Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1287 – 1299
© TÜBİTAK
doi:10.3906/mat-2006-93

General rotational $\xi-$surfaces in Euclidean spaces

Kadri ARSLAN, Yılmaz AYDIN, Betül BULCA*
Department of Mathematics, Faculty of Arts and Science, Bursa Uludağ University, Bursa, Turkey

Received: 22.06.2020 • **Accepted/Published Online:** 30.03.2021 • **Final Version:** 20.05.2021

Abstract: The general rotational surfaces in the Euclidean 4-space $\mathbb{R}^4$ was first studied by Moore (1919). The Vranceanu surfaces are the special examples of these kind of surfaces. Self-shrinker flows arise as special solution of the mean curvature flow that preserves the shape of the evolving submanifold. In addition, $\xi-$surfaces are the generalization of self-shrinker surfaces. In the present article we consider $\xi-$surfaces in Euclidean spaces. We obtained some results related with rotational surfaces in Euclidean $4-$space $\mathbb{R}^4$ to become self-shrinkers. Furthermore, we classify the general rotational $\xi-$surfaces with constant mean curvature. As an application, we give some examples of self-shrinkers and rotational $\xi-$surfaces in $\mathbb{R}^4$.

Key words: Mean curvature, self-shrinker, general rotational surface

1. Introduction

Let $x : M \to \mathbb{R}^m$ be an isometric immersion of an $n-$dimensional submanifold M $(m > n)$ into the Euclidean space $\mathbb{R}^m$. The position vector field x of M is very important object in differential geometry. It is the Euclidean vector $x = \overrightarrow{op}$ known as the radius vector of M, where $p \in M$ and $o \in \mathbb{R}^m$ is the arbitrary reference point. The position vector field x of M has a natural decomposition given by

$$x = x^T + x^N, \tag{1.1}$$

where $x^T \in T_pM$ and $x^N \in T_p^{\perp}M$ [9].

The mean curvature vector field $\overrightarrow{H}$ is one of the most important invariants of the submanifold M. In physics, the average curvature vector field is the torsion field applied to the submanifold originated from $\mathbb{R}^m$. The mean curvature flow is the gradient flow of the area functional on the space of the submanifold M. The self-shrinker flows arise as special solution of the mean curvature flow that preserve the shape of the evolving submanifold [18]. The most important mean curvature flow is self-similar flow which is obtained by the following nonlinear elliptic system

$$\overrightarrow{H} + x^N = 0,$$

where x^N is the normal component of the position vector x [18].

*Correspondence: bbulca@uludag.edu.tr
2010 *AMS Mathematics Subject Classification:* 53C40, 53C42

In [14] Chang and Wei introduced a $\lambda-$hypersurfaces in Euclidean space giving a natural generalization of self-shrinkers in the hypersurface case. According to [14], a hypersurface $M \subset \mathbb{R}^{n+1}$ is called a $\lambda-$hypersurface and its mean curvature H satisfies

$$H + \langle x, N \rangle = \lambda,$$

for some real function λ, where N is the unit normal of the hypersurface. Recently, Li and Chang made a generalization of both self-shrinkers and $\lambda-$hypersurfaces, by introducing the concepts of $\xi-$submanifolds [20]. By definition an immersed submanifold M^n in $\mathbb{R}^m$ is called a $\xi-$submanifold if there is a parallel vector field ξ such that the mean curvature vector field $\overrightarrow{H}$ satisfies

$$\overrightarrow{H} + x^N = \xi.$$

This paper is organized as follows: In Section 2, we give some basic concepts of the second fundamental form and curvatures of the surfaces in $\mathbb{R}^m$. In Section 3, we give some well known results of self-shrinker surfaces in $\mathbb{R}^m$. Further, we give some well known examples satisfying the self-shrinking condition. In Section 4 we consider generalized rotational surfaces in $\mathbb{R}^4$. We obtained some results related with generalized rotational surfaces in Euclidean $4-$space $\mathbb{R}^4$ to become self-shrinkers. Furthermore, we classify the general rotational $\xi-$surfaces with constant mean curvature. As an application, we give some examples of self-shrinkers and rotational $\xi-$surfaces in $\mathbb{R}^4$.

2. Preliminaries

Let $x : M \to \mathbb{R}^m$ be an immersed surface in the Euclidean space $\mathbb{R}^m$. Denote $\chi(M)$ and $\chi^{\perp}(M)$ with the space of the smooth vector fields tangent and normal to M, respectively. Given any local orthonormal vector fields e_1, e_2 tangent to M, consider the second fundamental form

$$h(e_i, e_j) = \widetilde{\nabla}_{e_i} e_j - \nabla_{e_i} e_j, \qquad 1 \leq i, j \leq 2 \tag{2.1}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M and the Riemannian connection of $\mathbb{R}^m$, respectively. This map is well-defined, symmetric and bilinear [9]. For any arbitrary orthonormal frame field $\{N_1, N_2, ..., N_{m-n}\}$ of M, recall the shape operator

$$A_{N_\alpha} e_j = -\widetilde{\nabla}_{e_j} N_\alpha + D_{e_j} N_\alpha, \quad 1 \leq \alpha \leq m-n, 1 \leq j \leq 2 \tag{2.2}$$

where D denotes the normal connection of M. This operator is bilinear, selfadjoint and satisfies the following equation:

$$\langle A_{N_\alpha} e_j, e_i \rangle = \langle h(e_i, e_j), N_\alpha \rangle = h_{ij}^\alpha, \quad 1 \leq i, j \leq 2; \quad 1 \leq \alpha \leq m-n, \tag{2.3}$$

where h_{ij}^α are the coefficients of the second fundamental form.

Eqs. (2.1) and (2.2) are called Gaussian formula and Weingarten formula, respectively. In addition,

$$h(e_i, e_j) = \sum_{\alpha=1}^{m-n} h_{ij}^\alpha N_\alpha, \; 1 \leq i, j \leq 2 \tag{2.4}$$

holds. Consequently, the squares length $\|h\|^2$ of the second fundamental form h is defined by

$$\|h\|^2 = \sum_{i,j=1}^{2} \left(h_{ij}^{\alpha}\right)^2. \tag{2.5}$$

The Gaussian curvature K and mean curvature vector $\overrightarrow{H}$ of M are given by

$$K = \langle h(e_1, e_1), h(e_2, e_2)\rangle - \langle h(e_1, e_2), h(e_1, e_2)\rangle, \tag{2.6}$$

and

$$\overrightarrow{H} = \frac{1}{2}\left\{h(e_1, e_1) + h(e_2, e_2)\right\}, \tag{2.7}$$

respectively. The norm of the mean curvature vector $H = \left\|\overrightarrow{H}\right\|$ is called the mean curvature of M. Recall that a surface M is said to be flat (resp. minimal) if its Gauss curvature (resp. mean curvature vector) vanishes identically [9].

3. $\xi-$surfaces in Euclidean spaces

Let $x : M \to \mathbb{R}^m$ be an immersed surface in the Euclidean space $\mathbb{R}^m$. The mean curvature flow of x is a family $x_t : M \to \mathbb{R}^m$ that satisfies

$$\left(\frac{\partial}{\partial t} x_t(p)\right)^{\perp} = H(p,t),\ x_0 = x \tag{3.1}$$

where $H(p,t)$ is the mean curvature vector of $x_t(M)$ at $x_t(p)$ and $(.)^{\perp}$ denotes the projection into the normal space of $x_t(M)$ [18].

Definition 3.1 *An immersed surface M in the Euclidean space $\mathbb{R}^m$ is called self-shrinker solution of (3.1) if the curvature vector field $\overrightarrow{H}$ of M satisfies the following nonlinear elliptic system:*

$$\overrightarrow{H} + x^N = 0, \tag{3.2}$$

where x^N is the normal component of x [18].

In [23] K. Smooczyk proved the following results.

Theorem 3.2 *[23] Let $x : M \to \mathbb{R}^m$ be a closed self-shrinker then M is a minimal surface of the sphere $S^{m-1}(\sqrt{2})$ if and only if $\overrightarrow{H} \neq 0$ and $\nabla^{\perp}\upsilon = 0$, where $\upsilon = \frac{\overrightarrow{H}}{\left\|\overrightarrow{H}\right\|}$ is the principal normal of the surface M.*

Theorem 3.3 *[23] Let $x : M \to \mathbb{R}^m$ $(d \geq 1)$ be a $2-$ dimensional compact self-shrinker. Then M is spherical surface if and only if $\overrightarrow{H} \neq 0$ and $\nabla^{\perp}\upsilon = 0$ hold identically.*

Theorem 3.4 *[13] Let $x : M \to \mathbb{R}^{n+d}$ $(d = m - n)$ be a $2-$dimensional complete proper self-shrinker (i.e. $H \neq 0$) without boundary and with $H > 0$. If the principal normal $\upsilon = \frac{\overrightarrow{H}}{\left\|\overrightarrow{H}\right\|}$ is parallel in the normal bundle of M and the squared norm of the second fundamental form is constant then M is one of the following;*

(i) $S^k(\sqrt{k}) \times \mathbb{R}^{2-k}$, $1 \le k \le 2$, $\|h\|^2 = 1$,

(ii) the Boruvka sphere $S^k(\sqrt{m(m+1)})$ *in* $S^{2m}(\sqrt{2})$ *with*

$$d = 2m - 1 \text{ and } \|h\|^2 = 2 - \frac{2}{m(m+1)},$$

(iii) a compact flat, minimal surface in $S^{2m+1}(\sqrt{2})$ *with* $d = 2m$, $\|h\|^2 = 2$.

Definition 3.5 *An immersed surface* M *in* $\mathbb{R}^m$ *is called a* $\xi-$*surface if there is a parallel vector field* ξ *such that the mean curvature vector field* $\overrightarrow{H}$ *satisfies the following nonlinear elliptic system:*

$$\overrightarrow{H} + x^N = \xi, \tag{3.3}$$

where x^N *is the normal component of* x.

Identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ recall the Lagrangian submanifold M in $\mathbb{R}^{2n}$ as follows:

Definition 3.6 *A submanifold* $M \subset \mathbb{R}^{2n}$ *is called Lagrangian if* $J(T_pM) = T_p^{\perp}M$ *holds for any* $p \in M$, *where* J *is the complex structure of* $\mathbb{R}^{2n}$, T_pM *and* $T_p^{\perp}M$ *denote the tangent space and normal space at* p.

In [20] Li and Chang proved the following result.

Proposition 3.7 *[20] Let* $x : M^2 \to \mathbb{C}^2$ *be a compact orientable Lagrangian self-shrinker. If* $\|h\|^2 + \|H\|^2 \le 4$, *then* $\|h\|^2 + \|H\|^2 \equiv 4$ *and* $x(M^2) = S^1(1) \times S^1(1)$ *up to a holomorphic isometry on* $\mathbb{C}^2$.

4. General rotational $\xi-$surfaces

Rotational surfaces in $\mathbb{R}^4$ was first introduced by Moore in 1919 [22]. In the recent years some mathematicians have taken an interest in the rotational surfaces in $\mathbb{R}^4$; see for example [7, 16, 17]. In [17], the authors applied the invariance theory of surfaces in $\mathbb{R}^4$ to the class of general rotational surfaces whose meridians lie in two dimensional planes in order to find all minimal surfaces (see also [15, 26] for the rotational surfaces with constant Gaussian curvature in $\mathbb{R}^4$).

A general rotational surface M in $\mathbb{R}^4$ is defined by the parametrization (see, [22]);

$$X(u,v) = (f(u)\cos cv, f(u)\sin cv, g(u)\cos dv, g(u)\sin dv), \tag{4.1}$$

where $u \in J, 0 \le v < 2\pi$, and $\alpha(u) = (f(u), g(u))$ is the meridian curve of the rotation satisfying $c^2f^2+d^2g^2 > 0$ and $(f')^2 + (g')^2 > 0$.

The orthonormal frame field of M is given by

$$\begin{aligned} e_1 &= \frac{1}{\psi(u)}\frac{\partial}{\partial u}, \ e_2 = \frac{1}{\varphi(u)}\frac{\partial}{\partial v} \\ e_3 &= \frac{1}{\psi(u)}(g'(u)\cos cv, g'(u)\sin cv, -f'(u)\cos dv, -f'(u)\sin dv) \\ e_4 &= \frac{1}{\varphi(u)}(-dg(u)\sin cv, dg(u)\cos cv, cf(u)\sin dv, -cf(u)\cos dv) \end{aligned} \tag{4.2}$$

where

$$\psi(u) = \sqrt{(f'(u))^2 + (g'(u))^2}, \\ \varphi(u) = \sqrt{c^2 f^2(u) + d^2 g^2(u)}, \tag{4.3}$$

are the smooth functions on M [7]. With respect to this frame we can obtain the second fundamental maps;

$$\begin{aligned} h(e_1, e_1) &= \frac{\kappa}{\psi^3} e_3 \\ h(e_1, e_2) &= \frac{\eta}{\psi\varphi^2} e_4 \\ h(e_2, e_2) &= \frac{\beta}{\psi\varphi^2} e_3 \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \kappa &= f''g' - f'g'' \\ \lambda &= c^2 ff' + d^2 gg' \\ \beta &= c^2 f'g - d^2 fg' \\ \eta &= cd\,(f'g - fg') \\ \delta &= cd\,(ff' + gg') \end{aligned} \tag{4.5}$$

are the smooth functions on M. Consequently, by the use of Eqs. (2.6) and (2.7) with Eq. (4.4) the Gaussian curvature and mean curvature vector $\overrightarrow{H}$ of M become

$$K = \frac{1}{\psi^2\varphi^2}\left(\frac{\kappa\beta}{\psi^2} - \frac{\eta^2}{\varphi^2}\right) \tag{4.6}$$

and

$$\overrightarrow{H} = \frac{1}{2\psi}\left(\frac{\kappa}{\psi^2} + \frac{\beta}{\varphi^2}\right) e_3 \tag{4.7}$$

respectively [7].

From the orthogonal decomposition (1.1) of the position vector x of M we obtain

$$x^N = x - \frac{\rho'(u)}{\psi(u)} e_1 \tag{4.8}$$

where $\rho(u) = \frac{1}{2}\|x\|^2$ is the square norm of the distance function of the position vector x such that

$$\rho'(u) = f(u)f'(u) + g(u)g'(u) \tag{4.9}$$

holds. The gradient of the distance function is given by

$$grad\,(\|x\|) = \sum_{j=1}^{2} \frac{\langle x, e_j\rangle}{\|x\|} e_j = \frac{\rho'(u)}{\psi(u)\,\|x\|} e_1. \tag{4.10}$$

Due to [10] we obtain the following results.

Theorem 4.1 *Let M be a general rotational surface in $\mathbb{R}^4$ given with the parametrization (4.1). Then $x = x^N$ holds identically if and only if M is a spherical surface of $\mathbb{R}^4$.*

Proof Assume that M is a general rotational surface in $\mathbb{R}^4$ given with the parametrization (4.1). If $x = x^N$ holds identically, then $\rho'(u) = 0$ holds. Therefore, Eq. (4.10) yields the the distance function of M has zero gradient so by Example 4.1. of [11] M is a spherical surface in $\mathbb{R}^4$. Infact,

$$f_1(u)f_1'(u) + f_2(u)f_2'(u) = 0, \tag{4.11}$$

i.e. $f_1^2(u) + f_2^2(u) = r_0^2$ implies that the meridian curve α is an open part of a circle parametrized by

$$f_1(u) = r_0 \cos\left(\frac{u}{r_0}\right), f_2(u) = r_0 \sin\left(\frac{u}{r_0}\right), \tag{4.12}$$

where r_0 is a positive real number.

The converse is clear. □

Remark 4.2 *The general rotational surface given with the meridian curve (4.12) is $H-$parallel and minimal surface in $\mathbb{S}^3(r_0) \subset \mathbb{R}^4$.*

Theorem 4.3 *Let M be a general rotational surface in $\mathbb{R}^4$ given with the parametrization (4.1). Then $x = x^T$ holds identically if and only if M is a conic surface with the vertex at the origin.*

Proof Assume that M is a general rotational surface in $\mathbb{R}^4$ given with the parametrization (4.1). If $x = x^T$ holds identically, then $x = \frac{\rho'(u)}{\psi(u)}e_1$ holds identically. Therefore, Eq. (4.10) yields that gradient of the distance function has constant length

$$\|grad\,(\|x\|)\| = \frac{|\langle x, e_1\rangle|}{\|x\|} = 1. \tag{4.13}$$

By Proposition 5.2 of [11] M is a conic surface in $\mathbb{R}^4$ with the vertex at the origin. Infact, $x = \rho'(u)e_1$ yields $f'g - fg' = 0$. Consequently the meridian curve α is an open part of a straight line passing through origin. The converse is clear. □

Definition 4.4 *A surface M in the Euclidean space $\mathbb{R}^m$ is called self-shrinker if the curvature vector field $\overrightarrow{H}$ of M satisfies the following nonlinear elliptic system:*

$$\overrightarrow{H} + x^N = 0, \tag{4.14}$$

where x^N is the normal component of x.

It is well known that the Euclidean plane $\mathbb{R}^2$, the unit sphere $S^2(1)$, the cylinder $S^1(1) \times \mathbb{R}$ and the Clifford torus $S^1(1) \times S^1(1)$ are the canonical self-shrinkers in $\mathbb{R}^4$. Besides the standard examples there are many examples of complete self-shrinkers in $\mathbb{R}^4$. For examples, compact minimal surfaces in the sphere $S^3(2)$ are compact self-shrinkers in $\mathbb{R}^4$ [12]. One can get the following well-know examples.

Example 4.5 *Let $\Gamma_1(s) = (x_1(s),\ y_1(s))$, $0 \leq s < L_1$ and $\Gamma_2(t) = (x_2(t),\ y_2(t))$, $0 \leq t < L_2$ be two self-shrinker curves in $\mathbb{R}^2$ given with arclength parameters. Consider the Riemannian product $M = \Gamma_1(s) \times \Gamma_2(t)$ defined by*

$$\Gamma_1(s) \times \Gamma_2(t) = (x_1(s), x_2(t), y_1(s), y_2(t)).$$

In [12] the authors showed that $\Gamma_1(s) \times \Gamma_2(t)$ is a Lagrangian self-shrinker in $\mathbb{R}^4$ with vanishing Gaussian curvature.

Example 4.6 *Let $\Gamma(t) = (x_1(t),\ x_2(t))$, $0 \leq t < L_1$ be a closed self-shrinker curves in $\mathbb{R}^2$ then its curvature κ_Γ satisfies*

$$\kappa_{\Gamma_i} = c\frac{e^{\frac{r^2}{2}}}{r^2},\ r = \|\Gamma\|$$

with a positive constant

$$c^2 = r^4\left(1 - (r')^2\right)e^{r^2}.$$

In [3] Anciaux proved that the (rotational) surface

$$x(t,s) = (x_1(t)\cos s, x_1(t)\sin s, x_2(t)\cos s, x_2(t)\sin s)$$

defines a compact Lagrangian self-shrinkers in $\mathbb{R}^4$ which is called Anciaux torus. The squared norm of the mean curvature and the second fundamental form of the Anciaux torus are given by

$$\begin{aligned} \left\|\overrightarrow{H}\right\|^2 &= c^2\frac{e^{r^2}}{r^2}, \\ \|h\|^2 &= c^2\frac{e^{\frac{r^2}{2}}}{r^2}\left(r^4 + 2r^2 + 4\right). \end{aligned}$$

Example 4.7 *For positive integers m, n, $(m,n) = 1$, consider the surface M in $\mathbb{R}^4$ given with the parametrization*

$$x(t,s) = \left(\cos\sqrt{\frac{m}{n}}t\frac{\cos s}{\sqrt{n}}, \cos\sqrt{\frac{m}{n}}t\frac{\sin s}{\sqrt{n}}, \sin\sqrt{\frac{n}{m}}t\frac{\cos s}{\sqrt{n}}, \sin\sqrt{\frac{n}{m}}t\frac{\sin s}{\sqrt{n}}\right).$$

In [19] Li and Wang proved that this surface is self-shrinker. Therefore, it is called Li-Wang tori.

Example 4.8 *For any $n_1, n_2 \in \mathbb{N}$, $n_1 + n_2 = n$, the Clifford torus $S^{n_1}(\sqrt{n_1}) \times S^{n_2}(\sqrt{n_d})$ in $\mathbb{R}^{n+2}$ is a compact self-shrinker with $\|h\|^2 = 2$ (see, [8]).*

Example 4.9 *[8] The product of $n-$circles $S^1 \times ... \times S^1$ in $\mathbb{R}^{2n}$ is a compact self-shrinkers with $\|h\|^2 = n$.*

For the self-shrinker surface case we have the following result.

Theorem 4.10 *Let M be a general rotational surface in $\mathbb{R}^4$ given with the parametrization (4.1). If M is a self-shrinker then*

$$\frac{\kappa}{\psi^2} + \frac{\beta}{\varphi^2} = 2\eta \tag{4.15}$$

holds, where φ, ψ, κ *and* β *are smooth functions defined in (4.3) and (4.5) and*

$$\eta = f'g - fg'. \tag{4.16}$$

Proof Assume that M is a general rotational surface in $\mathbb{R}^4$ given with the parametrization (4.1). If M is a self-shrinker then by (4.9) and (4.14)

$$He_3 + x - \frac{\rho'(u)}{\psi(u)}e_1 = 0 \tag{4.17}$$

holds identically. Consequently substituting

$$H = \frac{1}{2\psi}\left(\frac{\kappa}{\psi^2} + \frac{\beta}{\varphi^2}\right) \tag{4.18}$$

and

$$\langle x, e_3 \rangle = \frac{fg' - f'g}{\psi} \tag{4.19}$$

into (4.17) we obtain (4.15). □

Corollary 4.11 *[3] Let* M *be a general rotational surface in* $\mathbb{R}^4$ *given with the parametrization (4.1) with* $c = d = 1$*. If* M *is a self-shrinker surface then it is an Anciaux torus in* $\mathbb{R}^4$*.*

Definition 4.12 *The Vranceanu surface in* $\mathbb{R}^4$ *is defined by the following parametrization;*

$$f(u) = r(u)\cos u, \ \ g(u) = r(u)\sin u, \ \ c = d = 1, \tag{4.20}$$

where $r(u)$ *is a real valued nonzero function [24].*

Consequently, substituting Eq. (4.20) into Eqs. (4.6) and (4.7) one can get

$$K = \frac{(r')^2 - rr''}{\left(r^2 + (r')^2\right)^2} \tag{4.21}$$

and

$$\overrightarrow{H} = \frac{rr'' - 3(r')^2 - 2r^2}{2\left(r^2 + (r')^2\right)^{\frac{3}{2}}}e_3 \tag{4.22}$$

respectively [5]. As a consequence of Eqs. (4.21) and (4.22) we get the following results.

Corollary 4.13 *[26] Let* M *be a Vranceanu rotation surface given with the parametrization (4.20). If* M *has vanishing Gaussian curvature, then*

$$r(u) = \lambda e^{\mu u}$$

holds, where λ *and* μ *are real constants.*

Corollary 4.14 *[6] Let M be a Vranceanu rotation surface given with the parametrization (4.20). If M is minimal then*

$$r(u) = \frac{\pm 1}{\sqrt{a \sin 2u - b \cos 2u}}, \tag{4.23}$$

where, a and b are real constants.

By the use of (4.15) with (4.20) we obtain the following result.

Corollary 4.15 *Let M be a Vranceanu surface given with the parametrization (4.20). If M is a self-shrinking surface then*

$$\frac{rr'' - 3(r')^2 - 2r^2}{2\left(r^2 + (r')^2\right)} + 1 = 0 \tag{4.24}$$

holds identically.

Example 4.16 *The flat Vranceanu surface given with $r(u) = 1$ is a Clifford torus, that is; it is the product of two plane circles with same radius. Consequently, it is easy to show that Eq. (4.24) holds. Therefore, the Clifford torus is a self-shrinking surface of Euclidean $4-$ space $\mathbb{R}^4$.*

Theorem 4.17 *[8] Let M be a compact self-shrinking surface in Euclidean 4-space $\mathbb{R}^4$. If $\|H\|$ is constant or $\|h\| \leq 0$ or $\|h\| \geq 0$ then M is a Clifford torus.*

In Example 4.16 we have shown that the converse statement of Theorem 4.17 is also valid.

Definition 4.18 *An immersed surface M in $\mathbb{R}^{2+d}$ is called a $\xi-$surface if there is a parallel vector field ξ such that the mean curvature vector field $\overrightarrow{H}$ satisfies the following nonlinear elliptic system:*

$$\overrightarrow{H} + x^N = \xi, \tag{4.25}$$

where x^N is the normal component of x.

We have the following results.

Lemma 4.19 *Let M be an immersed surface in $\mathbb{R}^{2+d}$. Then M is a $\xi-$surface if and only if for each $e_i \in T_pM$*

$$D_{e_i}\overrightarrow{H} = \sum_{j=1}^{2} \langle x, e_j \rangle \, h\left(e_i, e_j\right), \tag{4.26}$$

and

$$A_\xi e_i = e_i - \nabla_{e_i} x^T + A_H e_i \tag{4.27}$$

hold identically, where x^T is the tangent component of x.

Proof By the definition of a $\xi-$surface, we have $\overrightarrow{H} = \xi - x^N$. By the Gauss and Weingarten formulas it follows that, for any $v \in T_pM$,

$$\begin{aligned} A_H v &= -\widetilde{\nabla}_v \overrightarrow{H} + D_v \overrightarrow{H} \\ &= -\widetilde{\nabla}_v \xi + \widetilde{\nabla}_v x - \widetilde{\nabla}_v x^T + D_v \overrightarrow{H} \\ &= -\widetilde{\nabla}_v \xi + v - \nabla_{e_i} x^T - h\left(v, x^T\right) + D_v \overrightarrow{H} \end{aligned} \tag{4.28}$$

where $x^N = x - x^T$ and $\widetilde{\nabla}_v x = v$ are well-known relations. Consequently, the tangent and normal parts of (4.28) gives (4.27) and (4.26), respectively. □

Lemma 4.20 *Let M be a general rotational surface in $\mathbb{R}^4$ given with the parametrization (4.1). If M is a $\xi-$surface of $\mathbb{R}^4$ then*

$$\frac{\partial H}{\partial u} \eta \psi^2 - H \delta \kappa = 0, \tag{4.29}$$

holds identically.

Proof Assume that M is a general rotational surface in $\mathbb{R}^4$ differentiating (4.7) with respect to e_1, e_2 a straight-forward computation gives

$$D_{e_1} \overrightarrow{H} = \frac{1}{\psi} \frac{\partial H}{\partial u} e_3, \; D_{e_2} \overrightarrow{H} = \frac{H\delta}{\psi \varphi^2} e_4. \tag{4.30}$$

Since M is a $\xi-$surface in $\mathbb{R}^4$ then by Lemma 4.19 we get

$$\begin{aligned} D_{e_1} \overrightarrow{H} = \langle x, e_1 \rangle h\left(e_1, e_1\right) + \langle x, e_2 \rangle h\left(e_1, e_2\right) \\ D_{e_2} \overrightarrow{H} = \langle x, e_1 \rangle h\left(e_2, e_1\right) + \langle x, e_2 \rangle h\left(e_2, e_2\right). \end{aligned} \tag{4.31}$$

Further, substituting the equations in Eq. (4.9) with $\langle x, e_1 \rangle = \frac{\rho'}{\psi}$, $\langle x, e_2 \rangle = 0$ into Eq. (4.31) we obtain

$$D_{e_1} \overrightarrow{H} = \frac{\kappa \rho'}{\psi^4} e_3, \; D_{e_2} \overrightarrow{H} = \frac{\eta \rho'}{\psi^2 \varphi^2} e_4 \tag{4.32}$$

Hence, comparing Eq. (4.30) with Eq. (4.32) after some computation we get the result. □

As a consequence of Lemma 4.20 we get the following results.

Theorem 4.21 *Let M be a general rotational surface in $\mathbb{R}^4$ with constant mean curvature. If M is a $\xi-$surface of $\mathbb{R}^4$ then M is one of the following;*

(i) a minimal surface of $\mathbb{R}^4$, or

(ii) a spherical surface of $\mathbb{R}^4$, or

(iii) a rotational surface of $\mathbb{R}^4$ whose profile curve is a straight line.

Proof Assume that M is a general rotational surface in $\mathbb{R}^4$. If M is a $\xi-$surface of $\mathbb{R}^4$ then the equation (4.29) holds. Since M has constant mean curvature then $H\delta\kappa = 0$. So we have three possible cases, $H = 0$, or $\delta = 0$, or $\kappa = 0$. For the first case M is a minimal surface of $\mathbb{R}^4$. Further, from (4.5) $\delta = 0$ implies that $\rho' = 0$, i.e. $x = x^N$. Therefore, by Theorem 4.1 it is easy to deduce that M is a spherical surface of $\mathbb{R}^4$. Finally, $\kappa = 0$ implies that the profile curve is a straight line. □

Theorem 4.22 *Let M be a Vranceanu surface given with the parametrization (4.20). If M is a $\xi-$surface then*

$$r(u) = e^{\int \varphi(z)dz + c_2}$$

holds identically, where

$$u = \int \frac{e^{\int \varphi(z)dz + c_2}}{\sqrt{z - e^{\int \varphi(z)dz + 2c_2}}} dz + c_1$$

is the parametric function such that the smooth functions

$$\begin{aligned} z &= r(u)^2 + r'(u)^2, \\ \varphi(z) &= \frac{1}{2r(u)\left(r(u)^2 + r'(u)^2\right)} \end{aligned}$$

satisfy the equality

$$\frac{\partial}{\partial z}\varphi(z) = \frac{12z^2\varphi(z)^3 + z(3z-5)\varphi(z)^2 - (3z-1)\varphi(z)}{z^2}.$$

Proof Assume that M is a Vranceanu surface given with the parametrization (4.20). Then using (4.5), (4.22) with (4.20) we get

$$\begin{aligned} H &= \frac{r(u)r''(u) - 3(r'(u))^2 - 2r^2(u)}{2\left(r^2(u) + (r'(u))^2\right)^{\frac{3}{2}}}, \\ \kappa &= r(u)r''(u) - 2(r'(u))^2 - r^2(u), \\ \psi^2 &= r^2(u) + (r'(u))^2, \\ \delta &= r(u)r'(u), \\ \eta &= -r^2(u). \end{aligned} \tag{4.33}$$

Since M is a $\xi-$surface then (4.29) holds. Therefore, substituting (4.33) into (4.29) we get the following differential equation

$$r\left(r^2 + (r')^2\right)\left(\frac{rr'' - 3(r')^2 - 2r^2}{2\left(r^2 + (r')^2\right)^{\frac{3}{2}}}\right)_u + r'\left(rr'' - 2(r')^2 - r^2\right)\left(\frac{rr'' - 3(r')^2 - 2r^2}{2\left(r^2 + (r')^2\right)^{\frac{3}{2}}}\right) = 0.$$

By the use of the Maple programing command;

$$\begin{aligned} > H1(u) : &= r(u)*diff(r(u),u,u) - 3*diff(r(u),u)\hat{}2 - 2*r(u)\hat{}2 : \\ > z(u) : &= diff(r(u),u)\hat{}2 + r(u)\hat{}2 : \\ > H(u) : &= H1(u)/(2*z(u)\hat{}(3/2)) : \\ > k(u) : &= r(u)*diff(r(u),u,u) - 2*diff(r(u),u)\hat{}2 - r(u)\hat{}2 : \\ > ode1 : &= r(u)*z(u)\hat{}2*diff(H(u),u) + diff(r(u),u)*k(u)*H(u) = 0 : \end{aligned}$$

$$> dsolve(ode1);$$

we get

$$r(u) = e^{(\int g(f)df + c_2)}$$

where

$$\frac{d}{df}g(f) = 12g(f)^3 + \frac{(-5+3f)g(f)^2}{f} - \frac{1}{2}\frac{(3f-1)g(f)}{f^2}$$

$$f = \left(\frac{d}{du}r(u)\right)^2 + r(u)^2, \quad g(f) = \frac{1}{2}\frac{1}{r(u)\left(\frac{d^2}{du^2}r(u) + r(u)^2\right)}$$

$$u = \int \frac{g(f)e^{(\int g(f)df + c_2)}}{\sqrt{f - e^{(2\int g(f)df + 2c_2)}}}df + c_1, \quad r(u) = e^{(\int g(f)df + c_2)}$$

This completes the proof of the theorem. □

References

[1] Abresch U, Langer J. The normalized curve shortening flow and homothetic solutions. Journal of Differential Geometry 1986; 23: 175-196.

[2] Aminov Yu. The Geometry of Submanifolds. London, UK: Gordon and Breach Science Publication, 2001.

[3] Anciaux H. Construction of Lagrangian self-similar solutions to the mean curvature flow in $\mathbb{C}^n$. Geometriae Dedicata 2006; 120: 37-48.

[4] Arezzo C, Sun J. Self-shrinkers for the mean curvature flow in arbitrary codimension. Mathematische Zeitschrift 2013; 274: 993-1027.

[5] Arslan K, Bayram B, Bulca B, Kim YH, Murathan C et al. Rotational embeddings in $\mathbb{E}^4$ with pointwise 1-type Gauss map. Turkish Journal of Mathematics 2011; 35: 493-499.

[6] Arslan K, Bayram B, Bulca B, Öztürk G. General rotation surfaces in $\mathbb{E}^4$. Results in Mathematics 2012; 61 (3): 315-327.

[7] Arslan K, Bulca B, Kosova D. On generalized rotational surfaces in Euclidean spaces. Journal of the Korean Mathematical Society 2017; 54 (3): 999-1013.

[8] Castro I, Lerma AM. The Clifford torus as a self-shrinker for the Lagrangian mean curvature flow. International Mathematics Research Notices 2014; 16: 1515-1527.

[9] Chen BY. Geometry of Submanifolds. New York, NY, USA: Dekker, 1973.

[10] Chen BY. Differential geometry of rectifying submanifolds. International Electronic Journal of Geometry 2016; 9 (2):1-8.

[11] Chen BY. More on convolution of Riemannian manifolds. Beitrage zur Algebra und Geometrie 2003; 44: 9-24.

[12] Cheng QM, Hori H, Wei G. Complete Lagrangian self-shrinkers in $\mathbb{R}^4$. arXiv 2018; arXiv:1802.02396.

[13] Cheng QM, Peng Y. Complete self-shrinkers of the mean curvature flow. Calculus of Variations Partial Differential Equations 2015; 52: 497-506.

[14] Cheng QM, Wei G. Complete $\lambda-$hypersurfaces of the weighted volume-preserving mean curvature flow. arXiv 2015; arXiv:1403.3177.

[15] Coung DV. Surfaces of revolution with constant Gaussian curvature in four space. Asian-European Journal of Mathematics 2013; 6: 1350021.

[16] Dursun U, Turgay NC. General rotational surfaces in Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map. Mathematical Communications 2012; 17: 71-81.

[17] Ganchev G, Milousheva V. On the theory of surfaces in the four dimensional Euclidean space. Kodai Mathematical Journal 2008; 31: 183-198.

[18] Joyse D, Lee Y, Tsui MP. Self-similar solutions and translating solutions for Lagrangian mean curvature flow. Journal of Differential Geometry 2010; 84: 127-161.

[19] Li H, Wang X. New characterizations of the Clifford torus as a Lagrangian self-shrinker. The Journal of Geometric Analysis 2017; 27: 1393-1412.

[20] Li X, Chang X. A rigidity theorem of $\xi-$ submanifolds in $\mathbb{C}^2$. Geometriae Dedicata 2016; 185: 155-169.

[21] Li X, Li Z. Variational characterization of $\xi-$submanifolds in the Euclidean space $\mathbb{R}^{m+p}$. Annali di Mathematica Pura ed Applicata 2020; 199: 1491-1518.

[22] Moore C. Surfaces of rotations in a space of four dimensions. Annals of Mathematics 1919; 21: 81-93.

[23] Smoczyk K. Self-shrinkers of the mean curvature flow in arbitrary codimension. International Mathematics Research Notices 2005; 48: 1983-3004.

[24] Vranceanu G. Surfaces de rotation dans $\mathbb{E}^4$. Revue Roumaine de Mathematıques Pures et Appliquees 1977; 22: 857-862.

[25] Wong YC. Contributions to the theory of surfaces in 4-space of constant curvature. Transactions of the American Mathematical Society 1946; 59: 467-507.

[26] Yoon DW. Some properties of the Clifford torus as rotation surface. Indian Journal of Pure and Applied Mathematics 2003; 34: 907-915.