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# On 2-algebras: crossed R-modules and categorical R-algebras

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**Abstract:** In this work, we describe the category of categorical $R$-algebras and show that a categorical $R$-algebra is a category object in $\mathcal{C} = A\lg_R$. By using this property of categorical $R$-algebras, we can give an equivalency between the category of categorical $R$-algebras and the category of crossed $R$-modules and also the category of simplicial $R$-algebras.

**Key words:** Category object, monoid object, categorical $R$-algebra, crossed $R$-module, simplicial $R$-algebra

## 1. Introduction

The category objects (or internal categories) are a generalisation of the notion of small category, and are defined with respect to a fixed arbitrary category. If the arbitrary category is taken to be the category of sets then one recovers the theory of small categories.

The idea is that given a category $\mathcal{C}$, we obtain the definition of a 'category in $\mathcal{C}$' by expressing the definition of a usual (small) category completely in terms of commutative diagrams and then interpreting those diagrams in $\mathcal{C}$.

Category objects were introduced by Ehresmann [9] in the 1960s, and by now they are an important part of category theory [5]. Some other works about internal categories can be found in [4, 12].

Crossed module was defined as a model of homotopy 2-types by Whitehead [13]. In [6, 7], crossed modules were considered as 2-dimensional groups. The commutative algebra version of this construction has been adapted by Arvasi and Porter [1, 11].

Crossed modules have been applied in many areas such that group presentations, algebraic K-theory and also homological algebra. In [8] and the study of Porter *, crossed module theory has been analysed with detail. For the (commutative) algebraic version of crossed modules can be seen in [1, 10, 11]. Also, Arvasi and Odabaş have given computing crossed modules in [2, 3].

In this paper, we give a notion called categorical $R$-algebra and show that a categorical $R$-algebra is a category object in $\mathcal{C} = A\lg_R$ and also a monoid object in $\mathcal{C} = \mathbf{Cat}$. After thinking a cetegorical $R$-algebra as a category object and a monoid object, we give some equivalencies about crossed $R$-modules and simplicial $R$-algebras.

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*The crossed menagerie: an introduction to crossed gadgetry and cohomology in algebra and topology [online]. Website https://ncatlab.org/nlab/files/menagerie14.pdf [23 September 2019].

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## 2. Preliminaries

We recall some standart definitions will be used in this work. See [1, 4, 10, 11] and the study of Porter.*

**The category of category objects** ($\mathbf{Cat}(\mathcal{C})$.):

Let $\mathcal{C}$ be any category. A category object in $\mathcal{C}$ consists of:

- An object of object $O$
- An object of morphism $M$

together with

- source and target morphisms $s, t : M \longrightarrow O$
- an identity morphism $e : O \longrightarrow M$
- a composition morphism $c : M \times_O M \longrightarrow M$

such that the following diagrams commute, expressing the usual category laws:

(i) laws specifying the source and target of composite morphisms

$$\begin{tikzcd} M \arrow[d, "t"'] & M \times_O M \arrow[l, "Pr_2"'] \arrow[r, "Pr_1"] \arrow[d, "c"] & M \arrow[d, "s"] \\ O & M \arrow[l, "t"] \arrow[r, "s"'] & O \end{tikzcd}$$

(ii) laws specifying the source and target of identity morphism

$$\begin{tikzcd} O & O \arrow[l, "id_O"'] \arrow[r, "id_O"] \arrow[d, "e"] & O \\ & M \arrow[ul, "t"] \arrow[ur, "s"'] & \end{tikzcd}$$

(iii) the associative law for composition of morphisms

$$\begin{tikzcd} M \times_O M \times_O M \arrow[r, "c \times id_M"] \arrow[d, "id_M \times c"'] & M \times_O M \arrow[d, "c"] \\ M \times_O M \arrow[r, "c"'] & M \end{tikzcd}$$

(iv) the left and right unit laws for composition of morphisms

$$\begin{tikzcd} O \times_O M \arrow[r, "e\times id_M"] \arrow[dr, "Pr_2"'] & M \times_O M \arrow[d, "c"] & M \times_O O \arrow[l, "id_M\times e"'] \arrow[dl, "Pr_1"] \\ & M & \end{tikzcd}$$

The pullback $M \times_O M$ is defined via the square:

$$\begin{tikzcd} M \times_O M \arrow[r, "\rho_2"] \arrow[d, "\rho_1"'] & M \arrow[d, "s"] \\ M \arrow[r, "t"'] & O \end{tikzcd}$$

We denote this category object with $C = (O, M, s, t, e, c)$.

A functor from $C$ to $D$ consists of

- a morphism $F_0 : O \longrightarrow O'$
- a morphism $F_1 : M \longrightarrow M'$

such that

$$s^C o = ms^D,\ t^C o = mt^D,\ e^C m = oe^D \text{ and } c^C m = (m \sqcap m)c^D.$$

where $C,\ D$ are category objectd in any category $\mathcal{C}$.

Thus we have a category of category objects in $\mathcal{C}$ and this category is denoted by $\mathbf{Cat}(\mathcal{C})$.

**Monoidal category:**

A monoidal category is a category $\mathcal{C}$ equipped with

- a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ out of the product category of $\mathcal{C}$ with itself, called tensor product,
- an object $I$ called the unit object
- a natural isomorphism

$$\alpha : ((-) \otimes (-)) \otimes (-) \xrightarrow{\cong} (-) \otimes ((-) \otimes (-))$$

  with components of the form

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z)$$

  called the associator,

- a natural isomorphism

$$\lambda : (1 \otimes (-)) \xrightarrow{\cong} (-)$$

  with components of the form

$$\lambda_x : (1 \otimes x) \xrightarrow{\cong} x$$

  called the left unitor, and

- a natural isomorphism

  $$\rho : ((-) \otimes 1) \xrightarrow{\cong} (-)$$

  with componets of the form

  $$\rho_x : (x \otimes 1) \xrightarrow{\cong} x$$

  called the right unitor

such that the axioms known as triangle and pentagon axioms are satisfied.

Examples:

(i) $\mathcal{C}$ =**Set** is a cartesian monoidal category with the cartesian product and one-element sets serving as the unit.

(ii) $\mathcal{C}$ = **Cat** is a cartesian monoidal category with the product category and the category with one object and only its identity morphism serving as the unit.

(iii) $\mathcal{C} = \mathbf{Mod}_R$ ($R$ is a commutative ring) is a monoidal category with the tensor product of modules $\otimes_R$ serving as the monoidal product and the ring $R$ serving as the unit.

(iv) $\mathcal{C} = \mathbf{Alg}_R$ ($R$ is a commutative ring) is a monoidal category with the tensor product of algebras as the monoidal product and $R$ as the unit.

**The category of monoid objects** ($\mathbf{Mon}(\mathcal{C})$):

A monoid object in a monoidal category $(\mathcal{C}, \otimes, I)$ is an object $E$ in $\mathcal{C}$ together with two morphisms

$$\begin{array}{llcll} \mu : & E \otimes E & \longrightarrow & E & \text{called multiplication,} \\ \eta : & I & \longrightarrow & E & \text{called unit,} \end{array}$$

such that the following diagrams commutative:

(i)

$$\begin{tikzcd} (E \otimes E) \otimes E \arrow[r, "\alpha"] \arrow[d, "\mu\otimes 1"'] & E \otimes (E \otimes E) \arrow[r, "1\otimes\mu"] & E \otimes E \arrow[d, "\mu"] \\ E \otimes E \arrow[rr, "\mu"'] & & E \end{tikzcd}$$

(ii)

$$\begin{tikzcd} I \otimes E \arrow[r, "\eta\otimes 1"] \arrow[dr, "\lambda"'] & E \otimes E \arrow[d, "\mu"] & E \otimes I \arrow[l, "1\otimes\eta"'] \arrow[dl, "\rho"] \\ & E & \end{tikzcd}$$

In the above notions, $I$ is the unit element and $\alpha, \lambda$ and $\rho$ are respectively the associativity, the left identity and right identity of the monoidal category $\mathcal{C}$.

Examples:

(a) A monoid object in the category of sets $\mathcal{C} = \mathbf{Sets}$ is a monoid.

(b) A monoid object in the category of $R$-modules $\mathcal{C} = \mathbf{Mod}_R$ is an $R$-algebra.

(c) A monoid object in the category of simplicial $R$-modules $\mathcal{C}$ =$\mathbf{sMod}_R$ is a simplicial $R$-algebra.

(d) A monoid object in the category of categories $\mathcal{C}$ =**Cat** is a small category $\mathcal{C}$ with two functors $\mu : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and $\eta : \mathcal{C}^{\otimes 0} \longrightarrow \mathcal{C}$ where $\mathcal{C}^{\otimes 0}$ is a category with one object and only its identity morphism.

Let $C$ and $D$ be monoid objects in $\mathcal{C}$. A monoid morphism from $C$ to $D$ in $\mathcal{C}$ is a morphism $C \xrightarrow{\varphi} D$ that is compatible with the multiplication and unit, that is the diagrams

$$\begin{tikzcd} C \otimes C \arrow[r, "\mu^C"] \arrow[d, "\varphi\otimes\varphi"'] & C \arrow[d, "\varphi"] \\ D \otimes D \arrow[r, "\mu^D"'] & D \end{tikzcd}$$

and

$$\begin{tikzcd} I \arrow[r, "\eta^C"] \arrow[dr, "\eta^D"'] & C \arrow[d, "\varphi"] \\ & D \end{tikzcd}$$

are commutative. ($\mu^C\varphi = (\varphi \otimes \varphi)\mu^D$ and $\eta^C\varphi = \eta^D$)

Thus, we have a category of monoid objects which is denoted by $\mathbf{Mon}(\mathcal{C})$.

**Crossed modules:**

In this paper, all algebras will be commutative. Also, we accept that $k$ is a commutative ring, $R$ is a $k$-algebra with identity.

A crossed module, $(C, R, \partial)$, (or shortly crossed $R$-module) consists of an $R$-algebra $C$ and a morphism $\partial : C \longrightarrow R$ with actions $R$ on $C$, written $(r, c) \longmapsto r \cdot c$ for $r \in R,\ c \in C$, satisfying the following conditions:

$(CM1)$

$$\partial(r \cdot c) = r\partial(c)$$

for all $r \in R$ and $c \in C$.

$(CM2)$

$$\partial(c) \cdot c' = cc'$$

for all $c, c' \in C$.

A morphism, $(\alpha, \beta) : V \longrightarrow V'$ of crossed modules consists of morphisms $\alpha : C \longrightarrow C'$ and $\beta : R \longrightarrow R'$ such that

$$\begin{array}{ll} (i) & \partial\beta = \alpha\partial' \\ (ii) & \alpha(r \cdot c) = \alpha(c) \cdot \beta(r) \end{array}$$

for all $c \in C,\ r \in R$ where $V = (C, R, \partial)$ and $V' = (C', R', \partial')$ are crossed modules.

Thus, we have a category of crossed modules from the above definitions and it is denoted by **XMod**.

Examples:

(a) Let $A$ be an $R$-algebra and $I$ be an ideal of $A$. Then $(I, A, i)$ is a crossed module with the multiplication action of $A$ on $I$. Conversely, we induce an ideal from a given crossed module. Indeed, for a given crossed module $(C, R, \partial)$, $\partial(C)$ is an ideal of $R$.

(b) Let $M$ be an $R$-module. Then $(M, R, 0)$ is a crossed module. Conversely, for a crossed module $(C, R, \partial)$, one can get that $\ker \partial$ is an $R/\partial(C)$-module.

(c) Let $\partial : C \longrightarrow R$ be a surjective $R$-algebra homomorphism. Define the action of $R$ on $C$ by $r \cdot c = \widetilde{r}c$ where $\widetilde{r} \in \partial^{-1}(r)$. Then, $(C, R, \partial)$ is a crossed module with the defined action.

(d) Let $C$ be an $R$-algebra such that $Ann(C) = 0$ or $C^2 = C$ then $(C, \mu(C), \partial)$ is a crossed module, where $\mu(C)$ is the algebra of multipliers of $C$ and $\partial$ is the canonical homomorphism.

**Simplicial algebras:**

Let $R$ be a commutative ring with identity. We will use the term commutative algebra to mean a commutative algebra over $R$.

A simplicial (commutative) algebra $\mathbf{E}$ consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1}, \quad 0 \leq i \leq n, \ (n \neq 0)$ and $s_i = s_i^n : E_n \to E_{n+1}, \quad 0 \leq i \leq n$, satisfying the usual simplicial identities. It can be completely described as a functor $\Delta^{op} \to \mathbf{Alg}_{\mathrm{R}}$ where $\Delta$ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps. We have a category of simplicial algebras the above definitions and it is denoted by $\mathbf{sAlg}_{\mathrm{R}}$. We have for each $k \geq 0$ a subcategory $\Delta_{\leq k}$ determined by the objects $[j]$ of $\Delta$ with $j \leq k$. A $k$-truncated simplicial commutative algebra is a functor from $(\Delta_{\leq k}^{op})$ to $\mathbf{Alg}_{\mathrm{R}}$.

Consider the product $\Delta \times \Delta$ whose objects are pairs $([p],[q])$ and whose maps are pairs of weakly increasing maps. A functor $(\Delta \times \Delta)^{op} \to \mathbf{Alg}_{\mathrm{R}}$ is called a bisimplicial algebra with value in $\mathbf{Alg}_{\mathrm{R}}$. This functor consists of algebras $E_{p,q}$ and homomorphisms

$$\begin{array}{ll} d_i^h : E_{p,q} \to E_{p-1,q} & \\ s_i^h : E_{p,q} \to E_{p+1,q} & i : 0, \ldots, p \\ d_j^v : E_{p,q} \to E_{p,q-1} & \\ s_j^v : E_{p,q} \to E_{p,q+1} & j : 0, \ldots, q \end{array}$$

such that the maps $d_i^h, s_i^h$ commute with $d_j^v, s_j^v$ and $d_i^h, s_i^h$ (resp. $d_j^v, s_j^v$) satisfy the usual simplicial identities. Here let $d_i^h, s_i^h$ denote the horizontal operators and let $d_j^v, s_j^v$ denote the vertical operators. We have a category of bisimplicial algebras from the above definitions and it is denoted by $\mathbf{s^2\,Alg}_{\mathrm{R}}$.

## 3. Categorical R-algebras as category objects

In this section, we give a notion called categorical $R$-algebra and show that a categorical $R$-algebra is a category object in $\mathcal{C} = A\lg_R$.

**Definition 3.1** *A categorical $R$-algebra is a (small) category $\mathcal{C}$ equipped with*

- $R$-algebras $O = Ob\mathcal{C}$ and $M = Mor\mathcal{C}$ ;
- a functor
$$\mu^{\mathcal{C}} : \mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C}$$
produced by the multiplication morphisms
$$\mu^O : O \times O \longrightarrow O$$
and
$$\mu^M : M \times M \longrightarrow M.$$

If $\mathcal{C}$ and $\mathcal{D}$ are categorical $R$-algebras, a morphism $\mathcal{C}\xrightarrow{\varphi}\mathcal{D}$ is a functor. Then one has a category, of course. It will usually be denoted by $\mathbf{cAlg}_R$.

**Lemma 3.2** *Let $\mathcal{C}$ be a categorical $R$-algebra.*
*(a) The source, target and identity morphisms*

$$\begin{tikzcd} M \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O \arrow[l, bend left=40, "e"] \end{tikzcd}$$

*and the composition morphism*

$$c : M \times_O M \longrightarrow M$$

*are $R$-algebra morphisms.*
*(b) If $\mathcal{C}^{\otimes 0}$ is a small category with one object and only its identity morphism, then*

$$\eta^{\mathcal{C}} : \mathcal{C}^{\otimes 0} \longrightarrow \mathcal{C}$$

*is a functor.*

**Proof** (a) From $\mathcal{C}\otimes\mathcal{C}\xrightarrow{\mu^{\mathcal{C}}}\mathcal{C}$ is a functor, we have the following commutative diagrams:

$$\begin{tikzcd} M \otimes M \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\alpha"'] & O \otimes O \arrow[l, bend left=40, "e"] \arrow[d, "\beta"] \\ M \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O \arrow[l, bend left=40, "e"] \end{tikzcd}$$

where $\alpha = Mor\mu^{\mathcal{C}}$ and $\beta = Ob\mu^{\mathcal{C}}$ , that is

$$\begin{array}{rcl} \alpha s & = & s\beta \\ \alpha t & = & t\beta \\ \beta e & = & e\alpha \end{array}$$

and

$$\begin{tikzcd} \mathcal{M} \times_O \mathcal{M} \arrow[r, "c"] \arrow[d, "\alpha\times\alpha"'] & \mathcal{M} \arrow[d, "\alpha"] \\ M \times_O M \arrow[r, "c"'] & M \end{tikzcd}$$

that is,

$$c\alpha = (\alpha \times \alpha)c.$$

where $\mathcal{M} = Mor(\mathcal{C}\otimes\mathcal{C})$.

So, we can write

$$\begin{array}{rcl} \gamma s & = & \alpha s \\ & = & s\beta \\ & = & (s \otimes s)\delta \end{array}$$

and

$$\begin{aligned}\gamma t &= \alpha t\\ &= t\beta\\ &= (t\otimes t)\delta\end{aligned}$$

as well as

$$\begin{aligned}\delta e &= \beta e\\ &= e\alpha\\ &= (e\otimes e)\gamma.\end{aligned}$$

where $\gamma = \mu^M$ and $\delta = \mu^O$.

By considering the canonical isomorphism

$$\xi : (M \times_O M) \otimes (M \times_O M) \longrightarrow (M\otimes M) \times_O (M\otimes M)$$

we also have

$$\begin{aligned}\varepsilon c &= \xi(\gamma\times\gamma)c\\ &= \xi(\alpha\times\alpha)c\\ &= \xi c\alpha\\ &= (c\otimes c)\alpha,\end{aligned}$$

where $\varepsilon = \mu^{M\times_O M}$. Thus, $s,\ t,\ e$ and $c$ are $R$-algebra homomorphisms.

(b) According to (a), the structure morphisms $s,t,e$ and $c$ are $R$-algebra homomorphisms. Therefore, we have

$$\begin{aligned}\eta^M s &= \eta^O = s^{\otimes 0}\eta^O = s\eta^O,\\ \eta^M t &= \eta^O = t^{\otimes 0}\eta^O = t\eta^O,\\ \eta^O e &= \eta^M = e^{\otimes 0}\eta^M = e\eta^M\end{aligned}$$

and

$$\begin{aligned}(\eta^M \times_O \eta^M)c &= \eta^{M_t \sqcap_s M} c\\ &= c^{\otimes 0}\eta^M\\ &= c\eta^M,\end{aligned}$$

that is, $\eta^{\mathcal{C}}$ with $Ob(\eta^{\mathcal{C}}) := \eta^O$ and $Mor(\eta^{\mathcal{C}}) := \eta^M$ is a functor. □

Thus, a categorical $R$-algebra is a category with $R$-algebras $O$ and $M$ such that the source and target maps $s,t : M \longrightarrow O$, the identity map $e : O \longrightarrow M$, and the composition map $c : M \times_O M \longrightarrow M$ are $R$-algebra morphisms.

**Proposition 3.3** *If $\mathcal{C}$ is a categorical $R$-algebra, then* *(i) the composition can be given as*

$$\begin{array}{rccc} c: & M\times_O M & \longrightarrow & M \\ & (m,n) & \longmapsto & m-(et)(m)+n \end{array}$$

*with an inverse of* $m$ *up to composition is* $(et)(m)-m+(es)(m)$.

*(ii)* $KertKers=\{0\}$.

**Proof** (i) $t(m)=s(n)$ where $m,n\in M$ are composable morphisms. As $c$ and $e$ are $R$-algebra homomorphisms, we can write

$$\begin{aligned} c(m,n) &= c(m+0,0+n) \\ &= c(m+0,(et)(m)-(et)(m)+n) \\ &= c(m,(et)(m))+c(0,-(et)(m)+n) \\ &= m-(et)(m)+n. \end{aligned}$$

On the other hand, as

$$t((et)(m)-m+(es)(m))=t(et)(m)-t(m)+t(es)(m)=s(m)$$

$$s((et)(m)-m+(es)(m))=s(et)(m)-s(m)+s(es)(m)=t(m)$$

also

$$c(m,(et)(m)-m+(es)(m))=m-(et)(m)+(et)(m)-m+(es)(m)=(es)(m)$$

while

$$c((et)(m)-m+(es)(m),m)=(et)(m)-m+(es)(m)-(es)(m)+m=(et)(m),$$

we can get that $(et)(m)-m+(es)(m)$ is the inverse of $m$.

(ii) Let $m\in Kert,\ n\in Kers$. Then we have $t(m)=0=s(n)$ and according to (a) we can write

$$\begin{aligned} mn &= m-(et)(m)+n \\ &= m-(es)(n)+n \\ &= n-(et)(m)+m \\ &= n-(es)(n)+m \\ &= nm. \end{aligned}$$

Thus, we have $KertKers=\{0\}$. □

**Lemma 3.4** *If* $M \underset{t}{\overset{s}{\rightrightarrows}} O$ *is retraction (i.e,* $se=te=id$*) and* $KersKert=\{0\}$, *then there exists a categorical* $R$*-algebra* $\mathcal{C}$ *with* $Ob\mathcal{C}:=O,\ Mor\mathcal{C}:=M$, *and categorical structure morphisms* $s,\ t,\ e$.

**Proof** We have composite $c(m,n) := m-(et)(m)+n$, for elements $m,n \in M$ with $t(m)=s(n)$. On the other hand, for $m,n,m',n' \in M$ with $t(m)=s(n)$ and $t(m')=s(n')$

$$
\begin{aligned}
c(m,n)c(m',n') &= (m-(et)(m)+n)(m'-(et)(m')+n') \\
&= (m-(et)(m))m' + nm' + (m-(et)(m))(n'-(es)(n')) + n(n'-(es)(n')) \\
&= mm' - (et)(m)m' + nm' + mn' - (et)(m)n' - m(es)(n') + (et)(mm') + nn' \\
&\quad -n(es)(n') \\
&= c(mm', nn') + A + B
\end{aligned}
$$

where

$$
\begin{aligned}
A &= mn' - (et)(m)n' - m(es)(n') + (et)(m)(es)(n') \\
&= (m-(et)(m))(n'-(es)(n')) \in KertKers
\end{aligned}
$$

and

$$
\begin{aligned}
B &= nm' - (es)(n)m' - n(et)(m') + (es)(n)(et)(m') \\
&= (n-(es)(n))(m'-(et)(m')) \in KersKert.
\end{aligned}
$$

Since $KersKert = \{0\}$, $A = B = 0$. So,

$$c(m,n)c(m',n') = c(mm', nn') = c((m,n)(m',n')).$$

That is, the morphism ($R$-module homomorphism) $c$ is an $R$-algebra homomorphism. Now, we check the category axioms given in definition of category object in $\mathcal{C}$.

(i) We have

$$
\begin{aligned}
(sc)(m,n) &= s(m-(et)(m)+n) \\
&= s(m) - s(et)(m) + s(n) \\
&= s(m) - s(n) + s(n) \\
&= s(m)
\end{aligned}
$$

and

$$
\begin{aligned}
(tc)(m,n) &= t(m-(et)(m)+n) \\
&= t(m) - t(et)(m) + t(n) \\
&= t(m) - t(m) + t(n) \\
&= t(n)
\end{aligned}
$$

for all $m,n \in M$ with $t(m)=s(n)$.

(ii) The identities $se = te = id$ are given by assumption.

(iii) We have

$$\begin{aligned} c(k,c(m,n)) &= c(k,m-(et)(m)+n) \\ &= k-(et)(k)+m-(et)(m)+n \\ &= k-(es)(m)+m-(es)(n)+n \\ &= c(k-(es)(m)+m,n) \\ &= c((k,m),n) \end{aligned}$$

for all $k,m,n \in M$ with $t(k)=s(m)$ and $t(m)=s(n)$, therefore the composition is associative.

(iv) We have

$$\begin{aligned} c((es)(m),m) &= (es)(m)-(et)(es)(m)+m \\ &= (es)(m)-(es)(m)+m \\ &= m \end{aligned}$$

and

$$\begin{aligned} c(m,(et)(m)) &= m-(et)(m)+(et)(m) \\ &= m \end{aligned}$$

for $m \in M$.

Thus, $\mathcal{C}$ with $Ob\mathcal{C} := O$, $Mor\mathcal{C} := M$ and $s^{\mathcal{C}} = s$, $t^{\mathcal{C}} = t$, $e^{\mathcal{C}} = e$, $c^{\mathcal{C}} = c$ is a category object in $\mathbf{Alg}_R$. □

**Corollary 3.5** *A categorical $R$-algebra is a category object in $\mathcal{C} = \mathbf{Alg}_R$.*

**Proof** Let $\mathcal{C}$ be categorical $R$-algebra. Then $\mathcal{C}$ is a (small) category such that $O$ and $M$ are $R$-algebras. According to Lemma 3.2 (a), the categorical structure morphisms $s,t,e$ and $c$ are $R$-algebra morphisms. On the other hand, from the Proposition 3.3, the composition morphism is defined as

$$\begin{array}{rccc} c: & M\times_O M & \longrightarrow & M \\ & (m,n) & \longmapsto & m-(et)(m)+n. \end{array}$$

We will show that this composition map satisfy the diagrams of the definition of category object. The proofs of (i)-(iii)-(iv) are similar to the proofs given in Lemma 3.4. Hence we only show (ii).

(ii) For $m: x \to y$, we have

$$\begin{aligned} c(e(x),m) &= e(x)-es(m)+m \\ &= e(x)-e(x)+m \\ &= m \end{aligned}$$

and

$$\begin{aligned} c(m,e(y)) &= m-es(e(y))+e(y) \\ &= m-e(y)+e(y) \\ &= m. \end{aligned}$$

□

## 4. Equivalence between categorical R-algebras and crossed R-modules

In this section, we show that the categories **XMod** and $\mathbf{cAlg}_R$ are equivalent. By using Corollary 3.5, we can show this equivalency. Of course, this is the similar way of Porter works [11, 12].

Now, we remember that for the given crossed module $(C, R, \partial)$, the semidirect product $C \rtimes R$ is formed using the action of $R$ on $C$ the crossed module provides. Hence, we have

$$(c, r)(c', r') = (cc' + cr' + rc', rr')$$

for $(c, r), (c', r') \in C \rtimes R$ .

**Proposition 4.1** *Given a crossed module $(C, R, \partial)$, we have a categorical $R$-algebra $\mathcal{C}$, in which the objects and morphisms are given by*

$$Ob\mathcal{C} := R \text{ and } Mor\mathcal{C} := C \rtimes R,$$

*source, target and identity morphisms ($R$-module homomorphisms) are given by*

$$\begin{aligned} s(c, r) &= \partial(c) + r \\ t(c, r) &= r \\ e(r) &= (0, r) \end{aligned}$$

*for $(c, r) \in C \rtimes R$ and $r \in Ob\mathcal{C}$, the $R$-algebra of composable morphisms is*

$$\{(c_2, \partial(c_1) + r_1), (c_1, r_1)) \in Mor\mathcal{C} \times Mor\mathcal{C} \mid c_1, c_2 \in C,\ r_1 \in R\}$$

*and the composition in $\mathcal{C}$ is given by*

$$c((c_2, \partial(c_1) + r_1), (c_1, r_1)) := (c_2 + c_1, r_1)$$

*for $c_1, c_2 \in C,\ r_1 \in R$ .*

**Proof** Since $Ob\mathcal{C} = R$ and $Mor\mathcal{C} = C \rtimes R$ are $R$-algebras, firstly we show that $s, t$ and $e$ are $R$-algebra homomorphisms. We get

$$\begin{aligned} s((c, r)(c', r')) &= s(cc' + cr' + rc', rr') \\ &= \partial(cc' + cr' + rc') + rr' \\ &= \partial(cc') + \partial(cr') + \partial(rc') + rr' \end{aligned}$$

$$\begin{aligned} s(c, r)s(c', r') &= (\partial(c) + r)(\partial(c') + r') \\ &= \partial(c)\partial(c') + \partial(c)r' + r\partial(c') + rr' \\ &= \partial(cc') + \partial(cr') + \partial(rc') + rr' \end{aligned}$$

that is $s((c, r)(c', r')) = s(c, r)s(c', r')$ and

$$\begin{aligned} t((c, r)(c', r')) &= t(cc' + cr' + rc', rr') \\ &= rr' \\ &= t(c, r)t(c', r') \end{aligned}$$

for $(c,r),(c',r')\in Mor\mathcal{C}$ as well as

$$e(rr')=(0,rr')=(0,r)(0,r')=e(r)e(r')$$

for $r,r'\in Ob\mathcal{C}$.

The $R$-algebra of composable morphisms can be computed as follows:

$$\begin{aligned} Mor\mathcal{C}\times_O Mor\mathcal{C} &= \{((c_2,r_2),(c_1,r_1))\in MorC\times MorC \mid t(c_2,r_2)=s(c_1,r_1)\} \\ &= \{((c_2,r_2),(c_1,r_1))\in MorC\times MorC \mid r_2=\partial(c_1)+r_1\} \\ &= \{((c_2,\partial(c_1)+r_1),(c_1,r_1))\in MorC\times MorC \mid c_1,c_2\in C, r_1\in R\}. \end{aligned}$$

Thus $c$ is an $R$-algebra homomorphism since

$$\begin{aligned} & c(((c_2,\partial(c_1)+r_1),(c_1,r_1))((c_2',\partial(c_1')+r_1'),(c_1',r_1'))) \\ = \;& c((c_2,\partial(c_1)+r_1)(c_2',\partial(c_1')+r_1'),(c_1,r_1)(c_1',r_1')) \\ = \;& c((c_2c_2'+c_2(\partial(c_1')+r_1')+(\partial(c_1)+r_1)c_2',(\partial(c_1)+r_1)(\partial(c_1')+r_1')), \\ & (c_1c_1'+c_1r_1'+r_1c_1',r_1r_1')) \\ = \;& c((c_2c_2'+c_2c_1'+c_2r_1'+c_1c_2'+r_1c_2',\partial(c_1c_1')+\partial(c_1r_1')+\partial(r_1c_1')+r_1r_1'), \\ & (c_1c_1'+c_1r_1'+r_1c_1',r_1r_1')) \\ = \;& (c_2c_2'+c_2c_1'+c_2r_1'+c_1c_2'+r_1c_2'+c_1c_1'+c_1r_1'+r_1c_1',r_1r_1') \end{aligned}$$

and on the other hand

$$\begin{aligned} & c((c_2,\partial(c_1)+r_1),(c_1,r_1))c((c_2',\partial(c_1')+r_1'),(c_1',r_1'))) \\ = \;& (c_1+c_2,r_1)(c_1'+c_2',r_1') \\ = \;& (c_1c_1'+c_1c_2'+c_2c_1'+c_2c_2'+r_1c_1'+r_1c_2'+c_1r_1'+c_2r_1',r_1r_1'). \end{aligned}$$

Now, we have to check that $\mathcal{C}$ satisfies the axioms for a category object in $\mathcal{C}=\mathbf{Alg}_R$ given in its definition.

(i) We have

$$(se)(r)=s(e(r))=s(0,r)=\partial(0)+r=r$$

and

$$(te)(r)=t(e(r))=t(0,r)=r$$

for $r\in Ob\mathcal{C}$.

(ii) Given a pair of composable morphisms $((c_2,\partial(c_1)+r_1),(c_1,r_1))$ in $\mathcal{C}$, we have

$$\begin{aligned} (sc)((c_2,\partial(c_1)+r_1),(c_1,r_1)) &= s(c_2+c_1,r_1) \\ &= \partial(c_2+c_1)+r_1 \\ &= \partial(c_2)+\partial(c_1)+r_1 \\ &= s(c_2,\partial(c_1)+r_1) \end{aligned}$$

and

$$\begin{aligned} (tc)((c_2,\partial(c_1)+r_1),(c_1,r_1)) &= t(c_2+c_1,r_1) \\ &= r_1 \\ &= t(c_1,r_1) \end{aligned}$$

(iii) We have

$$\begin{aligned} & c((c_3, \partial(c_2) + \partial(c_1) + r_1), c((c_2, \partial(c_1) + r_1), (c_1, r_1)))) \\ = \;& c((c_3, \partial(c_2) + \partial(c_1) + r_1), (c_2 + c_1, r_1)) \\ = \;& (c_3 + c_2 + c_1, r_1) \\ = \;& c((c_3 + c_2, \partial(c_1) + r_1), (c_1, r_1)) \\ = \;& c(c(c_3, \partial(c_2) + \partial(c_1) + r_1), (c_2, \partial(c_1) + r_1), (c_1, r_1)) \end{aligned}$$

for $c_1, c_2, c_3 \in C, r_1 \in R$.

(iv) We get

$$\begin{aligned} c((es)(c,r),(c,r)) &= c(e(\partial(c)+r),(c,r)) \\ &= c((0,\partial(c)+r),(c,r)) \\ &= (0+c,r) = (c,r) \end{aligned}$$

and$\varphi$

$$\begin{aligned} c((c,r),(et)(c,r)) &= c((c,r),(0,r)) \\ &= (c+0,r) = (c,r) \end{aligned}$$

for $(c,r) \in Mor\mathcal{C}$.

Thus, $\mathcal{C}$ is a category object in $\mathbf{Alg}_R$. □

**Corollary 4.2** *We have a functor*

$$F : \mathbf{XMod} \longrightarrow \mathbf{cAlg}_R.$$

**Proposition 4.3** *For every categorical $R$-algebra $\mathcal{C}$, we have a crossed module $(C, R, \partial)$ with given by*

$$C := Kert, \;\; R := Ob\mathcal{C}$$

*structure morphism $\partial := s|_{Kert}$, where the action of the $R$ on $C$ is given by $r \cdot c := e(r)c$ for $r \in R, \;\; c \in C$.*

**Proof** Since $Mor\mathcal{C}$ is an $R$-algebra and $s$ is an $R$-algebra homomorphism, $Kert$ is an $R$-algebra and $s|_{Kert}$ is an $R$-algebra homomorphism. There is a well defined $Ob\mathcal{C}$-action on $Kert$ because of $e$ is an $R$-algebra homomorphism. Now, we will show that $(CM1)$ and $(CM2)$.

(CM1) We have

$$\begin{aligned} s(r \cdot c) &= s(e(r)c) \\ &= s(e(r))s(c) \\ &= rs(c) \end{aligned}$$

for $r \in R = Ob\mathcal{C}$ and all $c \in Mor\mathcal{C}$, and hence in particular

$$(s|_{Kert})(r \cdot c) = r(s|_{Kert(c)})$$

for $r \in Ob\mathcal{C}$, $c \in Kert$.

(CM2) We have

$$\begin{aligned} s|_{Kert}(n) \cdot m &= s(n) \cdot m \\ &= e(s(n))m \\ &= nm \end{aligned}$$

for all $n, m \in Kert$. □

**Corollary 4.4** *We have a functor*

$$G : \mathbf{cAlg}_R \longrightarrow \mathbf{XMod}.$$

**Theorem 4.5** *The categories* $\mathbf{XMod}$ *and* $\mathbf{c}\,\boldsymbol{Alg}_R$ *are equivalent.*

**Proof** For the functors $\mathbf{XMod} \xrightarrow{F} \mathbf{cAlg}_R$ and $\mathbf{cAlg}_R \xrightarrow{G} \mathbf{XMod}$ , we show that $GF \cong id_{cAlg_R}$ and $FG \cong id_{XMod}$. Let $(C, R, \partial)$ be a crossed module. We have

$$\begin{aligned} FG(C, R, \partial) &= F(R, C \rtimes R, s, t, e, c) \\ &= (Kert, R, s|_{Kert}) \end{aligned}$$

and

$$\begin{aligned} Kert &= \{(c, r) \mid t(c, r) = 0\} \\ &= \{(c, r) \mid r = 0\} \\ &= \{(c, 0) \mid c \in C\} \\ &= C \rtimes \{0\}. \end{aligned}$$

Also

$$\begin{array}{rcl} R \times Kert & \longrightarrow & R \\ (r, (c, 0)) & \longmapsto & r \cdot (c, 0) = e(r)(c, 0) = (0, r)(c, 0) = (r \cdot c, 0) \end{array}$$

and

$$s|_{Kert}(c, 0) = s(c, 0) = \partial(c) + 0 = \partial(c).$$

That is we obtain $FG(C, R, \partial) = (C \rtimes \{0\}, R, s|_{Kert}) \cong (C, R, \partial) = id_{XMod}(C, R, \partial)$.
On the other hand, let $C = (ObC, MorC, s, t, e, c)$ be a categorical $R$-algebra. We have

$$\begin{aligned} GF(C) &= G(Kert, ObC, s|_{Kert}) \\ &= (ObC, Kert \rtimes ObC, s, t, e, c) \end{aligned}$$

and for $(m, o) \in Kert \rtimes ObC$ we can write

$$\begin{aligned} s(m, o) &= s|_{Kert}(m) + o = s(m) + o \\ t(m, o) &= o \\ e(o) &= (0, o). \end{aligned}$$

Now, we must show $Kert \rtimes ObC \cong MorC$. Firstly, we define a function such that

$$\begin{array}{rlll} \varphi: & MorC & \longrightarrow & Kert \rtimes ObC \\ & m & \longmapsto & (m-(et)(m), t(m)). \end{array}$$

Since

$$\begin{aligned} t(m-(et)(m)) &= t(m) - t(et)(m) \\ &= t(m) - t(m) = 0 \end{aligned}$$

$m-(et)(m) \in Kert$, $\varphi$ is well defined. Also since for $m, n \in MorC$

$$\begin{aligned} \varphi(mn) &= (mn-(et)(mn), t(mn)) \\ &= (mn-(et)(m)(et)(n), t(mn)) \end{aligned}$$

and

$\varphi(m)\varphi(n) = (m-(et)(m), t(m))(n-(et)(n), t(n))$
$= ((m-(et)(m))(n-(et)(n)) + (m-(et)(m))t(n) + t(m)(n-(et)(n)), t(m)t(n))$
$= (mn - m(et)(n) - (et)(m)n + (et)(m)(et)(n) + (m-(et)(m))e(t(n)) + e(t(m))(n-(et(n))), t(mn))$
($\because$action of $ObC$ on $Kert$ )
$= (mn-(et)(m)(et)(n), t(mn),$

$\varphi$ is a homomorphism. Also, we have an antihomomorphism

$$\begin{array}{rlll} \varphi^{-1}: & Kert \rtimes ObC & \longrightarrow & MorC \\ & (m, o) & \longmapsto & e(o) + m \end{array}$$

and since

$$\begin{aligned} \varphi\varphi^{-1}(m, o) &= \varphi(\varphi^{-1}(m, o)) \\ &= \varphi(e(o) + m) \\ &= (e(o) + m - (et)(e(o) + m), t(e(o) + m)) \\ &= (e(o) + m - e(te)(o) - (et)(m), (te)(o) + t(m)) \\ &= (m, o) \ (\because te = id \text{ and } m \in Kert) \end{aligned}$$

and

$$\begin{aligned} \varphi^{-1}\varphi(m) &= \varphi^{-1}(\varphi(m)) \\ &= \varphi^{-1}(m-(et)(m), t(m)) \\ &= e(t(m)) + m - (et)(m) \\ &= m, \end{aligned}$$

$\varphi$ is an isomorphism. Thus $GF \cong id_{cAlg_R}$. □

## 5. Categorical R-algebras as monoid objects

**Proposition 5.1** *A categorical $R$-algebra is a monoid object in $\mathcal{C} = Cat$.*

**Proof** Let $\mathcal{C}$ be a categorical $R$-algebra. Then $O$ and $M$ are $R$-algebras, that is, we have

$$\begin{aligned}(1 \otimes \mu^A)\mu^A &= (\mu^A \otimes 1)\mu^A,\\ (\eta^A \otimes 1)\mu^A &= \mathrm{Pr}_2 \text{ and } (1 \otimes \eta^A)\mu^A = \mathrm{Pr}_1,\end{aligned}$$

for $A \in \{O, M\}$, cf. definition of monoid object in $\mathcal{C}$. Furthermore, by the definition of a categorical $R$-algebra, we have a functor $\mu^{\mathcal{C}}$ given by $Ob(\mu^{\mathcal{C}}) := \mu^O$ and $Mor(\mu^{\mathcal{C}}) := \mu^M$. Additionally, Lemma 3.2 (b) tells us that there is a functor $\eta^{\mathcal{C}}$ where $Ob(\eta^{\mathcal{C}}) := \eta^O$, $Mor(\eta^{\mathcal{C}}) := \eta^M$. This implies

$$\begin{aligned}(1 \otimes \mu^{\mathcal{C}})\mu^{\mathcal{C}} &= (\mu^{\mathcal{C}} \otimes 1)\mu^{\mathcal{C}},\\ (\eta^{\mathcal{C}} \otimes 1)\mu^{\mathcal{C}} &= \mathrm{Pr}_2 \text{ and } (1 \otimes \eta^{\mathcal{C}})\mu^{\mathcal{C}} = \mathrm{Pr}_1,\end{aligned}$$

that is, $\mathcal{C}$ together with the functors $\mu^{\mathcal{C}}$ and $\eta^{\mathcal{C}}$ is an monoid object in **Cat**. □

**Corollary 5.2** *The followings are equivalent:*

(1) *a crossed R-module* $(C, R, \partial)$

(2) *a categorical R-algebra*

(3) *a monoid object in* $Cat$

(4) *a simplicial R-algebra whose Moore complex is of lenght 1.*

**Proof** (1) $\iff$ (2) $\iff$ (3) are clear from the Theorem 4.5 and Proposition 5.1. Now we will show that (3) $\iff$ (4). We obtain a simplicial $R$-module by taking the nerve. This simplicial $R$-module is a simplicial $R$-algebra because the category is monoid object in $Cat$. Its Moore complex is

$$\ldots 1 \longrightarrow 1 \longrightarrow C \longrightarrow R,$$

which is of lenght 1.

Suppose that there is a simplicial algebra $\mathbf{E}$ whose Moore complex is of lenght 1, that is

$$\ldots 1 \longrightarrow 1 \longrightarrow \ker d_1 \longrightarrow E_0,$$

By choosing $C = E_1$ and $R =$ image of $E_0$ in $E_1$ by the degeneracy map and the structural morphisms $s = d_1$ and $t = d_0$, we get a categorical $R$-algabra. From the relations between face and degeneracy maps, we have $se = id = te$. On the other hand, to prove $KersKert = \{0\}$ it is sufficient to see that for $x \in Kerd_1$ and $y \in Kerd_0$ the element $[s_o(x), s_0(y) - s_1(y)]$ of $E_2$ is in fact in $Kerd_1 \cap Kerd_2$ and its image by $d_0$ is $[x, y]$. As $Kerd_1 \cap Kerd_2 = 0$, we have $[Kerd_0, Kerd_1] = 0$.

Therefore, from the Lemma 3.4 , we have a categorical $R$-algebra and use of the previous equivalence gives the monoid object in $Cat$ with $O = E_0$ and $M = E_1$. □

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