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Solving fractional differential equations using collocation method based on hybrid of block-pulse functions and Taylor polynomials

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Abstract: In this paper, a novel approach is proposed to solve fractional differential equations (FDEs) based on hybrid functions. The hybrid functions consist of block-pulse functions and Taylor polynomials. The exact formula for the Riemann–Liouville fractional integral of the hybrid functions is derived via Laplace transform. The FDE under consideration is converted into an algebraic equation with undetermined coefficients by using this formula. A set of linear or nonlinear equations are obtained through collocating the algebraic equation at Newton-Cotes nodes. The numerical solution of the FDE is achieved by solving the linear or nonlinear equations. Error analysis is performed on the proposed method. Several numerical examples are given, and the results have proven that the proposed method is effective.

Key words: Fractional differential equations, numerical solution, collocation method, hybrid functions, block-pulse functions, Taylor polynomials

1. Introduction

FDEs are generalizations of classical integer-order differential equations (IDEs) based on the concept of fractional calculus (FC) [1]. Due to the nonlocal and long-memory properties of FC, FDEs offer an excellent instrument to describe physical systems having history memory [2–4]. Finding solutions to FDEs are helpful for analyzing their dynamic properties. However, sometimes it is impossible or hard to obtain the analytical solutions of such equations. Therefore, establishing accurate numerical schemes for FDEs becomes an important issue. In this regard, many numerical methods for solving IDEs are further extended to FDEs, such as finite difference method [3], predictor-corrector method [5], Adomian decomposition method [6], variational iteration method [7], homotopy analysis method [8], and collocation method [9]. Most of the methods mentioned above obtain numerical solutions of FDEs by discretizing FDEs into algebraic equations. Besides from them, in recent time, several new techniques for discretizing FDEs have been proposed and developed. In [10], a fractional trapezoidal method is introduced to convert the FDE into a nonlinear equation. In [11], an improved scheme based on the Grünwald–Letnikov definition is presented for discretization and simulation of FDEs. In [12], the combining quadrature rules are adopted to reduce the underlying FDE to a system of algebraic equations. However, considering the drawbacks of these methods, such as high computational complexities, large time consumption and low order of accuracy, it is desirable to develop easy and effective methods to numerically solve FDEs.

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Orthogonal functions or polynomials, such as Haar wavelets [13], block-pulse functions (BPFs) [14], shifted Legendre polynomials [15] and Genocchi polynomials [16] have been widely adopted to handle different kinds of FDEs. Their foundation is to convert the FDE to a system of algebraic equations through an operational matrix, thus simplifying the problem. Among these functions or polynomials, BPFs are piecewise constants and orthogonal in the interval $[0,T)$, and can approximate any square integrable functions in $[0,T)$ with arbitrary precision. BPFs have been utilized to solve fractional partial differential equation [17], nonlinear system of fractional integral–differential equations [18] and fractional Mathieu equation [14]. However, BPFs are not smooth enough. This implies that more BPFs should be adopted in order to achieve higher approximation precision, which in turn increases the computation burden.

Recently, hybrid functions have received more and more attention in computational mathematics [19]. Compared to BPFs, hybrid functions are piecewise polynomials. That is, in each interval, a hybrid function is a polynomial instead of a constant. Thus, hybrid functions allow for a more accurate and efficient approximation of a function than a set of BPFs if the same number of basis functions are used. Maleknejad and Mahmoudi used hybrid of BPFs and Taylor polynomials (HBT) to numerically solve linear Fredholm integral equation in [20]. Mirzaee et al. proposed a method based on hybrid of BPFs and parabolic functions for solving a system of nonlinear stochastic Itô–Volterra integral equations of fractional order in [21]. Jafari Behbahani and Roodaki introduced a combination of Chebyshev polynomials and BPFs to seek the numerical solution of integral equations in [22]. Mashayekhi and Razzaghi presented hybrid of BPFs and Bernoulli polynomials to solve the distributed FDEs in [23]. Among these hybrid functions, the structure of HBT functions is simpler than that of others [24]. Consequently, it is easier to transform FDEs into algebraic systems using HBT functions without sacrificing the precision of solution. To the author's best knowledge, little work has been done to find the numerical solution of FDEs using HBT functions. Furthermore, for the purpose of solving differential equations by the hybrid functions mentioned above, the operational matrices of integration for hybrid functions are used to eliminate the integral operations. It is noted that none of them gives the exact formula of fractional integral of hybrid functions.

In this paper, we focus on solving the following FDE

$$F(t, f(t), D^{\alpha_0} f(t), D^{\alpha_1} f(t), \ldots, D^{\alpha_r} f(t)) = 0,$$

with initial conditions

$$f^{(k)}(0) = d_k. \quad k = 0, 1, \ldots, m_0 - 1, \tag{1.1}$$

where the fractional derivative are defined in Caputo sense, and $t \in [0,T]$. For ease of operation, assume that $\alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_r \geq 0$. $m_k - 1 < \alpha_k \leq m_k$, and m_k is an integer.

First, the proposed method derives the analytical formula for the Riemann–Liouville (R-L) fractional integral of HBT functions via Laplace transform. Then, the FDE is converted to an algebraic equation using the derived formula. A set of linear or nonlinear equations are obtained by collocating the algebraic equation at Newton-Cotes collocation points. The numerical solution of the FDE can be obtained by solving the linear or nonlinear equations. It is shown that the proposed method improves the accuracy without increasing the computational cost compared with the methods based on BPFs.

The remainder of this paper is organized as follows. In Section 2, some essential mathematical preliminaries and definitions of fractional calculus are briefly reviewed. Section 3 gives the definition of HBT functions. In Section 4, the R-L fractional integral of HBT functions is derived. The error analysis is given in Section

5. The proposed method is described in Section 6 in details. In Section 7, numerical examples are given to demonstrate the effectiveness of the proposed method. Finally, conclusions are draw in Section 8.

2. Preliminaries and notations

2.1. Fractional derivative and integral

First, some necessary definitions and mathematical preliminaries of the FC theory are briefly reviewed, so as to establish our results. In the literature, there are different kinds of definitions for fractional integration and derivative. Among them, the R-L fractional integration and the Caputo fractional derivative are investigated in this paper.

Definition 2.1 *The R-L fractional integral of order α is defined as [25]*

$$\begin{aligned} I^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s \\ &= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \end{aligned} \tag{2.1}$$

where $\alpha > 0$, $(0,t)$ is the integral interval, $\Gamma(\cdot)$ denotes the Gamma function and $*$ is the convolution operator.

Definition 2.2 *The Caputo fractional derivative of order α is defined as*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} \mathrm{d}s, \tag{2.2}$$

where $n-1 < \alpha \leq n$, and n is an integer.

For the Caputo fractional derivative and R-L fractional integral, the following properties hold [15, 26].

(1) $D^{\alpha}(I^{\alpha} f(t)) = f(t)$,

(2) If $\alpha > 0$, n is the smallest integer greater than α, then

$$I^{\alpha}(D^{\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}, \tag{2.3}$$

(3) If $f(t)$ is a constant, then $D^{\alpha} f(t) = 0$,

(4) $D^{\alpha} t^k = \begin{cases} 0, & k \in \mathbb{N}_0,\ k < n, \\ \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} t^{k-\alpha}, & k \in \mathbb{N}_0,\ k \geq n, \end{cases}$

where $\alpha > 0$, n is the smallest integer greater than α, and $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

3. HBT functions and function approximation

3.1. HBT functions

Definition 3.1 *A set of BPFs $b_i(t), i = 1, 2, \ldots, N$ on the interval $[0, T)$ are defined as*

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{N} T \leq t < \frac{i}{N} T, \\ 0, & \textit{otherwise.} \end{cases} \tag{3.1}$$

The BPFs are disjoint and orthogonal on $[0,T)$. For more details see [27]. Consider Taylor polynomials $T_j(t)=t^j$ on the interval $[0,T)$, the HBT functions are defined in the following form [20].

Definition 3.2 *For $i=1,2,\ldots,N$ and $j=0,1,2,\ldots,M-1$, the HBT functions are defined as*

$$h_{ij}(t)=\begin{cases} T_j(N/T-i+1), & \frac{i-1}{N}T\le t<\frac{i}{N}T, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.2}$$

Remark 3.3 *It can be noted that when $j=0$, the HBT functions degenerate to BPFs.*

3.2. Function approximation

Denote $H=L^2[0,T),\{h_{10}(t),h_{11}(t),\cdots,h_{N(M-1)}(t)\}\subset H$ as a set of HBT functions, and

$$Y=\mathrm{span}\{h_{10}(t),h_{11}(t),\cdots,h_{N(M-1)}(t)\}.$$

Since Y is a finite dimensional subspace of H, for an arbitrary $f(t)\in H$, it has the unique best approximation $f^*(t)\in Y$, that is

$$\forall y(t)\in Y,\quad \| f(t)-f^*(t)\|\le\| f(t)-y(t)\| .$$

Since $f^*(t)\in Y$, there is a set of coefficients $c_{10},c_{11},\cdots,c_{N(M-1)}$ such that

$$f(t)\approx f^*(t)=\sum_{i=1}^{N}\sum_{j=0}^{M-1}c_{ij}h_{ij}(t)=C^{\mathsf{T}}H(t), \tag{3.3}$$

where the superscript T denotes transposition,

$$C=[c_{10},\cdots,c_{1(M-1)},c_{20},\cdots,c_{2(M-1)},\cdots,c_{N0},\cdots,c_{N(M-1)}]^{\mathsf{T}} \tag{3.4}$$

and

$$H(t)=[h_{10}(t),\cdots,h_{1(M-1)}(t),h_{20}(t),\cdots,h_{2(M-1)}(t),\cdots,h_{N0}(t),\cdots,h_{N(M-1)}(t)]^{\mathsf{T}} \tag{3.5}$$

are column vectors with the dimension of $\hat{m}=N\times M$.

4. R-L factional integral of HBT functions

In this section, the R-L fractional integral of $H(t)$ defined in Eq. 3.5 is derived. The fractional integral of $H(t)$ can be expressed as

$$I^{\alpha}H(t)=\bar{H}(t,\alpha), \tag{4.1}$$

where

$$\bar{H}(t,\alpha)=\left[I^{\alpha}h_{10}(t),\cdots,I^{\alpha}h_{1(M-1)}(t),I^{\alpha}h_{20}(t),\cdots,I^{\alpha}h_{2(M-1)}(t),\cdots,I^{\alpha}h_{N0}(t),\cdots,I^{\alpha}h_{N(M-1)}(t)\right]^{\mathsf{T}}. \tag{4.2}$$

To obtain $I^{\alpha}h_{ij}(t)$, the Laplace transform is used. According to Eq. 2.1, one has

$$I^{\alpha}h_{ij}(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*h_{ij}(t). \tag{4.3}$$

Eq. 4.3 is the convolution of functions $t^{\alpha-1}$ and $h_{ij}(t)$. Taking Laplace transform on both sides of Eq. 4.3, yields

$$\mathcal{L}[I^{\alpha}h_{ij}(t)] = F_1(s)\cdot F_2(s), \tag{4.4}$$

where

$$F_1(s) = \mathcal{L}\left[\frac{1}{\Gamma(\alpha)}t^{\alpha-1}\right] = \frac{1}{s^{\alpha}} \tag{4.5}$$

and

$$F_2(s) = \mathcal{L}[h_{ij}(t)] = \frac{N^j j!}{s^{j+1}T^j}e^{-\frac{i-1}{N}Ts} - \sum_{k=0}^{j}\frac{N^k\Gamma(j+1)}{s^{k+1}\Gamma(j-k+1)T^k}e^{-\frac{i}{N}Ts} \tag{4.6}$$

can be obtained by using the definition of Laplace transform and integration by parts. Substituting Eq. 4.5 and Eq. 4.6 into Eq. 4.4, one gets

$$\begin{aligned}\mathcal{L}[I^{\alpha}h_{ij}(t)] &= \frac{N^j\Gamma(j+1)}{\Gamma(\alpha+j+1)T^j}\times\frac{\Gamma(\alpha+j+1)}{s^{\alpha+j+1}}e^{-\frac{i-1}{N}Ts}\\ &-\sum_{k=0}^{j}\frac{N^k\Gamma(j+1)}{\Gamma(\alpha+k+1)\Gamma(j-k+1)T^k}\times\frac{\Gamma(\alpha+k+1)}{s^{\alpha+k+1}}e^{-\frac{i}{N}Ts}.\end{aligned} \tag{4.7}$$

Next, computing the inverse Laplace transform of Eq. 4.7, yields

$$\begin{aligned}I^{\alpha}h_{ij}(t) &= \frac{N^j\Gamma(j+1)}{\Gamma(\alpha+j+1)T^j}\left(t-\frac{i-1}{N}T\right)^{\alpha+j}\mu\left(t-\frac{i-1}{N}T\right)\\ &-\sum_{k=0}^{j}\frac{N^k\Gamma(j+1)}{\Gamma(\alpha+k+1)\Gamma(j-k+1)T^k}\left(t-\frac{i}{N}T\right)^{\alpha+k}\mu\left(t-\frac{i}{N}T\right),\end{aligned} \tag{4.8}$$

where $\mu(t)$ is a unit step function, which is defined as

$$\mu(t) = \begin{cases} 1, & t\geq 0,\\ 0, & \text{otherwise.}\end{cases} \tag{4.9}$$

Then Eq. 4.8 can be rewritten as

$$I^{\alpha}h_{ij}(t) = \begin{cases} 0, & t<\frac{i-1}{N}T,\\ \left(t-\frac{i-1}{N}T\right)^{\alpha+j}d_j, & \frac{i-1}{N}T\leq t<\frac{i}{N}T,\\ \left(t-\frac{i-1}{N}T\right)^{\alpha+j}d_j-\left(t-\frac{i}{N}T\right)^{\alpha}\bar{d}_j, & t\geq\frac{i}{N}T.\end{cases}$$

where

$$d_j = \frac{N^j\Gamma(j+1)}{\Gamma(\alpha+j+1)T^j},$$

$$\bar{d}_j = \sum_{k=0}^{j}\frac{N^k\Gamma(j+1)}{\Gamma(\alpha+k+1)\Gamma(j-k+1)T^k}\left(t-\frac{i}{N}T\right)^k.$$

5. Error analysis

In this section, the approximation error of the R-L fractional integral of the hybrid functions with respect to L^2-norm is deduced. First, the upper error bound of function approximation is given.

Theorem 5.1 *Suppose* $f^{(M)}(t)$ *is continuous and bounded on* $[0,T)$, *say* $|f^{(M)}(t)| \leq K$, *and* $f^*(t) = \sum_{i=1}^{N}\sum_{j=0}^{M-1} c_{ij}h_{ij}(t)$ *is the best approximation of* $f(t)$ *out from* Y. *Then the following inequality holds*

$$\| f(t) - f^*(t) \|_{L^2[0,T)} \leq \sqrt{\frac{T}{(2M+1)}}\left(\frac{K}{M!}\right)\left(\frac{T}{N}\right)^M. \tag{5.1}$$

Proof The function $f(t)$ defined on $[0,T)$ can be presented as

$$f(t) = \sum_{i=1}^{N} f_i(t), \tag{5.2}$$

where

$$f_i(t) = \begin{cases} f(t), & \frac{i-1}{N}T \leq t < \frac{i}{N}T, \\ 0, & \text{otherwise.} \end{cases}$$

Approximating $f_i(t)$ with $f_i^{M-1}(t)$, which is the Taylor expansion of $f(t)$ on interval $[\frac{i-1}{N}T, \frac{i}{N}T)$, yields

$$\begin{aligned} f_i(t) \approx f_i^{M-1}(t) &= f\left(\frac{i-1}{N}T\right) + f'\left(\frac{i-1}{N}T\right)\left(t - \frac{i-1}{N}T\right) + \\ &\frac{f''\left(\frac{i-1}{N}T\right)}{2!}\left(t - \frac{i-1}{N}T\right)^2 + \cdots + \frac{f^{(M-1)}\left(\frac{i-1}{N}T\right)}{(M-1)!}\left(t - \frac{i-1}{N}T\right)^{M-1} \\ &= f\left(\frac{i-1}{N}T\right)h_{i0}(t) + f'\left(\frac{i-1}{N}T\right)\frac{T}{N}h_{i1}(t) + \frac{f''\left(\frac{i-1}{N}T\right)}{2!}\left(\frac{T}{N}\right)^2 h_{i2}(t) \\ &+ \cdots + \frac{f^{(M-1)}\left(\frac{i-1}{N}T\right)}{(M-1)!}\left(\frac{T}{N}\right)^{M-1} h_{i(M-1)}(t) \\ &= \sum_{j=0}^{M-1} d_{ij}h_{ij}(t), \end{aligned} \tag{5.3}$$

where $d_{ij} = \frac{f^{(j)}\left(\frac{i-1}{N}T\right)}{j!}\left(\frac{T}{N}\right)^j$. According to Taylor's mean value theorem, the approximating error between $f_i(t)$ and $f_i^{M-1}(t)$ is

$$f_i(t) - f_i^{M-1}(t) = \frac{f^{(M)}(\xi_i)}{M!}\left(t - \frac{i-1}{N}T\right)^M$$

with $\xi_i \in (\frac{i-1}{N}T, t)$. Let

$$y(t) = \sum_{i=1}^{N} f_i^{M-1}(t) = \sum_{i=1}^{N}\sum_{j=0}^{M-1} d_{ij}h_{ij}(t). \tag{5.4}$$

Obviously, $y(t) \in Y$. Since $f^*(t)$ is the best approximation of $f(t)$ out from Y, the following inequality

$$\begin{aligned}\| f(t) - f^*(t) \|^2_{L^2[0,T)} &\leq \| f(t) - y(t) \|^2_{L^2[0,T)} \\ &= \sum_{i=1}^{N} \| f_i(t) - f_i^{M-1}(t) \|^2_{L^2[\frac{i-1}{N}T, \frac{i}{N}T)} \\ &= \sum_{i=1}^{N} \int_{\frac{i-1}{N}T}^{\frac{i}{N}T} \left| \frac{f^{(M)}(\xi_i)}{M!} \left(t - \frac{i-1}{N}T \right)^M \right|^2 \mathrm{d}t\end{aligned} \tag{5.5}$$

holds. Because $f^{(M)}(t)$ is continuous and bounded on $[0,T)$, Eq. 5.5 can be rewritten as

$$\begin{aligned}\| f(t) - f^*(t) \|^2_{L^2[0,T)} &\leq \left(\frac{K}{M!} \right)^2 \sum_{i=1}^{N} \int_{\frac{i-1}{N}T}^{\frac{i}{N}T} \left(t - \frac{i-1}{N}T \right)^{2M} \mathrm{d}t \\ &= \frac{1}{2M+1} \frac{T^{2M+1}}{N^{2M}} \left(\frac{K}{M!} \right)^2 .\end{aligned} \tag{5.6}$$

This completes the proof. □

Second, the error between the R-L fractional integral of function $f(t)$ and its HBT approximation is analyzed.

Theorem 5.2 *Suppose $f^{(M)}(t)$ is continuous and bounded on $[0,T)$, say $|f^{(M)}(t)| \leq K$, and $f^*(t) = \sum_{i=1}^{N} \sum_{j=0}^{M-1} c_{ij} h_{ij}(t)$ is the best approximation of $f(t)$ out from Y. Then for $\alpha > 0$ the following inequality holds*

$$\| I^\alpha f(t) - I^\alpha f^*(t) \|_{L^2[0,T)} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \sqrt{\frac{T}{(2M+1)}} \left(\frac{K}{M!} \right) \left(\frac{T}{N} \right)^M . \tag{5.7}$$

Proof According to [23], one has

$$\| f * g \|_p \leq \| f \|_1 \| g \|_p .$$

Together with the definition of fractional integral 2.1, it is easy to obtain

$$\begin{aligned}\| I^\alpha f(t) - I^\alpha f^*(t) \|^2_{L^2[0,T)} &= \| I^\alpha (f(t) - f^*(t)) \|^2_{L^2[0,T)} \\ &= \| \frac{1}{t^{1-\alpha}\Gamma(\alpha)} * (f(t) - f^*(t)) \|^2_{L^2[0,T)} \\ &\leq \left(\frac{T^\alpha}{\alpha\Gamma(\alpha)} \right)^2 \| f(t) - f^*(t) \|^2_{L^2[0,T)} \\ &= \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \| f(t) - f^*(t) \|^2_{L^2[0,T)} .\end{aligned}$$

By using Eq. 5.6, one gets

$$\| I^\alpha f(t) - I^\alpha f^*(t) \|^2_{L^2[0,T)} \leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \frac{1}{2M+1} \frac{T^{2M+1}}{N^{2M}} \left(\frac{K}{M!} \right)^2 .$$

Then Eq. 5.7 can be obtained. □

Remark 5.3 *Eq. 5.1 shows that the upper error bound between* $f(t)$ *and its HBT approximation* $f^*(t)$ *is* $\sqrt{\frac{T}{(2M+1)}}\left(\frac{K}{M!}\right)\left(\frac{T}{N}\right)^M$*, which is superior to that of BPFs approximation [14]. This is also true for the upper error bound given in Eq. 5.7.*

Remark 5.4 *The upper error bound given in Eq. 5.1 depends on both* N *and* M*. If* $N > T$*, this upper error bound decreases with the increase of* N *and* M*, and even tends to zero when the values of* N *and* M *are sufficient large. It is also true for the upper error bound given in Eq. 5.7. In practice, for a given error bound, one can determine the values of* N *and* M*.*

6. The proposed method

Here, the HBT functions are used to solve Eq. 1.1. First, $D^{\alpha_0} f(t)$ is expanded onto HBT functions as

$$D^{\alpha_0} f(t) = A^{\mathsf{T}} H(t), \tag{6.1}$$

where

$$A = [a_{10}, a_{11}, \ldots, a_{1(M-1)}, a_{20}, a_{21}, \ldots, a_{2(M-1)}, \ldots, a_{N(M-1)}]^{\mathsf{T}}$$

and

$$H(t) = [h_{10}(t), h_{11}(t), \ldots, h_{1(M-1)}(t), h_{20}(t), h_{21}(t), \ldots, h_{2(M-1)}(t), \ldots, h_{N(M-1)}(t)]^{\mathsf{T}}.$$

Applying α_0 order R-L fractional integration to both sides of Eq. 6.1, using Eq. 2.3 and Eq. 4.1, one has

$$f(t) = A^{\mathsf{T}} \bar{H}(t, \alpha_0) + \sum_{k=0}^{m_0-1} \frac{t^k}{k!} d_k. \tag{6.2}$$

Based on Eq. 6.2, yields

$$D^{\alpha_i} f(t) = A^{\mathsf{T}} \bar{H}(t, \alpha_0 - \alpha_i) + \sum_{k=0}^{m_0-1} \frac{D^{\alpha_i}(t^k)}{k!} d_k. \tag{6.3}$$

Substituting Eqs. 6.1–6.3 into Eq. 1.1, then

$$\begin{aligned} F\Bigg(& t, A^{\mathsf{T}} \bar{H}(t, \alpha_0) + \sum_{k=0}^{m_0-1} \frac{t^k}{k!} d_k, A^{\mathsf{T}} \bar{H}(t, \alpha_0 - \alpha_1) + \sum_{k=0}^{m_0-1} \frac{D^{\alpha_1}(t^k)}{k!} d_k, \ldots, \\ & A^{\mathsf{T}} \bar{H}(t, \alpha_0 - \alpha_r) + \sum_{k=0}^{m_0-1} \frac{D^{\alpha_r}(t^k)}{k!} d_k \Bigg) = 0. \end{aligned} \tag{6.4}$$

is obtained. Taking the Newton-Cotes nodes

$$t_l = \frac{2l+1}{2\hat{m}} T, \quad l = 0, 1, \ldots, \hat{m} - 1, \tag{6.5}$$

as collocation points and substituting these collocation points into Eq. 6.4, yields

$$\begin{aligned} F\Bigg(& t_l, A^{\mathsf{T}} \bar{H}(t_l, \alpha_0) + \sum_{k=0}^{m_0-1} \frac{t_l^k}{k!} d_k, A^{\mathsf{T}} \bar{H}(t_l, \alpha_0 - \alpha_1) + \sum_{k=0}^{m_0-1} \frac{D^{\alpha_1}(t_l^k)}{k!} d_k, \ldots, \\ & A^{\mathsf{T}} \bar{H}(t_l, \alpha_0 - \alpha_r) + \sum_{k=0}^{m_0-1} \frac{D^{\alpha_r}(t_l^k)}{k!} d_k \Bigg) = 0. \end{aligned} \tag{6.6}$$

Eq. 6.6 is a set of linear or nonlinear algebraic equations, which can be used to find the unknown vector A. After finding A, $f(t)$ can be obtained according to Eq. 6.2.

Remark 6.1 *The Newton-Cotes nodes given in Eq. 6.5 can be obtained as follows. The interval $[0,T]$ is divided into $\hat{m}$ equidistant subintervals, and the midpoint of each subinterval is taken as the Newton-Cotes nodes. For example, when the HBT functions with $N=1, M=3$ is selected to solve Eq. 1.1, one gets $\hat{m}=3$. In this case, the Newton-Cotes nodes are taken as $t_0=\frac{1}{6}T, t_1=\frac{3}{6}T, t_2=\frac{5}{6}T$.*

7. Numerical examples

In this section, the proposed method is used to solve three FDEs, including linear and nonlinear ones to show its effectiveness. Moreover, the numerical solutions of underlying FDEs obtained by the proposed method are compared with those obtained by other methods in the literatures.

7.1. Example 1

The following FDE [28]

$$D^2 f(t)+3D^{1.5}f(t)+2D^{\alpha_2}f(t)+D^{\alpha_1}f(t)+5f(t)=g(t), \quad t\in[0,1], \tag{7.1}$$

with the initial conditions $f(0)=1, f'(0)=0$ is considered, where $g(t)=1+3t+\frac{2}{\Gamma(3-\alpha_2)}t^{2-\alpha_2}+\frac{1}{\Gamma(3-\alpha_1)}t^{2-\alpha_1}+5\left(1+\frac{1}{2}t^2\right)$ and $\alpha_1=0.0159$, $\alpha_2=0.1379$. The analytical solution of Eq. 7.1 is $f(t)=1+\frac{1}{2}t^2$. In the proposed method, $N=1, M=3$ is selected. Let

$$D^2 f(t)=A^{\mathsf{T}}H(t)=a_{10}h_{10}(t)+a_{11}h_{11}(t)+a_{12}h_{12}(t). \tag{7.2}$$

According to Eq. 6.2, one obtains

$$f(t)=A^{\mathsf{T}}\bar{H}(t,2)+1. \tag{7.3}$$

By using Eq. 6.3, one gets

$$Df(t)=A^{\mathsf{T}}\bar{H}(t,1) \tag{7.4}$$

and

$$D^{\alpha_i}f(t)=A^{\mathsf{T}}\bar{H}(t,2-\alpha_i). \tag{7.5}$$

Substituting Eqs. 7.2–7.5 into Eq. 7.1, one has

$$\begin{aligned} &A^{\mathsf{T}}H(t)+3A^{\mathsf{T}}\bar{H}(t,1)+2A^{\mathsf{T}}\bar{H}(t,2-\alpha_2)+A^{\mathsf{T}}\bar{H}(t,2-\alpha_1)\\ &+5A^{\mathsf{T}}\bar{H}(t,2)-g(t)=0. \end{aligned} \tag{7.6}$$

By collocating Eq. 7.6 at the Newton-Cotes nodes given in Eq. 6.5, the following linear algebraic equations are obtained

$$\begin{aligned} 1.62418a_{10}+0.21534a_{11}+0.03260a_{12}&=1.62418,\\ 3.56446a_{10}+1.05502a_{11}+0.42051a_{12}&=3.56446,\\ 6.39515a_{10}+2.69051a_{11}+1.61656a_{12}&=6.39515. \end{aligned} \tag{7.7}$$

By solving the above linear equations, one gets

$$a_{10}=1, \quad a_{11}=0, \quad a_{12}=0. \tag{7.8}$$

Substituting coefficients obtained in Eq. 7.8 into Eq. 7.3, the solution can be obtained as

$$f(t) = 1 + \frac{1}{2}t^2,$$

which is the same as the analytical solution. The result demonstrates that the proposed HBT estimation of R-L fractional integration of the function is precise.

7.2. Example 2

Considering a FDE

$$D^{\alpha} f(t) + f(t) = 0, \quad 0 < \alpha < 2, \tag{7.9}$$

with the initial conditions $f(0) = 1, f'(0) = 0$. $t \epsilon [0, 1]$ This FDE was first considered by Diethelm et al. in [29] and also solved by Saadatmandi and Dehghan in [15] using the Legendre operational matrix. The analytical solution of Eq. 7.9 is

$$f(t) = \sum_{k=0}^{\infty} \frac{(-t^{\alpha})^k}{\Gamma(\alpha k + 1)}. \tag{7.10}$$

Firstly, $0 < \alpha < 1$ is considered. In this case, let $\alpha = 0.85$. Table 1 lists the maximum absolute errors of solutions obtained by the proposed method with $M = 2$ and different N, the BPFs method with $N = 6$ and the shifted Legendre polynomials method [15] with $M_1 = 2$. It can be seen from Table 1 that our proposed method gives the most accurate solution. Also, the error becomes smaller by increasing N and M.

Table 1. Maximum absolute errors of Example 2 when $\alpha = 0.85$.

Method	Maximum absolute error
Present method	
$M = 2, N = 1$	1.1e−002
$M = 2, N = 2$	3.3e−003
$M = 2, N = 3$	1.3e−003
BPFs method	
$N = 6$	1.1e−001
Shifted Legendre polynomials method [15]	
$M_1 = 2$	2.6e−002

Secondly, $1 < \alpha < 2$ is considered. In this case, let $\alpha = 1.6$. Once again, Table 2 provides the maximum absolute errors obtained by our proposed method with $M = 5$ and different N, the BPFs method with $N = 10$, and the shifted Legendre polynomials method with degree of shifted Legendre polynomials $M_1 = 10$ [15]. It demonstrates that the proposed method gives more accurate solutions than the other two methods and also the error becomes smaller by increasing the values of N and M.

7.3. Example 3

The nonlinear FDE

$$D^{\alpha} f(t) = \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)} t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1) + \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^3 - [f(t)]^{\frac{3}{2}}, \tag{7.11}$$

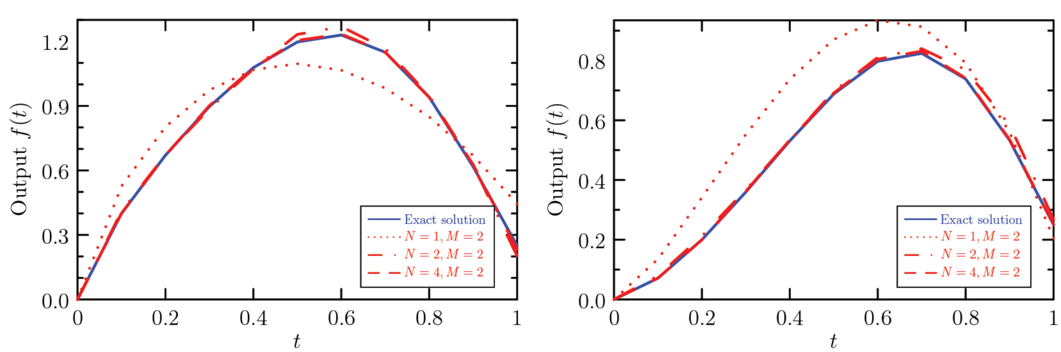
Table 2. Maximum absolute errors of Example 2 when $\alpha = 1.6$.

Method	Maximum absolute error
Present method	
$M = 5, N = 2$	3.8e−006
$M = 5, N = 4$	1.2e−006
$M = 5, N = 8$	2.1e−007
BPFs method	
$N = 10$	3.8e−002
Shifted Legendre polynomials method [15]	
$M_1 = 10$	3.1e−004

with the initial conditions $f(0) = f'(0) = 0$ is considered, where $0 < \alpha < 2$, $t \epsilon [0, 1]$ The analytical solution of Eq. 7.11 is $f(t) = t^8 - 3t^{4+\alpha/2} + \frac{9}{4}t^{\alpha}$ [30].

Eq. 7.11 can be transformed into $N \times M$ nonlinear algebraic equations following our proposed method. Table 3 lists the absolute errors between the analytical solution and numerical solutions obtained by our proposed method with $N = 2, M = 5$, BPFs method with $N = 10$, and method in [15] with degree of $M_3 = 10$ for different α. Obviously, the numerical solution obtained by our proposed method is more accurate than the other two methods.

Moreover, Figure shows the curves of analytical solutions and the solutions obtained by our proposed method with $N = 2, 3, 4, M = 2$ when $\alpha = 0.75, 1.5$. It is evident that as N, M increases the solutions of our method tend to the analytical solution.



(a) $\alpha = 0.75$ (b) $\alpha = 1.5$

Figure. Solutions of Example 3.

8. Conclusion

In this paper, a new method based on HBT functions is proposed to solve FDEs. Instead of deriving the fractional integral operation matrix of the hybrid functions, the analytical expression for the R-L fractional integral of HBT functions is deduced and used to reduce the FDE under consideration to an algebraic one, which is collocated at Newton-Cotes nodes, a set of linear or nonlinear equations are obtained. The numerical solution of the FDE is obtained through solving the set of equations. Numerical results verify the effectiveness

Table 3. Absolute errors of Example 3 when different values are taken for α.

α	Method	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
0.2	Present method	2.2e−005	1.4e−005	4.2e−004	2.3e−004	6.2e−004
	BPFs method	2.0e−004	9.4e−003	4.5e−002	1.0e−001	1.5e−001
	Method in [15]	2.2e−001	2.3e−001	3.6e−002	5.3e−001	1.7e−000
0.4	Present method	1.2e−005	9.6e−006	3.2e−004	1.6e−004	4.0e−004
	BPFs method	2.0e−004	6.5e−003	3.0e−002	6.9e−002	8.7e−002
	Method in [15]	6.3e−002	6.0e−002	2.4e−002	1.2e−001	3.0e−001
0.6	Present method	8.4e−006	3.4e−008	2.1e−004	1.2e−004	2.4e−004
	BPFs method	1.0e−004	4.1e−003	1.8e−002	4.0e−002	4.6e−002
	Method in [15]	1.5e−002	1.3e−002	9.6e−003	2.1e−002	3.7e−002
0.8	Present method	3.2e−005	1.6e−005	1.2e−004	1.4e−004	1.7e−004
	BPFs method	1.0e−004	2.3e−003	9.7e−003	2.1e−002	2.1e−002
	Method in [15]	2.9e−003	2.1e−003	2.3e−003	2.5e−003	2.1e−003
1.2	Present method	2.6e−006	2.9e−006	2.6e−006	2.2e−006	1.8e−006
	BPFs method	3.6e−005	3.0e−004	8.4e−004	1.5e−003	7.6e−004
	Method in [15]	1.9e−003	1.6e−003	2.8e−002	2.9e−003	1.6e−002
1.4	Present method	2.4e−006	3.5e−006	3.8e−006	3.7e−006	3.3e−006
	BPFs method	1.9e−005	1.8e−004	1.1e−003	2.4e−003	2.5e−003
	Method in [15]	2.0e−004	1.6e−003	7.6e−003	4.9e−003	3.3e−002
1.6	Present method	1.8e−006	3.3e−006	4.2e−006	4.7e−006	4.7e−006
	BPFs	7.6e−006	4.4e−004	2.1e−003	4.3e−003	4.1e−003
	Method in [15]	6.3e−005	7.3e−004	1.7e−003	2.3e−003	1.3e−002
1.8	Present method	9.2e−007	2.2e−006	3.2e−006	3.9e−006	4.3e−006
	BPFs method	1.3e−005	5.7e−004	2.5e−003	5.3e−003	4.9e−003
	Method in [15]	3.8e−005	2.0e−004	2.6e−004	5.9e−004	2.8e−003

of the proposed method, and comparisons demonstrate that the proposed method can provide more accurate solutions than other existing methods.

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