Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2021) 45: 1310 – 1318
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doi:10.3906/mat-2102-97

Multipliers and $\mathcal{I}$-core for sequences

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Received: 20.02.2021 • **Accepted/Published Online:** 02.04.2021 • **Final Version:** 20.05.2021

Abstract: In this paper we mainly deal with $\mathcal{I}_c^{(q)}-$convergence. In particular we study bounded multipliers of bounded $\mathcal{I}_c^{(q)}-$convergent sequences. We also give some $\mathcal{I}-$core results and characterize the inclusion $K-core\{Ax\} \subseteq \mathcal{I}-core\{x\}$ for bounded sequences $x = (x_n)$.

Key words: Statistical convergence, ideal, ideal convergence, multipliers, Knopp's core theorem

1. Introduction

Kostyrko et al. [19] introduced and studied the concept of $\mathcal{I}$-convergence of sequences in metric spaces, where $\mathcal{I}$ is an ideal of subsets of the set $\mathbb{N}$ of positive integers and extended this concept to $\mathcal{I}$-convergence of a sequence of real functions defined on a metric space and proved some basic properties of these concepts.

In 1951 Fast [8] and Steinhaus [25] introduced the concept of statistical convergence independently and established a relationship with summability (see also [9, 21, 22]). Some applications of statistical convergence in number theory and mathematical analysis can be found in [3, 4, 8, 10, 11, 25]. In 2011 Gogolo et al. studied the properties of ideals $\mathcal{I}_c^{(q)}$ related to the notion of $\mathcal{I}$-convergence and they showed that $\mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)*}$-convergences are equivalent [15].

In the present work we mainly deal with $\mathcal{I}_c^{(q)}-$convergence. In Section 2 we study results motivated by those of [7]. In particular we study bounded multipliers of bounded $\mathcal{I}_c^{(q)}-$convergent sequences. Section 3 is devoted to $\mathcal{I}-$core results. We characterize the inclusion

$$K - core\{Ax\} \subseteq \mathcal{I} - core\{x\}$$

for bounded sequences $x = (x_n)$. In the last section we give an ideal version of Choudhary's theorem on Knopp's core.

Now we recall some notation and terminology of an ideal.

Ideal $\mathcal{I}$ on X $(X \neq \varnothing)$ is a family of subsets, satisfying the following conditions: if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$ and if $A \in \mathcal{I}$, $B \subset A$ then $B \in \mathcal{I}$. Filter $\mathcal{F}$ on X $(X \neq \varnothing)$ is a nonempty family of subsets, satisfying the following conditions: $\varnothing \notin \mathcal{F}$; if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ and if $A \in \mathcal{F}, A \subset B$ then

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2010 *AMS Mathematics Subject Classification:* 40A35; Secondary 40G15

$B \in \mathcal{F}$. An ideal $\mathcal{I}$ is called nontrivial if $\mathcal{I} \neq \varnothing$ and $X \notin \mathcal{I}$. Ideal $\mathcal{I}$ on X is a nontrivial ideal if and only if $\mathcal{F}(\mathcal{I}) = \{X \backslash A : A \in \mathcal{I}\}$ is a filter on X. A nontrivial ideal $\mathcal{I}$ on X containing all singletons is called admissible.

An admissible ideal $\mathcal{I}$ on $\mathbb{N}$ is said to satisfy property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2 ...\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2 ...\}$ such that $A_j \vartriangle B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$ [19].

Let $\mathcal{I}$ be a nontrivial ideal on $\mathbb{N}$. Then a sequence $x = (x_n)$ of real numbers is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ and we write $\mathcal{I} - \lim x = L$, if and only if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$. Another type of convergence closely related to the ideal $\mathcal{I}$ is $\mathcal{I}^*$-convergence. A sequence $x = (x_n)$ of real numbers is said to be $\mathcal{I}^*$-convergent to $L \in \mathbb{R}$ and we write $\mathcal{I}^* - \lim x = L$, if and only if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < ... < m_k < ...\}$ such that $\lim_{k \to \infty} x_{m_k} = L$ (see e.g., [19]).

For every admissible ideal $\mathcal{I}$, Kostyrko et al. [19] proved that

$$\mathcal{I}^* - \lim x = L \Rightarrow \mathcal{I} - \lim x = L.$$

Let an admissible ideal $\mathcal{I}$ on $\mathbb{N}$. For an arbitrary sequence $x = (x_n)$ of real numbers and each $L \in \mathbb{R}$, $\mathcal{I} - \lim x = L$ implies $\mathcal{I}^* - \lim x = L$, then $\mathcal{I}$ has property (AP) [19].

For any $q \in (0, 1]$ the set

$$\mathcal{I}_c^{(q)} = \left\{ E \subseteq \mathbb{N} : \sum_{i \in E} \frac{1}{i^q} < \infty \right\}$$

is an admissible ideal. Gogolo et al. [15] proved that $\mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)*}$-convergences are equivalent. Recall that $\mathcal{I}_c^{(q)} \subset \mathcal{I}_d$ (see e.g., [15]), where the ideal $\mathcal{I}_d$ is the class of all subsets of positive integers that has asymptotic density zero.

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. Motivated by that of [13] the $\mathcal{I}$-limit superior of a real number sequence x is defined by

$$\mathcal{I}\!-\limsup x = \begin{cases} \sup B_x & , \quad \text{if } B_x \neq \varnothing \\ -\infty & , \quad \text{if } B_x = \varnothing \end{cases}$$

where $B_x := \{b \in \mathbb{R} : \{n : x_n > b\} \notin \mathcal{I}\}$ (see e.g., [6]). The real number sequence $x = (x_n)$ is said to be $\mathcal{I}$-bounded if there is a number B such that $\{n : |x_n| > B\} \in \mathcal{I}$.

By $c^{\mathcal{I}(q)}, c^{\mathcal{I}(q)}(b)$ we denote the set of all $\mathcal{I}_c^{(q)}-$ convergent sequences , the set of all bounded $\mathcal{I}_c^{(q)}$-convergent sequences, respectively.

2. Bounded multipliers

Gogolo et al. [15] introduced the class $\mathcal{T}_q$ of lower triangular nonnegative matrices as follows:

A matrix $T = (t_{nk})$ belongs to the class $\mathcal{T}_q$ if and only if it satisfies the following conditions:

(I) $\lim_{n \to \infty} \sum_{k=1}^{n} t_{nk} = 1$

(q) $E \subset \mathbb{N}$ and $E \in \mathcal{I}_c^{(q)}$, then $\lim_{n \to \infty} \sum_{k \in E} t_{nk} = 0$, $q \in (0, 1]$.

Note that every matrix of the class $\mathcal{T}_q$ is regular.

Let l_∞ be the space of all real bounded sequences and let $T = (t_{nk})$ be an infinite matrix with real entries. Tx is the sequence whose nth term is given by $(Tx)_n = \sum_{k=1}^{\infty} t_{nk} x_k$ whenever the series converges for each n. From now on the bounded summability field of the matrix $T \in \mathcal{T}_q$ will be denoted by $c_T(b)$, i.e.

$$c_T(b) = \left\{ x \in l_\infty : \lim_n (Tx)_n \text{ exists} \right\}$$

(see [7]).

Theorem 2.1 ([15]) *Let $q \in (0,1]$. Then the bounded sequence $x = (x_n)$ of real numbers $\mathcal{I}_c^{(q)}-$converges to $L \in \mathbb{R}$ if and only if it is $T-$summable to $L \in \mathbb{R}$ for each matrix $T \in \mathcal{T}_q$ (i.e. $c^{\mathcal{I}(q)}(b) = \bigcap_{T \in \mathcal{T}_q} c_T(b)$).*

We say that x is strongly $T-$summable to a real number a if

$$\lim_n \sum_{k=1}^{\infty} t_{nk} |x_k - a| = 0.$$

In this case, the bounded strong summability field of the matrix $T \in \mathcal{T}_q$ is given by

$$W_b(T) = \left\{ x \in l_\infty : \lim_n \sum_{k=1}^{\infty} t_{nk} |x_k - a| = 0 \text{ for some } a \right\}.$$

Assume that two sequence spaces, E and F, are given. We say that a sequence u is a bounded multiplier of E into F, and we write $M(E,F)$, if $u.x \in F$ whenever $x \in E$, i.e.

$$M(E,F) = \{ u \in l_\infty : u.x \in F \text{ for all } x \in E \}$$

where the multiplication is coordinatewise. If $F = E$, then we write $M(E)$ instead of $M(E,E)$.

It is known that

$$M(c_T(b)) = W_b(T) \tag{2.1}$$

provided that $t_{nk} \geq 0$ for all n and k (see [17]).

We also mention the following result of [17] that we need in the sequel.

Theorem 2.2 *If T is a regular matrix, then the bounded sequence x is strongly $T-$summable to a if and only if there exists a subset Z of $\mathbb{N}$ such that $\chi_{\mathbb{N} \setminus Z}$ is strongly $T-$summable to zero and $\lim_{n \in Z} x_n = a$.*

This section addresses the bounded multiplier space of $c^{\mathcal{I}(q)}(b)$ and then give an analogue of Theorem 2.1 for bounded multipliers.

Theorem 2.3 *$x \in M(c^{\mathcal{I}(q)}(b))$ if and only if $x \in c^{\mathcal{I}(q)}(b)$.*

Proof Let $x \in M(c^{\mathcal{I}(q)}(b))$. Since $\chi_{\mathbb{N}} \in c^{\mathcal{I}(q)}(b)$ and $x \in M(c^{\mathcal{I}(q)}(b))$ we get $\chi_{\mathbb{N}}.x \in c^{\mathcal{I}(q)}(b)$; hence $x \in c^{\mathcal{I}(q)}(b)$.

Conversely, assume that $x \in c^{\mathcal{I}(q)}(b)$. We claim that $x.y \in c^{\mathcal{I}(q)}(b)$ for an arbitrary $y \in c^{\mathcal{I}(q)}(b)$. As in Proposition 4.3 of [23] since $x, y \in c^{\mathcal{I}(q)}(b)$ for each $\varepsilon > 0$ the sets $\{n \in \mathbb{N} : |x_n - L_x| \leq \varepsilon\}$ and $\{n \in \mathbb{N} : |y_n - L_y| \leq \varepsilon\}$ belong to the filter $\mathcal{F}\left(\mathcal{I}_c^{(q)}\right)$.

On the other hand we get

$$\begin{aligned} |x_n y_n - L_x L_y| &= |x_n y_n - x_n L_y + x_n L_y - L_x L_y| \\ &\leq |x_n| \, |y_n - L_y| + |x_n - L_x| \, |L_y| \\ &\leq 2 \, \|x\| \, \varepsilon \end{aligned}$$

where $\|x\| = \sup_n |x_n|$.

Then for each $\varepsilon > 0$ we have

$$\{n \in \mathbb{N} : |x_n y_n - L_x L_y| \leq \varepsilon\} \in \mathcal{F}\left(\mathcal{I}_c^{(q)}\right).$$

Thus $x.y \in c^{\mathcal{I}(q)}(b)$. □

Now using the same technique as in Theorem 6 of Demirci and Orhan [7], we give an analogue of Theorem 2.1 for bounded multiplier space, but first we need some properties of $\beta\mathbb{N}$, the Stone-Čech compactification of positive integers $\mathbb{N}$. For each $B \subseteq \mathbb{N}$, let $cl_{\beta\mathbb{N}}B$ be the closure of B in $\beta\mathbb{N}$ and let $B^* = (cl_{\beta\mathbb{N}}B) \setminus B$. It is well-known [14] that the sets $\{B^* : B \subseteq \mathbb{N}\}$ form a basis for the topology of $\beta\mathbb{N} \setminus B$.

Recall that, for a regular matrix A, the support set K_A is defined by

$$K_A = \bigcap \{B^* : B \subseteq \mathbb{N} \text{ and } \chi_B \text{ is } A-\text{summable to } 1\}$$

which is nonempty compact subset of $\beta\mathbb{N} \setminus B$ [1]. Observe that if $B^* \supseteq K_A$, then χ_B is $A-$summable to 1. Furthermore, the intersection of any sequence of neighborhoods of K_A is again a neighborhood of K_A. Also, if B and D are infinite subsets of $\mathbb{N}$, then D^* is contained in B^* if and only if $D \setminus B$ is finite [26].

Now we give the main result of this section.

Theorem 2.4 $M(c^{\mathcal{I}(q)}(b)) = \bigcap\limits_{T \in \mathcal{T}_q} M(c_T(b))$.

Proof Let $x \in \bigcap\limits_{T \in \mathcal{T}_q} M(c_T(b))$. We claim that $x.y \in c^{\mathcal{I}(q)}(b)$ for an arbitrary $y \in c^{\mathcal{I}(q)}(b)$. By Theorem 2.1, $c^{\mathcal{I}(q)}(b) \subseteq c_T(b)$ for every $T \in \mathcal{T}_q$. This implies that $x.y \in c_T(b)$. Hence $x.y \in \bigcap\limits_{T \in \mathcal{T}_q} c_T(b)$. By Theorem 2.1 we get that $x.y \in c^{\mathcal{I}(q)}(b)$. Thus $\bigcap\limits_{T \in \mathcal{T}_q} M(c_T(b)) \subseteq M(c^{\mathcal{I}(q)}(b))$.

In order to prove the converse inclusion, assume that $x \in M(c^{\mathcal{I}(q)}(b))$ and $T \in \mathcal{T}_q$. Then by previous theorem we have $x \in c^{\mathcal{I}(q)}(b)$. Let $K(k) := \left\{n \in \mathbb{N} : |x_n - a| < \frac{1}{k}\right\}$. Then $K^c(k) \in \mathcal{I}_c^{(q)}$. Hence $\chi_{K(k)} \in c^{\mathcal{I}(q)}(b)$ and $\chi_{K(k)}$ is $\mathcal{I}_c^{(q)}-$convergent to 1. Therefore Theorem 2.1 implies that, for each k, $\chi_{K(k)} \in \bigcap_{T \in \mathcal{T}_q} c_T(b)$. Thus $\chi_{K(k)}$ is $T-$summable to 1 (for each k). This means that $K^*(k) \supseteq K_T$ for each k and $\bigcap_{k=1}^{\infty} K^*(k) \supseteq K_T$. As K_T is compact and the sets $\{B^* : B \subseteq \mathbb{N}\}$ form a basis for the topology of $\beta\mathbb{N} \setminus \mathbb{N}$, there exists a set $K \subseteq \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} K^*(k) \supseteq K^* \supseteq K_T$. Moreover $\lim_n (T\chi_K)_n = 1$. Since $K^*(k) \supseteq K^*$ for each k, there are at most a finite number of members of K not in $K(k)$. Hence $|x_n - a| < \frac{1}{k}$ for all but a finite number of $n \in K$. As k is arbitrary, one can conclude that

$$\lim_{n \in K} x_n = a. \tag{2.2}$$

Now we have

$$\lim_n \sum_{j=1}^{\infty} t_{nj} = \lim_n \sum_{j \in K} t_{nj} + \lim_n \sum_{j \in \mathbb{N} \setminus K} t_{nj}.$$

Since T is regular and $\lim_n (T\chi_K)_n = 1$, we get $\lim_n \sum_{j \in \mathbb{N} \setminus K} t_{nj} = 0$. So $\chi_{\mathbb{N} \setminus K}$ is strongly $T-$summable to zero. Combining this with (2.2), we conclude by Theorem 2.2 that x is strongly $T-$summable to a. Now (2.1) implies $x \in M(c_T(b))$. This implies $x \in \bigcap_{T \in \mathcal{T}_q} M(c_T(b))$ for every T in $\mathcal{T}_q$ from which the result follows. $\square$

3. $\mathcal{I}$-core

Fridy and Orhan [13] introduced the concept of statistical core for a sequence and proved the statistical core theorem. Later on Demirci [6] extended this concept to $\mathcal{I}-$core. In this section using a result of Kolk [18] we prove Demirci's result in the necessary and sufficient form.

Throughout Sections 3 and 4 the spaces of all bounded and convergent complex sequences will be denoted by l_∞ and c, respectively. In the sequel $A = (a_{nk})$ be an infinite matrix with complex entries.

In [16] the Knopp core of the sequence x is defined by

$$K - core\{x\} := \cap_{n \in \mathbb{N}} C_n(x),$$

where $C_n(x)$ is the closed convex hull of $\{x_k\}_{k \geq n}$.

If x and y are sequences such that $\{n \in \mathbb{N} : x_n = y_n\} \notin \mathcal{I}$, then we write "$x_n = y_n$, for $\mathcal{I} - a.a.k$". Also in this and next sections x, y and z will denote complex number sequences. By $c^{\mathcal{I}}$, we denote the set of all $\mathcal{I}$-convergent sequences, where $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$.

Definition 3.1 ([6]) *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. For any complex sequence x let $H_{\mathcal{I}}(x)$ be the collection of all closed half-planes that contain x_n for $\mathcal{I} - a.a.k$; i.e.*

$$H_{\mathcal{I}}(x) := \{H : H \text{ is a closed half-plane, } \{n \in \mathbb{N} : x_n \notin H\} \in \mathcal{I}\},$$

then the $\mathcal{I}$-core of x is given by

$$\mathcal{I} - core\{x\} := \cap_{H \in H_{\mathcal{I}}(x)} H.$$

It is easy to see that $\mathcal{I} - core\{x\} \subseteq K - core\{x\}$ for all x.

The next result is due to Demirci [6] that gives some sufficient conditions under which we have the core inclusion $K - core\{Ax\} \subseteq \mathcal{I} - core\{x\}$ for $x \in l_\infty$.

Theorem 3.2 ([6]) *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If the matrix A satisfies $\sup_n \sum_k |a_{nk}| < \infty$ and the following conditions*

(i) A is regular and $\lim_n \sum_{k\in E} |a_{nk}| = 0$ whenever $E \in \mathcal{I}$

(ii) $\lim_n \sum_{k=1}^{\infty} |a_{nk}| = 1$,

then $K - core\{Ax\} \subseteq \mathcal{I} - core\{x\}$ for every $x \in l_\infty$.

In this section we show that these conditions are also necessary. But we first recall a version of a result of Kolk [18].

Proposition 3.3 ([18]) *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If A maps $c^{\mathcal{I}} \cap l_\infty$ into c and leaves the $\mathcal{I}$-limit invariant then*

(i) A is regular,

(ii) $\lim_n \sum_{k\in E} |a_{nk}| = 0$ for every $E \in \mathcal{I}$.

If $\mathcal{I}$ has property (AP), then the conditions are also sufficient for A in order to map $c^{\mathcal{I}} \cap l_\infty$ into c leaving the $\mathcal{I}$-limit invariant.

Now we have the next result which is the converse of Theorem 3.2.

Theorem 3.4 ($\mathcal{I}$-core theorem) *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If the matrix A satisfies $\sup_n \sum_k |a_{nk}| < \infty$, then*

$$K - core\{Ax\} \subseteq \mathcal{I} - core\{x\} \;\; for\ every\ x \in l_\infty \tag{3.1}$$

if and only if the following conditions hold:

(i) A is regular and $\lim_n \sum_{k\in E} |a_{nk}| = 0$ whenever $E \in \mathcal{I}$;

(ii) $\lim_n \sum_{k=1}^{\infty} |a_{nk}| = 1$.

Proof (Necessity) Assume that (3.1) hold and let $x \in c$ with $\lim_n x = L$, this yields $x \in c^{\mathcal{I}}$. We know that

$$\mathcal{I} - core\{x\} \subseteq K - core\{x\} \tag{3.2}$$

for all x. Since (3.1) hold and $K - core\{Ax\} \neq \varnothing$, it follows that $K - core\{Ax\} = \{L\}$. Hence A is regular and $A \in (c^{\mathcal{I}} \cap l_\infty, c; p)$ so by Proposition 3.3 we get $\lim_n \sum_{k\in E} |a_{nk}| = 0$ for every $E \in \mathcal{I}$.

From (3.1) and (3.2), observe that $K - core\{Ax\} \subseteq K - core\{x\}$ for every $x \in l_\infty$. Hence Knopp's core theorem (see e.g., [20]) yields that $\lim_n \sum_{k=1}^{\infty} |a_{nk}| = 1$.

Sufficiency follows from Theorem 3.2. □

The next result is an analog of Theorem 6.3.II of Cooke ([5, p. 144]).

Proposition 3.5 *Let I be an admissible ideal on $\mathbb{N}$. If $x = (x_k)$ and $y = (y_k)$ are $\mathcal{I}$-bounded sequences and*

$$\mathcal{I} - \limsup_k |x_k - y_k| = 0$$

then $\mathcal{I} - core\,\{x\} = \mathcal{I} - core\,\{y\}$.

Proof For each $z \in \mathbb{C}$, let

$$B_x(z) = \{w \in \mathbb{C} : |w - z| \leq \mathcal{I} - \limsup_k |x_k - z|\}.$$

Then $\mathcal{I} - core\,\{x\} := \cap_{z\in\mathbb{C}} B_x(z)$ (see [6, 12, 24]).

Let $w \in B_x(z)$. Then for any $z \in \mathbb{C}$ one can get that

$$\begin{aligned} |w - z| &\leq \mathcal{I} - \limsup_k |x_k - z| \\ &\leq \mathcal{I} - \limsup_k |x_k - y_k| + \mathcal{I} - \limsup_k |y_k - z| \\ &= \mathcal{I} - \limsup_k |y_k - z|, \end{aligned}$$

hence $w \in B_y(z)$ which implies that $B_x(z) \subseteq B_y(z)$. Interchanging the roles of x and y one can also observe that $B_y(z) \subseteq B_x(z)$. This yields that $\mathcal{I} - core\,\{x\} = \mathcal{I} - core\,\{y\}$. □

4. Core comparisons of two matrix transformations

Choudhary [2] extended Knopp's core theorem to the case in which the cores of two transformations are compared, i.e. the conclusion is so that replacing B by the identity matrix yields Knopp's theorem. In [12] Fridy and Orhan proved a statistical analogue of Choudhary's theorem. In this Section we get an ideal version of Choudhary's theorem.

Lemma 4.1 ([2]) *Consider a fixed n. In order that, whenever Bx is bounded, $(Ax)_n$ should be defined for that particular n, it is necessary and sufficient that*

(v) $c_{nk} = \sum_{j=k}^{\infty} a_{nj} b_{jk}^{-1}$ exist for all k;

(vi) $\sum_{k=0}^{\infty} |c_{nk}| < \infty$;

(iv) for any fixed n, $\lim_v \sum_{k=0}^{v} \left| \sum_{j=v+1}^{\infty} a_{nj} b_{jk}^{-1} \right| = 0$.

If these conditions are satisfied then, for bounded $y = Bx$,

$$(Ax)_n = \sum_{k=0}^{\infty} c_{nk} y_k.$$

Theorem 4.2 *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$, let B be a normal matrix (i.e. triangular with nonzero diagonal entries), and denote its triangular inverse by $B^{-1} = \left[b_{nk}^{-1}\right]$. For an arbitrary matrix A, in order that,* *whenever* $Bx \in l_\infty$, Ax *should exist and be bounded and satisfy*

$$K - core\{Ax\} \subseteq \mathcal{I} - core\{Bx\} \tag{4.1}$$

it is necessary and sufficient that the following conditions hold:

(i) $C := AB^{-1}$ *exists;*

(ii) C *is regular and* $\lim_n \sum_{k\in E} |c_{nk}| = 0$ *whenever* $E \in \mathcal{I}$;

(iii) $\lim_n \sum_{k=1}^{\infty} |c_{nk}| = 1$;

(iv) for any fixed n, $\lim_v \sum_{k=0}^{v} \left| \sum_{j=v+1}^{\infty} a_{nj} b_{jk}^{-1} \right| = 0$.

Proof (Necessity) If $(Ax)_n$ exist for every n whenever $Bx \in l_\infty$, then by Lemma 4.1 it follows immediately that (i) and (iv) hold. By that same Lemma we also have $Ax = Cy$, where $y = Bx$. Since $Ax \in l_\infty$ we have $Cy \in l_\infty$. Therefore (4.1) implies that $K - core\{Cy\} \subseteq \mathcal{I} - core\{y\}$. Now $\mathcal{I}-$core theorem (Theorem 3.4) implies that (ii) and (iii) hold.

(Sufficiency) Properties (i)–(iv) obviously imply the four conditions of Lemma 4.1, so it follows by the Lemma that $Cy \in l_\infty$, hence $Ax \in l_\infty$. Now $\mathcal{I}-$Core Theorem implies that $K - core\{Cy\} \subseteq \mathcal{I} - core\{y\}$, and since $y = Bx$ and $Cy = Ax$ we have $K - core\{Ax\} \subseteq \mathcal{I} - core\{Bx\}$. □

Note that the sequences and matrices in Choudhary's paper [2] have real entries. But a careful checking shows that Choudhary's results remain true when the sequences and matrices have the complex entries.

By Theorem 4.2, the fact that $\mathcal{I} - core\{Ax\} \subseteq K - core\{Ax\}$ gives us following corollary.

Corollary 4.3 *If* A *and* B *satisfy conditions (i)-(iv) of Theorem 4.2, then*

$$\mathcal{I} - core\{Ax\} \subseteq \mathcal{I} - core\{Bx\}$$

for every x *such that* $Bx \in l_\infty$.

Acknowledgment

The authors wish to thank anonymous referees for their helpful suggestions that improved the presentation of the paper.

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