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Existence results and Ulam–Hyers stability to impulsive coupled system fractional differential equations

Hadjer BELBALI[1], **Maamar BENBACHIR**[2,*]
[1]Laboratory of Mathematics and Applied Sciences, University of Ghardaia, Bounoura, Algeria
[2]Faculty of Sciences, Saad Dahlab University, Blida 1, Algeria

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Abstract: In this paper, the existence and uniqueness of the solutions to impulsive coupled system of fractional differential equations with Caputo–Hadamard are investigated. Furthermore, Ulam's type stability of the proposed coupled system is studied. The approach is based on a Perov type fixed point theorem for contractions.

Key words: Fractional differential equations, Caputo–Hadamard derivative, fixed point theorems, coupled system, Ulam stability, convergent matrix

1. Introduction

The fractional calculus becomes now a very attractive subject to mathematicians, as an important field of investigation due to its extensive applications in numerous branches of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer, rheology, etc., one can consult [3, 25, 38, 42] and references therein. Some authors proposed a new type of fractional derivatives possessing different kernels, because the most used definitions proposed by Riemann–Liouville and the first Caputo version has the weakness that their kernel had singularity [8]. Definition of Hadamard's fractional derivative introduced in 1892 differs significantly from both the Riemann–Liouville type and the Caputo type [24] . In particular, the integral's kernel in the definition of Hadamard's fractional derivative contains a logarithmic function of so-called arbitrary exponent. There are several articles describing the properties and applications of Hadamard derivative [10–12, 33, 37, 47]. A recent new definition of fractional derivative has been provided by modifying the Hadamard derivative with the Caputo one, known as Caputo–Hadamard derivative, which was first studied by Jarad et al. [19], it is obtained from the Hadamard derivative by changing the order of its differentiation and integration [1], in addition, existence and uniqueness of solutions of fractional differential equations involving Caputo–Hadamard were considered, see for examples [2, 16].

The classical Banach contraction principle is a very useful tool in nonlinear analysis with many applications to operational equations, fractal theory, optimization theory and other topics. The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metrics by Perov in 1964 [40] and Perov and Kibenko [41]. In 1966 Perov formulated a fixed point theorem which extends the well-known contraction mapping principle for the case when the metric d takes values in $\mathbb{R}^m$, that is, in the case when we have a generalized metric space.

*Correspondence: mbenbachir2001@gmail.com
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Another important aspect of the research was that which attracted researcher's attention is Ulam stability and their various types. The abovementioned stability was first introduced by Ulam [27] in 1940 and then was confirmed by Hyers in 1941 [28]. Rassias generalized the Ulam–Hyers stability by considering variables. Thereafter, mathematicians extended the work mentioned above to functional, differential, integrals and FDEs. Wang [48] was the first mathematician who investigated the Ulam–Hyers stability for the impulsive ordinary differential equations in 2012. In the same line, he also obtained the aforesaid stability for the evolution equations [49]. For more details on the recent advances on the Ulam–Hyers stability and the Ulam–Hyers–Rassias stability of differential equations, one can see the monographs [14, 20] and the research papers [21, 26, 31, 35, 44, 50, 51]. We also note that Ulam stability has excellent applications in numerical analysis, optimization, economic, physics, biochemistry, and biological phenomena, and it does provide an effective way to seek the exact solution for the original equation.

Ali et al. [6] studied the Ulam–Hyers stability of the following

$$\begin{cases} {}^cD^p y(t) - f(t, z(t)) = 0; \quad t \in [0,1], \\ {}^cD^q z(t) - g(t, y(t)) = 0; \quad t \in [0,1], \\ y(t)|_{t=0} = 0, \;\; y(t)|_{t=1} = 0, \;\; I_T^{\gamma} h(y) = \frac{1}{\Gamma(\gamma)} \int_0^T (t-\varsigma)^{\gamma-1} h(y(\varsigma))\, d\varsigma, \\ z(t)|_{t=0} = 0, \;\; z(t)|_{t=1} = 0, \;\; I_T^{\delta} j(z) = \frac{1}{\Gamma(\delta)} \int_0^T (t-\varsigma)^{\delta-1} j(z(\varsigma))\, d\varsigma, \end{cases}$$

where $p, q, \gamma, \delta \in (1,2]$, $h, j \in L[0,1]$ are boundary functions and $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Ali et al. [5] studied the Ulam–Hyers stability of the following

$$\begin{cases} {}^cD^p y(t) - f(t, z(t), {}^cD^p y(t)) = 0; \quad p \in (2,3]; \; t \in \mathcal{J}, \\ {}^cD^q z(t) - g(t, y(t), {}^cD^p z(t)) = 0; \quad q \in (2,3]; \; t \in \mathcal{J}, \\ y^{'}(t)|_{t=0} = y^{''}(t)|_{t=0} = 0, \;\; y^{'}(t)|_{t=1} = \lambda y(\eta), \;\; \lambda, \eta \in (0,1), \\ z^{'}(t)|_{t=0} = z^{''}(t)|_{t=0} = 0, \;\; z^{'}(t)|_{t=1} = \lambda z(\eta), \;\; \lambda, \eta \in (0,1), \end{cases}$$

where $\mathcal{J} = [0,1]$ and $f, g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Zada et al. [52] studied the Ulam–Hyers stability of the following

$$\begin{cases} {}^cD^{\alpha} x(t) + h(t, {}^cD^a x(t), {}^cD^b y(t)) = 0; \quad t \neq t_m, \; m = 1, 2, \ldots n, \\ {}^cD^{\beta} y(t) + w(t, {}^cD^a x(t), {}^cD^b y(t)) = 0; \quad t \neq t_m, \; m = 1, 2, \ldots n, \\ \Delta x|_{t=t_m} = M_{1m}(x(t_m)), \; \Delta x^{'}|_{t=t_m} = N_{1m}(x(t_m)), \; \Delta x^{''}|_{t=t_m} = O_{1m}(x(t_m)), \\ \Delta y|_{t=t_m} = M_{2m}(x(t_m)), \; \Delta y^{'}|_{t=t_m} = N_{2m}(x(t_m)), \; \Delta y^{''}|_{t=t_m} = O_{2m}(x(t_m)), \\ x(0) = x'(0) = 0, \; {}^cD^{\epsilon} x(\Omega) = x''(1), \\ y(0) = y'(0) = 0, \; {}^cD^{p} y(\Phi) = y''(1), \end{cases}$$

where $t \in J = [0,1]$, $2 < \alpha, \beta \leq 3$, $0 < a, b, \epsilon, \mathit{\Omega}, p, \Phi < 1$, and $h, w : J \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions. $M_{1m}, M_{2m}, N_{1m}, N_{2m}, O_{1m}, O_{2m} \in C(\mathbb{R}, \mathbb{R})$.

In this paper, we study the existence, uniqueness and Ulam's type stability of a impulsive coupled system of fractional differential equations of the form :

$$\begin{cases} \left(D_t^{\alpha} x\right)(t) & = f_1(t,x,y) \quad t \in [a,T], \quad t \neq t_k, \; k = 1, ..., m, \\ \left(D_t^{\beta} y\right)(t) & = f_2(t,x,y) \quad t \in [a,T], \quad t \neq t_k, \; k = 1, ..., m, \\ \Delta x(t_k) & = I_k(x(t_k^-), y(t_k^-)), \qquad k = 1, ..., m, \\ \Delta y(t_k) & = \overline{I}_k(x(t_k^-), y(t_k^-)), \qquad k = 1, ..., m, \\ x(a) = x_a, & \\ y(a) = y_a, & \end{cases} \tag{1.1}$$

where D_a^α, D_a^β, are the Caputo–Hadamard fractional derivative of order α and β, $0<\alpha,\beta<1$, $a>0$. Here $a=t_0\leq t_1\leq\cdots\leq t_m\leq t_{m+1}=T$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, $x(t_k^+)=\lim_{h\to 0}x(t_k+h)$ and $x(t_k^-)=\lim_{h\to 0}x(t_k-h)$ represent the right and left limits of $x(t)$ at $t=t_k$ respectively. $x_a, y_a\in\mathbb{R}$, $f_1,f_2: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and I_k , $\overline{I}_k\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ are a given functions.

The plan of the paper is as follows. In Section 2, some definitions, theorems, lemmas and results are given which will be required for the later sections. In Section 3, we built up some appropriate conditions for the existence and uniqueness of solutions to the considered problem (1.1) using the Perov fixed point theorem. In Section 4, we study the Ulam–Hyers stability. In the last section, an example is given to demonstrate our main theoretical result.

2. Preliminaries

Definition 2.1 *[9, 25] The Hadamard fractional integral of order α for a function $h:[a;b]\to\mathbb{R}$ where $a,b\geq 0$ is defined by*

$$I_a^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds,\quad Re(\alpha)>0, \tag{2.1}$$

provided the integral exists.

Definition 2.2 *[9, 19] Let $AC_\delta^n[a,b]=\left\{g:[a,b]\to\mathbb{C}\ |\delta^{n-1}g\in AC[a,b]\right\}$ where $\delta=t\frac{d}{dt}$, $0<a<b<\infty$ and let $\alpha\in\mathbb{C}$, such that $Re(\alpha)\geq 0$. For a function $g\in AC_\delta^n[a,b]$ the Caputo type Hadamard derivative of fractional order α is defined as follows:*

(i) if $\alpha\leq\mathbb{N}$, then for $n-1<[Re(\alpha)]<n$, where $[Re(\alpha)]$ denotes the integer part of $Re(\alpha)$,

$$D_a^\alpha g(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{n-\alpha-1}\delta^n\frac{g(s)}{s}ds, \tag{2.2}$$

(ii) if $\alpha\in\mathbb{N}$, then $(D_a^\alpha g)(t)=\delta^n g(t)$.

Lemma 2.3 *[4] Suppose that f is continuous. Then the initial value problem (IVP)*

$$\begin{cases} D_a^\alpha x(t)=f(t,x,y), & t>a,\ a>0,\ 0<\alpha<1\\ x(a)=x_a, \end{cases}$$

is equivalent to the following Volterra integral equation

$$x(t)=x_a+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,x(s),y(s))}{s}ds.$$

Lemma 2.4 *Let $0<\alpha<1$ and let $f\in AC[J\times\mathbb{R}\times\mathbb{R}]$, a function x is a solution of the fractional integral equation*

$$x(t)=\begin{cases} x_a+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,x(s),y(s))}{s}ds & if\, t\in[a,t_1]\\ x_a+\frac{1}{\Gamma(\alpha)}\sum\limits_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{f(s,x(s),y(s))}{s}ds & \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,x(s),y(s)}{s}ds+\sum\limits_{i=1}^{k}I_i(x(t_i^-),y(t_i^-)) & if\, t\in(t_k,t_{k+1}],\quad k=1,...,m. \end{cases} \tag{2.3}$$

if and only if x is a solution of the impulsive fractional IVP

$$\Big(D_a^{\alpha}x\Big)(t) = f(t,x,y) \qquad \text{for each } t \in J, \tag{2.4}$$

$$\Delta x(t_k) = I_k(x(t_k^-), y(t_k^-)),\ k = 1, ..., m, \tag{2.5}$$

$$x(a) = x_a, \tag{2.6}$$

Proof Assume x satisfies (2.4)–(2.6). Using conditions (2.5), (2.6) and Lemma 2.3, we obtain:
If $t \in [a, t_1]$, then

$$x(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds.$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned}
x(t) &= x(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\
&= \Delta x(t_1) + x(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\
&= I_1(x(t_1^-), y(t_1^-)) + x_a + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds.
\end{aligned}$$

If $t \in (t_2, t_3]$, then

$$\begin{aligned}
x(t) &= x(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\
&= \Delta x(t_2) + x(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\
&= x_a + I_1(x(t_1^-), y(t_1^-)) + I_2(x(t_2^-), y(t_2^-)) + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds.
\end{aligned}$$

Repeating the same process for $t \in [t_k, t_{k+1}]$ and $k = 3, ..., m$, then we get

$$\begin{aligned}
x(t) = x_a &+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln\frac{t_i}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s))}{s} ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),y(s)}{s} ds \\
&+ \sum_{i=1}^{k} I_i(x(t_i^-), y(t_i^-)).
\end{aligned}$$

Conversely, assume that x satisfies the impulsive fractional integral equation (2.3). If $t \in [a, t_1]$ then $x(a) = x_a$ and using the fact that D_a^{α} is the left inverse of I_a^{α} and using the fact that $D_a^{\alpha}C = 0$, where C is a constant, we obtain

$$D_a^\alpha x(t) = f(t, x, y) \text{ for all } t \in [a, t_1] \cup [t_k, t_{k+1}], k = 1, \dots, m.$$

Also, we can easily show that $\Delta x|_{t=t_k} = I_k(x(t_k^-, y(t_k^-))$ for $k = 1, \dots, m$. □

In the following, we define generalized metric space (or vector metric spaces) and prove some properties.

If, $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, by $x \le y$ we mean $x_i \le y_i$ for all $i = 1, \dots, n$. Also $|x| = (|x_1|, \dots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \dots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \le c$ means $x_i \le c$ for each $i = 1, \dots, n$. For $x \in \mathbb{R}^n$, $(x)_i = x_i$, $i = 1, \dots, n$.

Definition 2.5 *[32] Let X be a nonempty set. By a generalized metric on X (or vector-valued metric) we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u, v) \ge 0$ *for all* $u, v \in X$*; if* $d(u, v) = 0$ *then* $u = v$.

(ii) $d(u, v) = d(v, u)$ *for all* $u, v \in X$.

(iii) $d(u, v) \le d(u, w) + d(w, v)$ *for all* $u, v, w \in X$.

Note that for any $i \in \{1, \dots, n\}$ $(d(u, v))_i = d_i(u, v)$ *is a metric space in* X.

We call the pair (X, d) a generalized metric space. For $r = (r_1, r_2, \dots, r_n) \in \mathbb{R}_+^n$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

the open ball centered at x_0 with radius r and

$$B(x_0, r) = \{x \in X : d(x_0, x) \le r\}$$

the closed ball centered at x_0 with radius $r = (r_1, \dots, r_n) > 0$, $r_i > 0$, $i = 1, \dots, n$.

Remark 2.6 *In generalized metric space in the sense of Perov, the notions of convergence sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.*

Definition 2.7 *[32] A square matrix A of real numbers is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$.*

Lemma 2.8 *(see [17]) Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Then the following statements are equivalent:*

- *A is a matrix convergent to zero;*
- *The eigenvalues of A are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$; where I denote the unit matrix of $\mathcal{M}_{m,m}(\mathbb{R}_+)$,*
- *The matrix $I - A$ is nonsingular and $(I - A)^{-1} = I + A + \cdots + A^n + \cdots$;*
- *The matrix $I - A$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements;*
- *$A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for any $q \in \mathbb{R}^m$.*

Definition 2.9 *Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a matrix A convergent to zero such that*

$$d(N(x), N(y)) \leq Ad(x, y), \quad \forall x, y \in X$$

Theorem 2.10 *[39, 40](Perov's fixed point theorem) . Let (X, d) be a complete generalized metric space and $N : X \to X$ be a contractive operator with Lipschitz matrix A. Then N has a unique fixed point x^* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x^*) \leq A^k(I - A)^{-1} d(x_0, N(x_0)) \quad \forall k \in \mathbb{N}.$$

3. Existence and uniqueness solution

For a given $T > a > 0$, let $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$. In order to define a solution for problem (1.1), consider the following space of picewise continuous functions

$$\begin{aligned} PC(J, \mathbb{R}) = \big\{ & y\colon [a, T] \to \mathbb{R},\ y_k \in C(J_k, \mathbb{R}) \text{ for } k = 0, \ldots m+1, \\ & \text{and there exist } \ y(t_k^-) \ \text{ and } \ y(t_k^+) \ \text{ with } \ y(t_k) = y(t_k^-), \ \ k = 1, \cdots, m \big\}. \end{aligned}$$

This set is a Banach space with the norm $\|y\|_{PC} = \sup_{t \in [a,T]} |y(t)|$.
Set $J' = J \backslash \{t_1, \ldots, t_m\}$.

Definition 3.1 *A function $(x, y) \in PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$ is said to be a solution of (1.1) if and only if*

$$\begin{cases} x(t) = & x_a + \frac{1}{\Gamma(\alpha)} \sum\limits_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f_1(s, x(s), y(s))}{s} ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f_1(s, x(s), y(s))}{s} ds + \sum\limits_{i=1}^{k} I_i(x(t_i), y(t_i)), t \in (t_k, t_{k+1}], \quad k = 1, ..., m. \\ y(t) = & y_a + \frac{1}{\Gamma(\beta)} \sum\limits_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\beta-1} \frac{f_2(s, x(s), y(s))}{s} ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\beta-1} \frac{f_2(s, x(s), y(s))}{s} ds + \sum\limits_{i=1}^{k} \overline{I}_i(x(t_i), y(t_i)), t \in (t_k, t_{k+1}], \quad k = 1, ..., m. \end{cases} \tag{3.1}$$

The following assumptions are needed in the sequel.

(H_1) There exist constants $k_i > 0$, $i = 1, \cdots, 4$, such that

$$|f_1(t, x, y) - f_1(t, \overline{x}, \overline{y})| \leq k_1 |x - \overline{x}| + k_2 |y - \overline{y}|, \quad \text{for all} \quad x, \overline{x}, y, \overline{y} \in \mathbb{R},$$

and

$$|f_2(t, x, y) - f_2(t, \overline{x}, \overline{y})| \leq k_3 |x - \overline{x}| + k_4 |y - \overline{y}|, \quad \text{for all} \quad x, \overline{x}, y, \overline{y} \in \mathbb{R}.$$

(H_2) There exist constants a_{1i}, a_{2i}, b_{1i}, $b_{2i} \geq 0$, $i = 1, \cdots, m$, such that

$$|I_i(x, y) - I_i(\overline{x}, \overline{y})| \leq a_{1i} |x - \overline{x}| + a_{2i} |y - \overline{y}|, \quad \text{for all} x, \overline{x}, y, \overline{y} \in \mathbb{R},$$

and

$$|\overline{I}_i(x, y) - \overline{I}_i(\overline{x}, \overline{y})| \leq b_{1i} |x - \overline{x}| + b_{2i} |y - \overline{y}|, \quad \text{for all} x, \overline{x}, y, \overline{y} \in \mathbb{R}.$$

We will use the Perov fixed point theorem to prove the existence of a solution of the problem (1.1).

Theorem 3.2 *Assume that* (H_1)*–*(H_2) *are satisfied and the matrix*

$$A = \begin{pmatrix} A_\alpha k_1 + \sum_{i=1}^{k} a_{1i} & A_\alpha k_2 + \sum_{i=1}^{k} a_{2i} \\ A_\beta k_3 + \sum_{i=1}^{k} b_{1i} & A_\beta k_4 + \sum_{i=1}^{k} b_{2i} \end{pmatrix}, k = 1, ..., m \tag{3.2}$$

converges to zero. Then the problem (1.1) has a unique solution.

Proof First, we put $A_\alpha = \frac{2}{\Gamma(\alpha+1)} \left(\ln \frac{T}{a}\right)^\alpha$, $A_\beta = \frac{2}{\Gamma(\beta+1)} \left(\ln \frac{T}{a}\right)^\beta$.
Consider operator $T : PC \times PC \to PC \times PC$ defined by

$$T(x, y) = (T_1(x, y), T_2(x, y)),$$

where

$$\begin{aligned} T_1(x,y)(t) =& x_a + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} \frac{f_1(s, x(s), y(s))}{s} ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f_1(s, x(s), y(s))}{s} ds + \sum_{i=1}^{k} I_i(x(t_i), y(t_i)), t \in (t_k, t_{k+1}], k = 1, ..., m \end{aligned}$$

and

$$\begin{aligned} T_2(x,y)(t) =& y_a + \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\beta-1} \frac{f_2(s, x(s), y(s))}{s} ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\beta-1} \frac{f_2(s, x(s), y(s))}{s} ds + \sum_{i=1}^{k} \overline{I}_i(x(t_i), y(t_i)), t \in (t_k, t_{k+1}], k = 1, ..., m. \end{aligned}$$

Now, we first show that T is well defined. Given $(x,y)\in PC\times PC$, $t\in[a,T]$, we have

$$\begin{aligned}\|T_1(x,y)\|_{PC}\leq&|x_a|+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{(k_1\|x\|_{PC}+k_2\|y\|_{PC})}{s}ds\\&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{\|f_1(s,0,0)\|_\infty}{s}ds\\&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{(k_1\|x\|_{PC}+k_2\|y\|_{PC})}{s}ds\\&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{\|f_1(s,0,0)\|_\infty}{s}ds\\&+\sum_{i=1}^{k}|I_i(x(t_i),y(t_i)|\\&\leq|x_a|+\frac{2}{\Gamma(\alpha+1)}(\ln\frac{T}{a})^\alpha((k_1\|x\|_{PC}+k_2\|y\|_{PC})+M_1)+\sum_{i=1}^{k}[a_{1i}\|x\|_{PC}+a_{2i}\|y\|_{PC}],\end{aligned}$$

where $M_1=\|f_1(s,0,0)\|$ and $k=1,...,m$.
and, we can also proof as below that:

$$\|T_2(x,y)\|_{PC}\leq|y_a|+\frac{2}{\Gamma(\beta+1)}(\ln\frac{T}{a})^\alpha((k_3\|x\|_{PC}+k_4\|y\|_{PC})+M_2)+\sum_{i=1}^{k}[b_{1i}\|x\|_{PC}+b_{2i}\|y\|_{PC}],$$

where $M_2=\|f_2(s,0,0)\|$ and $k=1,...,m$.
Thus

$$\begin{aligned}\begin{pmatrix}\|T_1(x,y)\|_{PC}\\\|T_2(x,y)\|_{PC}\end{pmatrix}&=\begin{pmatrix}|x_a|+A_\alpha M_1\\|y_a|+A_\beta M_2\end{pmatrix}\\&+\begin{pmatrix}A_\alpha k_1+\sum_{i=1}^{k}a_{1i}&A_\alpha k_2+\sum_{i=1}^{k}a_{2i}\\A_\beta k_3+\sum_{i=1}^{k}b_{1i}&A_\beta k_4+\sum_{i=1}^{k}b_{2i}\end{pmatrix}\begin{pmatrix}\|x\|_{PC}\\\|y\|_{PC}\end{pmatrix},k=1,...,m.\end{aligned}$$

This implies that T is well defined.

Clearly, fixed points of T are solutions of problem (1.1). We show that T is a contraction. Let

$(x,y),(x,y)\in PC\times PC$. Then (H_1) and (H_2) imply

$$\|T_1(x,y)-T_1(\overline{x},\overline{y})\|_{PC}\leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{|f_1(s,x(s),y(s))-f_1(s,\overline{x(s)},\overline{y(s)})|_{PC}}{s}ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{|f_1(s,x(s),y(s))-f_1(s,\overline{x(s)},\overline{y(s)})|_{PC}}{s}ds$$
$$+\sum_{i=1}^{k}|I_i(x(t_i),y(t_i)-\overline{I_i}(\overline{x(t_i)},\overline{y(t_i)})|_{PC}$$
$$\leq \frac{1}{\Gamma(\alpha)}(k_1\|x-\overline{x}\|_{PC}+k_2\|y-\overline{y}\|_{PC})\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{ds}{s}$$
$$+(k_1\|x-\overline{x}\|_{PC}+k_2\|y-\overline{y}\|_{PC})\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}$$
$$+\sum_{i=1}^{k}(a_{1i}\|x-\overline{x}\|_{PC}+a_{2i}\|y-\overline{y}\|_{PC})$$
$$\leq\left(A_\alpha k_1+\sum_{i=1}^{k}a_{1i}\right)\|x-\overline{x}\|_{PC}+\left(A_\alpha k_2+\sum_{i=1}^{k}a_{2i}\right)\|y-\overline{y}\|_{PC},$$

where $t\in(t_k,t_{k+1}]$, and $k=1,...,m$.
Similarly, we have

$$\|T_2(x,y)-T_2(\overline{x},\overline{y})\|_{PC}\leq\left(A_\beta k_3+\sum_{i=1}^{k}b_{1i}\right)\|x-\overline{x}\|_{PC}+\left(A_\beta k_4+\sum_{i=1}^{k}b_{2i}\right)\|y-\overline{y}\|_{PC}.$$

where $t\in(t_k,t_{k+1}]$, and $k=1,...,m$.
It follows that

$$\|T(x,y)-T(\overline{x},\overline{y})\|_{PC}\leq\quad A\begin{pmatrix}\|x-\overline{x}\|_{PC}\\ \|y-\overline{y}\|_{PC}\end{pmatrix},\quad\text{for all}\quad(x,y),(\overline{x},\overline{y})\in PC\times PC.$$

Hence, by Theorem (2.10), the problem (1.1) has a unique solution.

□

4. Ulam–Hyers stability

In this section, we introduce Ulam's type stability concepts for Eq. (1.1). Let $\epsilon=(\epsilon_\alpha,\epsilon_\beta)>0$, $\psi_{\alpha,\beta}=(\psi_\alpha,\psi_\beta)\geq 0$ and $\varphi_{\alpha,\beta}=(\varphi_\alpha,\varphi_\beta)\in PC(J,\mathbb{R}^+)$ is nondecreasing. We consider the following inequalities

$$\begin{cases}|\left(D_t^\alpha u\right)(t)-f_1(t,u,v)|\leq\epsilon_\alpha & t\in J'\\ |\Delta u(t_k)-I_k(u(t_k),v(t_k))|\leq\epsilon_\alpha, & k=1,...,m,\\ |\left(D_t^\beta v\right)(t)-f_2(t,u,v)|\leq\epsilon_\beta & t\in J'\\ |\Delta v(t_k)-\overline{I}_k(u(t_k),v(t_k))|\leq\epsilon_\beta & k=1,...,m,\end{cases}\tag{4.1}$$

$$\begin{cases} |\left(D_t^{\alpha}u\right)(t) - f_1(t,u,v)| \leq \varphi_\alpha(t) & t \in J' \\ |\Delta u(t_k) - I_k(u(t_k), v(t_k))| \leq \psi_\alpha, & k = 1,...,m, \\ |\left(D_t^{\beta}v\right)(t) - f_2(t,u,v)| \leq \varphi_\beta(t) & t \in J' \\ |\Delta v(t_k) - \overline{I}_k(u(t_k), v(t_k))| \leq \psi_\beta & k = 1,...,m, \end{cases} \tag{4.2}$$

and

$$\begin{cases} |\left(D_t^{\alpha}u\right)(t) - f_1(t,u,v)| \leq \epsilon_\alpha \varphi_\alpha & t \in J' \\ |\Delta u(t_k) - I_k(u(t_k), v(t_k))| \leq \epsilon_\alpha \psi_\alpha, & k = 1,...,m, \\ |\left(D_t^{\beta}v\right)(t) - f_2(t,u,v)| \leq \epsilon_\beta \varphi_\beta & t \in J' \\ |\Delta v(t_k) - \overline{I}_k(u(t_k), v(t_k))| \leq \epsilon_\beta \psi_\beta & k = 1,...,m, \end{cases} \tag{4.3}$$

We adopt the following definitions from [45]

Definition 4.1 *Eq. (1.1) is Ulam–Hyers stable if there exists a real number $\lambda_{\alpha,\beta} = (\lambda_\alpha, \lambda_\beta) > 0$ such that for each $\epsilon = (\epsilon_\alpha, \epsilon_\beta) > 0$ and for each solution $(u,v) \in PC^1(J,\mathbb{R})$ of inequality (4.1) there exists a solution $(x,y) \in PC^1(J,\mathbb{R})$ of Eq. (1.1) with*

$$|(u,v) - (x,y)| \leq \epsilon.\lambda_{\alpha,\beta}$$

Definition 4.2 *Eq. (1.1) is generalized Ulam–Hyers stable if there exists $\varphi_{\alpha,\beta} = (\varphi_\alpha, \varphi_\beta) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\varphi_{\alpha,\beta}(0) = 0$ such that for each solution $(u,v) \in PC^1(J,\mathbb{R})$ of inequality (4.1) there exists a solution $(x,y) \in PC^1(J,\mathbb{R})$ of Eq. (1.1) with*

$$|(u,v) - (x,y)| \leq \varphi_{\alpha,\beta}(\epsilon)$$

Definition 4.3 *Eq. (1.1) is Ulam–Hyers–Rassias stable with respect to $(\varphi_{\alpha,\beta}, \psi_{\alpha,\beta})$ if there exists $\lambda_{\varphi,\psi} > 0$, such that for each $\varepsilon > 0$ and for each solution $(u,v) \in PC^1(J,\mathbb{R})$ of inequality (4.3) there exists a solution $(x,y) \in PC^1(J,\mathbb{R})$ of Eq. (1.1) with*

$$|(u,v) - (x,y)| \leq \epsilon.\lambda_{\varphi,\psi}(\varphi_{\alpha,\beta}(t) + \psi_{\alpha,\beta})$$

Definition 4.4 *Eq. (1.1) is generalized Ulam–Hyers–Rassias stable with respect to $(\varphi_{\alpha,\beta}, \psi_{\alpha,\beta})$ if there exists $\lambda_{\varphi,\psi} > 0$, such that f for each solution $(u,v) \in PC^1(J,\mathbb{R})$ of inequality (4.2) there exists a solution $(x,y) \in PC^1(J,\mathbb{R})$ of Eq. (1.1) with*

$$|(u,v) - (x,y)| \leq \lambda_{\varphi,\psi}(\varphi_{\alpha,\beta}(t) + \psi_{\alpha,\beta}) \quad t \in J$$

Lemma 4.5 *A function $(u,v) \in PC^1(J,\mathbb{R})$ is a solution of inequality (4.1) if and only if there is $(g_1, g_2) \in PC(J,\mathbb{R})$ and a sequence g_{1k}, g_{2k}, $k = 1,2,\ldots,m$ (which depend on (u,v)) such that*

(i) $|g_1(t)| \leq \epsilon_\alpha$, $|g_2(t)| \leq \epsilon_\beta$, $|g_{1k}(t)| \leq \epsilon_\alpha$, $|g_{2k}(t)| \leq \epsilon_\beta$, $k = 1,2,\ldots,m$,

(ii)

$$\begin{cases} (D_t^{\alpha}u(t) = f_1(t,u,v) + g_1(t) & t \in J' \\ \left(D_t^{\beta}v\right)(t) = f_2(t,u,v)| + g_2(t) & t \in J' \\ \Delta u(t_k) = I_k(u(t_k), v(t_k)) + g_{1k}, & k = 1,...,m, \\ \Delta v(t_k) = \overline{I}_k(u(t_k), v(t_k)) + g_{2k} & k = 1,...,m, \end{cases}$$

Proof According to (ii), we have

$$\begin{cases} |(D_a^\alpha u(t) - f_1(t,u,v)| = |g_1(t)| & t \in J' \\ |\left(D_a^\beta v\right)(t) - f_2(t,u,v)| = |g_2(t)| & t \in J' \\ |\Delta u(t_k) - I_k(u(t_k),v(t_k))| = |g_{1k}|, & k = 1,...,m, \\ |\Delta v(t_k) - \overline{I}_k(u(t_k),v(t_k))| = |g_{2k}| & k = 1,...,m, \end{cases}$$

By using (i), we have

$$\begin{cases} |(D_a^\alpha u(t) - f_1(t,u,v)| \leq \epsilon_\alpha & t \in J' \\ |\left(D_a^\beta v\right)(t) - f_2(t,u,v)| \leq \epsilon_\beta & t \in J' \\ |\Delta u(t_k) - I_k(u(t_k),v(t_k))| \leq \epsilon_\alpha, & k = 1,...,m, \\ |\Delta v(t_k) - \overline{I}_k(u(t_k),v(t_k))| \leq \epsilon_\beta & k = 1,...,m, \end{cases}$$

Then , (u,v) is a solution of inequality (4.1). □

One can have similar lemma for inequalities (4.2) and (4.3).

Lemma 4.6 *Suppose (u,v) is the solution of the inequality (4.1), then we have the system of inequalities given as*

$$\begin{cases} \left|u(t) - u_a - \frac{1}{\Gamma(\alpha)}\sum\limits_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{f_1(s,u(s),v(s))}{s}ds - \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f_1(s,u(s),v(s))}{s}ds - \sum\limits_{i=1}^{k}I_i(u(t_i),v(t_i))\right| \leq \lambda_\alpha\varepsilon_\alpha, \\ \left|v(t) - v_a - \frac{1}{\Gamma(\beta)}\sum\limits_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\beta-1}\frac{f_2(s,u(s),v(s))}{s}ds - \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\beta-1}\frac{f_2(s,u(s),v(s))}{s}ds - \sum\limits_{i=1}^{k}\overline{I}_i(u(t_i),v(t_i))\right| \leq \lambda_\beta\varepsilon_\beta, \end{cases}$$

where $t \in (t_k,t_{k+1}]$ and $k = 1,...,m$.

Proof By using Lemma 4.5, we have

$$\begin{cases} \left(D_t^\alpha u\right)(t) = f_1(t,u,v) + g_1(t) & t \in J' \\ \left(D_t^\beta v\right)(t) = f_2(t,u,v) + g_2(t) & t \in J' \\ \Delta u(t_k) = I_k(u(t_k),v(t_k)) + g_{1k}, & k = 1,...,m, \\ \Delta v(t_k) = \overline{I}_k(u(t_k),v(t_k)) + g_{2K} & k = 1,...,m, \end{cases} \tag{4.4}$$

Then, the solution of (4.4) is given by

$$\begin{cases} u(t) &= u_a + \frac{1}{\Gamma(\alpha)}\sum\limits_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{f_1(s,u(s),v(s))+g_1(s)}{s}ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f_1(s,u(s),v(s))+g_1(s)}{s}ds \\ &+ \sum\limits_{i=1}^{k}I_i(u(t_i),v(t_i)) + g_{1i}, t \in (t_k,t_{k+1}] \text{ and } k = 1,...,m, \\ v(t) &= v_a + \frac{1}{\Gamma(\beta)}\sum\limits_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\beta-1}\frac{f_2(s,u(s),v(s))+g_2(s)}{s}ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f_2(s,u(s),v(s))+g_2(s)}{s}ds \\ &+ \sum\limits_{i=1}^{k}\overline{I}_i(u(t_i),v(t_i)) + g_{2i}, t \in (t_k,t_{k+1}] \text{ and } k = 1,...,m. \end{cases} \tag{4.5}$$

From first equation of the system (4.5), we have

$$\begin{aligned} &\left|u(t) - u_a - \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{f_1(s,u(s),v(s))}{s}ds - \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f_1(s,u(s),v(s))}{s}ds - \sum_{i=1}^{k}I_i(u(t_i),v(t_i))\right| \\ &\quad\leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{|g_1(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{|g_1(s)|}{s}ds + \sum_{i=1}^{k}|g_{1i}| \\ &\quad\leq \frac{2\varepsilon_\alpha}{\Gamma(\alpha+1)}\left(\ln\frac{T}{a}\right)^\alpha + k\varepsilon_\alpha \\ &\quad\leq \left(\frac{2}{\Gamma(\alpha+1)}\left(\ln\frac{T}{a}\right)^\alpha + k\right)\varepsilon_\alpha = \lambda_\alpha\varepsilon_\alpha,\ k = 1,...,m. \end{aligned}$$

Repeating the same procedure for second equation of the system (4.5), we have

$$\left|v(t)-v_a-\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\beta-1}\frac{f_2(s,u(s),v(s))}{s}ds-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\beta-1}\frac{f_2(s,u(s),v(s))}{s}ds-\sum_{i=1}^{k}\overline{I}_i(u(t_i),v(t_i))\right|\leq\lambda_\beta\varepsilon_\beta.$$

where $\frac{2}{\Gamma(\beta+1)}\left(\ln\frac{T}{a}\right)^{\beta}+k=\lambda_\beta$, $k=1,...,m$. □

Let us set

$\Lambda_1:=A_\alpha k_1+\sum\limits_{i=1}^{k}a_{1i}$, $\Lambda_2:=A_\alpha k_2+\sum\limits_{i=1}^{k}a_{2i}$,

$\Lambda_1^*:=A_\beta k_3+\sum\limits_{i=1}^{k}b_{1i}$, $\Lambda_2^*:=A_\beta k_4+\sum\limits_{i=1}^{k}b_{2i}$.

Theorem 4.7 *If the assumptions (H1)–(H2) hold, and suppose that*

$$\Lambda_1<1,\ \Lambda_2^*<1 \text{ and } \Lambda:=1-\frac{\Lambda_2\Lambda_1^*}{(1-\Lambda_1)(1-\Lambda_2^*)}\neq 0.$$

Then (1.1) is Ulam–Hyers and generalized Ulam–Hyers stable.

Proof Let $(u,v)\in PC^1(J,\mathbb{R})$ be any solution of the inequality (4.1) and let $(x,y)\in PC^1(J,\mathbb{R})$ be the unique solution of the following:

$$\begin{cases}\left(D_t^{\alpha}x\right)(t) & =f_1(t,x,y)\quad t\in[a,T], & t\neq t_k,\ k=1,...,m,\\ \left(D_t^{\beta}y\right)(t) & =f_2(t,x,y)\quad t\in[a,T], & t\neq t_k,\ k=1,...,m,\\ \Delta x(t_k) & =x(t_k^+)-x(t_k^-)=I_k(x(t_k),y(t_k)), & k=1,...,m,\\ \Delta y(t_k) & =y(t_k^+)-y(t_k^-)=\overline{I}_k(x(t_k),y(t_k)), & k=1,...,m,\\ x(a)=x_a, & \\ y(a)=y_a, & \end{cases}\tag{4.6}$$

Then, in view of Lemma 2.4, the solution of (4.6) is provided by

$$\begin{cases}x(t)= & u_a+\frac{1}{\Gamma(\alpha)}\sum\limits_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{f_1(s,x(s),y(s))}{s}ds\\ & +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f_1(s,x(s),y(s))}{s}ds+\sum\limits_{i=1}^{k}I_i(x(t_i),y(t_i))\\ y(t)= & v_a+\frac{1}{\Gamma(\alpha)}\sum\limits_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\frac{f_2(s,x(s),y(s))}{s}ds\\ & +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f_2(s,x(s),y(s))}{s}ds+\sum\limits_{i=1}^{k}\overline{I}_i(x(t_i),y(t_i))\end{cases}$$

Hence for each $t\in(t_k,t_{k+1}]$, it follows

$$\begin{aligned}\big\|u-x\big\|_{PC} \leq & \Big|u(t)-u_a-\tfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\tfrac{t_i}{s}\right)^{\alpha-1}\tfrac{f_1(s,u(s),v(s))}{s}ds-\tfrac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\tfrac{t}{s}\right)^{\alpha-1}\tfrac{f_1(s,u(s),v(s))}{s}ds \\ & -\sum_{i=1}^{k}I_i(u(t_i),v(t_i))\Big|+\tfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\tfrac{t_i}{s}\right)^{\alpha-1}\tfrac{\left|f_1(s,u(s),v(s))-f_1(s,x(s),y(s))\right|}{s}ds \\ & +\tfrac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\tfrac{t}{s}\right)^{\alpha-1}\tfrac{\left|f_1(s,u(s),v(s))-f_1(s,x(s),y(s))\right|}{s}ds+\sum_{i=1}^{k}\left|I_i(u(t_i),v(t_i))-I_i(x(t_i),y(t_i))\right| \\ & \leq \lambda_\alpha\varepsilon_\alpha+A_\alpha\left(k_1\|u-x\|_{PC}+k_2\|v-y\|_{PC}\right)+\sum_{i=1}^{k}\left(a_{1i}\|u-x\|_{PC}+a_{2i}\|v-y\|_{PC}\right) \\ & \leq \lambda_\alpha\varepsilon_\alpha+\left[A_\alpha k_1+\sum_{i=1}^{k}a_{1i}\right]\|u-x\|_{PC}+\left[A_\alpha k_2+\sum_{i=1}^{k}a_{2i}\right]\|v-y\|_{PC} \\ & \leq \lambda_\alpha\varepsilon_\alpha+\Lambda_1\|u-x\|_{PC}+\Lambda_2\|v-y\|_{PC}\end{aligned}$$

Thus, we get

$$\big\|u-x\big\|_{PC} \leq \frac{\lambda_\alpha\varepsilon_\alpha}{1-\Lambda_1}+\frac{\Lambda_2}{1-\Lambda_1}\|v-y\|_{PC}. \tag{4.7}$$

In addition, for each $t\in(t_k,t_{k+1}]$, it follows

$$\begin{aligned}\big\|v-y\big\|_{PC} \leq & \Big|v(t)-v_a-\tfrac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\tfrac{t_i}{s}\right)^{\beta-1}\tfrac{f_2(s,u(s),v(s))}{s}ds-\tfrac{1}{\Gamma(\beta)}\int_{t_k}^{t}\left(\ln\tfrac{t}{s}\right)^{\beta-1}\tfrac{f_2(s,u(s),v(s))}{s}ds \\ & -\sum_{i=1}^{k}\overline{I}_i(u(t_i),v(t_i))\Big|+\tfrac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\ln\tfrac{t_i}{s}\right)^{\beta-1}\tfrac{\left|f_2(s,u(s),v(s))-f_2(s,x(s),y(s))\right|}{s}ds \\ & +\tfrac{1}{\Gamma(\beta)}\int_{t_k}^{t}\left(\ln\tfrac{t}{s}\right)^{\beta-1}\tfrac{\left|f_2(s,u(s),v(s))-f_2(s,x(s),y(s))\right|}{s}ds+\sum_{i=1}^{k}\left|\overline{I}_i(u(t_i),v(t_i))-\overline{I}_i(x(t_i),y(t_i))\right| \\ & \leq \lambda_\beta\varepsilon_\beta+A_\beta\left(k_3\|u-x\|_{PC}+k_4\|v-y\|_{PC}\right)+\sum_{i=1}^{k}\left(b_{1i}\|u-x\|_{PC}+b_{2i}\|v-y\|_{PC}\right) \\ & \leq \lambda_\beta\varepsilon_\beta+\left[A_\beta k_3+\sum_{i=1}^{k}b_{1i}\right]\|u-x\|_{PC}+\left[A_\beta k_4+\sum_{i=1}^{k}b_{2i}\right]\|v-y\|_{PC} \\ & \leq \lambda_\beta\varepsilon_\beta+\Lambda_1^*\|u-x\|_{PC}+\Lambda_2^*\|v-y\|_{PC}.\end{aligned}$$

Thus, we get

$$\big\|v-y\big\|_{PC} \leq \frac{\lambda_\beta\varepsilon_\beta}{1-\Lambda_2^*}+\frac{\Lambda_1^*}{1-\Lambda_2^*}\|v-y\|_{PC}. \tag{4.8}$$

The equivalent matrix of Eqs. (4.7) and (4.8) is given as:

$$\begin{bmatrix} 1 & -\frac{\Lambda_2}{1-\Lambda_1} \\ -\frac{\Lambda_1^*}{1-\Lambda_2^*} & 1 \end{bmatrix}\begin{bmatrix}\big\|u-x\big\|_{PC} \\ \big\|v-y\big\|_{PC}\end{bmatrix} \leq \begin{bmatrix}\frac{\lambda_\alpha\varepsilon_\alpha}{1-\Lambda_1} \\ \frac{\lambda_\beta\varepsilon_\beta}{1-\Lambda_2^*}\end{bmatrix}$$

Solving the above inequality, we get

$$\begin{bmatrix}\big\|u-x\big\|_{PC} \\ \big\|v-y\big\|_{PC}\end{bmatrix} \leq \begin{bmatrix}\frac{1}{\Lambda} & \frac{\Lambda_1^*}{\Lambda(1-\Lambda_2^*)} \\ \frac{\Lambda_2}{\Lambda(1-\Lambda_1)} & \frac{1}{\Lambda}\end{bmatrix}\begin{bmatrix}\frac{\lambda_\alpha\varepsilon_\alpha}{1-\Lambda_1} \\ \frac{\lambda_\beta\varepsilon_\beta}{1-\Lambda_2^*}\end{bmatrix}$$

Further simplification of above system gives

$$\|u-x\|_{PC} \leq \frac{\lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1)} + \frac{\Lambda_1^* \lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)^2},$$
$$\|v-y\|_{PC} \leq \frac{\lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)} + \frac{\Lambda_2 \lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1^2)},$$

from which we have

$$\|u-x\|_{PC} + \|v-y\|_{PC} \leq \frac{\lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1)} + \frac{\Lambda_1^* \lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)^2} + \frac{\lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)} + \frac{\Lambda_2 \lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1^2)}. \tag{4.9}$$

Let $\varepsilon = \max\{\varepsilon_\alpha, \varepsilon_\beta\}$, then from (4.9) we have

$$\|(u,v)-(x,y)\|_{PC} \leq \lambda_{\alpha,\beta}\varepsilon,$$

where

$$\lambda_{\alpha,\beta} = \left[\frac{\lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1)} + \frac{\Lambda_1^* \lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)^2} + \frac{\lambda_\beta \varepsilon_\beta}{\Lambda(1-\Lambda_2^*)} + \frac{\Lambda_2 \lambda_\alpha \varepsilon_\alpha}{\Lambda(1-\Lambda_1^2)}\right].$$

Hence, problem (1.1) is Ulam–Hyers stable.
Over and above, if we write

$$\|(u,v)-(x,y)\|_{PC} \leq \lambda_{\alpha,\beta}\psi(\varepsilon), \quad \text{where } \psi(0)=0$$

then problem (1.1) is generalized Ulam–Hyers stable. □

5. Example

Example 5.1 *Consider the following differential equation system*

$$\begin{cases} \left(D_t^{\frac{1}{2}} x\right)(t) = \frac{\sin(x+y)}{20(\ln t+1)}, & t\in[1,e], \ \ t\neq \frac{5}{3}, \\ \left(D_t^{1/2} y\right)(t) = \frac{\arctan t}{3+|x+y|}, & t\in[1,e], \ \ t\neq \frac{5}{3}, \\ \Delta x(\frac{5}{3}) = \exp^{-\frac{5}{3}}\left(\sin x(\frac{5}{3}) + y(\frac{5}{3})\right), \\ \Delta y(\frac{5}{3}) = \frac{|x(\frac{5}{3})+y(\frac{5}{3})|}{10}, \\ x(1) = \frac{1}{2}, \\ y(1) = \frac{3}{2}, \end{cases} \tag{5.1}$$

Here, we have

$$f_1(t,x,y) = \frac{\sin(x+y)}{20(\ln t+1)}, \qquad f_2(t,x,y) = \frac{\arctan t}{3+|x+y|}$$

and we simply check that

$$\forall x,y,\overline{x},\overline{x}\in\mathbb{R}; \quad \left|f_1(t,x,y) - f_1(t,\overline{x},\overline{y})\right| \leq \frac{1}{20}\left|x-\overline{x}\right| + \frac{1}{20}\left|y-\overline{y}\right|, \quad \forall t\in[1,e],$$

$$\forall x, y, \overline{x}, \overline{x} \in \mathbb{R}; \quad \Big|f_2(t,x,y) - f_2(t,\overline{x},\overline{y})\Big| \leq \frac{\pi}{18}\Big|x-\overline{x}| + \frac{\pi}{18}\Big|y-\overline{y}\Big|, \quad \forall t \in [1,e],$$

$$\left|I\left(x(\frac{5}{3}), y(\frac{5}{3})\right) - I\left(\overline{x}(\frac{5}{3}), \overline{y}(\frac{5}{3})\right)\right| \leq e^{\frac{-5}{3}}\Big|x-\overline{x}\Big| + e^{\frac{-5}{3}}\Big|y-\overline{y}\Big|,$$

$$\left|\overline{I}\left(x(\frac{5}{3}), y(\frac{5}{3})\right) - \overline{I}\left(\overline{x}(\frac{5}{3}), \overline{y}(\frac{5}{3})\right)\right| \leq \frac{1}{10}\Big|x-\overline{x}\Big| + \frac{1}{10}\Big|y-\overline{y}\Big|.$$

Therefore the matrix

$$A = \begin{pmatrix} 0.23. & 0.23 \\ 0.3 & 0.3 \end{pmatrix}$$

converges to zero since its eigenvalues are $\Big|\lambda\Big| = \sqrt{0.5} =< 1$. *From Theorem (3.2), the problem (5.1) has a unique solution.*

On the other hand, we have $\Lambda_2 = \Lambda_1 = 0.24$, $\Lambda_1^* = \Lambda_2^* = 0.31$. *Therefore*

$$\Lambda = 1 - \frac{0.24 * 0.31}{(1-0.24)(1-0.31)} = 0.86 \neq 0$$

Therefore, the coupled system (5.1) is Ulam–Hyers stable, generalized Ulam–Hyers stable.

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