or dx^{K-2p+2} depending on θ . As a conclusion, if the differential equation is $2p - th$ order and K is the transformation order, then the IELDTM leads to $(K - 2p + 1) - th$ or $(K - 2p + 2) - th$ order method depending on the selection of the direction parameter θ .

3.2. Stability analysis

As explained in the last subsection, the stability and global errors of the present IELDTM for BVP (3.1–3.2) with the selection of (3.15) clearly depend on the structure of the matrix A defined in (3.27). Thus, the invertibility and the norm of the matrix A determine the stability of the present method. We analyse the bound value L_1 stated in the last subsection with the inequality $\|A^*\| = \|dx A^{-1}\| \leq L_1$ depending on p, q, dx, θ and K . In Figure 1, we demonstrate the effect of the parameter values p and q to the matrix norm $\|A^*\|_\infty$. As seen in Figure 1, the bound value L_1 is finite and does not have much negative effect on the stability as clearly seen in Equation (3.27). As stated in the error analysis, the norm bound of $\|A^*\|$ is almost independent of the spatial step size dx . In Figure 2, we have demonstrated the effect of the various values of K and θ to the norm value $\|A^*\|_\infty$. Asymptotic behaviour can be seen with changing values of K and a little effect of the changing values of θ can be observed in Figure 2.

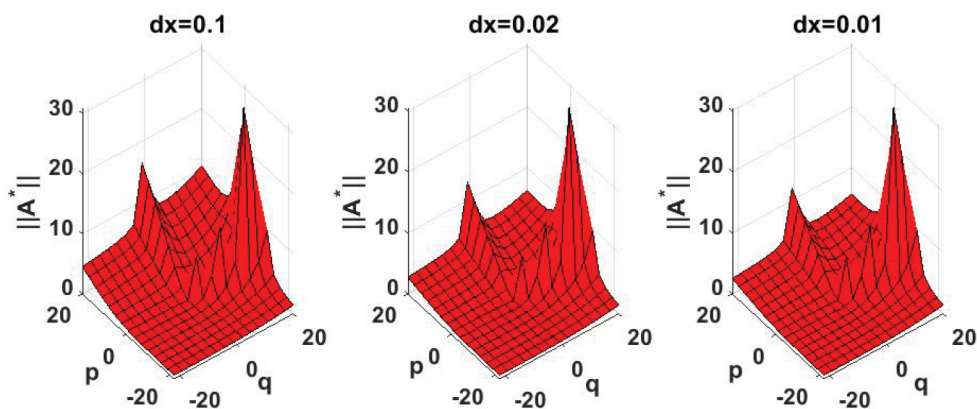


Figure 1. $\|A^*\|_\infty$ norm values as a function of the problem parameters p and q .

4. Numerical experiments

Numerical illustrations of the current method through various kinds of boundary value problems are presented here. We considered linear, nonlinear, singular, singularly perturbed and high order boundary value problems to illustrate the versatility of the IELDTM. The produced results are compared with the exact solutions, DTM and LDTM results. To measure errors of the present results, we prefer to use the following maximum error norm $\|E\|_\infty$ and absolute pointwise errors E_i

$$\|E\|_{L_\infty} = \max_i |y_i^{numerical} - y_i^{exact}|$$

$$E_i = |y_i^{numerical} - y_i^{exact}|.$$

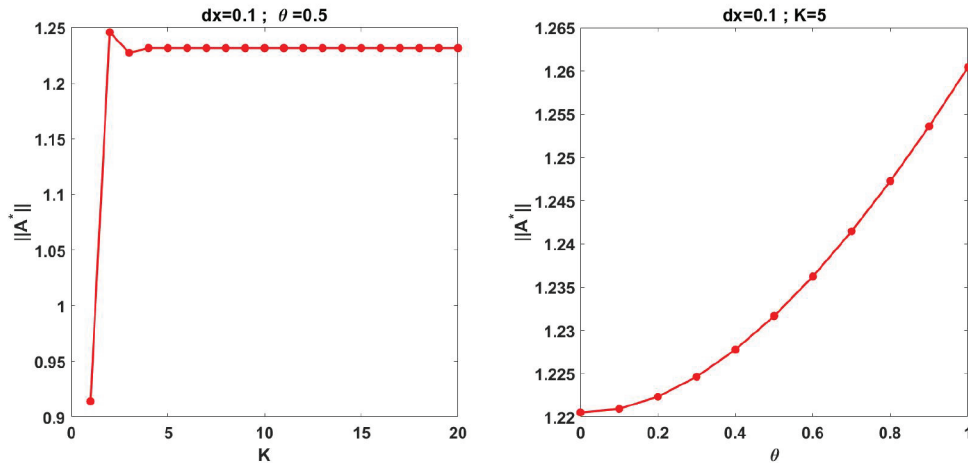


Figure 2. Effects of K and θ values on $\|A^*\|_\infty$ norm for the problem parameter values $p = 3$ and $q = -3$.

Problem 4.1 [2] Consider the following thick pressure vessel problem represented by

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = 0, \quad 5 \leq x \leq 8 \tag{4.1}$$

with the boundary conditions

$$y(5) = 0.0038731 \quad \text{and} \quad y(8) = 0.0030770 \tag{4.2}$$

where the exact solution is $y(x) = (10501x)/78000000 + 77999/(4875000x)$. The differential transform of the equation (4.1) is

$$Y_i(k+2) = \frac{1}{(k+1)(k+2)} \left[\sum_{n=0}^k (-k-n+1) F(n, x_i) Y_i(k-n+1) + G(n, x_i) Y_i(k-n) \right] \tag{4.3}$$

where Y_i is the transformed form of the local function $y_i(x)$. $F(n, x_i)$ and $G(n, x_i)$ can be defined as follows

$$F(n, x_i) = \frac{(-1)^n}{(x_i)^{n+1}} \quad \text{and} \quad G(n, x_i) = \frac{(-1)^n n + 1}{(x_i)^{n+2}}, \quad i = 0, 1, \dots, N \tag{4.4}$$

where the nodes are defined as $x_i = 5 + idx$ and $dx = \frac{3}{N}$. The boundary conditions lead to the following equations,

$$Y_0(0) = 0.0038731 \quad \text{and} \quad Y_N(0) = 0.0030770. \tag{4.5}$$

For a typical element $[x_i, x_{i+1}]$ the present IELDTM yields the following equations,

$$\sum_{k=0}^K Y_{i+1}(k) (-\theta dx)^k = \sum_{k=0}^K Y_i(k) ((1-\theta)dx)^k, \tag{4.6}$$

$$\sum_{k=1}^K Y_{i+1}(k) k (-\theta dx)^{k-1} = \sum_{k=1}^K Y_i(k) k ((1-\theta)dx)^{k-1}, \tag{4.7}$$

where $i = 0, 1, \dots, N - 1$ and the coefficients $Y_i(k)$ can be calculated from the relation (4.3) in terms of $Y_i(0)$ and $Y_i(1)$.

Problem 4.2 [14] Consider the following singularly perturbed nonlinear BVP,

$$-\varepsilon \frac{d^2 y}{dx^2} + y + y^2 = e^{\frac{-2x}{\sqrt{\varepsilon}}}, \quad 0 \leq x \leq 1 \tag{4.8}$$

with the following boundary conditions

$$y(0) = 1 \quad \text{and} \quad y(1) = e^{\frac{-1}{\sqrt{\varepsilon}}} \tag{4.9}$$

where $\varepsilon \ll 1$ and the exact solution is $y(x) = e^{\frac{-x}{\sqrt{\varepsilon}}}$. The transform of Equation (4.8) is

$$Y_i(k+2) = \frac{1}{\varepsilon(k+1)(k+2)} [Y_i(k) + \bar{Y}_i(2; 0, 0; k) - F(k, x_i)] \tag{4.10}$$

where Y_i is the transformed form of the local function $y_i(x)$. $F(k, x_i)$ can be defined as follows:

$$F(k, x_i) = \frac{1}{k!} \left(\frac{-2}{\sqrt{\varepsilon}} \right)^k e^{\frac{-2x_i}{\sqrt{\varepsilon}}}, \tag{4.11}$$

where the nodes are expressed by $x_i = idx$ and $dx = \frac{1}{N}$. The boundary conditions lead to the following equations,

$$Y_0(0) = 1 \quad \text{and} \quad Y_N(0) = e^{\frac{-1}{\sqrt{\varepsilon}}}. \tag{4.12}$$

The present IELDTM yields the same expressions defined in (4.6-4.7) and the required coefficients can be obtained from Equation (4.10) in terms of $Y_i(0)$ and $Y_i(1)$.

Problem 4.3 [12] Consider the following BVP

$$\frac{d^4 y}{dx^4} = \sin(x) + \sin^2(x) - \left(\frac{d^2 y}{dx^2} \right)^2, \quad 0 \leq x \leq 1 \tag{4.13}$$

with the following boundary conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y(1) = \sin(1) \quad \text{and} \quad y'(1) = \cos(1), \tag{4.14}$$

where the exact solution is $y(x) = \sin(x)$. The transform of Equation (4.13) is

$$Y_i(k+4) = \frac{1}{(k+1)(k+2)(k+3)(k+4)} \left[F(k, x_i) + \sum_{n=0}^k F(n, x_i) F(k-n, x_i) - \bar{Y}_i(0; 2, 2; k) \right] \tag{4.15}$$

where Y_i is the transformed form of the local function $y_i(x)$, \bar{Y}_i is defined in Section 2. $F(k, x_i)$ can be defined by

$$F(k, x_i) = \frac{1}{k!} \left. \frac{d^k(\sin x)}{dx^k} \right|_{x=x_i}, \tag{4.16}$$

where the nodes are defined by $x_i = idx$ and $dx = \frac{1}{N}$. The boundary conditions lead to the following equations,

$$Y_0(0) = 0, \quad Y_0(1) = 1, \quad Y_N(0) = \sin(1) \quad \text{and} \quad Y_N(1) = \cos(1). \tag{4.17}$$

For a typical element $[x_i, x_{i+1}]$ the present IELDTM yields

$$\begin{aligned} \sum_{k=0}^K Y_{i+1}(k) (-\theta dx)^k &= \sum_{k=0}^K Y_i(k) ((1-\theta)dx)^k, \\ \sum_{k=1}^K Y_{i+1}(k) k (-\theta dx)^{k-1} &= \sum_{k=1}^K Y_i(k) k ((1-\theta)dx)^{k-1}, \\ \sum_{k=2}^K Y_{i+1}(k) k(k-1) (-\theta dx)^{k-2} &= \sum_{k=2}^K Y_i(k) k(k-1) ((1-\theta)dx)^{k-2}, \\ \sum_{k=3}^K Y_{i+1}(k) k(k-1)(k-2) (-\theta dx)^{k-3} &= \sum_{k=3}^K Y_i(k) k(k-1)(k-2) ((1-\theta)dx)^{k-3}, \end{aligned} \tag{4.18}$$

where $i = 0, 1, \dots, N-1$ and the coefficients can be calculated from equation (4.15) in terms of $Y_i(0)$, $Y_i(1)$, $Y_i(2)$ and $Y_i(3)$.

Problem 4.4 [17] Consider the following second order nonlinear BVP,

$$\frac{d^2 y}{dx^2} = \frac{1}{8} \left(32 + 2x^3 - y \frac{dy}{dx} \right), \quad 1 \leq x \leq 2 \tag{4.19}$$

with the boundary conditions

$$y(1) = 17 \quad \text{and} \quad y'(2) = 0 \tag{4.20}$$

where the exact solution is $y(x) = x^2 + \frac{16}{x}$. The transform of Equation (4.19) is

$$Y_i(k+2) = \frac{1}{8(k+1)(k+2)} [32\delta(k) + 2 F(k, x_i) - \bar{Y}_i(1; 1, 1; k)] \tag{4.21}$$

where Y_i is the transformed form of the local function $y_i(x)$, δ is Kronecker delta, \bar{Y}_i is defined in Section 2. $F(k, x_i)$ is defined by

$$F(k, x_i) = \begin{cases} \binom{3}{k} x_i^{3-k}, & k = 0, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases} \tag{4.22}$$

where the nodes are defined by $x_i = 1 + idx$ and $dx = \frac{1}{N}$. The boundary conditions lead to the following equations,

$$Y_0(0) = 17 \quad \text{and} \quad Y_N(1) = 0. \tag{4.23}$$

The present IELDTM yields the same expressions defined in (4.6-4.7) and the required coefficients can be obtained from Equation (4.21) in terms of $Y_i(0)$ and $Y_i(1)$.

Problem 4.5 [17] Consider the following nonlinear BVP,

$$\frac{d^2y}{dx^2} = \frac{1}{x^3} \left(\frac{dy}{dx} \right)^2 - 9\frac{y^2}{x^5} + 4x, \quad 1 \leq x \leq 2 \tag{4.24}$$

with the boundary conditions

$$y(1) = 0 \quad \text{and} \quad y'(2) = 4 + 12\ln(2) \tag{4.25}$$

where the exact solution is $y(x) = x^3\ln(x)$. The transform of equation (4.19) is

$$Y_i(k+2) = \frac{1}{(k+1)(k+2)} \left[\sum_{n=0}^k (F(n, x_i) \bar{Y}_i(0; 1, 2; k-n) - 9G(n, x_i) \bar{Y}_i(2; 0, 0; k-n)) + 4H(k, x_i) \right] \tag{4.26}$$

where Y_i is the transformed form of the local function $y_i(x)$, δ is Kronecker delta, \bar{Y}_i is defined in Section 2. $F(n, x_i)$, $G(n, x_i)$ and $H(k, x_i)$ are defined by

$$F(n, x_i) = \frac{(-1)^n (n+1)(n+2)}{2(x_i)^{n+3}}, \quad G(n, x_i) = \frac{(-1)^n (n+1)(n+2)(n+3)(n+4)}{24(x_i)^{n+5}} \tag{4.27}$$

$$H(k, x_i) = \begin{cases} \binom{1}{k} x_i^{1-k}, & k = 0, 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.28}$$

where the nodes are defined by $x_i = 1 + idx$ and $dx = \frac{1}{N}$. The boundary conditions lead to the following equations,

$$Y_0(0) = 0 \quad \text{and} \quad Y_N(1) = 4 + 12\ln(2). \tag{4.29}$$

The present IELDTM yields the same expressions defined in (4.6-4.7) and the coefficients can be evaluated from equation (4.26) in terms of $Y_i(0)$ and $Y_i(1)$.

Problem 4.6 [36] Consider the following nonlinear singular BVP,

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = -y^5, \quad 0 < x < 1 \tag{4.30}$$

with the boundary conditions

$$y'(0) = 0 \quad \text{and} \quad y(1) = \frac{\sqrt{3}}{2}. \tag{4.31}$$

where the exact solution is $y(x) = (1 + x^2/3)^{-1/2}$. The transform of equation (4.6) is

$$Y_0(k+2) = \frac{1}{(k+1)(k+2)} [-\bar{Y}_0(5; 0, 0; k)] \tag{4.32}$$

$$Y_i(k+2) = \frac{1}{(k+1)(k+2)} \left[\sum_{n=0}^k (F(n, x_i) Y_i(k-n) - \bar{Y}_i(5; 0, 0; k)) \right], \quad i = 1, 2, \dots, N \quad (4.33)$$

where Y_i is the transformed form of the local function $y_i(x)$, \bar{Y}_i is defined in Section 2 and $F(n, x_i)$ is defined as

$$F(n, x_i) = \frac{(-1)^n}{(x_i)^{n+1}} \quad i = 0, 1, \dots, N \quad (4.34)$$

The nodes are defined by $x_i = idx$ and $dx = \frac{1}{N}$. The boundary conditions lead to the following equations,

$$Y_0(1) = 0 \quad \text{and} \quad Y_N(0) = \frac{\sqrt{3}}{2}. \quad (4.35)$$

The present IELDTM yields the same expressions defined in (4.6-4.7) and the coefficients can be evaluated from equation (4.32-4.33) in terms of $Y_i(0)$ and $Y_i(1)$.

In Figures 3–7, we demonstrate the effectiveness of the current IELDTM and the DTM on various BVP types, taking into account pointwise errors. As clearly seen in the figures, the present IELDTM with $\theta = 0.5$ is more accurate than the LDTM ($\theta = 0$) and the DTM for all types of the linear and nonlinear BVPs. With the consideration of the same transformation order K , numerically constructed IELDTM is seen to be far more efficient than the semianalytical DTM. While the DTM produces acceptable results for Problem 4.1 and Problem 4.3, the DTM results are seen to be divergent for Problem 4.2 and Problems 4.4–4.5 as shown in Figures 4, 8 and 9. Thus, the convergence, accuracy and versatility of the IELDTM are much more reliable than the classical DTM and the LDTM. To demonstrate the effect of the spatial step size dx and the transformation order K , the maximum errors produced with various parameter values are presented in Tables 1–5. In line with our theoretical analysis, the accuracy of the present IELDTM is increasing with either dx -refinement or K -refinement for all problems. Both the semianalytical DTM and the numerical LDTM [34, 35, 43] suffer from instability, especially in solving stiff BVPs. Just at this time, the IELDTM is here constructed for elimination of this serious drawback. Since Problem 4.2 has stiff nature for $\varepsilon \ll 1$, the IELDTM and the rival methods are compared with the calculation of $\|E\|_{L^\infty}$ error norms for various challenging ε values in Table 6. As observed from the table, the DTM is completely divergent and the LDTM produce divergent results for $\varepsilon < 0.001$. The IELDTM with $\theta = 0$ and $\theta = 0.5$ have ability to produce accurate numerical solutions up to $\varepsilon = 0.00001$ -with $dx=0.01$. To produce Table 6, the LDTM and the DTM are treated as a shooting method as previously considered in literature [38, 43] and the IELDTM with $\theta = 0$ and $\theta = 0.5$ are treated as a direct method as mathematically derived in Section 3.

Numerical nature of the current IELDTM depends heavily on the choice of the direction parameter value θ . Figure 10 has shown how the maximum error norms vary with respect to the direction parameter θ . It is obvious that the selections $\theta \cong 0.5$ provide more accurate results as we already proved in error analysis. In Figure 11, processing times of the present IELDTM algorithms for solving Problems 4.4–4.5 are illustrated with the changing values of the transformation order K . It is possible to observe that the computational cost increases linearly according to the transformation order K . In Table 7, the theoretical order expectations and experimental order averages of the IELDTM are compared with the various values of K . The highly matching experiment results are seen in the table and the theoretical expectations stated in the error analysis are verified. The experimental order is calculated as $p = \log(E_1/E_2)/\log(dx_1/dx_2)$ where dx_1 and dx_2 are

spatial increments and E_1 and E_2 are the corresponding absolute errors. By evaluating the experimental orders for various dx_1 and dx_2 values, the experimental average order is obtained for each problem. In addition to the above and those suggested here, also high order shooting methods and finite difference methods (FDM) have been widely used in the solution of various versions of BVPs in the literature [2–10, 36, 37]. To prove the efficiency of the proposed IELDTM, the $\|E\|_{L_\infty}$ error norms of the present algorithm, the sixth order shooting method [36] and the fourth order FDM [37] are compared in Table 8. The IELDTM of order four and six are used for computations with $\theta = 0.5$. As clearly observed from the table, our sixth order method produces far more accurate results than the sixth order shooting method [36]. Similarly, the present fourth order IELDTM offers far more accurate results than the fourth order FDM [37]. The CPU times required to produce the present results are also demonstrated in Table 8. As we illustrated qualitatively and quantitatively, the present IELDTM is an effective numerical method in terms of reliability, versatility and computational efficiency.

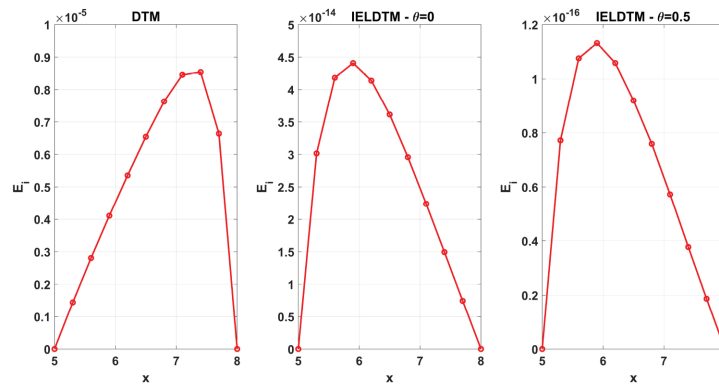


Figure 3. Pointwise errors produced by the IELDTM and the DTM with the fixed parameter values $dx = 0.1$ and $K = 9$ for Problem 4.1.

Table 1. Maximum errors of the IELDTM with $\theta = 0.5$ and various values of the parameters K and N for Problem 4.1.

K/N	N = 5	N = 10	N = 15	N = 20	N = 25	N = 30	N = 35	N = 40
K = 3	6.62E-07	1.63E-07	7.22E-08	4.06E-08	2.60E-08	1.80E-08	1.32E-08	1.01E-08
K = 4	7.84E-09	4.76E-10	9.46E-11	2.99E-11	1.22E-11	5.89E-12	3.18E-12	1.87E-12
K = 5	2.51E-09	1.51E-10	2.99E-11	9.45E-12	3.86E-12	1.86E-12	1.01E-12	5.89E-13
K = 6	4.39E-11	6.75E-13	5.87E-14	1.04E-14	2.73E-15	9.12E-16	3.63E-16	1.65E-16
K = 7	8.89E-12	1.33E-13	1.15E-14	2.04E-15	5.33E-16	1.78E-16	6.98E-17	2.82E-17
K = 8	2.06E-13	7.91E-16	3.12E-17	3.47E-18	2.17E-18	2.60E-18	3.04E-18	4.34E-18

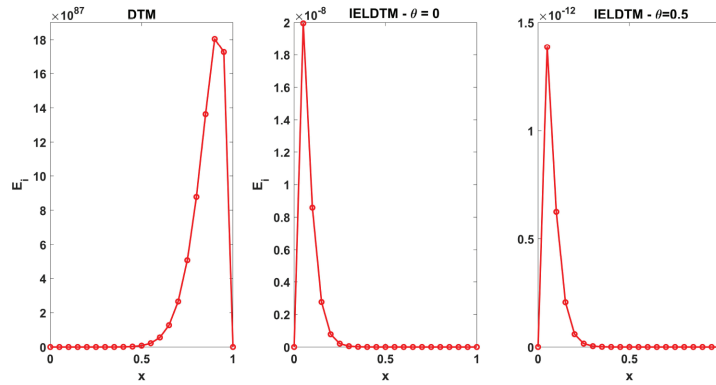


Figure 4. Pointwise errors produced by the IELDTM and the DTM with the fixed parameter values $dx = 0.05$, $\varepsilon = 0.001$ and $K = 13$ for Problem 4.2.

Table 2. Maximum errors of the IELDTM with $\theta = 0.5$ and various values of the parameters K and N for Problem 4.2.

K/N	N = 20	N = 25	N = 30	N = 35	N = 40	N = 45	N = 50
K = 5	4.26E-04	1.86E-04	9.12E-05	4.88E-05	2.80E-05	1.70E-05	1.14E-05
K = 6	2.93E-05	8.32E-06	2.87E-06	1.14E-06	5.05E-07	2.44E-07	1.33E-07
K = 7	6.63E-06	1.82E-06	6.14E-07	2.40E-07	1.05E-07	5.03E-08	2.73E-08
K = 8	3.60E-07	6.54E-08	1.57E-08	4.56E-09	1.54E-09	5.87E-10	2.59E-10
K = 9	5.91E-08	1.03E-08	2.40E-09	6.87E-10	2.30E-10	8.67E-11	3.80E-11
K = 10	2.66E-09	3.08E-10	5.11E-11	1.09E-11	2.83E-12	8.50E-13	3.04E-13
K = 11	3.42E-10	3.78E-11	6.10E-12	1.28E-12	3.28E-13	9.77E-14	3.47E-14
K = 12	1.31E-11	9.72E-13	1.12E-13	1.74E-14	3.44E-15	7.77E-16	3.33E-16

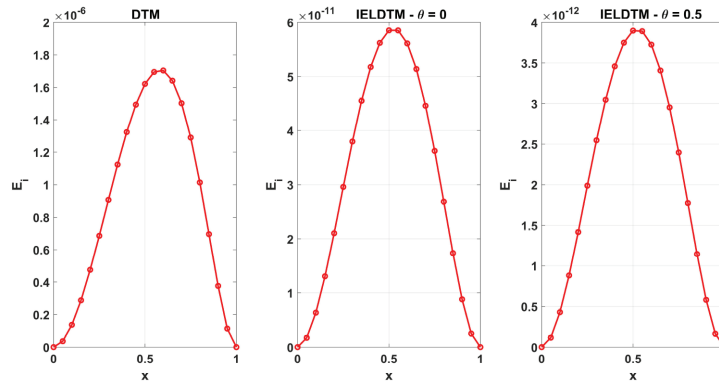


Figure 5. Pointwise errors produced by the IELDTM and the DTM with the fixed parameter values $dx = 0.05$ and $K = 7$ for Problem 4.3.

5. Conclusion

In this study, the implicit-explicit local differential transform method (IELDTM) has been produced for solving BVPs. Based on the DTM and the local DTM idea; a high-order, direction free and stability-preserved numerical

Table 3. Maximum errors of the IELDTM with $\theta = 0.5$ and various values of the parameters K and N for Problem 4.3.

K/N	N = 5	N = 10	N = 15	N = 20	N = 25	N = 30	N = 35	N = 40
K = 5	1.90E-06	4.99E-07	2.22E-07	1.25E-07	8.02E-08	5.56E-08	4.09E-08	3.13E-08
K = 6	2.83E-09	1.86E-10	3.70E-11	1.17E-11	4.81E-12	2.32E-12	1.25E-12	7.34E-13
K = 7	9.51E-10	6.23E-11	1.24E-11	3.90E-12	1.60E-12	7.73E-13	4.17E-13	2.45E-13
K = 8	1.13E-12	1.83E-14	1.67E-15	4.44E-16	2.22E-16	5.55E-16	4.44E-16	3.33E-16
K = 9	2.26E-13	3.83E-15	3.33E-16	1.11E-16	2.22E-16	3.33E-16	3.33E-16	2.22E-16
K = 10	3.33E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	4.44E-16	3.33E-16	3.33E-16

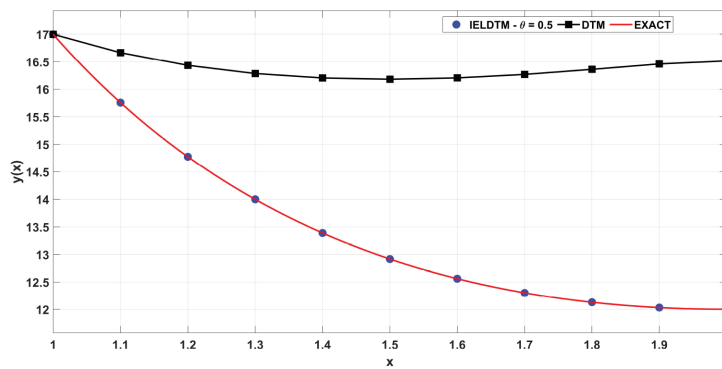


Figure 6. Pointwise errors produced by the IELDTM with the fixed parameter values $dx = 0.1$ and $K = 13$ for Problem 4.4.

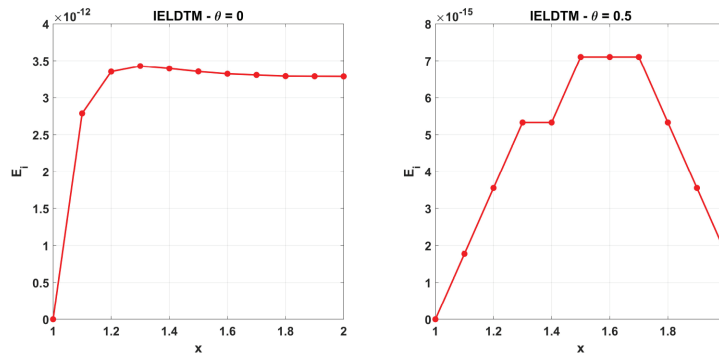


Figure 7. Pointwise errors produced by the IELDTM with the fixed parameter values $dx = 0.1$ and $K = 9$ for Problem 4.5.

method has been presented. Puri global error analysis has been done for the second order test problem and the order conditions of the IELDTM have been determined according to the direction parameter. Stability properties of the IELDTM have been explained with the use of approximation matrices. Various kinds of linear and nonlinear BVPs have been considered as test problems including singular and singularly perturbed BVPs. As proven theoretically and experimentally throughout the study, the IELDTM is a numerically reliable, high-order, stable and computationally efficient method for solving BVPs. One of the main advantages of the method

Table 4. Maximum errors of the IELDTM with $\theta = 0.5$ and various values of the parameters K and N for Problem 4.4

K/N	N = 5	N = 10	N = 15	N = 20	N = 25	N = 30	N = 35	N = 40
K = 3	4.84E-02	1.18E-02	5.20E-03	2.92E-03	1.87E-03	1.30E-03	9.51E-04	7.28E-04
K = 4	1.16E-03	6.85E-05	1.34E-05	4.22E-06	1.73E-06	8.31E-07	4.48E-07	2.63E-07
K = 5	3.71E-04	2.13E-05	4.14E-06	1.30E-06	5.32E-07	2.56E-07	1.38E-07	8.10E-08
K = 6	1.41E-05	2.01E-07	1.73E-08	3.06E-09	7.99E-10	2.67E-10	1.06E-10	4.74E-11
K = 7	2.96E-06	3.92E-08	3.33E-09	5.87E-10	1.53E-10	5.11E-11	2.03E-11	9.08E-12
K = 8	1.56E-07	5.29E-10	2.00E-11	1.97E-12	3.53E-13	8.88E-14	2.66E-14	1.95E-14
K = 9	2.57E-08	7.66E-11	2.82E-12	2.93E-13	4.09E-14	2.84E-14	1.24E-14	1.95E-14
K = 10	1.73E-09	1.38E-12	2.31E-14	1.95E-14	1.95E-14	1.24E-14	2.31E-14	1.95E-14

Table 5. Maximum errors of the IELDTM with $\theta = 0.5$ and various values of the parameters K and N for Problem 4.5.

K/N	N = 5	N = 10	N = 15	N = 20	N = 25	N = 30	N = 35	N = 40
K = 3	3.13E-03	7.77E-04	3.45E-04	1.94E-04	1.24E-04	8.62E-05	6.34E-05	4.85E-05
K = 4	1.13E-06	6.29E-08	1.22E-08	3.83E-09	1.56E-09	7.53E-10	4.06E-10	2.38E-10
K = 5	1.51E-06	9.23E-08	1.82E-08	5.74E-09	2.35E-09	1.13E-09	6.11E-10	3.58E-10
K = 6	9.62E-09	1.41E-10	1.22E-11	2.16E-12	5.63E-13	1.94E-13	7.73E-14	3.64E-14
K = 7	2.42E-09	3.54E-11	3.07E-12	5.53E-13	1.41E-13	4.88E-14	2.40E-14	1.20E-14
K = 8	3.95E-11	1.38E-13	9.77E-15	7.11E-15	7.11E-15	4.44E-15	8.44E-15	6.22E-15
K = 9	6.47E-12	2.93E-14	3.55E-15	4.44E-15	2.66E-15	3.55E-15	4.44E-15	6.22E-15

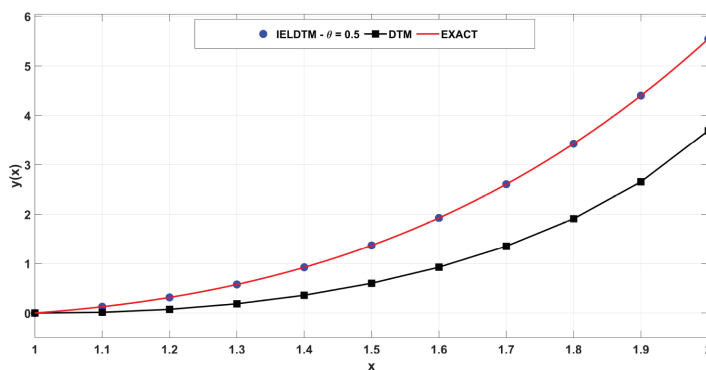


Figure 8. Comparison of numerical results produced by the IELDTM and the DTM with the fixed parameter values $dx = 0.1$ and $K = 11$ for Problem 4.4.

proposed here is that the degrees of freedom are not affected by the order of the method, i.e. the IELDTM is an optimized numerical technique. It has been proven that the degrees of freedom depend only on the order of the differential equation concerned. If the differential equation is of order p and N nodes are used then the degrees of freedom is $2pN$ irrespective of the local transformation order K . It has thus been proven that the convergence rate of the IELDTM increases with either p -refinement or h -refinement. If the number of terms used in local approximations is increased, then the p -refinement procedure can be easily applied for the

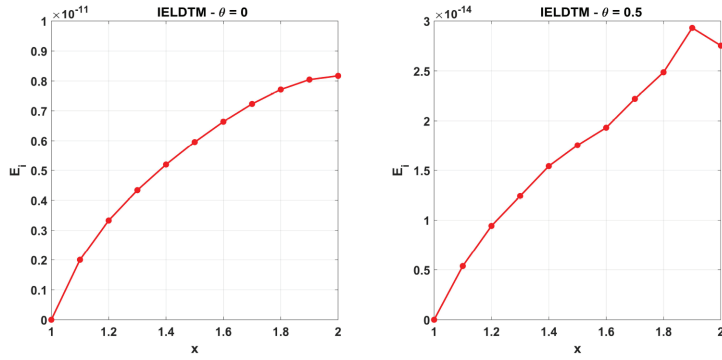


Figure 9. Comparison of numerical results produced by the IELDTM and the DTM with the fixed parameter values $dx = 0.1$ and $K = 9$ for Problem 4.5.

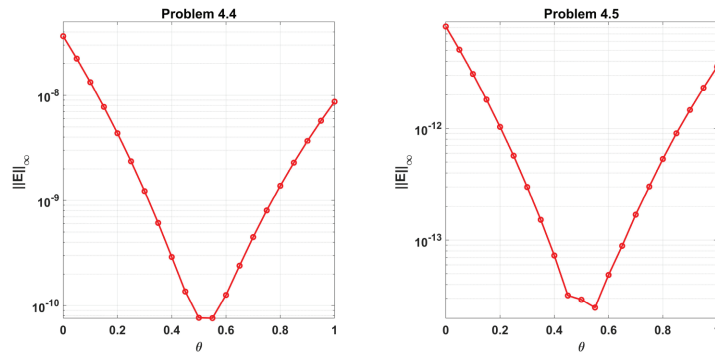


Figure 10. Effect of the direction parameter θ to maximum errors produced by the IELDTM with the parameter values $dx = 0.1$ and $K = 9$.

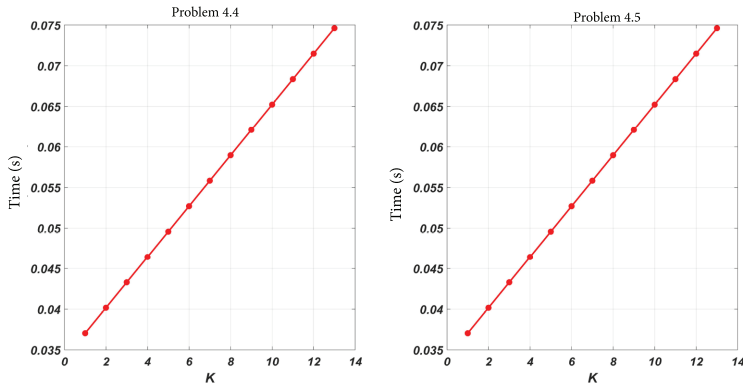


Figure 11. Processing times with respect to varying values of the transformation orders K with the parameter values $dx = 0.1$ and $\theta = 0.5$.

IELDTM. The central approach with $\theta = 0.5$ has been proven to be more accurate than both the other explicit ($\theta = 0$) and implicit ($\theta \neq 0$) selections.

Table 6. Comparison of the $\|E\|_{L_\infty}$ norms of the IELDTM, LD TM and DT M with $K = 6$ and $dx = 0.01$ for Problem 4.2.

ε	DTM	LD TM	IELDTM($\theta = 0$)	IELDTM($\theta = 0.5$)
1	divergent	2.15E-14	2.15E-14	4.44E-16
0.5	divergent	7.22E-14	7.22E-14	2.66E-15
0.4	divergent	1.38E-14	1.38E-14	8.88E-16
0.3	divergent	3.17E-13	3.17E-13	3.33E-16
0.2	divergent	9.76E-13	9.76E-13	6.66E-13
0.1	divergent	6.10E-12	6.10E-12	2.00E-15
0.01	divergent	2.08E-09	2.08E-09	2.16E-12
0.0015	divergent	6.29E-06	2.67E-07	6.38E-10
0.001	divergent	1.00E-02	7.67E-07	2.15E-09
0.0001	divergent	divergent	4.12E-04	2.10E-08
0.00001	divergent	divergent	4.09E-01	0.90E-02

Table 7. Validation of the approximation orders of the present IELDTM with the direction parameter $\theta = 0.5$ for Problems 4.4–4.5.

K	Problem 4.4		Problem 4.5	
	Experimental average order	Theoretical expectation	Experimental average order	Theoretical expectation
K = 3	2.016	2	2.003	2
K = 4	4.030	4	4.062	4
K = 5	4.047	4	4.013	4
K = 6	6.054	6	6.042	6
K = 7	6.095	6	6.049	6

Table 8. Comparison of the IELDTM and the rival higher order methods with $\|E\|_{L_\infty}$ norms for Problem 4.6.

N	Shooting method [36]	FDM [37]	EILD TM ($\theta = 0.5$) $K = 4$	CPU time	EILD TM ($\theta = 0.5$) $K = 6$	CPU time
8	1.23E-06		4.36E-07	0.025 s	9.85E-10	0.035 s
16	3.47E-08	4.78E-04	2.72E-08	0.048 s	1.54E-11	0.084 s
32	8.40E-10	3.31E-05	1.70E-09	0.127 s	2.43E-13	0.194 s
64	2.05E-11	2.12E-06	1.06E-10	0.539 s	4.21E-15	0.559 s

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6. Appendix

Complete error analysis of the IELDTM requires the bound of $\|G(\theta, dx)\|$ in terms of the direction parameter θ . Then, the following two cases are considered:

Case 1: Assume that the direction parameter $\theta = 0.5$ and K is even. In this case, the following general conditions are reached for $G_1(0.5, dx)$

$$\begin{aligned} (G_1(0.5, dx))_i &= \left[(0.5)^{K+1} Y_{i_*}(K+1) - (-0.5)^{K+1} (Y_{i_*}(K+1) + dx(K+2)Y_{i_*}(K+2)) \right] dx^K, \\ &= \left(\frac{1}{2} \right)^K Y_{i_*}(K+1) dx^K + O(dx^{K+1}). \end{aligned}$$

Defining $M_1 = \max_i \left| \left(\frac{1}{2} \right)^K Y_{i_*}(K+1) \right|$ and neglecting the higher order terms yield

$$\|G_1(0.5, dx)\| \leq M_1 dx^K.$$

Then the general term of $G_2(0.5, dx)$ becomes,

$$\begin{aligned} (G_2(0.5, dx))_i &= \left[(K+1)(0.5)^K Y_{i_*}(K+1) - (K+1)(-0.5)^K (Y_{i_*}(K+1) + dx(K+2)Y_{i_*}(K+2)) \right] dx^{K-1} \\ &= -(K+1)(K+2) \left(\frac{1}{2} \right)^K Y_{i_*}(K+2) dx^K + O(dx^{K+1}). \end{aligned}$$

Defining $M_2 = \max_i \left| (K+1)(K+2) \left(\frac{1}{2} \right)^K Y_{i_*}(K+2) \right|$ and neglecting the higher order terms yield

$$\|G_2(0.5, dx)\| \leq M_2 dx^K.$$

Remembering $G(0.5, dx) = G_1(0.5, dx) + G_2(0.5, dx)$ and taking $L_2 = \max(M_1, M_2)$ lead to the estimate $\|G(0.5, dx)\| \leq L_2 dx^K$.

Case 2: Assume that the conditions of Case 1 are not satisfied. Then, the following general terms are found for $G_1(\theta, dx)$

$$\begin{aligned} (G_1(\theta, dx))_i &= \left[(1-\theta)^{K+1} Y_{i_*}(K+1) - (-\theta)^{K+1} (Y_{i_*}(K+1) + dx(K+2)Y_{i_*}(K+2)) \right] dx^K \\ &= \left[(1-\theta)^{K+1} - (-\theta)^{K+1} \right] Y_{i_*}(K+1) dx^K + O(dx^{K+1}). \end{aligned}$$

Defining $M_3 = \max_i \left| \left[(1-\theta)^{K+1} - (-\theta)^{K+1} \right] Y_{i_*}(K+1) \right|$ and neglecting the higher order terms give rise to

$$\|G_1(\theta, dx)\| \leq M_3 dx^K.$$

Thus the general term of $G_2(\theta, dx)$ becomes,

$$\begin{aligned} (G_2(\theta, dx))_i &= \left[(K+1)(1-\theta)^K Y_{i_*}(K+1) - (K+1)(-\theta)^K (Y_{i_*}(K+1) + dx(K+2)Y_{i_*}(K+2)) \right] dx^{K-1} \\ &= \left[(K+1)(1-\theta)^K - (K+1)(-\theta)^K \right] Y_{i_*}(K+1) dx^{K-1} + O(dx^K). \end{aligned}$$

Defining $M_4 = \max_i \left| \left[(K+1)(1-\theta)^K - (K+1)(-\theta)^K \right] Y_{i_*} (K+1) \right|$ and dropping the higher order terms lead to

$$\|G_2(\theta, dx)\| \leq M_4 dx^{K-1}.$$

Since $G(\theta, dx) = G_1(\theta, dx) + G_2(\theta, dx)$, taking $L_2 = \max(M_3 dx, M_4)$ leads to the estimate $\|G(\theta, dx)\| \leq L_2 dx^{K-1}$.