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The 2-rank of the class group of some real cyclic quartic number fields II

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Abstract: In this paper, we determine the 2-rank of the class group of certain classes of real cyclic quartic number fields. Precisely, we consider the case in which the quadratic subfield is $\mathbb{Q}(\sqrt{\ell})$ with $\ell = 2$ or a prime congruent to 1 (mod 8).

Key words: Real cyclic quartic number field, 2-rank, 2-class group, quadratic fields, quartic extensios

1. Introduction

Let K be a number field and H its p-class group, that is the Sylow p-subgroup of the ideal class group $\mathrm{Cl}(K)$ of K in the wide sense, where p is a prime integer. Class groups of number fields have been studied for a long time, and there are many very interesting problems concerning their behavior. A particular quantity of interest is the rank $r_p(H)$ of the p-class group H defined as the number of cyclic p-groups appearing in the decomposition of $\mathrm{Cl}(K)$, i.e. the dimension of the $\mathbb{F}_p$-vector space $\mathrm{Cl}(K)/\mathrm{Cl}(K)^p$, where $\mathbb{F}_p$ is the field of p elements.

For $p = 2$, many mathematicians are interested in determining $r_2(H)$ and the power of 2 dividing the class number of K. Hasse [13], Bauer [5] and others gave methods for determining the exact power of 2 dividing the class number of a quadratic numbers field. These methods were developed by C. J. Parry and his co-authors to determine $r_2(H)$ and the power of 2 dividing the class number of some cyclic quartic numbers fields K having a quadratic subfield k with odd class number (e.g., [6, 7, 22–24]). To accomplish their task, they needed a suitable genus theory convenient to their situation. Hence they showed that, the theory firstly developed by Hilbert ([16]), assuming an imaginary base field k, can be adapted to the situation where K is a totally imaginary quartic cyclic extension of a totally real quadratic subfield k. In reality, this theory can be applied to any quartic number field K having a quadratic subfield k of odd class number ([22]).

The 2-rank of the class group of any biquadratic number field K is determined (partially or totally) in many papers ([2, 3, 6, 7, 20, 21]) up to the case: K is a real quartic cyclic extension of the rational number field $\mathbb{Q}$. This paper is devoted to investigate the 2-rank of the class group of this class of number fields. We will focus on the case where the unique quadratic subfield of K is $k = \mathbb{Q}(\sqrt{\ell})$ with ℓ is a prime congruent to 1 (mod 8) or $\ell = 2$, and it is well known that the norm of the fundamental unit of k in these cases is -1. Note that the case ℓ congruent to 5 (mod 8) is studied separately.

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An outline of the paper is as follows. In § 2 we summarize preliminary results on quartic cyclic number fields and the ambiguous class numbers formula, and in § 3, we recall the definition of the quadratic norm residue symbol and some of its properties. The main theorems are presented in § 4 and § 5. In § 6 we characterize, for a prime ℓ congruent to 1 (mod 8) or $\ell = 2$, all the real cyclic quartic number fields $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ whose 2-class group is trivial, cyclic, of rank 2 or 3.

Notations

Throughout this paper, we adopt the following notations.

- $\mathbb{Q}$: the rational field.
- ℓ: a prime integer congruent to 1 modulo 8 or $\ell = 2$.
- $k = \mathbb{Q}(\sqrt{\ell})$: a quadratic field.
- ϵ_0: the fundamental unit of k.
- n: a square-free positive integer relatively prime to ℓ.
- $\delta = 1$ or 2.
- $d = n\epsilon_0\sqrt{\ell}$.
- $\mathbb{K} = k(\sqrt{d})$: a real quartic cyclic number field.
- $\mathcal{O}_k$ (resp. $\mathcal{O}_{\mathbb{K}}$): the ring of integers of k (resp. $\mathbb{K}$).
- H: the 2-class group of $\mathbb{K}$.
- $r_2(H)$: the rank of H.
- $\mathfrak{2}_i$, $i \in \{1; 2\}$: the prime ideals of k above 2 if $\ell \equiv 1 \pmod 8$.
- $\mathbb{K}^*$, k^*: the nonzero elements of the fields $\mathbb{K}$ and k respectively.
- $N_{\mathbb{K}/k}(\mathbb{K})$: elements of k which are norm from $\mathbb{K}$.
- p, q, p_i, q_j: odd prime integers.
- $(\frac{x,y}{p})$: quadratic norm residue symbol over k.
- $[\frac{\alpha}{\beta}]$: quadratic residue symbol for k.
- $(\frac{a}{b})$: quadratic residue (Legendre) symbol.
- $(\frac{a}{b})_4$: the rational 4-th power residue symbol.

2. Preliminary results

Let K be a cyclic quartic extension of the rational number field $\mathbb{Q}$. By [15, Theorem 1], it is known that K can be expressed uniquely in the form $K = \mathbb{Q}(\sqrt{a(\ell + b\sqrt{\ell})})$, where a, b, c and ℓ are integers satisfying the conditions: a is odd and square-free, $\ell = b^2 + c^2$ is square free positive and relatively prime to a, with $b > 0$ and $c > 0$. Note that K possesses a unique quadratic subfield $k = \mathbb{Q}(\sqrt{\ell})$. Assuming the class number of k is odd and $N_{k/\mathbb{Q}}(\epsilon_0) = -1$, where ϵ_0 is the fundamental unit of k, then one can, by [17, 25], deduce that there exists an integer n such that $K = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ and:

$$n = \begin{cases} 2a & \text{if } \ell \equiv 1 \pmod 4 \text{ and } b \equiv 1 \pmod 2 \\ a & \text{otherwise.} \end{cases}$$

We need also the following theorem which gives the conductor f_K of K.

Theorem 2.1 ([15]) *The conductor f_K of the (real or imaginary) cyclic quartic field $K = \mathbb{Q}(\sqrt{a(\ell + b\sqrt{\ell})}$, where a is an odd square-free integer, $\ell = b^2 + c^2$ is a square-free positive integer relatively prime to a, with $b > 0$ and $c > 0$, is given by $f_K = 2^e|a|\ell$, where e is defined by:*

$$e = \begin{cases} 3, \ if\ \ell \equiv 2 \pmod 8 \ or\ \ell \equiv 1 \pmod 4 \ and\ b \equiv 1 \pmod 2, \\ 2, \ if\ \ell \equiv 1 \pmod 4, \ b \equiv 0 \pmod 2, \ a + b \equiv 3 \pmod 4, \\ 0, \ if\ \ell \equiv 1 \pmod 4, \ b \equiv 0 \pmod 2, \ a + b \equiv 1 \pmod 4. \end{cases}$$

Recall that the extensions $K/\mathbb{Q}$ were investigated by Hasse in a paper ([14]) prior to that of Leopoldt ([19]) on the arithmetic interpretation of the class number of real abelian fields. They were also investigated by M. N. Gras [10–12, ...] and others. By [14], the field K can be real as it can be imaginary. Precisely, we have the following result that specify the number of real (resp. imaginary) cyclic quartic fields sharing the same conductor and the same quadratic subfield.

Lemma 2.2 ([14]) *For a given square-free positive integer $\ell = p_0 p_1 \ldots p_m$, where p_i is a prime integer for all i, and for a given conductor f_K, the number of real (resp. imaginary) cyclic quartic fields K having the same conductor f_K and the same quadratic subfield $k = \mathbb{Q}(\sqrt{\ell})$ is equal to 2^m if $f_K \equiv 0 \pmod 8$. But if $f_K \not\equiv 0 \pmod 8$, then the number of real (resp. imaginary) cyclic quartic fields K with the conductor f_K and the same quadratic subfield $k = \mathbb{Q}(\sqrt{\ell})$ is equal to 2^m or 0 (resp. 0 or 2^m). Moreover, a cyclic quartic field K is real if and only if $S = \prod_{p|f_K} s_p = +1$, where $s_2 = -1$, $s_p = (-1)^{\frac{p-1}{e_p}}$ for odd prime integer p with e_p is the ramification index of p in K.*

Remark 2.3 *Keeping notations above, K is then real if and only if $a > 0$ (equivalently $n > 0$).*

The field K also satisfies the following lemma.

Lemma 2.4 ([26]) *Let a, b and c be positive integers, $\ell = b^2 + c^2$, with a and c odd, then $\mathbb{Q}(\sqrt{2a(\ell + b\sqrt{\ell})}) = \mathbb{Q}(\sqrt{a(\ell + c\sqrt{\ell})}$.*

We end this section by recalling the number of ambiguous ideal classes of a quadratic extension K/k. This result will allow us to investigate the 2-rank of the class group of K.

Theorem 2.5 ([1, 23]) *Let K/k be a cyclic extension of prime degree p. Denote by $A_{K/k}$ the number of ambiguous ideal classes of K with respect to k, then:*

$$A_{K/k} = h(k)p^{\mu+r^*-(r+c+1)},$$

where:
r is the number of fundamental units of k,
μ is the number of prime ideals of k (finite or infinite) which ramify in K,
$p^{r^*} = [N_{K/k}(K^*) \cap E_k : E_k^p]$ with E_k is the group of units of k,
$c = 1$ if k contains a primitive p-th root of unity and $c = 0$ otherwise.
Furthermore, for $p = 2$ and if the class number of k is odd, then the 2-rank of the class group of K is

$$\mu + r^* - (r + c + 1).$$

Remark 2.6 *Since -1 and ϵ_0 generate the unit group of the quadratic field k, so*

- *$r^* = 0$, if $-1, \epsilon_0$ and $-\epsilon_0$ are not in $N_{K/k}(K^*)$.*
- *$r^* = 1$, if ($-1 \in N_{K/k}(K^*)$ and ϵ_0 is not) or (-1 is not in $N_{K/k}(K^*)$ and ϵ_0 or $-\epsilon_0$ is).*
- *$r^* = 2$, if -1 and ϵ_0 are in $N_{K/k}(K^*)$.*

To compute $r_2(H)$, the rank of the 2-class group H of K, we will distinguish many cases. For this, let p_1, p_2, $\cdots$, p_t, q_1, $\cdots$, q_s be positive prime integers. Put $\delta = 1$ or 2.

3. Quadratic norm residue symbol

Since we frequently use the quadratic norm residue symbol, we have to recall its definition and some of its properties (cf. [9, Chapter II, Theorem 3.1.3]). Let k be a number field and $\beta \in k^*$ a square-free element in k. Let f be the conductor of $k(\sqrt{\beta})/k$. For any prime $\mathfrak{p}$ of k (finite or infinite), we denote by $f_\mathfrak{p}$ the largest power of $\mathfrak{p}$ dividing f. Let $\alpha \in k^*$, according to the approximation theorem there exists $\alpha_0 \in k$ such that

$$\alpha_0 \equiv \alpha \pmod{f_\mathfrak{p}} \quad \text{and} \quad \alpha_0 \equiv 1 \pmod{\tfrac{f}{f_\mathfrak{p}}}.$$

If $(\alpha_0) = \mathfrak{p}^n\mathfrak{P}$ with $n \in \mathbb{Z}$ and $(\mathfrak{P}, \mathfrak{p}) = 1$ ($n = 0$ if $\mathfrak{p}$ is infinite), set

$$\left(\frac{\alpha, k(\sqrt{\beta})}{\mathfrak{p}}\right) = \left(\frac{k(\sqrt{\beta})}{\mathfrak{P}}\right),$$

where $\left(\frac{k(\sqrt{\beta})}{\mathfrak{P}}\right)$ is the Artin map applied to $\mathfrak{P}$. For $\alpha \in k^*$ and a prime (finite or infinite) $\mathfrak{p}$ of k, the quadratic norm residue symbol is defined by

$$\left(\frac{\alpha, \beta}{\mathfrak{p}}\right) = \frac{\left(\frac{\alpha, k(\sqrt{\beta})}{\mathfrak{p}}\right)(\sqrt{\beta})}{\sqrt{\beta}} \in \{\pm 1\}.$$

If the prime $\mathfrak{p}$ is unramified in $k(\sqrt{\beta})/k$, we set

$$\left(\frac{\beta}{\mathfrak{p}}\right) = \frac{\left(\frac{k(\sqrt{\beta})}{\mathfrak{p}}\right)(\sqrt{\beta})}{\sqrt{\beta}} \in \{\pm 1\}.$$

Note that the norm residue symbol may be defined more generally for an extension $k(\sqrt[m]{\beta})/k$, where $m \in \mathbb{N}^*$, k is a number field containing the m-th root of unity and $\beta \in k^*$.
The quadratic norm residue symbol verifies the following properties that we shall use later.

1. $\left(\frac{\alpha_1\alpha_2,\,\beta}{\mathfrak{p}}\right) = \left(\frac{\alpha_1,\,\beta}{\mathfrak{p}}\right)\left(\frac{\alpha_2,\,\beta}{\mathfrak{p}}\right)$,
2. $\left(\frac{\alpha,\,\beta}{\mathfrak{p}}\right) = \left(\frac{\beta,\,\alpha}{\mathfrak{p}}\right)$,
3. If $\mathfrak{p}$ is unramified in $k(\sqrt{\beta})/k$ and appears with exponent e in the decomposition of (α), then $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right) = \left(\frac{\beta}{\mathfrak{p}}\right)^e$,
4. If $\mathfrak{p}$ is unramified in $k(\sqrt{\beta})/k$ and does not appear in the decomposition of (α), then $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right) = 1$,
5. $\prod_{\mathfrak{p}\in Pl}\left(\frac{\alpha,\,\beta}{\mathfrak{p}}\right) = 1$, where Pl is the set of all finite and infinite primes of k,
6. Let k_1 be a finite extension of k, $\alpha \in k_1^*$ and $\beta \in k^*$. Denote by $\mathfrak{p}$ a prime ideal of k and by $\mathfrak{B}$ a prime ideal of k_1 above $\mathfrak{p}$. Thus
$$\prod_{\mathfrak{B}|\mathfrak{p}}\left(\frac{\alpha,\beta}{\mathfrak{B}}\right) = \left(\frac{N_{k_1/k}(\alpha),\beta}{\mathfrak{p}}\right).$$

4. The case $\ell \equiv 1 \pmod 8$

Let ℓ be a prime integer congruent to $1 \pmod 8$ and n a square free positive integer relatively prime to ℓ. Let $\mathbb{K} = k(\sqrt{n\epsilon_0\sqrt{\ell}})$ and $k = \mathbb{Q}(\sqrt{\ell})$, where ϵ_0 is the fundamental unit of k.

Remark 4.1 *As n is relatively prime to ℓ, so the prime integers dividing n don't ramify in k, and 2 splits completely in k since $\ell \equiv 1 \pmod 8$. Moreover, e_ℓ, the ramification index of ℓ in $\mathbb{K}$, is 4. Thus, $s_\ell = (-1)^{\frac{\ell-1}{4}} = 1$.*

Remark 4.2 *In what follows, we consider $\ell = b^2 + c^2$ where b and c are two positive integers with c odd. As $\ell \equiv 1 \pmod 8$, so $b \equiv 0 \pmod 4$ (e.g., [8, page 2]). Recall, as mentioned in the beginning of section 2, that there exists an odd square-free integer a relatively prime to ℓ such that $\mathbb{K} = \mathbb{Q}(\sqrt{a(\ell + b\sqrt{\ell})})$ with*

$$a = \begin{cases} \frac{n}{2} & \text{if } \ell \equiv 1 \pmod 4 \text{ and } b \equiv 1 \pmod 2, \\ n & \text{otherwise.} \end{cases}$$

We also need the following lemma.

Lemma 4.3 ([4]) *Let* ℓ *be a prime integer congruent to* $1 \pmod 8$ *and* ϵ_0 *the fundamental unit of* $k=\mathbb{Q}(\sqrt{\ell})$. *Then* $\epsilon_0\sqrt{\ell}\equiv 1 \pmod 4$ *in* k.

4.1. Case $n=1$

Theorem 4.4 *Let* $\mathbb{K}=\mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ *be a real cyclic quartic number field, where* $\ell\equiv 1 \pmod 8$ *is a positive prime integer,* n *a square-free positive integer relatively prime to* ℓ *and* ϵ_0 *the fundamental unit of* $k=\mathbb{Q}(\sqrt{\ell})$. *If* $n=1$, *then* $r_2(H)=0$.

Proof Since $a=n=1$, $a+b\equiv 1+0\equiv 1 \pmod 4$, which implies, by Theorem 2.1, that $f_{\mathbb{K}}=\ell\not\equiv 0 \pmod 8$. But $S=s_\ell=+1$, then Lemma 2.2 ensures the existence of real number field $\mathbb{K}$ having as conductor $f_{\mathbb{K}}$ and as quadratic subfield k. Moreover, the only prime ideal of k that ramifies in $\mathbb{K}$ is $(\sqrt{\ell})$, i.e. $\mu=1$. To prove the theorem, we have to compute the integer r^* (see Theorem 2.5) by applying Remark 2.6, and then we call the theorem 2.5. We have:

$$\left(\frac{-1,d}{(\sqrt{\ell})}\right)=\left[\frac{-1}{(\sqrt{\ell})}\right]=\left(\frac{-1}{\ell}\right)=1 \quad\text{and}\quad \left(\frac{\epsilon_0,d}{(\sqrt{\ell})}\right)=\left[\frac{\epsilon_0}{(\sqrt{\ell})}\right]=\left[\frac{u}{(\sqrt{\ell})}\right]=\left(\frac{u}{\ell}\right)=\left(\frac{\ell}{u}\right)=\left(\frac{v^2\ell}{u}\right)=1,$$

indeed $\epsilon_0=u+v\sqrt{\ell}$, so $-1=u^2-v^2\ell$ and thus $v^2\ell\equiv 1 \pmod u$. Hence $r^*=2$, which implies that: $r_2(H)=\mu+r^*-3=1+2-3=0$. □

4.2. Case $n=2$

Theorem 4.5 *Let* $\mathbb{K}=\mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ *be a real cyclic quartic number field, where* $\ell\equiv 1 \pmod 8$ *is a positive prime integer,* n *a square-free positive integer relatively prime to* ℓ *and* ϵ_0 *the fundamental unit of* $k=\mathbb{Q}(\sqrt{\ell})$. *For* $n=2$, *we have:*

1. *If* $\left(\frac{2}{\ell}\right)_4=(-1)^{\frac{l-1}{8}}$, *then* $r_2(H)=2$.
2. *If* $\left(\frac{2}{\ell}\right)_4\neq(-1)^{\frac{l-1}{8}}$, *then* $r_2(H)=1$.

Proof Since $n=2$, n is even and according to Lemma 2.4 we get $\mathbb{K}=\mathbb{Q}(\sqrt{2a(\ell+b\sqrt{\ell})})=\mathbb{Q}(\sqrt{a(\ell+c\sqrt{\ell})})$ with $a=\frac{n}{2}=1$. As $c\equiv 1 \pmod 2$, so, by Theorem 2.1, $f_{\mathbb{K}}=2^3a\ell=2^3\ell\equiv 0 \pmod 8$; thus, there exists as many real cyclic fields as imaginary ones having as a conductor $f_{\mathbb{K}}$ and as a quadratic subfield k. The prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$ and $\mathfrak{z}_i$, $i\in\{1,2\}$, where $2\mathcal{O}_k=\mathfrak{z}_1\mathfrak{z}_2$ is the decomposition of 2 in k, i.e. $\mu=3$. We have

$$\left(\frac{-1,d}{\mathfrak{z}_1}\right)=\left(\frac{-1,d}{\mathfrak{z}_2}\right)=\left(\frac{-1,2\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{-1,2}{\mathfrak{z}_1}\right)\left(\frac{-1,\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left[\frac{-1}{\mathfrak{z}_1}\right]=\left(\frac{-1}{2}\right)=\left(\frac{-1,2}{2}\right)=1,$$

indeed $\left(\frac{-1,\,\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left[\frac{\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right]^0=1$ since $\mathfrak{z}_1$ don't ramify in $\mathbb{Q}(\sqrt{\epsilon_0\sqrt{\ell}})$ (see Theorem 2.1).

$$\left(\frac{\epsilon_0,d}{\mathfrak{z}_1}\right)=\left(\frac{\epsilon_0,d}{\mathfrak{z}_2}\right)=\left(\frac{\epsilon_0,2\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{\epsilon_0,2}{\mathfrak{z}_1}\right)\left(\frac{\epsilon_0,\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{\epsilon_0,2}{\mathfrak{z}_1}\right)=\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}} \quad \text{(see [2]).}$$

Using the previous results, we complete the following table:

Unit\ Character	$\sqrt{\ell}$	$\mathfrak{z}_i$
-1	$+$	$+$
ϵ_0	$+$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\epsilon_0$	$+$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

a. If $\left(\frac{2}{\ell}\right)_4 = (-1)^{\frac{\ell-1}{8}}$, then $r^* = 2$, which implies that: $r_2(H) = \mu + r^* - 3 = 3 + 2 - 3 = 2$.

b. If $\left(\frac{2}{\ell}\right)_4 \neq (-1)^{\frac{\ell-1}{8}}$, then $r^* = 1$, which implies that: $r_2(H) = \mu + r^* - 3 = 3 + 1 - 3 = 1$. □

4.3. Case $n = \prod_{i=1}^{t} p_i$ and, for all i, $p_i \equiv 1 \pmod 4$

Theorem 4.6 *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell \equiv 1 \pmod 8$ is a positive prime integer, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of $k = \mathbb{Q}(\sqrt{\ell})$. Let $n = \prod_{i=1}^{i=t} p_i$ with $p_i \equiv 1 \pmod 4$ for all $i \in \{1, \ldots, t\}$ and t is a positive integer.*

1. *If, for all i, $(\frac{p_i}{\ell}) = -1$, then $r_2(H) = t$.*

2. *If, for all i, $(\frac{p_i}{\ell}) = 1$, then:*

   a. *If $\left(\frac{p_i}{\ell}\right)_4 \neq \left(\frac{\ell}{p_i}\right)_4$ for at least one $i \in \{1, \ldots, t\}$, then $r_2(H) = 2t - 1$.*

   b. *If $\left(\frac{p_i}{\ell}\right)_4 = \left(\frac{\ell}{p_i}\right)_4$ for all $i \in \{1, \ldots, t\}$, then $r_2(H) = 2t$.*

Moreover, for $n = \prod_{i=1}^{i=t_1} p_i \prod_{j=1}^{i=t_2} q_j$ with $(\frac{p_i}{\ell}) = -(\frac{q_j}{\ell}) = -1$ and $p_i \equiv q_j \equiv 1 \pmod 4$ for all $i \in \{1, \ldots, t_1\}$ and for all $j \in \{1, \ldots, t_2\}$, we have:

a. *If $\left(\frac{q_j}{\ell}\right)_4 \neq \left(\frac{\ell}{q_j}\right)_4$ for at least one $j \in \{1, \ldots, t_2\}$, then $r_2(H) = t_1 + 2t_2 - 1$.*

b. *If $\left(\frac{q_j}{\ell}\right)_4 = \left(\frac{\ell}{q_j}\right)_4$ for all $j \in \{1, \ldots, t_2\}$, then $r_2(H) = t_1 + 2t_2$.*

Proof Note first that $b \equiv 0 \pmod 2$, and since $\ell \equiv 1 \pmod 8$, so $b \equiv 0 \pmod 4$ (see Remark 4.2). On the other hand, $n = \prod_{i=1}^{i=t} p_i$, with $p_i \equiv 1 \pmod 4$ for all $i = 1, \ldots, t$, then $n = a = \prod_{i=1}^{i=t} p_i \equiv 1 \pmod 4$, so $a + b \equiv 1 + 0 \equiv 1 \pmod 4$. Thus, by Theorem 2.1, $f_{\mathbb{K}} = a\ell = \prod_{i=1}^{i=t} p_i\ell$, which implies that $f_{\mathbb{K}} \not\equiv 0 \pmod 8$. Thus, $\mathbb{K}$ is either real or imaginary cyclic quartic field. As the ramification index in $\mathbb{K}$ of each prime p_i is $e_{p_i} = 2$, then $\frac{p_i-1}{2} \equiv 0 \pmod 2$, this implies that $s_{p_i} = (-1)^{\frac{p_i-1}{2}} = +1$. Hence, $S = \prod_{q|f} s_q = \prod_{i=1}^{i=t} s_{p_i} s_\ell = +1$, and Lemma 2.2 ensures the existence of real number field $\mathbb{K}$, for all non-zero positive integer t, having as conductor $f_{\mathbb{K}}$ and as quadratic subfield k.

1. If $(\frac{p_i}{\ell}) = -1$, for all $i = 1, \ldots, t$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$ and the prime ideals $\mathfrak{p}_i$, $i = 1, \ldots, t$, where $\mathfrak{p}_i$ is the prime ideal of k above p_i, this implies that the number of prime ideals of k ramifying in $\mathbb{K}$ is $\mu = t + 1$. Hence, for all, $i = 1, \ldots, t$, we have

$$\left(\frac{-1,\, d}{\mathfrak{p}_i}\right) = \left[\frac{-1}{\mathfrak{p}_i}\right] = \left(\frac{1}{p_i}\right) = 1 \text{ and } \left(\frac{\epsilon_0,\, d}{\mathfrak{p}_i}\right) = \left[\frac{\epsilon_0}{\mathfrak{p}_i}\right] = \left(\frac{-1}{p_i}\right) = 1.$$

Since $\left(\frac{-1,\,d}{(\sqrt{\ell})}\right)=\left(\frac{\epsilon_0,\,d}{(\sqrt{\ell})}\right)=1$, so $r^*=2$, from which we infer that: $r_2(H)=\mu+r^*-3=t+1+2-3=t$.

2. If $(\frac{p_i}{\ell})=1$, for all $i=1,\dots,t$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, and the prime ideals $\wp_i$ and $\bar{\wp}_i$ with $p_i\mathcal{O}_k=\wp_i\bar{\wp}_i$, $i=1,\dots,t$, in this case $\mu=2t+1$. Hence, for all, $i=1,\dots,t$, we have

$$\left(\frac{-1,d}{\wp_i}\right)=\left(\frac{-1,d}{\bar{\wp}_i}\right)=\left[\frac{-1}{\wp_i}\right]=\left(\frac{-1}{p_i}\right)=1,$$
$$\left(\frac{\epsilon_0,d}{\wp_i}\right)=\left(\frac{\epsilon_0,d}{\bar{\wp}_i}\right)=\left[\frac{\epsilon_0}{\wp_i}\right]=\left(\frac{\epsilon_0,p}{\wp_i}\right)=\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4 \text{(see [2]).}$$

Using the previous results, we get the table:

Unit\ Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_i$
-1	$+$	$+$	$+$
ϵ_0	$+$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$
$-\epsilon_0$	$+$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$

From which we infer that:

a. If $\left(\frac{p_i}{\ell}\right)_4\neq\left(\frac{\ell}{p_i}\right)_4$ for at least one $i\in\{1,\dots,t\}$, then $r^*=1$ and $r_2(H)=\mu+r^*-3=2t+1+1-3=2t-1$.

b. If $\left(\frac{p_i}{\ell}\right)_4=\left(\frac{\ell}{p_i}\right)_4$ for all $i\in\{1,\dots,t\}$, then $r^*=2$ and $r_2(H)=\mu+r^*-3=2t+1+2-3=2t$.

If $n=\prod_{i=1}^{i=t_1}p_i\prod_{j=1}^{j=t_2}q_j$ with $p_i\equiv q_j\equiv 1 \pmod 4$ and $\left(\frac{p_i}{\ell}\right)=-\left(\frac{q_j}{\ell}\right)=-1$ for all $i\in\{1,\dots,t_1\}$ and for all $j\in\{1,\dots,t_2\}$, then according to the two cases above we have:

a. If $\left(\frac{q_j}{\ell}\right)_4\neq\left(\frac{\ell}{q_j}\right)_4$ for at least one $j\in\{1,\dots,t_2\}$, then $r^*=1$ and $r_2(H)=\mu+r^*-3=t_1+2t_2+1+1-3=t_1+2t_2-1$.

b. If $\left(\frac{q_j}{\ell}\right)_4=\left(\frac{\ell}{q_j}\right)_4$ for all $j\in\{1,\dots,t_2\}$, then $r^*=2$ and $r_2(H)=\mu+r^*-3=t_1+2t_2+1+2-3=t_1+2t_2$.

□

4.4. Case $n=2\prod_{i=1}^{t}p_i$, and for all i, $p_i\equiv 1 \pmod 4$

Theorem 4.7 *Let $\mathbb{K}=\mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell\equiv 1 \pmod 8$ is a positive prime integer, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of $k=\mathbb{Q}(\sqrt{\ell})$. Let $n=2\prod_{i=1}^{i=t}p_i$ with $p_i\equiv 1 \pmod 4$ for all $i\in\{1,\dots,t\}$ and t is a positive integer.*

1. *Assume, for all i, $(\frac{p_i}{\ell})=-1$, then:*

   a. *If $\left(\frac{2}{\ell}\right)_4=(-1)^{\frac{\ell-1}{8}}$, then $r_2(H)=t+2$.*

   b. *If $\left(\frac{2}{\ell}\right)_4\neq(-1)^{\frac{\ell-1}{8}}$, then $r_2(H)=t+1$.*

2. *Assume, for all* i, $(\frac{p_i}{\ell})=1$, *then*

   a. *If* $\left(\frac{2}{\ell}\right)_4=(-1)^{\frac{\ell-1}{8}}$ *and* $\left(\frac{p_i}{\ell}\right)_4=\left(\frac{\ell}{p_i}\right)_4$ *for all* $i\in\{1,\ldots,t\}$, *then* $r_2(H)=2t+2$.

   b. *If* $\left(\frac{2}{\ell}\right)_4\neq(-1)^{\frac{\ell-1}{8}}$ *or* $\left(\frac{p_i}{\ell}\right)_4\neq\left(\frac{\ell}{p_i}\right)_4$ *for at least one* $i\in\{1,\ldots,t\}$, *then* $r_2(H)=2t+1$.

Moreover, if $n=2\prod_{i=1}^{i=t_1}p_i\prod_{j=1}^{j=t_2}q_j$ *with* $p_i\equiv q_j\equiv 1 \pmod 4$ *and* $(\frac{p_i}{\ell})=-(\frac{q_j}{\ell})=-1$ *for all* $i\in\{1,\ldots,t_1\}$ *and for all* $j\in\{1,\ldots,t_2\}$, *then:*

a. *If* $\left(\frac{2}{\ell}\right)_4=(-1)^{\frac{\ell-1}{8}}$ *and* $\left(\frac{q_j}{\ell}\right)_4=\left(\frac{\ell}{q_j}\right)_4$ *for all* $j\in\{1,\ldots,t_2\}$, *then* $r_2(H)=t_1+2t_2+2$.

b. *If* $\left(\frac{2}{\ell}\right)_4\neq(-1)^{\frac{\ell-1}{8}}$ *or* $\left(\frac{q_j}{\ell}\right)_4\neq\left(\frac{\ell}{q_j}\right)_4$ *for at least one* $j\in\{1,\ldots,t_2\}$, *then* $r_2(H)=t_1+2t_2+1$.

Proof For $n=2\prod_{i=1}^{i=t}p_i$, with $p_i\equiv 1 \pmod 4$ for all $i=1,\ldots,t$, we have, according to Lemma 2.4, $\mathbb{K}=\mathbb{Q}(\sqrt{2a(\ell+b\sqrt{\ell}})=\mathbb{Q}(\sqrt{a(\ell+c\sqrt{\ell}})$ with $a=\frac{n}{2}=\prod_{i=1}^{i=t}p_i$. As $c\equiv 1 \pmod 2$ and $a=\frac{n}{2}=\prod_{i=1}^{i=t}p_i\equiv 1 \pmod 4$, so, by Theorem 2.1, $f_{\mathbb{K}}=2^3a\ell=2^3\ell\prod_{i=1}^{i=t}p_i\equiv 0 \pmod 8$, this implies, by Lemma 2.2, that there are as many real cyclic fields as imaginary ones $\mathbb{K}$ having as a conductor $f_{\mathbb{K}}$ and as quadratic subfield k.

1. If $(\frac{p_i}{\ell})=-1$, for all $i=1,\ldots,t$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\mathfrak{z}_i$, $i\in\{1,2\}$, and the prime ideals $\mathfrak{p}_i$, $i=1,\ldots,t$, where $\mathfrak{p}_i$ is the prime ideal of k above p_i and $2\mathcal{O}_k=\mathfrak{z}_1\mathfrak{z}_2$ is the decomposition of 2 in k, this implies that the number of prime ideals of k ramifying in $\mathbb{K}$ is $\mu=t+3$. Therefore,

$$\left(\frac{-1,d}{\mathfrak{z}_1}\right)=\left(\frac{-1,d}{\mathfrak{z}_2}\right)=\left(\frac{-1,2\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{-1,2}{\mathfrak{z}_1}\right)\left(\frac{-1,\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{-1,2}{2}\right)=1,$$
$$\left(\frac{\epsilon_0,d}{\mathfrak{z}_1}\right)=\left(\frac{\epsilon_0,d}{\mathfrak{z}_2}\right)=\left(\frac{\epsilon_0,2\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{\epsilon_0,2}{\mathfrak{z}_1}\right)\left(\frac{\epsilon_0,\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{\epsilon_0,2}{\mathfrak{z}_1}\right)=\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}},$$

   indeed $\mathfrak{z}_i$ don't ramifie in $\mathbb{Q}(\sqrt{\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}})$.

   Using the above results, we get the table:

Unit \ Character	$\sqrt{\ell}$	$\mathfrak{p}_i$	$\mathfrak{z}_i$
-1	$+$	$+$	$+$
ϵ_0	$+$	$+$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\epsilon_0$	$+$	$+$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

   a. If $\left(\frac{2}{\ell}\right)_4=(-1)^{\frac{\ell-1}{8}}$, then $r^*=2$, which implies that: $r_2(H)=\mu+r^*-3=t+3+2-3=t+2$.

   b. If $\left(\frac{2}{\ell}\right)_4\neq(-1)^{\frac{\ell-1}{8}}$, then $r^*=1$, which implies that: $r_2(H)=\mu+r^*-3=t+3+1-3=t+1$.

2. If $(\frac{p_i}{\ell})=1$, for all $i=1,\ldots,t$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\mathfrak{z}_i$, $i\in\{1,2\}$ and the prime ideals $\wp_i$ and $\bar{\wp}_i$ with $p_i\mathcal{O}_k=\wp_i\bar{\wp}_i$, $i=1,\ldots,t$, in this case $\mu=2t+3$. So using the results of the above cases we get:

Unit\ Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_i$	$\mathfrak{2}_i$
-1	$+$	$+$	$+$	$+$
ϵ_0	$+$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\epsilon_0$	$+$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

a. If $\left(\frac{2}{\ell}\right)_4 = (-1)^{\frac{\ell-1}{8}}$ and $\left(\frac{p_i}{\ell}\right)_4 = \left(\frac{\ell}{p_i}\right)_4$ for all $i \in \{1,\dots,t\}$ then $r^* = 2$, so: $r_2(H) = \mu + r^* - 3 = 2t+3+2-3 = 2t+2$.

b. If $\left(\frac{2}{\ell}\right)_4 \neq (-1)^{\frac{\ell-1}{8}}$ or $\left(\frac{p_i}{\ell}\right)_4 \neq \left(\frac{\ell}{p_i}\right)_4$ for at least one $i \in \{1,\dots,t\}$ then $r^* = 1$, which implies that: $r_2(H) = \mu + r^* - 3 = 2t+3+1-3 = 2t+1$.

Finally, if $n = 2\prod_{i=1}^{i=t_1} p_i \prod_{j=1}^{j=t_2} q_j$ with $p_i \equiv q_j \equiv 1 \pmod 4$ and $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$ for all $i \in \{1,\dots,t_1\}$ and for all $j \in \{1,\dots,t_2\}$, then according to the two cases above:

a. If $\left(\frac{2}{\ell}\right)_4 = (-1)^{\frac{\ell-1}{8}}$ and $\left(\frac{q_j}{\ell}\right)_4 = \left(\frac{\ell}{q_j}\right)_4$ for all $j \in \{1,\dots,t_2\}$, then $r^* = 2$ and $r_2(H) = t_1 + 2t_2 + 2\cdot$

b. If $\left(\frac{2}{\ell}\right)_4 \neq (-1)^{\frac{\ell-1}{8}}$ or $\left(\frac{q_j}{\ell}\right)_4 \neq \left(\frac{\ell}{q_j}\right)_4$ for at least one $j \in \{1,\dots,t_2\}$, then $r^* = 1$ and $r_2(H) = t_1 + 2t_2 + 1\cdot$

□

4.5. Case $n = \delta\prod_{i=1}^{i=t} p_i$ with t odd, and for all i, $p_i \equiv 3 \pmod 4$

Theorem 4.8 *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell \equiv 1 \pmod 8$ is a positive prime integer, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of $k = \mathbb{Q}(\sqrt{\ell})$. Assume $n = \delta\prod_{i=1}^{i=t} p_i$, $p_i \equiv 3 \pmod 4$ for all $i = 1,\dots,t$ and t is a positive odd integer.*

1. *If, for all i, $\left(\frac{p_i}{\ell}\right) = -1$, then $r_2(H) = t$.*

2. *If, for all i, $\left(\frac{p_i}{\ell}\right) = 1$, then $r_2(H) = 2t$.*

Moreover, if $n = \prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$, where $p_i \equiv q_j \equiv 3 \pmod 4$ and $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, for all $i \in \{1,\dots,t_1\}$, and for all $j \in \{1,\dots,t_2\}$ with $t_1 + t_2$ is odd, then $r_2(H) = t_1 + 2t_2$.

Proof Assume $n = \prod_{i=1}^{i=t} p_i$, since t is odd, we have $a = \prod_{i=1}^{i=t} p_i \equiv 3 \pmod 4$, so $a + b \equiv 3 + 0 \equiv 3 \pmod 4$. Thus, by Theorem 2.1, $f_{\mathbb{K}} = 2^2 a\ell = 2^2\ell\prod_{i=1}^{i=t} p_i \not\equiv 0 \pmod 8$. But $S = s_2 s_\ell \prod_{i=1}^{i=t} s_{p_i} = +1$. Indeed for $i = 1,\dots,t$, we have $e_{p_i} = 2$ and $p_i \equiv 3 \pmod 4$, then $\frac{p_i - 1}{2} \equiv 1 \pmod 2$, so $s_{p_i} = -1$. Therefore, the real number field $\mathbb{K}$ having as a conductor $f_{\mathbb{K}}$ and as quadratic subfield k exists.
If $n = 2\prod_{i=1}^{i=t} p_i$, $p_i \equiv 3 \pmod 4$ for all $i = 1,\dots,t$, with t odd, then by Lemma 2.4 we get $\mathbb{K} = \mathbb{Q}(\sqrt{2a(\ell + b\sqrt{\ell})}) = \mathbb{Q}(\sqrt{a(\ell + c\sqrt{\ell})})$ with $a = \frac{n}{2} = \prod_{i=1}^{i=t} p_i$. As $c \equiv 1 \pmod 2$ and $\ell \equiv 1 \pmod 4$, so $f_{\mathbb{K}} = 2^3\ell\prod_{i=1}^{i=t} p_i \equiv 0 \pmod 8$, then there exist as many real cyclic quartic number fields as imaginary ones sharing the conductor $f_{\mathbb{K}}$ and the quadratic subfield k.

1. If $\left(\frac{p_i}{\ell}\right)=-1$, for all $i\in\{1,\dots,t\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{z}_i$, $i\in\{1,2\}$, $\mathfrak{p}_i$ and $(\sqrt{\ell})$, where $\mathfrak{p}_i$ is the prime ideal of k above p_i and $2\mathcal{O}_k=\mathfrak{z}_1\mathfrak{z}_2$ the decomposition of 2 in k.

   a. For the case $n=\prod_{i=1}^{i=t}p_i$, $i\in\{1,\dots,t\}$, we have:

$$\left(\frac{-1,d}{\mathfrak{p}_i}\right)=\left[\frac{-1}{\mathfrak{p}_i}\right]=\left(\frac{1}{p_i}\right)=1,\qquad \left(\frac{\epsilon_0,d}{\mathfrak{p}_i}\right)=\left[\frac{\epsilon_0}{\mathfrak{p}_i}\right]=\left(\frac{-1}{p_i}\right)=-1,$$
$$\left(\frac{-1,d}{\mathfrak{z}_1}\right)=\prod_{i=1}^{i=t}\left(\frac{-1,p_i}{\mathfrak{z}_1}\right)\left(\frac{-1,\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\prod_{i=1}^{i=t}\left(\frac{-1,p_i}{\mathfrak{z}_1}\right)=(-1)^t=-1,$$
$$\left(\frac{\epsilon_0,d}{\mathfrak{z}_1}\right)=\left(\frac{\epsilon_0,-1}{\mathfrak{z}_1}\right)\left(\frac{\epsilon_0,-\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{\epsilon_0,-1}{\mathfrak{z}_1}\right)=1,\ \text{since } \mathfrak{z}_1 \text{ don't ramify in } \mathbb{Q}(\sqrt{-\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}).$$

   Using the above results and the product formula, we get the following table:

Unit \ Character	$(\sqrt{\ell})$	$\mathfrak{p}_i$	$\mathfrak{z}_1$	$\mathfrak{z}_2$
-1	$+$	$+$	$-$	$-$
ϵ_0	$+$	$-$	$+$	$-$
$-\epsilon_0$	$+$	$-$	$-$	$+$

   b. For the case $n=2\prod_{i=1}^{i=t}p_i$, $i\in\{1,\dots,t\}$, we have:

$$\left(\frac{-1,d}{\mathfrak{z}_1}\right)=\left(\frac{-1,2\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{-1,2}{\mathfrak{z}_1}\right)\prod_{i=1}^{i=t}\left(\frac{-1,p_i}{\mathfrak{z}_1}\right)\left(\frac{-1,\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=(-1)^t=-1,$$
$$\left(\frac{\epsilon_0,d}{\mathfrak{z}_1}\right)=\left(\frac{-1,2\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{\epsilon_0,2}{\mathfrak{z}_1}\right)\left(\frac{\epsilon_0,-\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}.$$

   So we need just to change the two columns in the previous table:

$\mathfrak{z}_1$	$\mathfrak{z}_2$
$-$	$-$
$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

   Hence for the two cases we have: $r^*=0$, which implies: $r_2(H)=\mu+r^*-3=t+3+0-3=t$.

2. If $\left(\frac{p_i}{\ell}\right)=1$ for all $i\in\{1,\dots,t\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{z}_i$, $i\in\{1,2\}$, $(\sqrt{\ell})$, $\wp_i$ and $\bar{\wp}_i$, where $p_i\mathcal{O}_k=\wp_i\bar{\wp}_i$, $i=1,\dots,t$ and $2\mathcal{O}_k=\mathfrak{z}_1\mathfrak{z}_2$.

   a. For the case $n=\prod_{i=1}^{i=t}p_i$, $i\in\{1,\dots,t\}$, we have for all $i\in\{1,\dots,t\}$:

$$\left(\frac{-1,d}{\wp_i}\right)=\left(\frac{-1,d}{\bar{\wp}_i}\right)=\left[\frac{-1}{\wp_i}\right]=\left(\frac{-1}{p_i}\right)=-1\quad\text{and}\quad\left(\frac{\epsilon_0,d}{\wp_i}\right)=\left[\frac{\epsilon_0}{\wp_i}\right].$$

   To compute the last unity, put $p_i^{h_0}=\wp_i\bar{\wp}_i$ and $\wp_i=a_i+b_i\sqrt{\ell}$ and $\bar{\wp}_i=a_i-b_i\sqrt{\ell}$, for all i. According to [7] we have $\left[\frac{\epsilon_0\sqrt{\ell}}{\wp_i}\right]=\left[\frac{\epsilon_0\sqrt{\ell}}{\bar{\wp}_i}\right]=\left(\frac{p_i}{\ell}\right)_4$. Thus $\left[\frac{\epsilon_0}{\wp_i}\right]=\left(\frac{p_i}{\ell}\right)_4\left[\frac{\sqrt{\ell}}{\wp_i}\right]$. On the other hand,

$$\left[\frac{\sqrt{\ell}}{\wp_i}\right]=\left[\frac{b_i^2\sqrt{\ell}}{\wp_i}\right]=\left[\frac{b_i(-a_i+a_i+b_i\sqrt{\ell})}{\wp_i}\right]=\left[\frac{-a_ib_i}{\wp_i}\right]=-\left(\frac{a_i}{p_i}\right)\left(\frac{b_i}{p_i}\right).$$

As $p_i^{h_0} = a_i^2 - b_i^2\ell$, so $b_i^2\ell \equiv a_i^2 \pmod{p_i}$. Since ℓ and p_i are relatively prime, then $b_i^2\ell^2 \equiv \ell a_i^2 \pmod{p_i}$, so $\left(\frac{b_i}{p_i}\right) = \left(\frac{b_i\ell}{p_i}\right) = \left(\frac{b_i^2\ell^2}{p_i}\right)_4 = \left(\frac{\ell a_i^2}{p_i}\right)_4 = \left(\frac{\ell}{p_i}\right)_4\left(\frac{a_i}{p_i}\right)$. Finally,

$$\left[\frac{\epsilon_0}{\wp_i}\right] = -\left(\frac{p_i}{\ell}\right)_4\left(\frac{a_i}{p_i}\right)\left(\frac{b_i}{p_i}\right) = -\left(\frac{p_i}{\ell}\right)_4\left(\frac{a_i}{p_i}\right)\left(\frac{\ell}{p_i}\right)_4\left(\frac{a_i}{p_i}\right) = -\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4.$$

Proceeding similarly, we get $\left[\frac{\epsilon_0}{\bar{\wp}_i}\right] = \left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$ using the fact $\left[\frac{\sqrt{\ell}}{\bar{\wp}_i}\right] = -\left[\frac{-b_i^2\sqrt{\ell}}{\bar{\wp}_i}\right]$.

Then we get the following table by using the above results and the product formula.

Unit/Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_1$	$\mathfrak{z}_1$	$\mathfrak{z}_2$
-1	$+$	$-$	$-$	$-$	$-$
ϵ_0	$+$	$-\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$+$	$-$
$-\epsilon_0$	$+$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-$	$+$

b. For the case $n = 2\prod_{i=1}^{i=t} p_i$, $i \in \{1, \dots, t\}$, we need just to change the last two columns in the previous table by the following two ones:

$\mathfrak{z}_1$	$\mathfrak{z}_2$
$-$	$-$
$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

Hence for the two cases $-1, \epsilon_0$ and $-\epsilon_0$ are not norms in $\mathbb{K}$, thus $r_2(H) = \mu + r^* - 3 = 2t + 3 + 0 - 3 = 2t$.

Finally, if $n = \delta\prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$, with $p_i \equiv q_j \equiv 3 \pmod 4$ and $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$ for all $i \in \{1, \dots, t_1\}$ and for all $j \in \{1, \dots, t_2\}$ with $t_1 + t_2$ is odd, then, according to the two cases above, there are $t_1 + 2t_2 + 3$ prime ideals of k which ramify in $\mathbb{K}$ and $r^* = 0$. Thus, $r_2(H) = t_1 + 2t_2 + 3 + 0 - 3 = t_1 + 2t_2$. □

4.6. Case $n = \delta\prod_{i=1}^{t} p_i$, with t even, and for all i $p_i \equiv 3 \pmod 4$

Theorem 4.9 *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell \equiv 1 \pmod 8$ is a positive prime integer, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of $k = \mathbb{Q}(\sqrt{\ell})$. Let $n = \delta\prod_{i=1}^{i=t} p_i$ with $p_i \equiv 3 \pmod 4$ for all $i \in \{1, \dots, t\}$ and t is an even positive integer ($\delta = 1$ or 2).*

1. *If, for all i, $(\frac{p_i}{\ell}) = -1$, then $r_2(H) = t - 1 + 2(\delta - 1)$.*

2. *If, for all i, $(\frac{p_i}{\ell}) = 1$, then $r_2(H) = 2t - 2 + 2(\delta - 1)$.*

Moreover, for $n = \delta\prod_{i=1}^{i=t_1} p_i \prod_{j=1}^{j=t_2} q_j$ with $(\frac{p_i}{\ell}) = -(\frac{q_j}{\ell}) = -1$ and $p_i \equiv q_j \equiv 3 \pmod 4$ for all $i \in \{1, \dots, t_1\}$ and for all $j \in \{1, \dots, t_2\}$ with $t_1 + t_2$ is even, we have $r_2(H) = t_1 + 2t_2 - 2 + 2(\delta - 1)$.

Proof For $n = \prod_{i=1}^{i=t} p_i \equiv 1 \pmod 4$, with $p_i \equiv 3 \pmod 4$ for all $i = 1, \dots, t$, it is easy to see that $f_{\mathbb{K}} = a\ell = \prod_{i=1}^{i=t} p_i\ell \not\equiv 0 \pmod 8$, and for $n = 2\prod_{i=1}^{i=t} p_i$, we prove that $f_{\mathbb{K}} = 2^3\ell\prod_{i=1}^{i=t} p_i \equiv 0 \pmod 8$. We proceed as in the cases above to prove, for any even nonzero positive integer t, the existence of real number fields $\mathbb{K}$ having $f_{\mathbb{K}}$ as a conductor and k as a quadratic subfield.

1. Assume $(\frac{p_i}{\ell})=-1$ for all i.

• For $\delta=1$, the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$ and $\mathfrak{p}_i$, the prime ideals of k above p_i, thus $\mu=t+1$. As above we get the following table:

Unit \ Character	$\sqrt{\ell}$	$\mathfrak{p}_i$
-1	$+$	$+$
ϵ_0	$+$	$-$
$-\epsilon_0$	$+$	$-$

Then $r^*=1$, from which we infer that: $r_2(H)=\mu+r^*-3=t+1+1-3=t-1$.

• For $\delta=2$, the prime ideals of k ramifying in $\mathbb{K}$ are $(\sqrt{\ell})$, $\mathfrak{z}_i$, $i\in\{1,2\}$, and $\mathfrak{p}_i$, the prime ideal of k above p_i, thus $\mu=t+3$. We have: $\left(\dfrac{-1,\, d}{\mathfrak{z}_1}\right)=\left(\dfrac{-1,\, d}{\mathfrak{z}_2}\right)=\left(\dfrac{-1,\, 2\prod_{i=1}^{i=t}p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right)=(-1)^t=1$. Using the above results, we get the following table.

Unit/Character	$\sqrt{\ell}$	$\mathfrak{z}_i$	$\mathfrak{p}_i$
-1	$+$	$+$	$+$
ϵ_0	$+$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$-$
ϵ_0	$+$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$-$

Hence $r^*=1$, which implies that: $r_2(H)=\mu+r^*-3=t+3+1-3=t+1$.

2. Assume $(\frac{p_i}{\ell})=1$ for all i.

• For $\delta=1$, the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\wp_i$ and $\bar{\wp}_i$ with $p_i\mathcal{O}_k=\wp_i\bar{\wp}_i$, thus $\mu=2t+1$. As above, we have the following table:

Unit\ Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_i$
-1	$+$	$-$	$-$
ϵ_0	$+$	$-\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$
$-\epsilon_0$	$+$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$

So $r^*=0$, from which we infer that: $r_2(H)=\mu+r^*-3=2t+1+0-3=2t-2$.

• For $\delta=2$, then the prime ideals of k ramifying in $\mathbb{K}$ are $(\sqrt{\ell})$, $\mathfrak{z}_i$, $i\in\{1,2\}$, $\wp_i$ and $\bar{\wp}_i$, where $p\mathcal{O}_k=\wp_i\bar{\wp}_i$ and $2\mathcal{O}_k=\mathfrak{z}_1\mathfrak{z}_2$. Using the above results we get:

Unit/Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_i$	$\mathfrak{z}_i$
-1	$+$	$-$	$-$	$+$
ϵ_0	$+$	$-\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\epsilon_0$	$+$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

Hence $r^*=0$, so: $r_2(H)=\mu+r^*-3=2t+3+0-3=2t$.

According to previous cases, if $n = \delta\prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$ with t_1+t_2 even and $p_i \equiv q_j \equiv 3 \pmod 4$, $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, for all $i \in \{1,\dots,t_1\}$ and for all $j \in \{1,\dots,t_2\}$, then $r^* = 0$ and $r_2(H) = t_1+2t_2-2+2(\delta-1)$. □

4.7. Case $n = \delta\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, where $p_i \equiv -q_j \equiv 1 \pmod 4$ for all (i,j) and s is odd

Theorem 4.10 *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell \equiv 1 \pmod 8$ is a positive prime integer, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of the quadratic subfield $k = \mathbb{Q}(\sqrt{\ell})$. Assume $n = \delta\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ with s odd, where $p_i \equiv -q_j \equiv 1 \pmod 4$, for all $(i,j) \in \{1,\dots,t\}\times\{1,\dots,s\}$, are prime integers. Denote by h the number of prime ideals of k above all the $p_i's$, $i \in \{1,\dots,t\}$.*

1. *If, for all j, $\left(\frac{q_j}{\ell}\right) = -1$, then $r_2(H) = h+s$.*
2. *If, for all j, $\left(\frac{q_j}{\ell}\right) = 1$, then $r_2(H) = h+2s$.*

Moreover, if $\prod_{j=1}^{j=s} q_j = \prod_{j'=1}^{j'=s_1} q_{j'} \prod_{j=1}^{j=s_2} q_j$ with $\left(\frac{q_{j'}}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, for all $j' = 1,\dots,s_1$ and $j = 1,\dots,s_2$, with s_1+s_2 is odd, then $r_2(H) = h+s_1+2s_2$.

Proof Assume $n = \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, where $p_i \equiv -q_j \equiv 1 \pmod 4$, for $i = 1,\dots,t$ and $j = 1,\dots,s$. As $b \equiv 0 \pmod 2$ and $a = n \equiv 3 \pmod 4$ (since s is odd), so $a+b \equiv 3+0 \equiv 3 \pmod 4$. Thus $f_{\mathbb{K}} = 2^2\ell a \not\equiv 0 \pmod 8$. But $S = s_2 s_\ell \prod_{i=1}^{i=t} s_{p_i} \prod_{j=1}^{j=s} s_{q_j} = +1$, this implies that the real number $\mathbb{K}$ exists.

Assume $n = 2\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, where $p_i \equiv -q_j \equiv 1 \pmod 4$, for $i = 1,\dots,t$ and $j = 1,\dots,s$. Then $\mathbb{K} = \mathbb{Q}(\sqrt{2a(\ell+b\sqrt{\ell}}) = \mathbb{Q}(\sqrt{a(\ell+c\sqrt{\ell}})$ and $a = \frac{n}{2}$. As $c \equiv 1 \pmod 2$, so $f_{\mathbb{K}} = 2^3\ell a \equiv 0 \pmod 8$, and there exists as many real cyclic quartic number fields as imaginary ones.

1. If $\left(\frac{p_i}{\ell}\right) = \left(\frac{q_j}{\ell}\right) = -1$, for all $i = 1,\dots,t$ and $j = 1,\dots,s$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{z}_i$, $i \in \{1;2\}$, $(\sqrt{\ell})$, $\mathfrak{p}_i$, $\mathfrak{q}_j$ and $2\mathcal{O}_k = \mathfrak{z}_1\mathfrak{z}_2$.

   a. For the case $n = \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ we have:

$$\begin{aligned}\left(\frac{-1,d}{\mathfrak{z}_1}\right) &= \prod_j^s \left(\frac{-1,q_j}{\mathfrak{z}_1}\right)\left(\frac{-1,\prod_i^t p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right) = (-1)^s = -1,\\ \left(\frac{\epsilon_0,d}{\mathfrak{z}_1}\right) &= \left(\frac{\epsilon_0,\prod_j^s q_i\prod_i^t p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right) = \left(\frac{\epsilon_0,-1}{\mathfrak{z}_1}\right)\left(\frac{\epsilon_0,-\prod_j^s q_i\prod_i^t p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right) = 1,\end{aligned}$$

   indeed $\mathfrak{z}_1$ don't ramify in $\mathbb{Q}\left(\sqrt{-\prod_j^s q_i\prod_i^t p_i\epsilon_0\sqrt{\ell}}\right)$ since $-\prod_j^s q_i\prod_i^t p_i \equiv 1 \pmod 4$. The other characters are computed as above; using the product formula, we get the following table:

Unit/Character	$\sqrt{\ell}$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{p}_i$	$\mathfrak{q}_j$
-1	+	−	−	+	+
ϵ_0	+	−	+	+	−
$-\epsilon_0$	+	−	+	+	−

b. For the case $n = 2\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ we have:

$$\left(\frac{-1,d}{\mathfrak{z}_1}\right) = \left(\frac{-1,2}{\mathfrak{z}_1}\right)\prod_j^s\left(\frac{-1,q_j}{\mathfrak{z}_1}\right)\left(\frac{-1,\prod_i^t p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right) = -1,$$
$$\left(\frac{\epsilon_0,d}{\mathfrak{z}_1}\right) = \left(\frac{\epsilon_0,2}{\mathfrak{z}_1}\right)\left(\frac{\epsilon_0,-1}{\mathfrak{z}_1}\right)\left(\frac{\epsilon_0,-\prod_j^s q_i\prod_i^t p_i\epsilon_0\sqrt{\ell}}{\mathfrak{z}_1}\right) = \left(\frac{\epsilon_0,2}{\mathfrak{z}_1}\right) = \left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$$

So we need just to change the two columns in the previous table:

$\mathfrak{z}_1$	$\mathfrak{z}_2$
$-$	$-$
$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

Hence, for the two cases we have: $r^* = 0$, so $r_2(H) = \mu + r^* - 3 = t + s + 3 + 0 - 3 = t + s$.

2. If $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = 1$, for all $i = 1, \ldots, t$ and $j = 1, \ldots, s$, then, the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{z}_i$, $i \in \{1;2\}$, $(\sqrt{\ell})$, $\wp_i$, $\bar{\wp}_i$ and $\mathfrak{q}_j$, where $p_i\mathcal{O}_k = \wp_i\bar{\wp}_i$ and $2\mathcal{O}_k = \mathfrak{z}_1\mathfrak{z}_2$.

a. For the case $n = \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, proceeding as above, we obtain:

Unit/Character	$\sqrt{\ell}$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\wp_i$	$\bar{\wp}_i$	$\mathfrak{q}_j$
-1	$+$	$-$	$-$	$+$	$+$	$+$
ϵ_0	$+$	$+$	$-$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-$
$-\epsilon_0$	$+$	$-$	$+$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-$

b. For the case $n = 2\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ we need just to change the two columns in the previous table:

$\mathfrak{z}_1$	$\mathfrak{z}_2$
$-$	$-$
$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

Hence for the two cases $r^* = 0$, so $r_2(H) = \mu + r^* - 3 = 2t + s + 3 + 0 - 3 = 2t + s$.

3. If $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$; for all $i = 1, \ldots, t$ and $j = 1, \ldots, s$, then, the prime ideals of k, which ramify in $\mathbb{K}$ are $\mathfrak{z}_i$, $i \in \{1;2\}$, $(\sqrt{\ell})$, $\mathfrak{p}_i$, ρ_j, and $\bar{\rho}_j$, where $q_j\mathcal{O}_k = \rho_j\bar{\rho}_j$ and $2\mathcal{O}_k = \mathfrak{z}_1\mathfrak{z}_2$.

a. For the case $n = \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, proceeding as above, we obtain:

Unit/Character	$\sqrt{\ell}$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{p}_i$	ρ_j	$\bar{\rho}_j$
-1	$+$	$-$	$-$	$+$	$-$	$-$
ϵ_0	$+$	$+$	$-$	$+$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$
$-\epsilon_0$	$+$	$-$	$+$	$+$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$

b. For the case $n = 2\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, we need just to change the two columns in the previous table:

$\mathfrak{2}_1$	$\mathfrak{2}_2$
$-$	$-$
$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

For the two cases we have: $r^*=0$, so $r_2(H)=\mu+r^*-3=t+2s+2+0-3=t+2s-1$.

4. If $\left(\frac{p_i}{\ell}\right)=\left(\frac{q_j}{\ell}\right)=1$, for all $i=1,\ldots,t$ and $j=1,\ldots,s$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{2}_i$, $i\in\{1;2\}$, $(\sqrt{\ell})$, $\wp_i$, $\bar{\wp}_i$, ρ_j and $\bar{\rho}_j$, where $p_i\mathcal{O}_k=\wp_i\bar{\wp}_i$ and $2\mathcal{O}_k=\mathfrak{2}_1\mathfrak{2}_2$, $q_j\mathrm{O}_k=\rho_j\bar{\rho}_j$.

   (a) For the case $n=\prod_{i=1}^{i=t}p_i\prod_{j=1}^{j=s}q_j$, as above we get:

Unit/Char	$\sqrt{\ell}$	$\mathfrak{2}_1$	$\mathfrak{2}_2$	$\wp_i$	$\bar{\wp}_i$	ρ_j	$\bar{\rho}_j$
-1	$+$	$-$	$-$	$+$	$+$	$-$	$-$
ϵ_0	$+$	$+$	$-$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$
$-\epsilon_0$	$+$	$-$	$+$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$

   (b) For the case $n=2\prod_{i=1}^{i=t}p_i\prod_{j=1}^{j=s}q_j$ we need just to change the two columns in the previous table:

$\mathfrak{2}_1$	$\mathfrak{2}_2$
$-$	$-$
$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$
$-\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$	$\left(\frac{2}{\ell}\right)_4(-1)^{\frac{\ell-1}{8}}$

Hence for the two cases we have: $r^*=0$, so $r_2(H)=\mu+r^*-3=2t+2s+3+0-3=2t+2s$.

In general, if $\prod_{j=1}^{j=s}q_j=\prod_{j'=1}^{j'=s_1}q_{j'}\prod_{j=1}^{j=s_2}q_j$ with $\left(\frac{q_{j'}}{\ell}\right)=-\left(\frac{q_j}{\ell}\right)=-1$, for all $j'=1,\ldots,s_1$ and $j=1,\ldots,s_2$, with s_1+s_2 is odd, then $r^*=0$ and $r_2(H)=h+s_1+2s_2+3+0-3=h+s_1+2s_2$. □

4.8. Case $n=\delta\prod_{i=1}^{i=t}p_i\prod_{j=1}^{j=s}q_j$, where $p_i\equiv -q_j\equiv 1 \pmod 4$ for all (i,j) and s is even

Theorem 4.11 *Let $\mathbb{K}=\mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell\equiv 1 \pmod 8$ is a positive prime integer, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of $k=\mathbb{Q}(\sqrt{\ell})$. Assume $n=\prod_{i=1}^{i=t}p_i\prod_{j=1}^{j=s}q_j$ with s an even positive integer and $p_i\equiv -q_j\equiv 1 \pmod 4$, for all $(i,j)\in\{1,\ldots,t\}\times\{1,\ldots,s\}$ are prime integers. Denote by h the number of prime ideals in k above all the $p_i's$, $i\in\{1,\ldots,t\}$.*

1. *If, for all j, $\left(\frac{q_j}{\ell}\right)=-1$, then $r_2(H)=h+s-1+2(\delta-1)$.*

2. *If, for all j, $\left(\frac{q_j}{\ell}\right)=1$, then $r_2(H)=h+2s-2+2(\delta-1)$.*

Moreover, if $\prod_{j=1}^{j=s}q_j=\prod_{j'=1}^{j'=s_1}q_{j'}\prod_{j=1}^{j=s_2}q_j$ with $\left(\frac{q_{j'}}{\ell}\right)=-\left(\frac{q_j}{\ell}\right)=-1$, for all $j'=1,\ldots,s_1$ and $j=1,\ldots,s_2$, s_1+s_2 is even, then $r_2(H)=h+s_1+2s_2-2+2(\delta-1)$.

Proof We proceed as in the previous cases. □

5. The Case $\ell = 2$

Let $\ell = 2$ and n a square-free positive integer relatively prime to 2. Let $\mathbb{K} = k(\sqrt{n\epsilon_0\sqrt{2}})$ with $k = \mathbb{Q}(\sqrt{2})$, where ϵ_0 is the fundamental unit of k. Since $\ell = 2$ we have $f_{\mathbb{K}} = 2^3 a\ell = 2^3\ell\prod_{i=1}^{i=t} p_i$, then $f_{\mathbb{K}} \equiv 0 \pmod 8$, which implies, by Lemma 2.2, that there are as many real cyclic fields as imaginary ones having as conductor $f_{\mathbb{K}}$ and as quadratic subfield k. On the other hand, the prime ideals of k, which ramify in $\mathbb{K}$ are $(\sqrt{2})$ and the prime ideals dividing n in k. Denote by μ the number of prime ideals of k ramifying in $\mathbb{K}$.

5.1. Case $n = 1$

Theorem 5.1 *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{2}})$ be a real cyclic quartic number field, where ϵ_0 is the fundamental unit of the quadratic subfield $k = \mathbb{Q}(\sqrt{2})$ and n a square-free positive integer relatively prime to 2. If $n = 1$, then $r_2(H) = 0$.*

Proof In this case only $(\sqrt{2})$ ramifies in $\mathbb{K}$, i.e. $\mu = 1$. Using the product formula, we must have: $\left(\frac{-1,d}{\sqrt{2}}\right) = \left(\frac{\epsilon_0,d}{\sqrt{2}}\right) = 1$. Hence $r^* = 2$, which implies that: $r_2(H) = \mu + r^* - 3 = 2 + 1 - 3 = 0$. □

5.2. Case $n = \prod_{i=1}^{t} p_i$ and, for all i, $p_i \equiv 1 \pmod 4$

Theorem 5.2 *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{2}})$ be a real cyclic quartic number field, where ϵ_0 is the fundamental unit of the quadratic subfield $k = \mathbb{Q}(\sqrt{2})$ and n a square-free positive integer relatively prime to 2. Let $n = \prod_{i=1}^{i=t} p_i$ with $p_i \equiv 1 \pmod 4$ for all $i \in \{1, \ldots, t\}$ and t is a positive integer.*

1. *Assume, for all i, $(\frac{2}{p_i}) = -1$, then $r_2(H) = t$.*

2. *Assume, for all i, $(\frac{2}{p_i}) = 1$.*

   a. *If $\left(\frac{2}{p_i}\right)_4 = \left(\frac{p_i}{2}\right)_4$, for all i, then $r_2(H) = 2t$.*

   b. *If $\left(\frac{2}{p_i}\right)_4 \neq \left(\frac{p_i}{2}\right)_4$ for at least one i, then $r_2(H) = 2t - 1$.*

Moreover, if $n = \prod_{i=1}^{i=t_1} p_i \prod_{j=1}^{j=t_2} q_j$ with $\left(\frac{2}{p_i}\right) = -\left(\frac{2}{q_j}\right) = -1$ for all $i \in \{1, \ldots, t_1\}$ and for all $j \in \{1, \ldots, t_2\}$, then:

a. *If $\left(\frac{2}{q_j}\right)_4 = \left(\frac{q_j}{2}\right)_4$ for all $j \in \{1, \ldots, t_2\}$, then $r_2(H) = t_1 + 2t_2$.*

b. *If $\left(\frac{2}{q_j}\right)_4 \neq \left(\frac{q_j}{2}\right)_4$ for at least one $j \in \{1, \ldots, t_2\}$, then $r_2(H) = t_1 + 2t_2 - 1$.*

Proof

1. If $(\frac{2}{p_i}) = -1$, for all $i = 1, \ldots, t$, then $\mu = t + 1$. Denote by $\mathfrak{p}_i$ the prime ideal of k above p_i, hence, for all

$i=1,\ldots,t$, we get

$$\left(\frac{-1,\, d}{\mathfrak{p}_i}\right)=\left[\frac{-1}{\mathfrak{p}_i}\right]=\left(\frac{1}{p_i}\right)=1,\quad \left(\frac{\epsilon_0,\, d}{\mathfrak{p}_i}\right)=\left[\frac{\epsilon_0}{\mathfrak{p}_i}\right]=\left(\frac{-1}{p_i}\right)=1$$
$$\text{and } \left(\frac{-1,\, d}{\sqrt{2}}\right)=\left(\frac{\epsilon_0,\, d}{\sqrt{2}}\right)=1 \text{ by the product formula.}$$

So $r^*=2$, from which we infer that: $r_2(H)=\mu+r^*-3=t+1+2-3=t$.

2. If $(\frac{2}{p_i})=1$, for all $i=1,\ldots,t$, then $\mu=2t+1$. Let $\wp_i$ and $\bar{\wp}_i$ be the prime ideals of k above p_i, $i=1,\ldots,t$. Hence

$$\left(\frac{-1,d}{\wp_i}\right)=\left(\frac{-1,d}{\bar{\wp}_i}\right)=\left[\frac{-1}{\wp_i}\right]=\left(\frac{-1}{p_i}\right)=1, \text{ for all } i=1,\ldots,t.$$
$$\left(\frac{\epsilon_0,d}{\wp_i}\right)=\left(\frac{\epsilon_0,d}{\bar{\wp}_i}\right)=\left[\frac{\epsilon_0}{\wp_i}\right]=\left(\frac{2}{p_i}\right)_4\left(\frac{p_i}{2}\right)_4 \text{ (see [18, Proposition 5.8, p 160]).}$$

The following table is completed by using the product formula:

Unit\ Character	$\sqrt{2}$	$\wp_i$	$\bar{\wp}_i$
-1	$+$	$+$	$+$
ϵ_0	$+$	$\left(\frac{2}{p_i}\right)_4\left(\frac{p_i}{2}\right)_4$	$\left(\frac{2}{p_i}\right)_4\left(\frac{p_i}{2}\right)_4$
$-\epsilon_0$	$+$	$\left(\frac{2}{p_i}\right)_4\left(\frac{p_i}{2}\right)_4)$	$\left(\frac{2}{p_i}\right)_4\left(\frac{p_i}{2}\right)_4$

So we have to discus two cases:

a. If $\left(\frac{2}{p_i}\right)_4=\left(\frac{p_i}{2}\right)_4$ for all $i=1,\ldots,t$, then $r^*=2$, thus $r_2(H)=\mu+r^*-3=2t+1+2-3=2t$.

b. If $\left(\frac{2}{p_i}\right)_4\neq\left(\frac{p_i}{2}\right)_4$ for at least one $i=1,\ldots,t$, then $r^*=1$, thuq $r_2(H)=\mu+r^*-3=2t+1+1-3=2t-1$.

In general, if $n=\prod_{i=1}^{i=t_1}p_i\prod_{j=1}^{j=t_2}q_j$ with $\left(\frac{2}{p_i}\right)=-\left(\frac{2}{q_j}\right)=-1$ for all $i\in\{1,\ldots,t_1\}$ and for all $j\in\{1,\ldots,t_2\}$, then according to the two cases above,

a. If $\left(\frac{2}{q_j}\right)_4=\left(\frac{q_j}{2}\right)_4$ for all $j=1,\ldots,t_2$, then $r^*=2$, thus $r_2(H)=\mu+r^*-3=t_1+2t_2+1+2-3=t_1+2t_2$.

b. If $\left(\frac{2}{q_j}\right)_4\neq\left(\frac{q_j}{2}\right)_4$ for at least one $j=1,\ldots,t_2$, then $r^*=1$, thus $r_2(H)=\mu+r^*-3=t_1+2t_2+1+1-3=t_1+2t_2-1$.

□

5.3. Case $n=\prod_{i=1}^{i=t}p_i$ with $p_i\equiv 3 \pmod 4$ for all i

Theorem 5.3 *Let $\mathbb{K}=\mathbb{Q}(\sqrt{n\epsilon_0\sqrt{2}})$ be a real cyclic quartic number field, where ϵ_0 is the fundamental unit of the quadratic subfield $k=\mathbb{Q}(\sqrt{2})$ and n a square-free positive integer relatively prime to 2. Assume $n=\prod_{i=1}^{i=t}p_i$, $p_i\equiv 3 \pmod 4$ for all $i=1,\ldots,t$ and t is a positive integer.*

1. *If, for all* i, $\left(\frac{2}{p_i}\right)=-1$, *then* $r_2(H)=t-1$.

2. *If, for all* i, $\left(\frac{2}{p_i}\right)=1$, *then* $r_2(H)=2t-2$.

Moreover, if $n=\prod_{i=1}^{t_1}p_i\prod_{j=1}^{t_2}q_j$, *where* $p_i\equiv q_j\equiv 3 \pmod 4$ *and* $\left(\frac{2}{p_i}\right)=-\left(\frac{2}{q_j}\right)=-1$, *for all* $i\in\{1,\dots,t_1\}$, *and for all* $j\in\{1,\dots,t_2\}$, *then* $r_2(H)=t_1+2t_2-2$.

Proof

1. If $\left(\frac{2}{p_i}\right)=-1$, for all $i\in\{1,\dots,t\}$, then $\mu=t+1$. For $\mathfrak{p}_i$ the prime ideal of k above p_i we have

$$\left(\frac{-1,d}{\mathfrak{p}_i}\right)=\left[\frac{-1}{\mathfrak{p}_i}\right]=\left(\frac{1}{p_i}\right)=1 \quad \text{and} \quad \left(\frac{\epsilon_0,d}{\mathfrak{p}_i}\right)=\left[\frac{\epsilon_0}{\mathfrak{p}_i}\right]=\left(\frac{-1}{p_i}\right)=-1.$$

Using the product formula, we obtain the following table:

Unit\ Character	$\sqrt{2}$	$\mathfrak{p}_i$
-1	$+$	$+$
ϵ_0	$(-1)^t$	$-$
$-\epsilon_0$	$(-1)^t$	$-$

Hence $r^*=1$, which implies that: $r_2(H)=\mu+r^*-3=t+1+1-3=t-1$.

2. If $\left(\frac{2}{p_i}\right)=1$ for all $i\in\{1,\dots,t\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{2})$, $\wp_i$ and $\bar{\wp}_i$, where $p_i\mathcal{O}_k=\wp_i\bar{\wp}_i$, $i=1,\dots,t$. Hence for all $i\in\{1,\dots,t\}$, we have:

$$\left(\frac{-1,d}{\wp_i}\right)=\left(\frac{-1,d}{\bar{\wp}_i}\right)=\left[\frac{-1}{\wp_i}\right]=\left(\frac{-1}{p_i}\right)=-1 \quad \text{and} \quad \left(\frac{\epsilon_0,d}{\wp_i}\right)=\left[\frac{\epsilon_0}{\wp_i}\right]=\left[\frac{1+\sqrt{2}}{\wp_i}\right].$$

To compute the last unity, put $p_i\mathcal{O}_k=\wp_i\bar{\wp}_i$ and $p_i=a_i^2-2b_i^2\equiv 7 \pmod 8$ with $\wp_i=a_i+b_i\sqrt{2}$ and $\bar{\wp}_i=a_i-b_i\sqrt{2}$, for all i (note that $\mathcal{O}_k$ is a principal ring).
We have:

$$\left[\frac{1+\sqrt{2}}{\wp_i}\right]=\left(\frac{b_i}{p_i}\right)\left[\frac{b_i+b_i\sqrt{2}}{\wp_i}\right]=\left(\frac{b_i}{p_i}\right)\left[\frac{b_i+b_i\sqrt{2}-\wp_i}{\wp_i}\right].$$

So

$$\left(\frac{\epsilon_0,d}{\wp_i}\right)=\left(\frac{b_i}{p_i}\right)\left[\frac{b_i-a_i}{\wp_i}\right]=\left(\frac{b_i}{p_i}\right)\left(\frac{b_i-a_i}{p_i}\right)=-\left(\frac{b_i}{p_i}\right)\left(\frac{a_i-b_i}{\wp_i}\right)=-\left(\frac{b_i}{p_i}\right)\left(\frac{a_i+b_i}{p_i}\right),$$

indeed: $(a_i-b_i)(a_i+b_i)=a_i^2-b_i^2\equiv a_i^2-b_i^2-p_i\equiv b_i^2 \pmod{p_i}$. Then:

$$\left(\frac{a_i-b_i}{p_i}\right)\left(\frac{a_i+b_i}{p_i}\right)=\left(\frac{b_i^2}{p_i}\right)=1 \text{ for all } i=1,\dots,t.$$

Since $p_i=a_i^2-2b_i^2$, one gets $\left(\frac{2b_i^2}{p_i}\right)_4=\left(\frac{a_i^2}{p_i}\right)_4=\left(\frac{a_i}{p_i}\right)=(-1)^{\frac{p_i+1}{8}}$ ([6, page 19]). Which implies:

$$\left(\frac{b_i}{p_i}\right)=\left(\frac{b_i^2}{p_i}\right)_4=\left(\frac{2}{p_i}\right)_4(-1)^{\frac{p_i+1}{8}}.$$

Let $a_i + b_i = 2^j c$ with c an odd integer (a_i and b_i must be odd). So:

$$\left(\frac{b_i + a_i}{p_i}\right) = \left(\frac{2^j c}{p_i}\right) = \left(\frac{2}{p_i}\right)^j \left(\frac{c}{p_i}\right) = \left(\frac{c}{p_i}\right) = \left(\frac{-p_i}{c}\right).$$

On the other hand, we have: $(a_i - b_i)(a_i + b_i) = a_i^2 - b_i^2 = p_i + b_i^2$, then $\left(\frac{-p_i}{c}\right) = \left(\frac{b_i^2}{c}\right) = 1$, which implies that $\left(\frac{b_i + a_i}{p_i}\right) = 1$ and $\left(\frac{\epsilon_0, d}{\wp_i}\right) = \left(\frac{2}{p_i}\right)_4 (-1)^{\frac{p_i+1}{8}}$.

In the same way we have:

$$\left(\frac{\epsilon_0, d}{\bar{\wp}_i}\right) = \left(\frac{b_i}{p_i}\right)\left[\frac{b_i + a_i}{\bar{\wp}_i}\right] = \left(\frac{b_i}{p_i}\right)\left(\frac{b_i + a_i}{p_i}\right) = \left(\frac{2}{q_i}\right)_4 (-1)^{\frac{q_i+1}{8}}, \text{ for all } i = 1, \dots, t.$$

Consequently, we have the following table.

Unit\ Character	$\sqrt{2}$	$\wp_i$	$\bar{\wp}_i$
-1	$+$	$-$	$-$
ϵ_0	$(-1)^t$	$-\left(\frac{2}{p_i}\right)_4 (-1)^{\frac{p_i+1}{8}}$	$\left(\frac{2}{q_i}\right)_4 (-1)^{\frac{p_i+1}{8}}$
$-\epsilon_0$	$(-1)^t$	$\left(\frac{2}{p_i}\right)_4 (-1)^{\frac{p_i+1}{8}}$	$-\left(\frac{2}{p_i}\right)_4 (-1)^{\frac{p_i+1}{8}}$

Hence $r^* = 0$, and we infer that $r_2(H) = \mu + r^* - 3 = 2t + 1 + 0 - 3 = 2t - 2$.

Finally, if $n = \prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$, with $p_i \equiv q_j \equiv 3 \pmod 4$ and $\left(\frac{2}{p_i}\right) = -\left(\frac{2}{p_i}\right) = -1$ for all $i \in \{1, \dots, t_1\}$ and for all $j \in \{1, \dots, t_2\}$, then according to the two cases above, there are $t_1 + 2t_2 + 1$ prime ideals of k which ramify in $\mathbb{K}$ and $r^* = 0$. Thus $r_2(H) = t_1 + 2t_2 + 1 + 0 - 3 = t_1 + 2t_2 - 2$. □

5.4. Case $n = \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, $p_i \equiv -q_j \equiv 1 \pmod 4$ $\forall (i, j)$

Theorem 5.4 *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{2}})$ be a real cyclic quartic number field, where n a square-free positive integer relatively prime to 2 and ϵ_0 the fundamental unit of $k = \mathbb{Q}(\sqrt{2})$. Assume $n = \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ with $p_i \equiv -q_j \equiv 1 \pmod 4$ for all $(i, j) \in \{1, \dots, t\} \times \{1, \dots, s\}$. Denote by h the number of prime ideals of k dividing all the $p_i's$, $i \in \{1, \dots, t\}$.*

1. *If, for all j, $\left(\frac{2}{q_j}\right) = -1$, then $r_2(H) = h + s - 1$.*

2. *If, for all j, $\left(\frac{2}{q_j}\right) = 1$, then $r_2(H) = h + 2s - 2$.*

Moreover, if $\prod_{j=1}^{j=s} q_j = \prod_{i=1}^{i=s_1} q_i \prod_{j=1}^{j=s_2} q_j$ with $q_i \equiv q_j \equiv 3 \pmod 4$ and $\left(\frac{2}{q_j}\right) = -\left(\frac{2}{q_j}\right) = -1, i \in \{1, \dots, s_1\}, j \in \{1, \dots, s_2\}$, then $r_2(H) = h + s_1 + 2s_2 - 2$.

Proof Proceeding as above and using the previous results, we prove the theorem. □

6. Applications

In this section, we will determine the integers n such that $r_2(H)$, the rank of the 2-class group H of $\mathbb{K}=\mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$, is equal to 0, 1, 2 or 3. For this we adopt the following notations: p and p_i (resp. q and q_i), $i\in\mathbb{N}^*$, are prime integers congruent to 1 (resp. 3) modulo 4. $\delta=1$ or 2. The following theorems are simple deductions from the results of previous subsections. For all the examples below, we use PARI/GP calculator version 2.11.2 (64bit), April 28, 2019.

6.1. Case $\ell\equiv 1 \pmod 8$

Theorem 6.1 *Let $\mathbb{K}=\mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell\equiv 1 \pmod 8$ is a positive prime integer, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of $k=\mathbb{Q}(\sqrt{\ell})$. The class number of $\mathbb{K}$ is odd if and only if $n=1$.*

Example 6.2 *For $n=1$ and $\ell=257\equiv 1 \pmod 8$, we have the class number of the class group H of $\mathbb{K}=\mathbb{Q}(\sqrt{\epsilon_0\sqrt{\ell}})$ is 3.*

Theorem 6.3 *Let $\mathbb{K}=\mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell\equiv 1 \pmod 8$ is a prime, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of $k=\mathbb{Q}(\sqrt{\ell})$. H is cyclic if and only if one of the following assertions holds:*

1. *$n=p$ and either $\left(\frac{p}{\ell}\right)=-1$ or $\left(\frac{p}{\ell}\right)=1$ and $\left(\frac{p}{\ell}\right)_4\neq\left(\frac{\ell}{p}\right)_4$.*

2. *$n=2$ and $\left(\frac{2}{\ell}\right)_4\neq(-1)^{\frac{\ell-1}{8}}$.*

3. *$n=\delta q$ and $\left(\frac{q}{\ell}\right)=-1$.*

4. *$n=q_1q_2$ and $\left(\frac{q_1}{\ell}\right)=-1$ or $\left(\frac{q_2}{\ell}\right)=-1$.*

Example 6.4

1. *For $n=p=89\equiv 1 \pmod 4$ and $\ell=41\equiv 1 \pmod 8$, $\left(\frac{p}{\ell}\right)=-1$ and H is cyclic of order 2. For $n=p=97\equiv 1 \pmod 4$ and $\ell=89\equiv 1 \pmod 8$, $\left(\frac{p}{\ell}\right)_4=-\left(\frac{\ell}{p}\right)_4=1$ and H is cyclic of order 2.*

2. *For $n=2$ and and $\ell=1913\equiv 1 \pmod 8$, $\left(\frac{2}{\ell}\right)_4=-(-1)^{\frac{\ell-1}{8}}=1$ and H is cyclic of order 2.*

3. *For $n=q=83\equiv 3 \pmod 4$ and $\ell=137\equiv 1 \pmod 8$, $\left(\frac{q}{\ell}\right)=-1$ and H is cyclic of order 2. For $n=2q=2.83\equiv 2 \pmod 4$ and $\ell=97\equiv 1 \pmod 8$, $\left(\frac{q}{\ell}\right)=-1$ and H is cyclic of order 2.*

4. *For $n=q_1q_2=71.83\equiv 1 \pmod 4$ and $\ell=97\equiv 1 \pmod 8$, $\left(\frac{q_1}{\ell}\right)=\left(\frac{q_2}{\ell}\right)=-1$ and H is cyclic of order 2. For $n=q_1q_2=79.83\equiv 1 \pmod 4$ and $\ell=41\equiv 1 \pmod 8$, $\left(\frac{q_1}{\ell}\right)=-\left(\frac{q_2}{\ell}\right)=-1$ and H is cyclic of order 2.*

Theorem 6.5 *Let $\mathbb{K}=\mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell \equiv 1 \pmod 8$ is a prime, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of $k=\mathbb{Q}(\sqrt{\ell})$. The rank $r_2(H)$ equals 2 if and only if n takes one of the following forms.*

1. $n=p_1p_2$ *and*
   - i. *either* $(\frac{p_i}{\ell})=-1$ *for all* $i\in\{1,2\}$
   - ii. *or* $(\frac{p_i}{\ell})=-(\frac{p_j}{\ell})=-1$ *and* $\left(\frac{p_j}{\ell}\right)_4 \neq \left(\frac{\ell}{p_j}\right)_4$, $i\neq j\in\{1,2\}$.
2. $n=p$, $(\frac{p}{\ell})=1$ *and* $\left(\frac{p}{\ell}\right)_4=\left(\frac{\ell}{p}\right)_4$.
3. $n=2p$, $(\frac{p}{\ell})=-1$ *and* $\left(\frac{2}{\ell}\right)_4\neq(-1)^{\frac{\ell-1}{8}}$.
4. $n=2$ *and* $\left(\frac{2}{\ell}\right)_4=(-1)^{\frac{\ell-1}{8}}$.
5. $n=\delta q$ *and* $(\frac{q}{\ell})=1$.
6. $n=q_1q_2$ *and* $(\frac{q_1}{\ell})=(\frac{q_2}{\ell})=1$.
7. $n=\delta pq$ *and* $(\frac{p}{\ell})=(\frac{q}{\ell})=-1$.
8. $n=pq_1q_2$, $(\frac{p}{\ell})=-1$ *and* $(\frac{q_1}{\ell})=-1$ *or* $(\frac{q_2}{\ell})=-1$.

Example 6.6

1. *For* $n=p_1p_2=89.97\equiv 1 \pmod 4$ *and* $\ell=41\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=(\frac{p_2}{\ell})=-1$ *and* H *is of type* $(2,2)$. *For* $n=p_1p_2=97.89\equiv 1 \pmod 4$ *and* $\ell=17\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=-(\frac{p_2}{\ell})=-1$ *and* $\left(\frac{p_2}{\ell}\right)_4=-\left(\frac{\ell}{p_2}\right)_4=1$, H *is of type* $(2,2)$.
2. $n=p=613\equiv 1 \pmod 4$ *and* $\ell=17\equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell})=1$, $\left(\frac{p}{\ell}\right)_4=\left(\frac{\ell}{p}\right)_4=1$ *and* H *is of type* $(4,4)$.
3. $n=2p=2.1994\equiv 2 \pmod 4$ *and* $\ell=1753\equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell})=-1$, $\left(\frac{2}{\ell}\right)_4=-(-1)^{\frac{\ell-1}{8}}=1$ *and* H *is of type* $(2,2)$.
4. *For* $n=2$ *and and* $\ell=1889\equiv 1 \pmod 8$, *we have* $\left(\frac{2}{\ell}\right)_4=(-1)^{\frac{\ell-1}{8}}=1$ *and* H *is of type* $(2,4)$.
5. *For* $n=q=79\equiv 3 \pmod 4$ *and* $\ell=97\equiv 1 \pmod 8$, *we have* $(\frac{q}{\ell})=1$ *and* H *is of type* $(2,2)$. *For* $n=2q=2.71\equiv 2 \pmod 4$ *and* $\ell=73\equiv 1 \pmod 8$, *we have* $(\frac{q}{\ell})=1$ *and* H *is of type* $(2,4)$.
6. *For* $n=q_1q_2=47.67\equiv 1 \pmod 4$ *and* $\ell=17\equiv 1 \pmod 8$, *we have* $(\frac{q_1}{\ell})=(\frac{q_2}{\ell})=1$ *and* H *is of type* $(2,4)$.

7. *For* $n = pq = 73.79 \equiv 3 \pmod 4$ *and* $\ell = 17 \equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell}) = (\frac{q}{\ell}) = -1$ *and* H *is of type* $(2,2)$. *For* $n = 2pq = 2.41.79 \equiv 2 \pmod 4$ *and* $\ell = 17 \equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell}) = (\frac{q}{\ell}) = -1$ *and* H *is of type* $(2,2)$.

8. *For* $n = pq_1q_2 = 97.71.79 \equiv 1 \pmod 4$ *and* $\ell = 41 \equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell}) = (\frac{q_1}{\ell}) = (\frac{q_2}{\ell}) = -1$ *and* H *is of type* $(2,2)$. *For* $n = pq_1q_2 = 97.79.83 \equiv 1 \pmod 4$ *and* $\ell = 41 \equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell}) = (\frac{q_1}{\ell}) = -(\frac{q_2}{\ell}) = -1$ *and* H *is of type* $(2,2)$.

Theorem 6.7 *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ *be a real cyclic quartic number field, where* $\ell \equiv 1 \pmod 8$ *is a prime,* n *a square-free positive integer relatively prime to* ℓ *and* ϵ_0 *the fundamental unit of* $k = \mathbb{Q}(\sqrt{\ell})$. *The rank* $r_2(H)$ *equals* 3 *if and only if* n *takes one of the following forms.*

1. $n = 2p$ *and one of the following cases holds:*

   i. $(\frac{p}{\ell}) = -1$ *and* $\left(\frac{2}{\ell}\right)_4 = (-1)^{\frac{\ell-1}{8}}$,

   ii. $(\frac{p}{\ell}) = 1$ *and* $\left(\frac{2}{\ell}\right)_4 \neq (-1)^{\frac{\ell-1}{8}}$ *or* $\left(\frac{p}{\ell}\right)_4 \neq \left(\frac{\ell}{p}\right)_4$.

2. $n = p_1p_2$ *and*

   i. *either* $(\frac{p_i}{\ell}) = 1$, *for all* $i \in \{1,2\}$ *and* $\left(\frac{p_i}{\ell}\right)_4 \neq \left(\frac{\ell}{p_i}\right)_4$ *for at least one* $i \in \{1,2\}$.

   ii. *or* $(\frac{p_1}{\ell}) = -(\frac{p_2}{\ell}) = -1$ *and* $\left(\frac{p_2}{\ell}\right)_4 = \left(\frac{\ell}{p_2}\right)_4$.

3. $n = 2p_1p_2$, $(\frac{p_i}{\ell}) = -1$ *for all* $i \in \{1,2\}$ *and* $\left(\frac{2}{\ell}\right)_4 \neq (-1)^{\frac{\ell-1}{8}}$.

4. $n = p_1p_2p_3$ *and*

   i. *either* $(\frac{p_i}{\ell}) = -1$ *for all* $i \in \{1,2,3\}$,

   ii. *or* $(\frac{p_1}{\ell}) = (\frac{p_2}{\ell}) = -(\frac{p_3}{\ell}) = -1$ *and* $\left(\frac{p_3}{\ell}\right)_4 \neq \left(\frac{\ell}{p_3}\right)_4$.

5. $n = \delta q_1q_2q_3$ *and* $(\frac{q_i}{\ell}) = -1$ *for all* $i \in \{1,2,3\}$

6. $n = q_1q_2q_3q_4$ *and there exist at most one of symbols* $(\frac{q_i}{\ell})$ *for* $i \in \{1,2,3,4\}$ *equal* 1.

7. $n = 2q_1q_2$ *and* $(\frac{q_i}{\ell}) = -1$ *for at least one* $i \in \{1,2\}$.

8. $n = \delta p_1p_2q$, $(\frac{p_i}{\ell}) = -1$ *for all* $i \in \{1,2\}$ *and* $(\frac{q}{\ell}) = -1$.

9. $n = \delta pq$ *and* $(\frac{p}{\ell}) \neq (\frac{q}{\ell})$.

10. $n = p_1p_2q_1q_2$, $(\frac{p_i}{\ell}) = -1$ *for all* $i \in \{1,2\}$ *and* $(\frac{q_i}{\ell}) = 1$ *for at most one* $i \in \{1,2\}$.

11. $n = pq_1q_2$ *and one of the following cases holds:*

    i. $(\frac{p}{\ell}) = 1$ *and at most one of the symbols* $(\frac{q_i}{\ell})$, $i \in \{1,2\}$, *is* 1.

ii. $(\frac{p}{\ell})=-1$ *and* $(\frac{q_i}{\ell})=1$ *for all* $i\in\{1,2\}$.

Example 6.8

1. *For* $n=p_1p_2p_3=37.41.61\equiv 1 \pmod 4$ *and* $\ell=89\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=(\frac{p_2}{\ell})=(\frac{p_3}{\ell})=-1$ *and* H *is of type* $(2,2,2)$. *For* $n=p_1p_2p_3=89.97.73\equiv 1 \pmod 4$ *and* $\ell=41\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=(\frac{p_2}{\ell})=-(\frac{p_3}{\ell})=-1$ *and* $\left(\frac{p_3}{\ell}\right)_4=-\left(\frac{\ell}{p_3}\right)_4=-1$, H *is of type* $(2,2,2)$.

2. *For* $n=p_1p_2=89.97\equiv 1 \pmod 4$ *and* $\ell=73\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=(\frac{p_2}{\ell})=1$ *and* $\left(\frac{p_2}{\ell}\right)_4=-\left(\frac{\ell}{p_2}\right)_4=-1$, H *is of type* $(2,4,4)$. *For* $n=p_1p_2=61.73\equiv 1 \pmod 4$ *and* $\ell=89\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=-(\frac{p_2}{\ell})=-1$ *and* $\left(\frac{p_2}{\ell}\right)_4=\left(\frac{\ell}{p_2}\right)_4=1$, H *is of type* $(2,2,4)$.

3. *For* $n=2p=2.97\equiv 2 \pmod 4$ *and* $\ell=41\equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell})=-1$ *and* $\left(\frac{2}{\ell}\right)_4=(-1)^{\frac{\ell-1}{8}}=-1$, H *is of type* $(2,2,2)$. *For* $n=2p=2.97\equiv 2 \pmod 4$ *and* $\ell=89\equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell})=1$, $\left(\frac{2}{\ell}\right)_4=-(-1)^{\frac{\ell-1}{8}}=1$ *and* $\left(\frac{p}{\ell}\right)_4=-\left(\frac{\ell}{p}\right)_4=1$, H *is of type* $(2,4,4)$.

4. *For* $n=2p_1p_2=2.73.97\equiv 1 \pmod 4$ *and* $\ell=17\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=(\frac{p_2}{\ell})=-1$ *and* $\left(\frac{2}{\ell}\right)_4=-(-1)^{\frac{\ell-1}{8}}=-1$, H *is of type* $(2,2,2)$.

5. *For* $n=q_1q_2q_3=67.71.83\equiv 3 \pmod 4$ *and* $\ell=97\equiv 1 \pmod 8$, *we have* $(\frac{q_1}{\ell})=(\frac{q_2}{\ell})=(\frac{q_3}{\ell})=-1$ *and* H *is of type* $(2,2,2)$. *For* $n=2q_1q_2q_3=59.67.83\equiv 2 \pmod 4$ *and* $\ell=97\equiv 1 \pmod 8$, *we have* $(\frac{q_1}{\ell})=(\frac{q_2}{\ell})=(\frac{q_3}{\ell})=-1$ *and* H *is of type* $(2,2,2)$.

6. *For* $n=q_1q_2q_3q_4=11.23.31.7\equiv 1 \pmod 4$ *and* $\ell=17\equiv 1 \pmod 8$, *we have* $(\frac{q_1}{\ell})=(\frac{q_2}{\ell})=(\frac{q_3}{\ell})=(\frac{q_3}{\ell})=-1$ *and* H *is of type* $(2,2,2)$.

7. *For* $n=2q_1q_2=71.83\equiv 2 \pmod 4$ *and* $\ell=97\equiv 1 \pmod 8$, *we have* $(\frac{q_1}{\ell})=(\frac{q_2}{\ell})=-1$ *and* H *is of type* $(2,2,4)$. *For* $n=2q_1q_2=71.83\equiv 2 \pmod 4$ *and* $\ell=41\equiv 1 \pmod 8$, *we have* $(\frac{q_1}{\ell})=-(\frac{q_2}{\ell})=-1$ *and* H *is of type* $(2,2,2)$.

8. *For* $n=p_1p_2q=89.97.79\equiv 3 \pmod 4$ *and* $\ell=41\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=(\frac{p_2}{\ell})=(\frac{q}{\ell})=-1$ *and* H *is of type* $(2,2,2)$. *For* $n=2p_1p_2q=2.89.97.79\equiv 3 \pmod 4$ *and* $\ell=41\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=(\frac{p_2}{\ell})=(\frac{q}{\ell})=-1$ *and* H *is of type* $(2,2,2)$.

9. *For* $n=pq=61.47\equiv 3 \pmod 4$ *and* $\ell=17\equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell})=-(\frac{q}{\ell})=-1$ *and* H *is of type* $(2,4,8)$. *For* $n=2pq=2.97.47\equiv 2 \pmod 4$ *and* $\ell=17\equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell})=-(\frac{q}{\ell})=1$ *and* H *is of type* $(2,2,2)$.

10. *For* $n=p_1p_2q_1q_2=61.73.71.79\equiv 1 \pmod 4$ *and* $\ell=17\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=(\frac{p_2}{\ell})=(\frac{q_1}{\ell})=(\frac{q_2}{\ell})=-1$ *and* H *is of type* $(2,2,2)$. *For* $n=p_1p_2q_1q_2=61.73.67.79\equiv 1 \pmod 4$ *and* $\ell=17\equiv 1 \pmod 8$, *we have* $(\frac{p_1}{\ell})=(\frac{p_2}{\ell})=-(\frac{q_1}{\ell})=(\frac{q_2}{\ell})=-1$ *and* H *is of type* $(2,2,2)$.

11. *For* $n = pq_1q_2 = 97.83.79 \equiv 1 \pmod 4$ *and* $\ell = 89 \equiv 1 \pmod 8$, *we have* $(\frac{p}{\ell}) = -(\frac{q_1}{\ell}) = (\frac{q_2}{\ell}) = 1$ *and* H *is of type* $(2,2,2)$.

6.2. Case $\ell = 2$

Theorem 6.9 *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{2}})$ *be a real cyclic quartic number field,* n *an odd square-free positive integer and* ϵ_0 *the fundamental unit of* $k = \mathbb{Q}(\sqrt{2})$. *The class number of* $\mathbb{K}$ *is odd if and only if* $n = 1$ *or* n *is a prime integer congruent to* $3 \pmod 4$.

Example 6.10 *For* $n = q = 59 \equiv 3 \pmod 4$, *we have* $\left(\frac{2}{q}\right) = -1$, H *has order* 5. *For* $n = q = 631 \equiv 3 \pmod 4$, *we have* $\left(\frac{2}{q}\right) = 1$, H *has order* 5.

Theorem 6.11 *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{2}})$ *be a real cyclic quartic number field,* n *an odd square-free positive integer and* ϵ_0 *the fundamental unit of* $k = \mathbb{Q}(\sqrt{2})$. H *is cyclic if and only if one of the following assertions holds:*

1. $n = p$ *and either* $(\frac{2}{p}) = -1$ *or* $(\frac{2}{p}) = 1$ *and* $\left(\frac{2}{p}\right)_4 \neq (\frac{p}{2})_4$.

2. $n = q_1q_2$ *and* $(\frac{2}{q_1}) = -1$ *or* $(\frac{2}{q_2}) = -1$.

3. $n = pq$ *and* $(\frac{2}{p}) = -1$.

Example 6.12

1. *For* $n = p = 61 \equiv 1 \pmod 4$, *we have* $(\frac{2}{p}) = -1$ *and* H *is cyclic of order* 2. *For* $n = p = 89 \equiv 1 \pmod 4$, $(\frac{2}{p}) = 1$, $\left(\frac{2}{p}\right)_4 = -\left(\frac{p}{2}\right)_4 = 1$ *and* H *is cyclic of order* 2.

2. *For* $n = q_1q_2 = 59.83 \equiv 1 \pmod 4$, $(\frac{2}{q_1}) = (\frac{2}{q_2}) = -1$ *and* H *is cyclic of order* 2. *For* $n = q_1q_2 = 67.71 \equiv 1 \pmod 4$, $(\frac{2}{q_1}) = -(\frac{2}{q_2}) = -1$ *and* H *is cyclic of order* 2.

3. *For* $n = pq = 61.59 \equiv 3 \pmod 4$, $(\frac{2}{p}) = (\frac{2}{q}) = -1$ *and* H *is cyclic of order* 2. *For* $n = pq = 61.47 \equiv 3 \pmod 4$, $(\frac{2}{p}) = -(\frac{2}{q}) = -1$ *and* H *is cyclic of order* 2.

Theorem 6.13 *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{2}})$ *be a real cyclic quartic number field,* n *an odd square-free positive integer and* ϵ_0 *the fundamental unit of* $k = \mathbb{Q}(\sqrt{2})$. *The rank* $r_2(H)$ *equals* 2 *if and only if* n *takes one of the following forms.*

1. $n = p$ *with* $(\frac{2}{p}) = 1$ *and* $\left(\frac{2}{p}\right)_4 = (\frac{p}{2})_4$

2. $n = p_1p_2$ *and:*

   a. *either* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = -1$

b. *or* $(\frac{2}{p_i}) = -(\frac{2}{p_j}) = -1$ *and* $\left(\frac{2}{p_j}\right)_4 \neq \left(\frac{p_j}{2}\right)_4$ *for* $i \neq j$ *in* $\{1, 2\}$.

3. $n = q_1q_2$ *with* $(\frac{2}{q_1}) = (\frac{2}{q_2}) = 1$.

4. $n = q_1q_2q_3$ *and at most one of the symbols* $(\frac{2}{q_1})$, $(\frac{2}{q_2})$, $(\frac{2}{q_3})$ *equals* 1.

5. $n = p_1p_2q$ *and* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = -1$.

6. $n = pq$ *and* $(\frac{2}{p}) = 1$.

7. $n = pq_1q_2$ *with* $(\frac{2}{p}) = -1$ *and* $(\frac{2}{q_1}) = -1$ *or* $(\frac{2}{q_2}) = -1$.

Example 6.14

1. *For* $n = p = 881 \equiv 1 \pmod 4$, $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 = 1$ *and* H *is of type* $(4, 4)$.

2. *For* $n = p_1p_2 = 877.997 \equiv 1 \pmod 4$, $(\frac{2}{p_1}) = (\frac{2}{p_2}) = -1$ *and* H *is of type* $(2, 2)$. *For* $n = p_1p_2 = 941.977 \equiv 1 \pmod 4$, *we have* $(\frac{2}{p_1}) = -(\frac{2}{p_2}) = -1$ *and* $\left(\frac{2}{p_2}\right)_4 = -\left(\frac{p_2}{2}\right)_4 = -1$, H *is of type* $(2, 2)$.

3. *For* $n = q_1q_2 = 47.79 \equiv 1 \pmod 4$, *we have* $(\frac{2}{q_1}) = (\frac{2}{q_2}) = 1$ *and* H *is of type* $(4, 4)$.

4. *For* $n = q_1q_2q_3 = 67.83.43 \equiv 3 \pmod 4$, *we have* $(\frac{2}{q_1}) = (\frac{2}{q_2}) = (\frac{2}{q_3}) = -1$ *and* H *is of type* $(2, 2)$. *For* $n = q_1q_2q_3 = 67.83.47 \equiv 3 \pmod 4$, *we have* $(\frac{2}{q_1}) = (\frac{2}{q_2}) = -(\frac{2}{q_3}) = -1$ *and* H *is of type* $(2, 2)$.

5. *For* $n = p_1p_2q = 53.61.83 \equiv 3 \pmod 4$, *we have* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = (\frac{2}{q_3}) = -1$ *and* H *is of type* $(2, 2)$. *For* $n = p_1p_2q_3 = 53.61.71 \equiv 3 \pmod 4$, *we have* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = -(\frac{2}{q_3}) = -1$ *and* H *is of type* $(2, 2)$.

6. *For* $n = pq = 73.83 \equiv 3 \pmod 4$, $(\frac{2}{p}) = -(\frac{2}{q}) = 1$ *and* H *is of type* $(2, 4)$. *For* $n = pq = 73.79 \equiv 3 \pmod 4$, $(\frac{2}{p}) = (\frac{2}{q}) = 1$ *and* H *is of type* $(2, 2)$.

7. *For* $n = pq_1q_2 = 61.43.67 \equiv 1 \pmod 4$, *we have* $(\frac{2}{p}) = (\frac{2}{q_1}) = (\frac{2}{q_2}) = -1$ *and* H *is of type* $(2, 2)$. *For* $n = pq_1q_2 = 61.59.71 \equiv 1 \pmod 4$, *we have* $(\frac{2}{p}) = (\frac{2}{q_1}) = -(\frac{2}{q_2}) = -1$ *and* H *is bicyclic and of type* $(2, 2)$.

Theorem 6.15 *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{2}})$ *be a real cyclic quartic number field,* n *an odd square-free positive integer and* ϵ_0 *the fundamental unit of* $k = \mathbb{Q}(\sqrt{2})$. *The rank* $r_2(H)$ *equals* 3 *if and only if* n *takes one of the following forms.*

1. $n = p_1p_2$ *and*

a. *either* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = 1$ *and* $\left(\frac{2}{p_i}\right)_4 \neq \left(\frac{p_i}{2}\right)_4$ *for at least one* $i \in \{1, 2\}$.

b. *or* $(\frac{2}{p_i}) = -(\frac{2}{p_j}) = -1$ *and* $\left(\frac{2}{p_j}\right)_4 = \left(\frac{p_j}{2}\right)_4$ *for* $i \neq j \in \{1, 2\}$.

2. $n = p_1p_2p_3$ *and*

   a. *either* $(\frac{2}{p_i}) = -1$ *for all* $i \in \{1,2,3\}$.

   b. *or* $(\frac{2}{p_i}) = (\frac{2}{p_j}) = -(\frac{2}{p_k}) = -1$ *and* $\left(\frac{2}{p_k}\right)_4 \neq \left(\frac{p_k}{2}\right)_4$ *for* i, j *and* k *different two by two in* $\{1,2,3\}$.

3. $n = q_1q_2q_3$ *and only one of the symbols* $(\frac{2}{q_i})$, $i \in \{1,2,3\}$, *equals* -1.

4. $n = q_1q_2q_3q_4$ *and at most one of the symbols* $(\frac{2}{q_i})$, $i \in \{1,2,3,4\}$, *is* 1.

5. $n = p_1p_2p_3q$ *and* $(\frac{2}{p_i}) = -1$ *for all* $i \in \{1,2,3\}$.

6. $n = pq_1q_2q_3$ *with* $(\frac{2}{p}) = -1$ *and at most one of the symbols* $(\frac{2}{q_i})$, $i = 1,2,3$, *equals* 1.

7. $n = p_1p_2q_1q_2$ *with* $(\frac{2}{p_i}) = -1$ *for all* $i \in \{1,2\}$ *and* $((\frac{2}{q_1})$ *or* $(\frac{2}{q_2}) = -1)$.

8. $n = pq_1q_2$ *and:*

   a. *either* $(\frac{2}{p}) = 1$ *and* $(\frac{2}{q_1}) = -1$ *or* $(\frac{2}{q_2}) = -1$.

   b. *or* $(\frac{2}{p}) = -1$ *and* $(\frac{2}{q_i}) = 1$ *for all* $i \in \{1,2\}$.

9. $n = p_1p_2q$ *with* $(\frac{2}{p_1}) \neq (\frac{2}{p_2})$.

Example 6.16

1. *For* $n = p_1p_2 = 769.977 \equiv 1 \pmod 4$, *we have* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = 1$, $\left(\frac{2}{p_1}\right)_4 = -\left(\frac{p_1}{2}\right)_4 = -1$ *and* $\left(\frac{2}{p_2}\right)_4 = -\left(\frac{p_2}{2}\right)_4 = -1$ H *is of type* $(2,2,4)$. *For* $n = p_1p_2 = 797.953 \equiv 1 \pmod 4$, *we have* $(\frac{2}{p_1}) = -(\frac{2}{p_2}) = -1$ *and* $\left(\frac{2}{p_2}\right)_4 = \left(\frac{p_2}{2}\right)_4 = -1$, H *is of type* $(2,4,4)$.

2. *For* $n = p_1p_2p_3 = 37.53.61 \equiv 1 \pmod 4$, *we have* $(\frac{2}{p_i}) = -1$ *for all* $i \in \{1,2,3\}$ *and* H *is of type* $(2,2,2)$. *For* $n = p_1p_2p_3 = 53.61.89 \equiv 1 \pmod 4$, *we have* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = -(\frac{2}{p_3}) = -1$ *and* $\left(\frac{2}{p_3}\right)_4 = -\left(\frac{p_3}{2}\right)_4 = -1$, H *is of type* $(2,2,2)$.

3. *For* $n = q_1q_2q_3 = 71.79.67 \equiv 3 \pmod 4$, *we have* $(\frac{2}{q_1}) = (\frac{2}{q_2}) = -(\frac{2}{q_3}) = 1$ *and* H *is of type* $(2,2,4)$.

4. *For* $n = q_1q_2q_3q_4 = 59.67.83.43 \equiv 1 \pmod 4$, *we have* $(\frac{2}{q_1}) = (\frac{2}{q_2}) = (\frac{2}{q_3}) = (\frac{2}{q_4}) = -1$ *and* H *is of type* $(2,2,2)$. *For* $n = q_1q_2q_3q_4 = 59.67.83.79 \equiv 1 \pmod 4$, *we have* $(\frac{2}{q_1}) = (\frac{2}{q_2}) = (\frac{2}{q_3}) = -(\frac{2}{q_4}) = -1$ *and* H *is of type* $(2,2,2)$.

5. *For* $n = p_1p_2p_3q = 37.53.61.67 \equiv 3 \pmod 4$, *we have* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = (\frac{2}{p_3}) = (\frac{2}{q}) = -1$ *and* H *is of type* $(2,2,2)$. *For* $n = p_1p_2p_3q = 37.53.61.71 \equiv 3 \pmod 4$, *we have* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = (\frac{2}{p_3}) = -(\frac{2}{q}) = -1$ *and* H *is of type* $(2,2,2)$.

6. *For* $n = pq_1q_2q_3 = 61.67.83.59 \equiv 3 \pmod 4$*, we have* $(\frac{2}{p}) = (\frac{2}{q_1}) = (\frac{2}{q_2}) = (\frac{2}{q_3}) = -1$ *and* H *is of type* $(2,2,2)$*. For* $n = pq_1q_2q_3 = 61.67.83.71 \equiv 3 \pmod 4$*, we have* $(\frac{2}{p}) = (\frac{2}{q_1}) = (\frac{2}{q_2}) = -(\frac{2}{q_3}) = -1$ *and* H *is of type* $(2,2,2)$*.*

7. *For* $n = p_1p_2q_1q_2 = 53.61.83.67 \equiv 1 \pmod 4$*, we have* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = (\frac{2}{q_1}) = (\frac{2}{q_2}) = -1$ *and* H *is of type* $(2,2,2)$*. For* $n = p_1p_2q_1q_2 = 53.61.83.79 \equiv 1 \pmod 4$*, we have* $(\frac{2}{p_1}) = (\frac{2}{p_2}) = (\frac{2}{q_1}) = -(\frac{2}{q_2}) = -1$ *and* H *is of type* $(2,2,2)$*.*

8. *For* $n = pq_1q_2 = 97.79.83 \equiv 1 \pmod 4$*, we have* $(\frac{2}{p}) = (\frac{2}{q_1}) = -(\frac{2}{q_2}) = 1$ *and* H *is of type* $(2,2,2)$*. For* $n = pq_1q_2 = 61.47.71 \equiv 1 \pmod 4$*, we have* $(\frac{2}{p}) = -(\frac{2}{q_1}) = -(\frac{2}{q_2}) = -1$ *and* H *is of type* $(2,2,4)$*.*

9. *For* $n = p_1p_2q = 61.73.83 \equiv 3 \pmod 4$*, we have* $(\frac{2}{p_1}) = -(\frac{2}{p_2}) = (\frac{2}{q}) = -1$ *and* H *is of type* $(2,2,4)$*.*

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