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The extension of step-N signatures

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Abstract: In 2009, Gyurko introduced Π-rough path which extends p-rough path. Inspired by this work we introduce the degree-(Π, N) signature which can be treated as the step-N signature for some Π. The degree-(Π, N) signature holds some algebraic properties which will be proven in this paper.

Key words: Rough path, signature, Lie group, Lie algebra

1. Introduction

The theory of rough path introduced by Lyons in [16]. The author generalizes a classical theory of controlled differential equations which is sufficiently robust. In particular Lyons developed differential equations of the form $dy_t = f(y_t)dx_t$ where x be a path of finite p-variation. In the pathwise sense, Lyons [17, 18] presents the definition of a wide class of stochastic differential equations. Since its introduction, the theory has discussed intensively. We refer the reader to study the papers [1–13, 16–18].

In [14, 15], the authors introduced the concept of rough paths of inhomogeneous degree of smoothness sketched by [16] which it is called geometric Π-rough paths. The authors have proved that the geometric Π-rough paths can be handle as p-rough paths for a sufficiently large p. Furthermore they also have proved the existence of integrals of one-forms under weaker conditions. Moreover, the authors presented differential equations driven by geometric Π-rough paths and proved the existence and uniqueness of solution.

In this paper we introduce the degree-(Π, N) signature which extend the step-N signature on $\mathbb{R}^d$ (see [10, Chapter 7]). This is inspired by Gyurko in 2009 which discussed Π-rough path. The first main result, we present Chen identity which it is basic property for Π-rough path. The structure of the paper will be as follows. In Section 2 we shortly introduce some definitions and notations about Π-rough paths. For more details we refer to [14, 15]. In Section 3, we present the main result of this paper which we introduce the degree-(Π, N) signature and the algebraic properties. Finally, in Section 4, we introduce Lie group $1 + \boldsymbol{t}^{\Pi,s}(R^I)$ and Lie Algebra $\boldsymbol{t}^{\Pi,s}(R^I)$ and show some properties in these spaces.

2. Preliminaries

In this section we recall some definitions and notations which have discussed in [14, 15].

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2.1. Notations

In this subsection we shortly introduce some notations. Let $\mathbb{R}_+$ be a set of real positive and $k \in \mathbb{N}$. Suppose $\Pi = (p_1, \cdots, p_k)$ be k-tuple of $\mathbb{R}_+$ where $p_i \geq 1$ for all $i \in \{1, \cdots, k\}$. We now give an index $R = (r_1, \cdots, r_l)$, called a k-multiindex, such that $r_j \in \{1, 2, \cdots, k\}$ for all $j \in \{1, 2, \cdots, l\}$ and $\|R\|$ denotes the length of R. We will write empty multiindex with ϵ. Moreover we denote by $\mathcal{A}^\Pi$ the set of all k-multiindexes of finite length. We now introduce some operators in $\mathcal{A}^\Pi$. Suppose $R = (r_1, r_2, \cdots, r_l) \in \mathcal{A}^\Pi$, the k-multiindexes ${}^-R$ and R^- given by

$$\begin{aligned} {}^-R &:= {}^-(r_1, r_2, \cdots, r_{l-1}, r_l) = (r_2, \cdots, r_{l-1}, r_l), \\ R^- &:= (r_1, r_2, \cdots, r_{l-1}, r_l)^- = (r_1, r_2, \cdots, r_{l-1}). \end{aligned}$$

Furthermore, the concatenation defined by

$$R * Q = (r_1, \cdots, r_l) * (q_1, \cdots, q_m) = (r_1, \cdots, r_l, q_1, \cdots, q_m),$$

where the multiindexes $R = (r_1, \cdots, r_l), Q = (q_1, \cdots, q_m) \in \mathcal{A}^\Pi$.

Example 2.1 *i. The tuples* $(1), (2), (1,2), (1,1,2), (2,2,1,2)$ *are* 2*-multiindexes.*

ii. The tuples $(1,3), (5,1)$ *are not* 2*-multiindexes, because* 3 *and* 5 *are greater than* 2*.*

iii. $\|(1)\| = 1, \|(2)\| = 1, \|(1,2)\| = 2, \|(1,1,2)\| = 3$ *and* $\|(2,2,1,2)\| = 4$*.*

iv. Suppose that $\Pi = (\frac{3}{2}, \frac{4}{3})$ *be* 2*-tuple, then*

$$(1), (2), (1,2), (1,1,2), (2,2,1,2) \in \mathcal{A}^\Pi \mathit{but}\ (1,3) \notin \mathcal{A}^\Pi.$$

iv. Suppose that Π *be* 2*-tuple. Let* $x^1 \in \mathbb{R}, x^2 \in \mathbb{R}^2$ *and* $\boldsymbol{x} = x^1 + x^2 \in \mathbb{R} \oplus \mathbb{R}^2$*, we have*

$$\begin{aligned} \boldsymbol{x}^{\otimes 0} &= \mathbf{1}, \textit{where } \mathbf{1} \textit{ is unit in } \mathbb{R} \oplus \mathbb{R}^2. \\ \boldsymbol{x}^{\otimes 1} &= x^1 + x^2 = \sum_{\substack{R=(r_1)\in\mathcal{A}^\Pi \\ \|R\|=1}} x^{r_1} \\ \boldsymbol{x}^{\otimes 2} &= x^1 \otimes x^1 + x^1 \otimes x^2 + x^2 \otimes x^1 + x^2 \otimes x^2 = \sum_{\substack{R=(r_1,r_2)\in\mathcal{A}^\Pi \\ \|R\|=2}} x^{r_1} \otimes x^{r_2} \\ \boldsymbol{x}^{\otimes 3} &= \sum_{r_1=1}^{2} \sum_{r_2=1}^{2} \sum_{r_3=1}^{2} x^{r_1} \otimes x^{r_2} \otimes x^{r_3} = \sum_{\substack{R=(r_1,r_2,r_3)\in\mathcal{A}^\Pi \\ \|R\|=3}} x^{r_1} \otimes x^{r_2} \otimes x^{r_3} \\ &\vdots \\ \boldsymbol{x}^{\otimes n} &= \sum_{r_1=1}^{2} \cdots \sum_{r_n=1}^{2} \boldsymbol{x}^{r_1} \otimes \cdots \otimes x^{r_n} = \sum_{\substack{R=(r_1,\cdots,r_n)\in\mathcal{A}^\Pi \\ \|R\|=n}} x^{r_1} \otimes \cdots \otimes x^{r_n}. \end{aligned}$$

Finally, we have

$$\sum_{n=0}^{\infty} \boldsymbol{x}^{\otimes n} = \sum_{n=0}^{\infty} \sum_{\substack{R=(r_1,\cdots,r_n)\in\mathcal{A}^{\Pi}\\ \|R\|=n}} x^{r_1}\otimes\cdots\otimes x^{r_n}$$

$$:= \sum_{R=(r_1,\cdots,r_l)\in\mathcal{A}^{\Pi}} x^{r_1}\otimes\cdots\otimes x^{r_l}.$$

We now introduce the space of series of tensor. Let $k\in\mathbb{N}$ and a family $\{V^i : i=1,\cdots,k\}$ of Banach spaces. Suppose $\Pi=(p_1,\cdots,p_k)\in(\mathbb{R}_+)^k$ be as before and $\mathcal{V}=V^1\oplus\cdots\oplus V^k$. We denote by $T((\mathcal{V}))$ the space of formal series of tensors $\mathcal{V}$ defined by

$$T((\mathcal{V})) := \bigoplus_{n=0}^{\infty} \mathcal{V}^{\otimes n} = \bigoplus_{(r_1,\cdots,r_l)\in\mathcal{A}^{\Pi}} V^{r_1}\otimes\cdots\otimes V^{r_l},$$

and the notation $\mathcal{V}^{\otimes R}$ defined by

$$\mathcal{V}^{\otimes R} = V^{r_1}\otimes\cdots\otimes V^{r_l},$$

where $R=(r_1,r_2,\cdots,r_l)\in\mathcal{A}^{\Pi}$. Moreover for $i\in\{1,\cdots,k\}$ we write the projections

$$\pi_R := \pi_{V^{r_1}\otimes\cdots\otimes V^{r_l}} \quad : \quad T((V)) \to V^{\otimes R},$$

$$\pi_{T((V^i))} \quad : \quad T((V)) \to T((V^i)).$$

Furthermore the notation $\boldsymbol{a}^{\otimes R}$ given by

$$\boldsymbol{a}^R := \pi_R(\boldsymbol{a}) = \pi_{V^{r_1}\otimes\cdots\otimes V^{r_l}}(\boldsymbol{a}),$$

and we can write

$$\boldsymbol{a} = \sum_{R\in\mathcal{A}^{\Pi}} \boldsymbol{a}^R.$$

Example 2.2 *Suppose that* Π *be* 2*-tuple. Let* $x^1\in\mathbb{R}, x^2\in\mathbb{R}^2$ *and* $\boldsymbol{x}=x^1+x^2\in\mathbb{R}\oplus\mathbb{R}^2$*. We can observe that* $\boldsymbol{a}=\sum_{n=0}^{\infty}\boldsymbol{x}^{\otimes n}\in T((\mathbb{R}\oplus\mathbb{R}^2))$*. We obtain*

$$\pi_R(\boldsymbol{a}) = \boldsymbol{a}^R = x^{r_1}\otimes\cdots\otimes x^{r_l}$$

for $R=(r_1,\cdots,r_l)\in\mathcal{A}^{\Pi}$*.*

We now introduce truncated tensor algebra of $\mathcal{V}$. We first define the function n_j for $j\in\{1,\cdots,k\}$ by

$$n_j(R) := \text{card}\{i|r_i=j, r_i\in R\}$$

and the degree-Π of R given by

$$deg_{\Pi}(R) = \sum_{j=1}^{k} \frac{n_j(R)}{p_j}.$$

Thus we have $deg_{\Pi}(\epsilon)=0$. Moreover, we introduce a function $\Gamma_{\Pi}:\mathcal{A}^{\Pi}\to[0,\infty)$ given by

$$\Gamma_{\Pi}(R)=\Gamma\left(\frac{n_1(R)}{p_1}+1\right)\times\cdots\times\Gamma\left(\frac{n_k(R)}{p_k}+1\right),\ \text{for } R\in\mathcal{A}^{\Pi},$$

where $\Gamma(x)$ is gamma function. Furthermore we denote by $\mathcal{A}^{\Pi}_s$ and B^{Π}_s the sets given by

$$\begin{aligned}\mathcal{A}^{\Pi}_s &:= \{R=(r_1,\cdots,r_l)|l\geq 1, deg_{\Pi}(R)\leq s\}\\ B^{\Pi}_s &:= \left\{\boldsymbol{a}\in T((\mathcal{V}))|\forall R\in\mathcal{A}^{\Pi}_s, \boldsymbol{a}^R=\boldsymbol{0}\right\}.\end{aligned}$$

where s be a real nonnegative. We can observe that B^{Π}_s is an ideal in $T((\mathcal{V}))$. Finally, we give the truncated tensor algebra of order (Π,s) which is given by

$$T^{(\Pi,s)}(\mathcal{V}):=T((\mathcal{V}))/B^{\Pi}_s.$$

Remark 2.3 *$T^{(\Pi,s)}(\mathcal{V})$ is isomorphic to $\oplus_{R\in\mathcal{A}^{\Pi}_s}\mathcal{V}^{\otimes R}$ equipped with the product*

$$a\otimes_{\Pi,s}b:=\left(\sum_{Q\in\mathcal{A}^{\Pi}_s}\boldsymbol{a}^Q\right)\otimes_{\Pi,s}\left(\sum_{R\in\mathcal{A}^{\Pi}_s}\boldsymbol{b}^R\right):=\sum_{Q*R\in\mathcal{A}^{\Pi}_s}\boldsymbol{a}^Q\otimes\boldsymbol{b}^R.$$

for $\boldsymbol{a},\boldsymbol{b}\in T^{(\Pi,s)}(\mathcal{V})$.

We can also write

$$\boldsymbol{a}\otimes_{\Pi,s}\boldsymbol{b}=\sum_{R\in\mathcal{A}^{\Pi}_s}\pi_R(\boldsymbol{a}\otimes_{\Pi,s}\boldsymbol{b}),$$

where

$$\pi_{R=(r_1,\cdots,r_l)}(\boldsymbol{a}\otimes_{\Pi,s}\boldsymbol{b})=\sum_{j=0}^{l}\pi_{(r_1,\cdots,r_j)}(\boldsymbol{a})\otimes\pi_{(r_{j+1},\cdots,r_l)}(\boldsymbol{b}).$$

2.2. Π-rough paths

In this section we introduce Π-rough paths which have discussed in [14, 15]. We first introduce control function which given by [18]. Let $T>0$, a control (function) ω is a nonnegative continuous function on $\Delta_T:=\{(t,u):0\leq t\leq u\leq T\}$ such that $\omega(s,t)+\omega(t,u)\leq\omega(s,u)$, for all $0\leq s\leq t\leq u\leq T$ and $\omega(t,t)=0$, for all $t\in[0,T]$. Let k, Π and $\mathcal{V}$ be as before and s be a positive real number. We define a continuous map $\boldsymbol{X}:\Delta_T\to T^{(\Pi,s)}(V)$, called a multiplicative functional of degree s, such that $\pi_{\varepsilon}(\boldsymbol{X}_{t,u})=1$ and satisfies Chen identity

$$\boldsymbol{X}_{t,v}=\boldsymbol{X}_{t,u}\otimes\boldsymbol{X}_{u,v}$$

for all $0\leq t<u<v\leq T$. Furthermore, $\boldsymbol{X}$ is of finite Π variation controlled by ω if

$$\left\|\boldsymbol{X}^R_{t,u}\right\|\leq\frac{\omega(t,u)^{deg_{\Pi}(R)}}{\beta^k\Gamma_{\Pi}(R)}$$

for all $(t,u)\in\Delta_T$ and for all k-multiindex $R\in\mathcal{A}^{\Pi}_s$. Next, we give the following definition which introduces Π-rough path.

Definition 2.4 (Π-rough paths) *Let k, Π, ω and $\mathcal{V}$ be as before and s be a positive real number. The Π-rough path in $\mathcal{V}$, denoted by $\Omega_\Pi(\mathcal{V})$, is the continuous multiplicative functional $\boldsymbol{X} : \Delta_T \to T^{(\Pi,1)}(\mathcal{V})$ controlled by ω with finite Π-variation. Furthermore we denote by $C_{0,\Pi}\left(\Delta_T, T^{(\Pi,s)}\right)$ the space of all continuous functions from Δ_T into $T^{(\Pi,s)}(\mathcal{V})$ which has finite Π-variation. The Π-variation metric on this linear space given by*

$$d^s_\Pi(\boldsymbol{X},\boldsymbol{Y}) := \max_{R\in\mathcal{A}^\Pi_s} \sup_{(t_j)\in D([0,T])} \left(\sum_j \left\| \boldsymbol{X}^R_{t_{j-1},t_j} - \boldsymbol{Y}^R_{t_{j-1},t_j} \right\|^{1/deg_\Pi(R)} \right)^{deg_\Pi(R)},$$

where $D([0,T])$ is the set of all partition of some interval $[0,T]$.

3. Degree-(Π, N) signatures

In [10], we see that the step-N signature on $\mathbb{R}^d$. We will generalize it on $\mathbb{R}^{i_1} \oplus \cdots \oplus \mathbb{R}^{i_k}$ where $i_1, \cdots, i_k \in \mathbb{N}$ and k denotes a fixed positive integer. Furthermore, we will verify whether both of the degree-(Π, N) signature and the step-N signature have the same properties. Therefore we introduce the following definition which relate with this domain.

Definition 3.1 *Let $I = (i_1, \cdots, i_k) \in \mathbb{N}^k$ be k-tuple natural number. We denote the space $\mathbb{R}^I := \mathbb{R}^{i_1} \oplus \cdots \oplus \mathbb{R}^{i_k}$ endowed with the following metric*

$$d_I(\boldsymbol{a},\boldsymbol{b}) = \sqrt{\sum_{j=1}^k |b_j - a_j|^2},$$

where $\boldsymbol{a} = a_1 + \cdots + a_k$, $\boldsymbol{b} = b_1 + \cdots + b_k \in \mathbb{R}^I$, and $|\cdot|$ is euclidean norm. We also define norm on $\mathbb{R}^I$ by

$$\|\boldsymbol{a}\|_I = \sqrt{\sum_{j=1}^k |a_j|^2}.$$

We now recall from [10] that, path $x : [0,T] \to \mathbb{R}^I$ is said to be

i. Hölder continuous with exponent $\alpha \geq 0$, or simply $\alpha - H\ddot{o}lder$, if

$$|x|_{\alpha-H\ddot{o}l;[0,T]} := \sup_{0\leq s<t\leq T} \frac{d_I(x_s, x_t)}{|t-s|^\alpha} < \infty;$$

ii. of finite p-variation for some $p > 0$ if

$$|x|_{p-var;[0,T]} := \left(\sup_{(t_j)\in D([0,T])} \sum_j d_I(x_{t_{j-1}}, x_{t_j})^p \right)^{1/p} < \infty,$$

where $D([0,T])$ is the set of all dissections of some interval $[0,T]$.

Furthermore we denote by $C^{\alpha-H\ddot{o}l}([0,T],\mathbb{R}^I)$ the set of all $\alpha - H\ddot{o}lder$ paths and $C^{p-var}([0,T],\mathbb{R}^I)$ the set of continuous paths of finite p-variation.

We can see that $\mathbb{R}^I$ and $\mathbb{R}^{i_1+\cdots+i_k}$ are isomorphic. Furthermore, if $\psi : \mathbb{R}^I \ni \boldsymbol{x} = \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_k \mapsto \boldsymbol{x} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k)^T \in \mathbb{R}^{i_1+\cdots+i_k}$ is an isomorphism then $\|\boldsymbol{x}\|_I = \|\psi(\boldsymbol{x})\|$ and $d_I(\boldsymbol{x}) = d(\psi(\boldsymbol{x}))$ where $\|\cdot\|$ and d are the euclidean norm and the metric, respectively. Therefore, we can recall Proposition 1.38 of [10], where $\mathbb{R}^d$ is replaced by $\mathbb{R}^I$.

Proposition 3.2 *Let I be as before and the function $x \in C^{1-var}([0,T], \mathbb{R}^I)$ is not constant, then there exists a continuous nondecreasing function $\varphi : [0,T] \to [0,1]$ and a path $y \in C^{1-H\ddot{o}l}([0,1], \mathbb{R}^I)$ such that $x = y \circ \varphi$. In particular, we have $\varphi(t) = |x|_{1-var;[0,t]} / |x|_{1-var;[0,T]}$ and $\|\dot{y}(r)\|_I =$(const) for a.e. $r \in [0,1]$. Furthermore, there exists $\dot{y} \in L^\infty([0,1], \mathbb{R}^I)$ such that $y(t) = \int_0^t \dot{y}(s)ds$ and*

$$\|\dot{y}(t)\|_I = |x|_{1-var;[0,T]} = |y|_{1-H\ddot{o}l;[0,1]}$$

for a.e. $t \in [0,1]$.

Proof The proof is analogous to the proof of Proposition 1.38 of [10]. □

In the next part we will use notations which have been introduced by [14] and the Section 2. We also replace $\mathcal{V} = V^1 \oplus \cdots \oplus V^k$ with $\mathbb{R}^I$. Throughout this section we denote by S^Π the infinite and countable set which can be listed in ascending order, i.e. $S^\Pi = \{s_0 = 0, s_1, s_2, \cdots\}$ where $s_0 < s_1 < \cdots$.

Lemma 3.3 *Let $\Pi = (p_1, \cdots, p_k)$ be real k-tuple and suppose $S^\Pi = \{s_0 = 0, s_1, \cdots\}$ listed in ascending order, then*

$$\left(\mathcal{A}^\Pi_{s_N} \backslash \{\epsilon\}\right) \subset \left(\mathcal{A}^\Pi_{s_{N+1}} \backslash \{\epsilon\}\right) \subset \{R * (m) | m = 1, \cdots, k; R \in \mathcal{A}^\Pi_{s_N}\}.$$

Proof We can observe that $\mathcal{A}^\Pi_{s_N} - \epsilon \subset \mathcal{A}^\Pi_{s_{N+1}} - \epsilon$ and

$$\{R * (m) | m = 1, \cdots, k; R \in \mathcal{A}^\Pi_{s_N}\} = \{R | R- \in \mathcal{A}^\Pi_{s_N}\}.$$

Furthermore, let $Q \in \mathcal{A}^\Pi_{s_{N+1}} - \epsilon$, then we have $deg_\Pi(Q) \leq s_{N+1}$. Therefore, we obtain $deg_\Pi(Q-) < s_{N+1}$. Because $S^\Pi = \{s_0 = 0, s_1, \cdots\}$ listed in ascending order then $deg_\Pi(Q-) \leq s_N$. Finally, we have $Q \in \{R * (m) | m = 1, \cdots, k; R \in \mathcal{A}^\Pi_{s_N}\}$ and we can conclude

$$\mathcal{A}^\Pi_{s_N} - \epsilon \subset \mathcal{A}^\Pi_{s_{N+1}} - \epsilon \subset \{R * (m) | m = 1, \cdots, k; R \in \mathcal{A}^\Pi_{s_N}\}.$$

□

We now give the definition of signatures on $\mathbb{R}^I$ in the following definition, .

Definition 3.4 (Degree-(Π, N) signatures) *Let Π and S^Π be as before and $I = (i_1, \cdots, i_k)$ be real k-tuple. The degree-(Π, N) signature of $x = x^1 + \cdots + x^k \in C^{1-var}([s,t], \mathbb{R}^I)$ is given by*

$$S_{\Pi,N}(x)_{s,t} := \sum_{\substack{R \in \mathcal{A}^\Pi_{s_N} \\ R=(r_1,\cdots,r_l)}} \int_{s<u_1<\cdots<u_l<t} dx^{r_1}_{u_1} \otimes \cdots \otimes dx^{r_l}_{u_l}.$$

We can observe that $S_{\Pi,N}(x)_{s,\cdot}$ takes value in $T^{\Pi,s_N}(\mathbb{R}^I)$. From Definition 3.4 we obtain the following theorem.

Theorem 3.5 *Let* Π, I *and* S^Π *be as before. If* $x = x^1 + \cdots + x^k \in C^{1-var}([s,t],\mathbb{R}^I)$, *then for fixed* $s \in [0,T)$,

$$\begin{aligned} dS_{\Pi,N}(x)_{s,t} &= \sum_{j=1}^{k} S_{\Pi,N}(x)_{s,t} \otimes dx_t^j \\ S_{\Pi,N}(x)_{s,s} &= 1. \end{aligned}$$

Proof Let $R = (r_1, \cdots, r_l)$ is k-multiindex in $\mathcal{A}^\Pi_{s_N}$, we can write

$$\begin{aligned} \pi_R\left(S_{\Pi,N}(x)_{s,t}\right) &= \int_{s<u_1<\cdots<u_l<t} dx_{u_1}^{r_1} \otimes \cdots \otimes dx_{u_l}^{r_l} \\ &= \int_s^t \left(\int_{s<u_1<\cdots<u_{l-1}<u_l} dx_{u_1}^{r_1} \otimes \cdots \otimes dx_{u_{l-1}}^{r_l} \right) \otimes dx_{u_l}^{r_l} \\ &= \int_s^t \pi_{R-}\left(S_{\Pi,N}(x)_{s,u_l}\right) \otimes dx_{u_l}^{r_l}. \end{aligned}$$

From Lemma 3.3, we have $\mathcal{A}^\Pi_{s_N} - \epsilon \subset \{R*(m) | m = 1, \cdots, k; R \in \mathcal{A}^\Pi_{s_N}\}$. Besides that, if $R_1 \neq R_2$ or $m \neq n$ then $R_1 * (m) \neq R_2 * (n)$. Thus,

$$\begin{aligned} \sum_{j=1}^{k} \int_s^t S_{\Pi,N}(x)_{s,u} \otimes dx_u^j &= \sum_{j=1}^{k} \int_s^t \sum_{R\in\mathcal{A}^\Pi_{s_N}} \pi_R\Big(S_{\Pi,N}(x)_{s,u}\Big) \otimes dx_u^j \\ &= \sum_{j=1}^{k} \sum_{R\in\mathcal{A}^\Pi_{s_N}} \pi_{R*(j)}\Big(S_{\Pi,N}(x)_{s,t}\Big) \\ &= \Bigg(\sum_{R\in\mathcal{A}^\Pi_{s_N}} \pi_R\Big(S_{\Pi,N}(x)_{s,t}\Big)\Bigg) - 1 = S_{\Pi,N}(x)_{s,t} - 1 \end{aligned}$$

where the third line follows from truncation beyond degree-(Π, N). Thus we have proved the theorem. □

Corollary 3.6 *Let* Π, I *and* S^Π *be as before. Let* $(x_n) \subset C^{1-var}([0,1],\mathbb{R}^I)$ *with* $\sup_n |x_n|_{1-var;[0,1]} < \infty$, *uniformly convergent to some* $x \in C^{1-var}([0,1],\mathbb{R}^I)$. *Then,* $S_{\Pi,N}(x_n)_{0,\cdot}$ *converges uniformly to* $S_{\Pi,N}(x)_{0,\cdot}$.

Proof The proof is analogous to the proof of Proposition 7.15 of [10]. □

Moreover the following theorem, we give Chen identity which is one of Π-rough paths property.

Theorem 3.7 (Chen identity) *Let* Π, I *and* S^Π *be as before. Given* $x = x^1 + \cdots + x^k \in C^{1-var}([s,t],\mathbb{R}^I)$, *then for* $0 \leq s < t < u \leq T$,

$$S_{\Pi,N}(x)_{s,u} = S_{\Pi,N}(x)_{s,t} \otimes S_{\Pi,N}(x)_{t,u}. \tag{3.1}$$

Proof We will show the theorem by induction. For $N=0$, we get $S_{\Pi,0}(x)_{s,u} = S_{\Pi,0}(x)_{s,t} = S_{\Pi,0}(x)_{t,u} = 1$, hence the statement is true. Suppose the identity (3.1) is true for N and we will prove the identity (3.1) is true for $N+1$. We split the proof into three parts.

i. From Lemma 3.3, we have $\mathcal{A}^{\Pi}_{s_{N+1}} - \epsilon \subset \{R*(m) | m = 1,\cdots,k; R \in \mathcal{A}^{\Pi}_{s_N}\}$. Besides that, if $R_1 \neq R_2$ or $m \neq n$ then $R_1 * (m) \neq R_2 * (n)$. By similar argument with Theorem 3.5, then

$$S_{\Pi,N+1}(x)_{s,u} = 1 + \sum_{j=1}^{k} \int_s^u S_{\Pi,N}(x)_{s,r} \otimes dx_r^j; \tag{3.2}$$

ii. Using the fact that

$$\sum_{\substack{R\in\mathcal{A}^{\Pi} \\ deg_{\Pi}(R)=s_{N+1}}} \pi_R\Big(S_{\Pi,N+1}(x)_{s,t}\Big) \otimes \left(\sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{t,r} \otimes dx_r^j\right) = 0,$$

we obtain

$$\begin{aligned} & S_{\Pi,N+1}(x)_{s,t} \otimes \left(\sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{t,r} \otimes dx_r^j\right) \\ = & \left(S_{\Pi,N}(x)_{s,t} + \sum_{\substack{R\in\mathcal{A}^{\Pi} \\ deg_{\Pi}(R)=s_{N+1}}} \pi_R\Big(S_{\Pi,N+1}(x)_{s,t}\Big)\right) \otimes \\ & \left(\sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{t,r} \otimes dx_r^j\right) \\ = & S_{\Pi,N}(x)_{s,t} \otimes \left(\sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{t,r} \otimes dx_r^j\right) \\ & + \sum_{\substack{R\in\mathcal{A}^{\Pi} \\ deg_{\Pi}(R)=s_{N+1}}} \pi_R\Big(S_{\Pi,N+1}(x)_{s,t}\Big) \otimes \left(\sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{t,r} \otimes dx_r^j\right) \\ = & S_{\Pi,N}(x)_{s,t} \otimes \left(\sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{t,r} \otimes dx_r^j\right). \end{aligned} \tag{3.3}$$

iii. We will use induction statement and (3.2), (3.3). Using (3.2), we have

$$\begin{aligned} S_{\Pi,N+1}(x)_{s,u} &= 1 + \sum_{j=1}^{k} \int_s^u S_{\Pi,N}(x)_{s,r} \otimes dx_r^j \\ &= 1 + \sum_{j=1}^{k} \int_s^t S_{\Pi,N}(x)_{s,r} \otimes dx_r^j + \sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{s,r} \otimes dx_r^j. \end{aligned}$$

By (3.2) and induction statement, we get

$$S_{\Pi,N+1}(x)_{s,u} = S_{\Pi,N+1}(x)_{s,t} + \sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{s,t} \otimes S_{\Pi,N}(x)_{t,r} \otimes dx_r^j.$$

From (3.3), we rewrite

$$\begin{aligned} S_{\Pi,N+1}(x)_{s,u} &= S_{\Pi,N+1}(x)_{s,t} + S_{\Pi,N}(x)_{s,t} \otimes \left(\sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{t,r} \otimes dx_r^j \right) \\ &= S_{\Pi,N+1}(x)_{s,t} + S_{\Pi,N+1}(x)_{s,t} \otimes \left(\sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{t,r} \otimes dx_r^j \right) \end{aligned}$$

By (3.2), we have

$$\begin{aligned} S_{\Pi,N+1}(x)_{s,u} &= S_{\Pi,N+1}(x)_{s,t} \otimes \left(1 + \sum_{j=1}^{k} \int_t^u S_{\Pi,N}(x)_{t,r} \otimes dx_r^j \right) \\ &= S_{\Pi,N+1}(x)_{s,t} \otimes S_{\Pi,N+1}(x)_{t,u}, \end{aligned}$$

Therefore, the proof may be completed by induction.

□

Corollary 3.8 *Let* Π, I *and* S^{Π} *be as before. Suppose* $\gamma = \gamma^1 + \cdots + \gamma^k \in C^{1-var}([0,T],\mathbb{R}^I)$, $\eta = \eta^1 + \cdots + \eta^k \in C^{1-var}([T,U],\mathbb{R}^I)$ *and*

$$(\gamma \sqcup \eta)_t := \begin{cases} \gamma_t & , t \in [0,T] \\ \eta_t - \eta_T + \gamma_T & , t \in [T,U] \end{cases}$$

such that $x := x_1 + \cdots + x_k := \gamma \sqcup \eta \in C^{1-var}([0,T],\mathbb{R}^I)$. *Then*

$$S_{\Pi,N}(x)_{0,U} = S_{\Pi,N}(\gamma)_{0,T} \otimes S_{\Pi,N}(\eta)_{T,U}.$$

Proof Let $y = y^1 + \cdots + y^k \in C^{1-var}([s,t],\mathbb{R}^I)$ and $c \in \mathbb{R}^I$, we obtain

$$\begin{aligned} S_{\Pi,N}(y+c)_{s,t} &= \sum_{\substack{R \in \mathcal{A}_{s_N}^{\Pi} \\ R=(r_1,\cdots,r_l)}} \int_{s<u_1<\cdots<u_l<t} d(y+c)_{u_1}^{r_1} \otimes \cdots \otimes d(y+c)_{u_l}^{r_l} \\ &= \sum_{\substack{R \in \mathcal{A}_{s_N}^{\Pi} \\ R=(r_1,\cdots,r_l)}} \int_{s<u_1<\cdots<u_l<t} dy_{u_1}^{r_1} \otimes \cdots \otimes dy_{u_l}^{r_l} = S_{\Pi,N}(y)_{s,t}. \end{aligned}$$

Therefore, using Theorem 3.7, we have

$$\begin{aligned} S_{\Pi,N}(x)_{0,U} &= S_{\Pi,N}(x)_{0,T} \otimes S_{\Pi,N}(x)_{T,U} \\ &= S_{\Pi,N}(\gamma)_{0,T} \otimes S_{\Pi,N}(\eta_{\cdot} - \eta_T + \gamma_T)_{T,U} \\ &= S_{\Pi,N}(\gamma)_{0,T} \otimes S_{\Pi,N}(\eta)_{T,U}. \end{aligned}$$

Thus we have proved the corollary. □

We now construct inverse of the (degree-(Π, N)) signature. We start with reparametrization of x under φ which can be seen the the following lemma.

Lemma 3.9 *Let* Π, I *and* S^{Π} *be as before. Let* $x = x^1 + \cdots + x^k \in C^{1-var}([s,t], \mathbb{R}^I)$ *and the function* $\varphi : [T_1, T_2] \to [0, T]$ *be nondecreasing surjection. Then, for all* $s, t \in [T_1, T_2]$,

$$S_{\Pi,N}(x)_{\varphi(s),\varphi(t)} = S_{\Pi,N}(x \circ \varphi)_{s,t}.$$

Proof We will show the lemma by induction for length of k-multiindex R in $\mathcal{A}^{\Pi}_{s_N}$. For $\|R\| = 0$, we get $\pi_R(S_{\Pi,N}(x)_{\pi(s),\pi(t)}) = \pi_R(S_{\Pi,N}(x \circ \varphi)_{s,t}) = 1$, hence the statement is true. Furthermore, we assume that it is true for $\|R\| = k$ and we will prove it is true for $\|R\| = k+1$. This case, we will truncate to $T^{\Pi, s_{N+1}}(\mathbb{R}^I)$, then we get some equations below. By proof of Theorem 3.5, we have

$$\begin{aligned}
\pi_R\Big(S_{\Pi,N}(x)_{\varphi(s),\varphi(t)}\Big) &= \int_{\varphi(s)}^{\varphi(t)} \pi_{R-}\Big(S_{\Pi,N}(x)_{\varphi(s),r}\Big) \otimes dx^j_r \\
&= \int_s^t \pi_{R-}\Big(S_{\Pi,N}(x)_{\varphi(s),\varphi(u)}\Big) \otimes dx^j_{\varphi(u)} \\
&= \int_s^t \pi_{R-}\Big(S_{\Pi,N}(x \circ \varphi)_{s,u}\Big) \otimes d(x \circ \varphi)^j_u \\
&= \pi_R\Big(S_{\Pi,N}(x \circ \varphi)_{s,t}\Big),
\end{aligned}$$

where we obtain the second equation using a change of variable $r = \varphi(u)$ for Riemann–Stieljes integral and the third equation can be obtained by using induction hypothesis. □

Next, we give the following theorem which explains that inverse of the (degree-(Π, N)) signature lift of $x = x^1 + \cdots + x^k \in C^{1-var}([0,T], \mathbb{R}^I)$ is the (degree-(Π, N)) signature lift of $\overleftarrow{x}$ where $\overleftarrow{x} = x_{T-t}$.

Theorem 3.10 (Inverse of the degree-(Π, N) signature) *Let* Π, I *and* S^{Π} *be as before. Let* $x = (x^1, x^2, \cdots, x^k) \in C^{1-var}([0,T], \mathbb{R}^I)$. *We denote by* $\overleftarrow{x}$ *the path* $\overleftarrow{x} = x_{T-t} \in \mathbb{R}^I$, *then*

$$S_{\Pi,N}(x)_{0,T} \otimes S_{\Pi,N}(\overleftarrow{x})_{0,T} = S_{\Pi,N}(\overleftarrow{x})_{0,T} \otimes S_{\Pi,N}(x)_{0,T} = 1.$$

Proof From Theorem 3.5, we have that $y_t = S_{\Pi,N}(x)_{0,t}$ is solution to the differential equation

$$dy_t = \sum_{j=1}^{k} y_t \otimes dx^j_t, \quad y_0 = 1.$$

Furthermore, Theorem 3.10 follows immediately from the result on the differential equation with time-reversed driving signal in Proposition 3.13 of [10]. □

The following definition, we introduce dilation map on $T^{(\Pi,s)}(\mathbb{R}^I)$.

Definition 3.11 *Let* Π *and* I *be as before. For* $\lambda > 0$, *we define the dilation map*

$$\delta_\lambda : T^{(\Pi,s)}(\mathbb{R}^I) \to T^{(\Pi,s)}(\mathbb{R}^I)$$

such that

$$\pi_R\Big(\delta_\lambda(g)\Big) = \lambda^{\|R\|}\pi_R(g)$$

for all k*-multiindex* R *such that* $deg_\Pi(R) \le s$ *and* $g \in T^{(\Pi,s)}(\mathbb{R}^I)$*.*

Remark 3.12 *We can observe that* $S_{\Pi,N}(\lambda x)_{t,u} = \delta_\lambda S_{\Pi,N}(x)_{t,u}$ *where* $\lambda \in \mathbb{R}$ *and* $x = x^1 + \cdots + x^k \in C^{1-var}([0,T],\mathbb{R}^I)$*.*

4. Lie group $1+\boldsymbol{t}^{\Pi,s}(R^I)$ and Lie Algebra $\boldsymbol{t}^{\Pi,s}(R^I)$

Friz and Victoir (in [10] Chapter 7) have explained that $\boldsymbol{t}^N(R^d)$ and $1+\boldsymbol{t}^N(R^d)$ are Lie algebra and Lie group, respectively. Moreover Gyurko [14] have discussed about Lie algebra $T^{(\Pi,s)}\Big(V\Big)$, exponential and logarithm function on $T\Big((U)\Big)$. Therefore, we will observe whether $\boldsymbol{t}^{\Pi,s}(R^I)$ and $1+\boldsymbol{t}^{\Pi,s}(R^I)$ are also Lie algebra and Lie group, respectively. Before we give it, we recall operator which is used in this section (see Definition 4.1). Furthermore, we also recall definition of Lie bracket, exponential function and logarithm function but the domain is $\boldsymbol{t}^{\Pi,s}(R^I)$ and $1+\boldsymbol{t}^{\Pi,s}(R^I)$ (see Definition 4.2).

Definition 4.1 *Let* k *be a positive integer and both of* $\Pi = (p_1,\cdots,p_k)$ *and* $I = (i_1,\cdots,i_k)$ *be real* k*-tuple. Let* $\lambda \in \mathbb{R}$ *and* $\boldsymbol{x},\boldsymbol{y} \in T^{(\Pi,s)}(\mathbb{R}^I)$*, we give the operations*

$$\begin{aligned}
\cdot \quad &: \quad \mathbb{R}\times T^{(\Pi,s)}(\mathbb{R}^I) \to T^{(\Pi,s)}(\mathbb{R}^I) \text{ by } \lambda\cdot\boldsymbol{x} = \sum_{R\in\mathcal{A}_s^\Pi} \lambda \boldsymbol{x}^R \\
+ \quad &: \quad T^{(\Pi,s)}(\mathbb{R}^I)\times T^{(\Pi,s)}(\mathbb{R}^I) \to T^{(\Pi,s)}(\mathbb{R}^I) \text{ by } \boldsymbol{x}+\boldsymbol{y} := \sum_{R\in\mathcal{A}_s^\Pi} \boldsymbol{x}^R + \boldsymbol{y}^R \\
\otimes \quad &: \quad T^{(\Pi,s)}(\mathbb{R}^I)\times T^{(\Pi,s)}(\mathbb{R}^I) \to T^{(\Pi,s)}(\mathbb{R}^I) \text{ by} \\
&\quad\ \boldsymbol{x}\otimes\boldsymbol{y} = \left(\sum_{Q\in\mathcal{A}_s^\Pi}\boldsymbol{x}^Q\right)\otimes\left(\sum_{Q\in\mathcal{A}_s^\Pi}\boldsymbol{y}^R\right) := \sum_{Q*R\in\mathcal{A}_s^\Pi}\boldsymbol{x}^Q\otimes\boldsymbol{y}^R
\end{aligned}$$

We can also write

$$\boldsymbol{x}\otimes\boldsymbol{y} = \sum_{R\in\mathcal{A}_s^\Pi}\pi_R(\boldsymbol{x}\otimes\boldsymbol{y})$$

where

$$\pi_{R=(r_1,\cdots,r_l)}(\boldsymbol{x}\otimes\boldsymbol{y}) = \sum_{j=0}^{l}\pi_{(r_1,\cdots,r_j)}(\boldsymbol{x})\otimes\pi_{(r_{j+1},\cdots,r_l)}(\boldsymbol{y}).$$

The space $\Big(T^{(\Pi,s)}(\mathbb{R}^I),+,\cdot,\otimes\Big)$ *is an associative algebra, where* $\Big(T^{(\Pi,s)}(\mathbb{R}^I),+,\cdot\Big)$ *and* $\Big(T^{(\Pi,s)}(\mathbb{R}^I),+,\otimes\Big)$ *are vector space and ring, respectively. Furthermore, unit element of* $T^{(\Pi,s)}(\mathbb{R}^I)$ *is* $\mathbf{1}$ *where* $\mathbf{1}^R = 0$ *for all* $R\in\mathcal{A}_s^\Pi$*.*

Analogous to [10], we denote $\boldsymbol{t}^{\Pi,s}(R^I)$ and $1+\boldsymbol{t}^{\Pi,s}(R^I)$ which can be seen the following definition.

Definition 4.2 *Let k be a positive integer and both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple. We define $\boldsymbol{t}^{\Pi,s}(R^I)$ and $1 + \boldsymbol{t}^{\Pi,s}(R^I)$ by*

$$\begin{aligned} \boldsymbol{t}^{\Pi,s}(R^I) &:= \{\boldsymbol{a} \in T^{(\Pi,s)}(R^I) : \pi_\epsilon(\boldsymbol{a}) = 0\}; \\ 1 + \boldsymbol{t}^{\Pi,s}(R^I) &:= \{\boldsymbol{a} \in T^{(\Pi,s)}(R^I) : \pi_\epsilon(\boldsymbol{a}) = 1\}. \end{aligned}$$

Remark 4.3 *It is obvious that if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in 1 + \boldsymbol{t}^{\Pi,s}(R^I)$ and $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{t}^{\Pi,s}(R^I)$ then $\boldsymbol{x}_1 \otimes \boldsymbol{x}_2 \in 1 + \boldsymbol{t}^{\Pi,s}(R^I)$ and $\boldsymbol{y}_1 \otimes \boldsymbol{y}_2 \in \boldsymbol{t}^{\Pi,s}(R^I)$.*

We first discuss about property of $1 + \boldsymbol{t}^{\Pi,s}(R^I)$, where we will prove that it is Lie group. Therefore, we will discuss that elements in $1 + \boldsymbol{t}^{\Pi,s}(R^I)$ is invertible which we will give in Theorem 4.6, but we will need the following Lemma to prove the theorem.

Lemma 4.4 *Let k be a positive integer and both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple. Given $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n \in \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ then $\pi_R(\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_n) = 0$ for all $n > \|R\|$ where R is k-multiindex in $\mathcal{A}_s^\Pi$. In particular, we have $\pi_R(\boldsymbol{x}^{\otimes n}) = 0$ for all $n > \|R\|$ where R is k-multiindex in $\mathcal{A}_s^\Pi$.*

Proof We can show the lemma by induction for length of k-multiindex $R = (r_1, \cdots, r_l)$ in $\mathcal{A}_s^\Pi$. For $l = \|R\| = 0$, we get $\pi_R(\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_n) = 0$ for all $n > 0$, hence the statement is true. Furthermore, we assume that it is true for $\|R\| = K$ and we will prove it is true for $\|R\| = K + 1$. Let $R = (r_1, \cdots, r_K, r_{K+1})$ is k-multiindex and $n > K + 1$, then

$$\begin{aligned} \pi_R(\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_n) &= \pi_R(\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_{n-1} \otimes \boldsymbol{x}_n) \\ &= \sum_{j=0}^{K+1} \pi_{(r_1,\cdots,r_j)}(\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_{n-1}) \otimes \pi_{(r_{j+1},\cdots,r_{K+1})}(\boldsymbol{x}_n). \end{aligned}$$

We can see that $n - 1 > K$ then by induction hypothesis $\pi_{(r_1,\cdots,r_j)}(\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_{n-1}) = 0$ for $j = 1, \cdots, K$. Therefore,

$$\pi_R(\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_n) = \pi_{(r_1,\cdots,r_{K+1})}(\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_{n-1}) \otimes \pi_\epsilon(\boldsymbol{x}_n) = 0.$$

Therefore, the proof may be completed by induction. □

Remark 4.5 *We know that for all k-multiindex R in $\mathcal{A}_s^\Pi$ has finite length. Let M denote $\sup_{R\in\mathcal{A}_s^\Pi} \|R\|$ then we have*

$$\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_n = \boldsymbol{x}^{\otimes n} = 0$$

for all $n > M$ and $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{x} \in \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$.

Theorem 4.6 (inverse in $1 + \boldsymbol{t}^{\Pi,s}(R^I)$) *Let k be a positive integer and both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple. Given $\boldsymbol{h} = 1 + \boldsymbol{g} \in 1 + \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ where $\boldsymbol{g} \in \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ then*

$$\boldsymbol{h}^{-1} = 1 + \sum_{k=1}^{M} (-1)^k \boldsymbol{g}^{\otimes k} \in 1 + \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I),$$

where $M = \sup_{R\in\mathcal{A}_s^\Pi} \|R\|$. It means that $\boldsymbol{h} \otimes \boldsymbol{h}^{-1} = \boldsymbol{h}^{-1} \otimes \boldsymbol{h} = \boldsymbol{1}$.

Proof According to Lemma 4.4, we have

$$\begin{aligned} \boldsymbol{h} \otimes \boldsymbol{h}^{-1} &= (1+\boldsymbol{g}) \otimes \left(1+\sum_{k=1}^{M}(-1)^k \boldsymbol{g}^{\otimes k}\right) = (1+\boldsymbol{g}) \otimes \left(\sum_{k=0}^{M}(-1)^k \boldsymbol{g}^{\otimes k}\right) \\ &= \left(\sum_{k=0}^{M}(-1)^k \boldsymbol{g}^{\otimes k}\right) + \left(\sum_{k=0}^{M}(-1)^k \boldsymbol{g}^{\otimes k+1}\right) \\ &= \left(\sum_{k=0}^{M}(-1)^k \boldsymbol{g}^{\otimes k}\right) + \left(\sum_{k=1}^{M+1}(-1)^{k+1} \boldsymbol{g}^{\otimes k}\right) \\ &= \mathbf{1} + (-1)^{M+2}\boldsymbol{g}^{\otimes M+1} = \mathbf{1}. \end{aligned}$$

For similar reasons, we have $\boldsymbol{h}^{-1} \otimes \boldsymbol{h} = \mathbf{1}$. Therefor, For any elements in $1+\boldsymbol{t}^{\Pi,s}(R^I)$ is invertible. □

Proposition 4.7 *Let k be a positive integer and both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple. The space $1+\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ is a Lie group with respect to tensor multiplication $\otimes$.*

Proof The proof is analogous to the proof of Proposition 7.17 of [10]. □

Next we introduce metric on $1+\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$. We denote basis of $\mathbb{R}^{i_1}, \cdots, \mathbb{R}^{i_k}$ by

$$\begin{aligned} \langle \{e_{(1,1)}, \cdots, e_{(i_1,1)}\} \rangle &= \mathbb{R}^{i_1}, \\ \langle \{e_{(1,2)}, \cdots, e_{(i_2,2)}\} \rangle &= \mathbb{R}^{i_2}, \\ &\vdots \\ \langle \{e_{(1,k)}, \cdots, e_{(i_k,k)}\} \rangle &= \mathbb{R}^{i_k}. \end{aligned}$$

Let $\boldsymbol{a} \in T^{(\Pi,s)}(\mathbb{R}^I)$ then $\pi_R(\boldsymbol{a}) \in \mathbb{R}^{i_{r_1}} \otimes \cdots \otimes \mathbb{R}^{i_{r_l}}$ for $R = (r_1, \cdots, r_l)$ and

$$\begin{aligned} \pi_R(\boldsymbol{a}) &= \left(\sum_{j_1=1}^{i_{r_1}} a_{(j_1,r_1)} e_{(j_1,r_1)}\right) \otimes \left(\sum_{j_2=1}^{i_{r_2}} a_{(j_2,r_2)} e_{(j_2,r_2)}\right) \otimes \\ &\quad \cdots \otimes \left(\sum_{j_1=1}^{i_{r_l}} a_{(j_l,r_l)} e_{(j_l,r_l)}\right) \\ &= \sum_{j_1=1}^{i_{r_1}} \cdots \sum_{j_l=1}^{i_{r_l}} a_{(j_1,r_1)} \cdot a_{(j_2,r_2)} \cdots a_{(j_l,r_l)} e_{(j_1,r_1)} \otimes e_{(j_2,r_2)} \otimes \cdots \otimes e_{(j_l,r_l)} \\ &= \sum_{j_1=1}^{i_{r_1}} \cdots \sum_{j_l=1}^{i_{r_l}} a^{(j_1,r_1)\cdots(j_l,r_l)} e_{(j_1,r_1)} \otimes e_{(j_2,r_2)} \otimes \cdots \otimes e_{(j_l,r_l)}. \end{aligned}$$

We define norm on $T^{(\Pi,s)}(\mathbb{R}^I)$ by

$$\|\boldsymbol{a}\|_{T^{(\Pi,s)}(\mathbb{R}^I)} := \max_{R \in \mathcal{A}_s^{\Pi}} \|\boldsymbol{a}\|_R \,, \tag{4.1}$$

where

$$\|\boldsymbol{a}\|_R := \sqrt{\sum_{j_1=1}^{i_{r_1}} \cdots \sum_{j_l=1}^{i_{r_l}} \left| a^{(j_1,r_1)\cdots(j_l,r_l)} \right|^2}.$$

Clearly, this is norm makes $T^{(\Pi,s)}(\mathbb{R}^I)$ a Banach space. Furthermore, we give metric on $1+\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$, i.e.

$$\rho(\boldsymbol{a},\boldsymbol{b}) := \|\boldsymbol{a}-\boldsymbol{b}\|_{T^{(\Pi,s)}(\mathbb{R}^I)} := \max_{\substack{R\in\mathcal{A}_s^\Pi \\ R\neq\epsilon}} \|\boldsymbol{a}-\boldsymbol{b}\|_R\,. \tag{4.2}$$

Clearly, $1+\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ is manifold topology on the metric ρ.

Proposition 4.8 *Let k be a positive integer and both of $\Pi=(p_1,\cdots,p_k)$ and $I=(i_1,\cdots,i_k)$ be real k-tuple. Given $\boldsymbol{h},\boldsymbol{h}_n\subset 1+\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$. Then, $\lim_{n\to\infty}\|\boldsymbol{h}_n-\boldsymbol{h}\|_{T^{(\Pi,s)}(\mathbb{R}^I)}=0$ if and only if $\lim_{n\to\infty}\left\|\boldsymbol{h}_n^{-1}\otimes\boldsymbol{h}-1\right\|_{T^{(\Pi,s)}(\mathbb{R}^I)}=0$.*

Proof The proof is analogous to the proof of Proposition 7.18 of [10]. □

Furthermore, we will recall Lie bracket from [14] that is $[\cdot,\cdot]: T\Big((V)\Big)\times T\Big((V)\Big)\to T\Big((V)\Big)$ which is defined by

$$[\boldsymbol{x},\boldsymbol{y}] := \boldsymbol{x}\otimes\boldsymbol{y}-\boldsymbol{y}\otimes\boldsymbol{x}$$

for $\boldsymbol{x},\boldsymbol{y}\in T\Big((V)\Big)$. Throughout this section, we use this Lie bracket which we replace $T\Big((V)\Big)$ be $\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$. Hence, we can see that $\Big(\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I),+,\cdot,[\cdot,\cdot]\Big)$ is a Lie algebra. We also give operator $(\text{ad }\boldsymbol{x}):\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)\to\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ for fix $\boldsymbol{x}\in\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ which is defined $(\text{ad }\boldsymbol{x})\boldsymbol{y}=[\boldsymbol{x},\boldsymbol{y}]$. Hence, by Lemma 4.4 we have $(\text{ad }\boldsymbol{x})^n\boldsymbol{y}=0$ where $n>M=\sup_{R\in\mathcal{A}_s^\Pi}\|R\|$ and denote $(\text{ad }\boldsymbol{x})^n\boldsymbol{y}=(\text{ad }\boldsymbol{x})\left((\text{ad }\boldsymbol{x})^{n-1}\boldsymbol{y}\right)$ for $n=1,2,\cdots\cdots$ and $(\text{ad }\boldsymbol{x})^0\boldsymbol{y}=\boldsymbol{y}$.

In the following definition, we also give definition of exponential and logarithm function which is recall from [14] but the domain is $\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I), 1+\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)\subset\boldsymbol{T}^{\Pi,s}(\mathbb{R}^I)$. In the definition, it is the sum of finite element because $\boldsymbol{x}^{\otimes n}=0$ and $(\boldsymbol{y}-\boldsymbol{1})^{\otimes n}=0$ for $n>M=\sup_{R\in\mathcal{A}_s^\Pi}\|R\|$, $\boldsymbol{x}\in\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ and $\boldsymbol{y}\in 1+\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$.

Definition 4.9 *Let k be a positive integer, $M:=\sup_{R\in\mathcal{A}_s^\Pi}\|R\|$ and both of $\Pi=(p_1,\cdots,p_k)$ and $I=(i_1,\cdots,i_k)$ be real k-tuple. The exponential function is defined by*

$$\exp:\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)\to 1+\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$$

where $\exp(\boldsymbol{x})=1+\sum_{j=1}^M\frac{1}{j!}\boldsymbol{x}^{\otimes j}$ for $\boldsymbol{x}\in\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$. Then, the logarithm function is defined by

$$\log:1+\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)\to\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$$

where $\log(\boldsymbol{y})=\sum_{j=1}^M\frac{(-1)^{j+1}}{j}(\boldsymbol{y}-\boldsymbol{1})^{\otimes j}$ for $\boldsymbol{y}\in\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$.

Next we emphasize that the same Campbell–Baker–Hausdorff formulas can be applied on $\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$. You can see it below.

Lemma 4.10 *Let k be a positive integer, $M := \sup_{R\in\mathcal{A}_s^\Pi}\|R\|$ and both of $\Pi = (p_1,\cdots,p_k)$ and $I = (i_1,\cdots,i_k)$ be real k-tuple. Given $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ we have*

$$\exp(\boldsymbol{x})\otimes\boldsymbol{y}\otimes\exp(-\boldsymbol{x}) = \exp(ad\ \boldsymbol{x})\boldsymbol{y}$$

where

$$\exp(ad\ \boldsymbol{x})\boldsymbol{y} := \sum_{j=0}^{M}\frac{1}{j!}(ad\ \boldsymbol{x})^j\boldsymbol{y}$$

and $M := \sup_{R\in\mathcal{A}_s^\Pi}\|R\|$.

Proof We define map $f:\mathbb{R}\to t^{\Pi,s}(\mathbb{R}^I)$ by

$$f(t) = \exp(t\boldsymbol{x})\otimes\boldsymbol{y}\otimes\exp(t\boldsymbol{x}).$$

By Taylor series, the map $f(t)$ can be rewrite as

$$f(t) = \sum_{j=0}^{\infty}\frac{1}{j!}f^{(j)}(0)t^j$$

where $f^{(n)}(0) = \frac{d^n f}{dt^n}\Big|_{t=0}$. Hence, we have

$$\begin{aligned}\frac{d}{dt}f(t) &= \frac{d}{dt}\exp(t\boldsymbol{x})\otimes\boldsymbol{y}\otimes\exp(t\boldsymbol{x}) = \boldsymbol{x}\otimes f(t) - f(t)\otimes\boldsymbol{x} = [\boldsymbol{x}, f(t)] \\ &= \left[\boldsymbol{x},\sum_{j=0}^{\infty}\frac{1}{j!}f^{(j)}(0)t^j\right] = \sum_{j=0}^{\infty}\frac{1}{j!}\left[\boldsymbol{x}, f^{(j)}(0)\right]t^j.\end{aligned}$$

In other words, we also have

$$\frac{d}{dt}f(t) = \frac{d}{dt}\sum_{j=0}^{\infty}\frac{1}{j!}f^{(n)}(0)t^n = \sum_{j=1}^{\infty}\frac{1}{(j-1)!}f^{(j)}(0)t^{j-1} = \sum_{j=0}^{\infty}\frac{1}{(j)!}f^{(j+1)}(0)t^j.$$

Therefore, we have

$$f^{(j+1)}(0) = \left[\boldsymbol{x}, f^{(j)}(0)\right]$$

where $f^0(0) = \boldsymbol{y}$. By recursively, we have

$$f^{(j)}(0) = (\mathrm{ad}\ \boldsymbol{x})^j(\boldsymbol{y}).$$

Finally, we get

$$\exp(\boldsymbol{x})\otimes\boldsymbol{y}\otimes\exp(-\boldsymbol{x}) = f(1) = \sum_{j=0}^{\infty}\frac{1}{j!}f^{(j)}(0) = \sum_{j=0}^{\infty}\frac{1}{j!}(\mathrm{ad}\ \boldsymbol{x})^j(\boldsymbol{y}) = \sum_{j=0}^{\infty}\frac{1}{j!}(\mathrm{ad}\ \boldsymbol{x})^j(\boldsymbol{y}).$$

□

Corollary 4.11 *Let k be a positive integer, $M := \sup_{R\in\mathcal{A}_s^{\Pi}} \|R\|$ and both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d} \in \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$, then*

$$\exp(ad\ \boldsymbol{z})\boldsymbol{d} = \exp(ad\ \boldsymbol{x}) \circ \exp(ad\ \boldsymbol{y})\boldsymbol{d}$$

where

$$\boldsymbol{z} = \log(\exp(\boldsymbol{x}) \otimes \exp(\boldsymbol{y})).$$

Proof By Lemma 4.10, we have

$$\begin{aligned}
\exp(\text{ad } \boldsymbol{z})\boldsymbol{d} &= \exp(\boldsymbol{z}) \otimes \boldsymbol{d} \otimes \exp(-\boldsymbol{z}) \\
&= \exp(\log(\exp(\boldsymbol{x}) \otimes \exp(\boldsymbol{y}))) \otimes \boldsymbol{d} \otimes \exp(-\log(\exp(\boldsymbol{x}) \otimes \exp(\boldsymbol{y}))) \\
&= \exp(\boldsymbol{x}) \otimes \exp(\boldsymbol{y}) \otimes \boldsymbol{d} \otimes \exp(-\boldsymbol{y}) \otimes \exp(-\boldsymbol{x}) \\
&= \exp(\boldsymbol{x}) \otimes (\exp(\boldsymbol{y}) \otimes \boldsymbol{d} \otimes \exp(-\boldsymbol{y})) \otimes \exp(-\boldsymbol{x}) \\
&= \exp(\boldsymbol{x}) \otimes (\exp(\text{ad } \boldsymbol{y})\boldsymbol{d}) \otimes \exp(-\boldsymbol{x}) \\
&= \exp(\text{ad } \boldsymbol{x}) \circ \exp(\text{ad } \boldsymbol{y})\boldsymbol{d}
\end{aligned}$$

□

Lemma 4.12 *Let k be a positive integer, $M := \sup_{R\in\mathcal{A}_s^{\Pi}} \|R\|$ and both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple. Let f_t be function on $\boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ is continuously differentiable and denote $\dot{f}_t = \frac{df_t}{dt}$ and*

$$g(\boldsymbol{x}) = \Big(\exp(\boldsymbol{x}) - 1\Big) \otimes (\boldsymbol{x}^{-1}) = \sum_{j=1}^{M} \frac{1}{j!}\boldsymbol{x}^{\otimes j-1}$$

for $\boldsymbol{x} \in \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$, then

$$\exp(f_t) \otimes \frac{d}{dt}\exp(-f_t) = -g(ad\ f_t)\dot{f}_t.$$

Proof We first observe that

$$\frac{d}{dt}\exp(f_t) = \int_0^1 \exp\Big((1-y)f_t\Big) \otimes \dot{f}_t \otimes \exp(yf_t)\ dy.$$

On the left hand side, we have

$$\begin{aligned}
\frac{d}{dt}\exp(f_t) &= \frac{d}{dt}\left(1 + \sum_{j=1}^{M}\frac{1}{j!}{f_t}^{\otimes j}\right) = \sum_{j=1}^{M}\frac{1}{j!}\frac{d}{dt}{f_t}^{\otimes j} \\
&= \sum_{j+h=0}^{M-1}\frac{1}{(j+h+1)!}f_t^{\otimes j} \otimes \dot{f}_t \otimes f_t^{\otimes h}.
\end{aligned}$$

On the right hand side, we find

$$
\begin{aligned}
& \int_0^1 \exp((1-y)f_t) \otimes \dot{f}_t \otimes \exp(yf_t)\ dy \\
= & \int_0^1 \left(\sum_{j=0}^{M} \frac{1}{j!}(1-y)^j {f_t}^{\otimes j} \right) \otimes \dot{f}_t \otimes \left(\sum_{h=0}^{M} \frac{1}{h!} y^h {f_t}^{\otimes h} \right)\ dy \\
= & \sum_{j=0}^{M} \sum_{h=0}^{M} \frac{1}{j!h!} \int_0^1 \left((1-y)^j {f_t}^{\otimes j}\right) \otimes \dot{f}_t \otimes \left(y^h {f_t}^{\otimes h} \right)\ dy \\
= & \sum_{j=0}^{M} \sum_{h=0}^{M} \frac{1}{j!h!} \int_0^1 (1-y)^j y^h\ dy\ \left({f_t}^{\otimes j} \otimes \dot{f}_t \otimes {f_t}^{\otimes h} \right) \\
= & \sum_{j=0}^{M} \sum_{h=0}^{M} \frac{1}{j!h!} \left(\frac{j!h!}{(j+h+1)!} \right)\ \left({f_t}^{\otimes j} \otimes \dot{f}_t \otimes {f_t}^{\otimes h} \right) \\
= & \sum_{j+h=0}^{M-1} \frac{1}{(j+h+1)!}\ \left({f_t}^{\otimes j} \otimes \dot{f}_t \otimes {f_t}^{\otimes h} \right),
\end{aligned}
$$

where we get the fifth equation by beta function and the sixth equation can be obtained by using Lemma 4.4. Hence, we have

$$
\begin{aligned}
\exp(-f_t) \otimes \frac{d}{dt} \exp(f_t) &= \int_0^1 \exp(-yf_t) \otimes \dot{f}_t \otimes \exp(yf_t)\ dy \\
&= \int_0^1 \sum_{j=0}^{M} \frac{1}{j!} (\mathrm{ad}\ (-yf_t))^j \dot{f}_t\ dy
\end{aligned}
$$

We then replace f_t be $-f_t$, we get

$$
\begin{aligned}
\exp(f_t) \otimes \frac{d}{dt} \exp(-f_t) &= -\int_0^1 \sum_{j=0}^{M} \frac{1}{j!} \Big(\mathrm{ad}\ (yf_t) \Big)^j \dot{f}_t\ dy \\
&= -\int_0^1 \sum_{j=0}^{M} \frac{1}{j!} y^j (\mathrm{ad}\ f_t)^j \dot{f}_t\ dy \\
&= -\sum_{j=0}^{M} \frac{1}{(j+1)!} (\mathrm{ad}\ f_t)^j \dot{f}_t\ dy = -g(\mathrm{ad}\ f_t)\dot{f}_t
\end{aligned}
$$

□

Theorem 4.13 (Campbell–Baker–Hausdorff) *Let k be a positive integer, $M := \sup_{R \in \mathcal{A}_s^{\Pi}} \|R\|$ and both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple. Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$ and denote*

$$
g(\boldsymbol{z}) = \Big(\log(\boldsymbol{z}) \Big) \otimes (\boldsymbol{z} - 1)^{-1} = \sum_{j=1}^{M} \frac{(-1)^{j+1}}{j} (\boldsymbol{z} - \mathbf{1})^{\otimes j-1}
$$

for $\boldsymbol{z} \in \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I)$*, then*

$$\log\Big[\exp(\boldsymbol{x}) \otimes \exp(\boldsymbol{y})\Big] = \boldsymbol{y} + \int_0^1 g\Big(\exp(t\ \ ad\ \boldsymbol{x}) \circ \exp(ad\ \boldsymbol{y})\Big)\boldsymbol{x} dt.$$

In particular, $\log\Big[\exp(\boldsymbol{x}) \otimes \exp(\boldsymbol{y})\Big]$ *equals a sum of iterated bracket of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$*, with universal coefficient.*

Proof The proof is analogous to the proof of Theorem 7.24 of [10]. The different is finite sum of $g(\boldsymbol{z})$. □

Furthermore, we will introduce free degree-(Π, N) nilpotent Lie algebra on the following definition. The definition is inspired by Definition 7.25 of [10] and Definition 3.3.3 of [14], but given more details definition.

Definition 4.14 *Let* k *be a positive integer, both of* $\Pi = (p_1, \cdots, p_k)$ *and* $I = (i_1, \cdots, i_k)$ *be real* k*-tuple. Let* $S^\Pi = \{s_0 = 0, s_1, \cdots\}$ *listed in ascending order. We denote the smallest sub-Lie algebra of* $\boldsymbol{t}^{\Pi,s_N}(\mathbb{R}^I)$ *which contains* $\pi_{(r_j)}\Big(\boldsymbol{t}^{\Pi,s_N}(\mathbb{R}^I)\Big) = \mathbb{R}^{i_j}$ *for all* $j = 1, \cdots, k$ *by* $\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)$ *which is called free degree-*(Π, N) *nilpotent Lie algebra. We also denote the set* $\mathcal{L}_{\Pi,R}(\mathbb{R}^I)$ *by*

$$\begin{aligned}
\mathcal{L}_{\Pi,(r_1)}(\mathbb{R}^I) &:= \mathbb{R}^{i_{r_1}} \\
\mathcal{L}_{\Pi,(r_1,r_2)}(\mathbb{R}^I) &:= \Big[\mathbb{R}^{i_{r_1}}, \mathbb{R}^{i_{r_2}}\Big] := Span\Big\{[a,b] | a \in \mathbb{R}^{i_{r_1}}, b \in \mathbb{R}^{i_{r_2}}\Big\} \\
&\ \ \vdots \\
\mathcal{L}_{\Pi,(r_1,\cdots,r_l)}(\mathbb{R}^I) &:= \Big[\mathbb{R}^{i_{r_1}}, \mathcal{L}_{\Pi,(r_1,\cdots,r_{l-1})}(\mathbb{R}^I)\Big] \\
&:= Span\Big\{[a,b] | a \in \mathbb{R}^{i_{r_1}}, b \in \mathcal{L}_{\Pi,(r_1,\cdots,r_{l-1})}(\mathbb{R}^I)\Big\}
\end{aligned}$$

Lemma 4.15 *Let* k *be a positive integer, both of* $\Pi = (p_1, \cdots, p_k)$ *and* $I = (i_1, \cdots, i_k)$ *be real* k*-tuple. Let* $S^\Pi = \{s_0 = 0, s_1, \cdots\}$ *listed in ascending order and*

$$H = \bigoplus_{\substack{R \in \mathcal{A}^\Pi_{s_N} \\ R \neq \epsilon}} \mathcal{L}_{\Pi,R}(\mathbb{R}^I),$$

then $H = \mathfrak{g}^{\Pi,N}(\mathbb{R}^I)$.

Proof It is easily seen that H is subset of all sub-Lie algebra of $\boldsymbol{t}^{\Pi,s_N}(\mathbb{R}^I)$ which contains $\pi_{(r_j)}\Big(\boldsymbol{t}^{\Pi,s_N}(\mathbb{R}^I)\Big) = \mathbb{R}^{i_j}$ for all $j = 1, \cdots, k$. We also easily see that $(H, +, \cdot)$ is vector space. Therefore, we sufficient show that $[\cdot,\cdot] : H \times H \to H$. We first give $\boldsymbol{x}, \boldsymbol{y} \in H$ which can be written as

$$\begin{aligned}
\boldsymbol{x} &= \sum_{\substack{R \in \mathcal{A}^\Pi_{s_N} \\ R \neq \epsilon}} \pi_R(\boldsymbol{x}), \text{where } \pi_R(\boldsymbol{x}) \in \mathcal{L}_{\Pi,R}(\mathbb{R}^I), \\
\boldsymbol{y} &= \sum_{\substack{R \in \mathcal{A}^\Pi_{s_N} \\ R \neq \epsilon}} \pi_R(\boldsymbol{y}), \text{where } \pi_R(\boldsymbol{y}) \in \mathcal{L}_{\Pi,R}(\mathbb{R}^I),
\end{aligned}$$

which we have

$$\begin{aligned}[\boldsymbol{x},\boldsymbol{y}] &= \boldsymbol{x}\otimes\boldsymbol{y}-\boldsymbol{y}\otimes\boldsymbol{x}\\ &= \sum_{\substack{R=(r_1,\cdots,r_l)\in\mathcal{A}^{\Pi}_{s_N}\\ l\neq 0}}\sum_{j=1}^{l-1}\pi_{(r_1,\cdots,r_j)}(\boldsymbol{x})\otimes\pi_{(r_{j+1},\cdots,r_l)}(\boldsymbol{y})\\ &\quad -\pi_{(r_{j+1},\cdots,r_l)}(\boldsymbol{y})\otimes\pi_{(r_1,\cdots,r_j)}(\boldsymbol{x}).\end{aligned}$$

Therefore we will prove that

$$\pi_{(r_1,\cdots,r_j)}(\boldsymbol{x})\otimes\pi_{(r_{j+1},\cdots,r_l)}(\boldsymbol{y})-\pi_{(r_{j+1},\cdots,r_l)}(\boldsymbol{y})\otimes\pi_{(r_1,\cdots,r_j)}(\boldsymbol{x})\in\mathcal{L}_{\Pi,R}(\mathbb{R}^I)$$

for all $j=1,\cdots,l-1$ by strong induction. For $j=1$, we have

$$\begin{aligned}\pi_{(r_1)}(\boldsymbol{x})\otimes\pi_{(r_2,\cdots,r_l)}(\boldsymbol{y})-\pi_{(r_2,\cdots,r_l)}(\boldsymbol{y})\otimes\pi_{(r_1)}(\boldsymbol{x}) &= \Big[\pi_{(r_1)}(\boldsymbol{x}),\pi_{(r_2,\cdots,r_l)}(\boldsymbol{y})\Big]\\ &\in \mathcal{L}_{\Pi,R}(\mathbb{R}^I).\end{aligned}$$

Furthermore, we assume that that it is true for $j=1,\cdots,K$ and we will prove it is true for $j=K+1$. For abbreviation, we write $\pi_{(r_{K+2},\cdots,r_l)}(\boldsymbol{y})=c$ and $\pi_{(r_1,\cdots,r_K,r_{K+1})}(\boldsymbol{x})=\sum_i a_i\otimes b_i-b_i\otimes a_i$ where $a_i\in\mathcal{L}_{\Pi,(r_1)}(\mathbb{R}^I)$, $b_i\in\mathcal{L}_{\Pi,(r_2,\cdots,r_{K+1})}(\mathbb{R}^I)$, and $c\in\mathcal{L}_{\Pi,(r_{K+2},\cdots,r_l)}(\mathbb{R}^I)$. Therefore, we have

$$\begin{aligned}&\pi_{(r_1,\cdots,r_{K+1})}(\boldsymbol{x})\otimes\pi_{(r_{K+2},\cdots,r_l)}(\boldsymbol{y})-\pi_{(r_{K+2},\cdots,r_l)}(\boldsymbol{y})\otimes\pi_{(r_1,\cdots,r_{K+1})}(\boldsymbol{x})\\ &=\sum_i(a_i\otimes b_i-b_i\otimes a_i)\otimes c-c\otimes(a_i\otimes b_i-b_i\otimes a_i)\\ &=\sum_i a_i\otimes b_i\otimes c-b_i\otimes a_i\otimes c-c\otimes a_i\otimes b_i-c\otimes b_i\otimes a_i\\ &=\sum_i a_i\otimes(b_i\otimes c-c\otimes b_i)-b_i\otimes(a_i\otimes c-c\otimes a_i)\\ &-(c\otimes a_i-a_i\otimes c)\otimes b_i+(c\otimes b_i-b_i\otimes c)\otimes a_i\\ &=\sum_i a_i\otimes(b_i\otimes c-c\otimes b_i)-(b_i\otimes c-c\otimes b_i)\otimes a_i\\ &-b_i\otimes(a_i\otimes c-c\otimes a_i)+(a_i\otimes c-c\otimes a_i)\otimes b_i\\ &=\sum_i\Big[a_i,[b_i,c]\Big]-\Big[b_i,[a_i,c]\Big].\end{aligned}$$

By induction hypothesis, we have

$$\begin{aligned}\sum_i\Big[a_i,[b_i,c]\Big] &\in \Big[\mathcal{L}_{\Pi,(r_1)}(\mathbb{R}^I),\big[\mathcal{L}_{\Pi,(r_2,\cdots,r_{K+1})}(\mathbb{R}^I),\mathcal{L}_{\Pi,(r_{K+2},\cdots,r_l)}(\mathbb{R}^I)\big]\Big]\\ &= \mathcal{L}_{\Pi,(r_1,\cdots,r_l)}(\mathbb{R}^I)\end{aligned}$$

and

$$\sum_i \Big[b_i, [a_i, c]\Big] \in \Big[\mathcal{L}_{\Pi,(r_2,\cdots,r_{K+1})}(\mathbb{R}^I), \big[\mathcal{L}_{\Pi,(r_1)}(\mathbb{R}^I), \mathcal{L}_{\Pi,(r_{K+2},\cdots,r_l)}(\mathbb{R}^I)\big]\Big]$$
$$= \mathcal{L}_{\Pi,(r_1,\cdots,r_l)}(\mathbb{R}^I).$$

Hence,

$$\pi_{(r_1,\cdots,r_{K+1})}(\boldsymbol{x}) \otimes \pi_{(r_{K+2},\cdots,r_l)}(\boldsymbol{y}) - \pi_{(r_{K+2},\cdots,r_l)}(\boldsymbol{y}) \otimes \pi_{(r_1,\cdots,r_{K+1})}(\boldsymbol{x}) \in \mathcal{L}_{\Pi,R}(\mathbb{R}^I).$$

Finally, the proof may be completed by strong induction. □

Corollary 4.16 *Let k be a positive integer, both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple. Let $S^\Pi = \{s_0 = 0, s_1, \cdots\}$ listed in ascending order then $\exp\left(\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)\right)$ is subgroup of $\left(\mathbf{1} + \boldsymbol{t}^{\Pi,s}(\mathbb{R}^I), \otimes\right)$.*

Proof We take arbitrary $\boldsymbol{x}, \boldsymbol{y} \in \exp\left(\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)\right)$ and we can write $\boldsymbol{x} = \exp(\boldsymbol{a})$, $\boldsymbol{y} = \exp(\boldsymbol{b})$ where $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{g}^{\Pi,N}(\mathbb{R}^I)$. According to Theorem 4.13, $\log[\exp(\boldsymbol{a}) \otimes \exp(-\boldsymbol{b})]$ equals a sum of iterated bracket of $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\log[\exp(\boldsymbol{a}) \otimes \exp(-\boldsymbol{b})] \in \mathfrak{g}^{\Pi,N}(\mathbb{R}^I)$. Therefore, $\boldsymbol{x} \otimes \boldsymbol{y}^{-1} \in \exp\left(\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)\right)$. □

Next we also emphasize that the Chow theorem can be applied on $\exp\left(\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)\right)$. You can see it in Theorem 4.19. But we first give Lemma 4.17 which will be used to prove Theorem 4.19.

Lemma 4.17 *Let k be a positive integer; both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple; and $S^\Pi = \{s_0 = 0, s_1, \cdots\}$ listed in ascending order. Let $x = x^1 + \cdots + x^k : [T_1, T_2] \to \mathbb{R}^I$ be piecewise linear path with $x_t = \boldsymbol{v}_j t + \boldsymbol{c}_j = (\boldsymbol{v}_j^1 t + \boldsymbol{c}_j^1) + \cdots + (\boldsymbol{v}_j^k t + \boldsymbol{c}_j^k)$ for $t_{j-1} \le t \le t_j$ where $T_1 = t_0 < t_1 < \cdots < t_n = T_2$ and $j = 1, \cdots, n$. Then we have*

$$S_{\Pi,N}(x)_{T_1,T_2} = \bigotimes_{j=1}^{n} \exp\Big((t_j - t_{j-1})v_j\Big).$$

Proof In order to prove the Lemma, we will do it by three steps.

Step 1. We will prove that $S_{\Pi,N}(y)_{0,1} = \exp\Big(\boldsymbol{v}\Big)$ with $y = y^1 + \cdots + y^k : [0,1] \ni t \mapsto \boldsymbol{v}t + \boldsymbol{c} = (\boldsymbol{v}^1 t + \boldsymbol{c}^1) + \cdots +$

$(\boldsymbol{v}^k t + \boldsymbol{c}^k) \in \mathbb{R}^I$ where $\boldsymbol{v}, \boldsymbol{c} \in \mathbb{R}^I$.

$$\begin{aligned} S_{\Pi,N}(y)_{0,1} &= \sum_{\substack{R \in \mathcal{A}^{\Pi}_{s_N} \\ R=(r_1,\cdots,r_l)}} \int_{0<u_1<\cdots<u_l<1} dy^{r_1}_{u_1} \otimes \cdots \otimes dy^{r_l}_{u_l} \\ &= \sum_{\substack{R \in \mathcal{A}^{\Pi}_{s_N} \\ R=(r_1,\cdots,r_l)}} \int_{0<u_1<\cdots<u_l<1} d(\boldsymbol{v}^{r_1} u_1) \otimes \cdots \otimes d(\boldsymbol{v}^{r_l} u_l) \\ &= \sum_{\substack{R \in \mathcal{A}^{\Pi}_{s_N} \\ R=(r_1,\cdots,r_l)}} \boldsymbol{v}^{r_1} \otimes \cdots \otimes \boldsymbol{v}^{r_l} \int_{0<u_1<\cdots<u_l<1} du_1 \cdots du_l \\ &= \sum_{\substack{R \in \mathcal{A}^{\Pi}_{s_N} \\ R=(r_1,\cdots,r_l)}} \frac{1}{l!} \boldsymbol{v}^{r_1} \otimes \cdots \otimes \boldsymbol{v}^{r_l} = \sum_{j=0}^{M} \frac{1}{j!} \boldsymbol{v}^{\otimes j} = \exp(\boldsymbol{v}). \end{aligned}$$

where $M := \sup_{R \in \mathcal{A}^{\Pi}_s} \|R\|$.

Step 2. We will prove that $S_{\Pi,N}(y)_{p,q} = \exp\Big((q-p)\boldsymbol{v}\Big)$ with $y = y^1 + \cdots + y^k : [p,q] \ni t \mapsto \boldsymbol{v}t + \boldsymbol{c} = (\boldsymbol{v}^1 t + \boldsymbol{c}^1) + \cdots + (\boldsymbol{v}^k t + \boldsymbol{c}^k) \in \mathbb{R}^I$ where $\boldsymbol{v}, \boldsymbol{c} \in \mathbb{R}^I$. We will use Step 1 and Lemma 3.9 where we take $\varphi(t) = \frac{t-p}{q-p}$ and $z = y \circ \varphi^{-1}$.

$$\begin{aligned} S_{\Pi,N}(y)_{p,q} &= S_{\Pi,N}(z \circ \varphi)_{p,q} = S_{\Pi,N}(z)_{\varphi(p),\varphi(q)} = S_{\Pi,N}(z)_{0,1} \\ &= S_{\Pi,N}\Big((q-p)\boldsymbol{v}t + s\boldsymbol{v} + \boldsymbol{c}\Big)_{0,1} = \exp\Big((q-p)\boldsymbol{v}\Big) \end{aligned}$$

Step 3. We will prove Lemma 4.17 by Chen identity and Step 2.

$$S_{\Pi,N}(x)_{T_1,T_2} = \bigotimes_{j=1}^{n} S_{\Pi,N}(x)_{t_{j-1},t_j} = \bigotimes_{j=1}^{n} \exp\Big((t_j - t_{j-1})v_j\Big).$$

□

Remark 4.18 *According to Lemma 4.17, we can see that* $\boldsymbol{h} \in \exp\big(\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)\big)$ *which has form*

$$\boldsymbol{h} = \bigotimes_{j=1}^{m} \exp\Big(\boldsymbol{v}_j\Big)$$

where $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m \in \mathbb{R}^I$, *then there exist piecewise linear path*

$$x = x^1 + x^2 + \cdots + x^k : [0,1] \to \mathbb{R}^I$$

with $S_{\Pi,N}(x)_{0,1} = \boldsymbol{h}$ *where*

$$x_t = \frac{1}{t_j - t_{j-1}} \boldsymbol{v}_j t + \boldsymbol{c}_j = \Big(\frac{1}{t_j - t_{j-1}} \boldsymbol{v}^1_j t + \boldsymbol{c}^1_j, \cdots, \frac{1}{t_j - t_{j-1}} \boldsymbol{v}^k_j t + \boldsymbol{c}^k_j\Big),$$

$t_{j-1} \le t \le t_j$, $0 = t_0 < t_1 < \cdots < t_m = 1$ *and* $j = 1, \cdots, m$.

Theorem 4.19 (Chow) *Let k be a positive integer; both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple; and $S^\Pi = \{s_0 = 0, s_1, \cdots\}$ listed in ascending order. If $\boldsymbol{h} \in \exp\left(\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)\right)$, then there exist $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m \in \mathbb{R}^I$ such that*

$$\boldsymbol{h} = \bigotimes_{j=1}^{m} \exp\left(\boldsymbol{v}_j\right).$$

Proof The proof is similar to the classical case in the book of Friz and Victoir [10]. The difference is space vector basis of nilpotent Lie algebra $\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)$.

□

Remark 4.20 *According to Lemma 4.17 and Theorem 4.19, we can see that if $\boldsymbol{h} \in \exp\left(\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)\right)$, then there exist vector $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m \in \mathbb{R}^I$, piecewise linear path $x = x^1 + x^2 + \cdots + x^k : [0,1] \to \mathbb{R}^I$ such that*

$$\boldsymbol{h} = \bigotimes_{j=1}^{m} \exp\left(\boldsymbol{v}_j\right) = S_{\Pi,N}(x)_{0,1}$$

where

$$x_t = \frac{1}{t_j - t_{j-1}}\boldsymbol{v}_j t + \boldsymbol{c}_j = \left(\frac{1}{t_j - t_{j-1}}\boldsymbol{v}_j^1 t + \boldsymbol{c}_j^1, \cdots, \frac{1}{t_j - t_{j-1}}\boldsymbol{v}_j^k t + \boldsymbol{c}_j^k\right), \textit{for } t_{j-1} \le t \le t_j$$

with $0 = t_0 < t_1 < \cdots < t_m = 1$ and $j = 1, \cdots, m$.

Corollary 4.21 *Let k be a positive integer; both of $\Pi = (p_1, \cdots, p_k)$ and $I = (i_1, \cdots, i_k)$ be real k-tuple; and $S^\Pi = \{s_0 = 0, s_1, \cdots\}$ listed in ascending order. Given $(\boldsymbol{g}_n) \subset \exp\left(\mathfrak{g}^{\Pi,N}(\mathbb{R}^I)\right)$ converges to $\mathbf{1}$. Let x_n denote the piecewise linear such that $S_{\Pi,N}(x_n) = \boldsymbol{g}_n$ as constructed in the proof of Theorem 4.19. Then the length of x_n converges to 0, i.e. $\int_0^1 |dx_n| \to 0$.*

Proof The proof is analogous to the proof of Theorem 7.29 of [10]. □

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