Classification of genus-1 holomorphic Lefschetz pencils

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Abstract: In this paper, we classify relatively minimal genus-1 holomorphic Lefschetz pencils up to smooth isomorphism. We first show that such a pencil is isomorphic to either the pencil on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 2)$ or a blow-up of the pencil on $\mathbb{P}^2$ of degree 3, provided that no fiber of a pencil contains an embedded sphere (note that one can easily classify genus-1 Lefschetz pencils with an embedded sphere in a fiber). We further determine the monodromy factorizations of these pencils and show that the isomorphism class of a blow-up of the pencil on $\mathbb{P}^2$ of degree 3 does not depend on the choice of blown-up base points. We also show that the genus-1 Lefschetz pencils constructed by Korkmaz-Ozbagci (with nine base points) and Tanaka (with eight base points) are respectively isomorphic to the pencils on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ above, in particular these are both holomorphic.

Key words: Lefschetz pencil, monodromy factorization, holed torus relation, braid monodromy

1. Introduction

Classification problems of Lefschetz fibrations up to smooth isomorphism have attracted a lot of interest since around 1980. The first result concerning the problems was given in [13, 18], in which Kas and Moishezon independently classified genus-1 Lefschetz fibrations over the 2-sphere. This classification result was extended to more general genus-1 fibrations: those with general base spaces and achiral singularities [12, 15, 16]. Furthermore, Siebert and Tian [28] classified genus-2 Lefschetz fibrations over the 2-sphere with transitive monodromies and no reducible fibers by showing that such fibrations are always holomorphic. Classifications up to stabilizations by fiber sums were also studied in [1–5].

Whereas there are various results on classifications of Lefschetz fibrations, very little is known about the corresponding results on Lefschetz pencils, aside from the classification of genus-0 pencils implicitly given in [26]. In this paper, we will deal with the classification problem of genus-1 Lefschetz pencils. We first show that a genus-1 holomorphic Lefschetz pencil is isomorphic to either of the standard ones given below:

Theorem 1.1 *Let* $f : X \dashrightarrow \mathbb{P}^1$ *be a genus-1 relatively minimal holomorphic Lefschetz pencil. Suppose that no fibers of* f *contain an embedded sphere. Then either of the following holds:*

- *f is smoothly isomorphic to the one obtained by blowing-up the Lefschetz pencil $f_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, which is the composition of the Veronese embedding $v_3 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^9$ of degree 3 and a generic projection $\mathbb{P}^9 \dashrightarrow \mathbb{P}^1$.*

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- f is smoothly isomorphic to the Lefschetz pencil $f_s : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$, which is the composition of the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, the Veronese embedding $v_2 : \mathbb{P}^3 \hookrightarrow \mathbb{P}^9$ of degree 2, and a generic projection $\mathbb{P}^9 \dashrightarrow \mathbb{P}^1$.

The subscripts "n" and "s" for the Lefschetz pencils f_n and f_s represent the properties "nonspin" and "spin", respectively. Note that, needless to say, the blow-ups of f_s also give Lefschetz pencils. Theorem 1.1 implies that such pencils are isomorphic to the blow-ups of f_n. The assumption of relative minimality and the additional requirement that no fibers contain an embedded sphere with any self-intersection number should not be confused. The latter is required to exclude inessential Lefschetz pencils. For more detail, see Remark 3.2.

Although the isomorphism classes of the pencils f_n and f_s do not depend on the choice of generic projections $\mathbb{P}^9 \dashrightarrow \mathbb{P}^1$ (cf. Remark 2.2), one cannot deduce immediately from Theroem 1.1 that the isomorphism class of a blow-up of f_n does not depend on the choice of blown-up base points (one can indeed find in [10] examples of a pair of nonisomorphic pencils that are obtained by blowing-up a common pencil at the same number but different combinations of base points). We next address this issue by examining the monodromies of f_n and f_s.

It is a standard fact in the literature that there is one-to-one correspondence between the isomorphism classes of genus-g Lefschetz pencils with m critical points and k base points and the Hurwitz equivalence classes of factorizations

$$t_{c_m} \cdots t_{c_1} = t_{\delta_1} \cdots t_{\delta_k}$$

of the boundary multitwist $t_{\delta_1} \cdots t_{\delta_k}$ as products of positive Dehn twists in the mapping class group of a k-holed surface of genus g. Here, δ_i stands for a simple closed curve parallel to the i-th boundary component. Such a factorization is called a monodromy factorization in general, or also a k-holed torus relation when $g = 1$. Relying on the theory of braid monodromies due to Moishezon-Teicher [19–23] we determine the monodromy factorizations of f_n and f_s. We further analyze the Hurwitz equivalence classes of the factorizations, and eventually show the following:

Theorem 1.2 *Let $f : X \dashrightarrow \mathbb{P}^1$ be a relatively minimal genus-1 holomorphic Lefschetz pencil without embedded spheres in fibers. The monodromy factorization of f is Hurwitz equivalent to that of one of the pencils in Table 1 (the curves in the table are given in Figure 1). In particular, the isomorphism class of a blow-up of f_n does not depend on the choice of blown-up base points.*

Note that according to the aforementioned works of Kas and Moishezon [13, 18] the only genus-1 Lefschetz fibration that admits a (-1)-section is the well-known rational elliptic fibration $E(1) \to \mathbb{P}^1$, whose monodromy factorization is $(t_a t_b)^6 = 1$, where a is the meridian and b is the longitude of the torus. Thus, any genus-1 Lefschetz pencil, even a nonholomorphic one (if exists), must descend to this fibration after blowing-up all the base points. This is clearly reflected in Table 1, where once more blowing-up of the pencil $f_n \sharp 8 \overline{\mathbb{P}}^2$ results in $E(1) = \mathbb{P}^2 \sharp 9 \overline{\mathbb{P}}^2$ and $(t_a t_b)^6 = 1$.

Examples of explicit k-holed torus relations have also been discovered in purely topological and combinatorial ways, without relying on the knowledge from complex geometry. Korkmaz and Ozbagci [14] systematically constructed k-holed torus relations up to the largest possible $k = 9$ for the first time. Then an 8-holed torus relation was constructed by Tanaka [31]. In both of the studies, the authors obtained those relations by combining certain known relations (i.e. the 2-chain relation and the lantern relation) in the mapping class groups.

Table 1. Classification of the genus-1 holomorphic Lefschetz pencils. The boundary multitwist $t_{\delta_1} \cdots t_{\delta_k}$ is represented by ∂_k.

Pencil	Number of base points	Monodromy factorization	Total space
f_n	9	$t_{a_1} t_{b_1} t_{b_2} t_{b_3} t_{a_4} t_{b_4} t_{b_5} t_{b_6} t_{a_7} t_{b_7} t_{b_8} t_{b_9} = \partial_9$	$\mathbb{P}^2$
f_s	8	$t_{a_1} t_{b_1} t_{b_2} t_{a_3} t_{b_3} t_{b_4} t_{a_5} t_{b_5} t_{b_6} t_{a_7} t_{b_7} t_{b_8} = \partial_8$	$\mathbb{P}^1 \times \mathbb{P}^1$
$f_n \sharp \overline{\mathbb{P}}^2$	8	$t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{b_3} t_{a_4} t_{b_4} t_{b_5} t_{b_6} t_{a_7} t_{b_7} t_{b_8} = \partial_8$	$\mathbb{P}^2 \sharp \overline{\mathbb{P}}^2$
$f_n \sharp 2\overline{\mathbb{P}}^2$	7	$t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{b_4} t_{a_5} t_{b_5} t_{b_6} t_{a_7} t_{b_7} = \partial_7$	$\mathbb{P}^2 \sharp 2\overline{\mathbb{P}}^2$
$f_n \sharp 3\overline{\mathbb{P}}^2$	6	$t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4} t_{a_5} t_{b_5} t_{a_6} t_{b_6} = \partial_6$	$\mathbb{P}^2 \sharp 3\overline{\mathbb{P}}^2$
$f_n \sharp 4\overline{\mathbb{P}}^2$	5	$t_{a_1}^2 t_{b_1} t_{a_2}^2 t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4} t_{a_5} t_{b_5} = \partial_5$	$\mathbb{P}^2 \sharp 4\overline{\mathbb{P}}^2$
$f_n \sharp 5\overline{\mathbb{P}}^2$	4	$t_{a_1}^2 t_{b_1} t_{a_2}^2 t_{b_2} t_{a_3}^2 t_{b_3} t_{a_4}^2 t_{b_4} = \partial_4$ $\sim (t_{a_1} t_{a_3} t_b t_{a_2} t_{a_4} t_b)^2 = \partial_4$	$\mathbb{P}^2 \sharp 5\overline{\mathbb{P}}^2$
$f_n \sharp 6\overline{\mathbb{P}}^2$	3	$t_{a_1}^3 t_{b_1} t_{a_2}^3 t_{b_2} t_{a_3}^3 t_{b_3} = \partial_3$ $\sim (t_{a_1} t_{a_2} t_{a_3} t_b)^3 = \partial_3$	$\mathbb{P}^2 \sharp 6\overline{\mathbb{P}}^2$
$f_n \sharp 7\overline{\mathbb{P}}^2$	2	$(t_{a_1} t_b t_{a_2})^4 = \partial_2$	$\mathbb{P}^2 \sharp 7\overline{\mathbb{P}}^2$
$f_n \sharp 8\overline{\mathbb{P}}^2$	1	$(t_{a_1} t_b)^6 = \partial_1$	$\mathbb{P}^2 \sharp 8\overline{\mathbb{P}}^2$
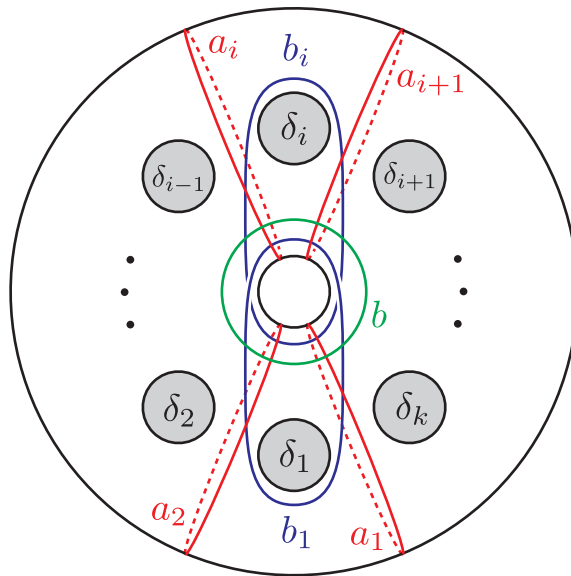
Figure 1. The curves on the k-holed torus Σ_1^k.

Ozbagci [25] later observed that the two 8-holed torus relations by Korkmaz-Ozbagci and by Tanaka are not Hurwitz equivalent. Some other 8-holed torus relations are also constructed in [9, 10], though it turns out that they are Hurwitz equivalent to either Korkmaz-Ozbagci's or Tanaka's. We will show that the k-holed torus relations of Korkmaz-Ozbagci and Tanaka are Hurwitz equivalent to the monodromy factorizations in Table 1, in particular we conclude that the Lefschetz pencils associated with their relations are holomorphic (Theorems 5.1 and 5.2).

The virtue of our presentations of the k-holed torus relations in Table 1 is that the curves involved are remarkably simple as they are well-organized lifts of the meridian and longitude of a closed torus. As the

k-holed torus relations are fundamentally important to construct relations in the mapping class groups of even higher genera, having simpler expressions may help those who try to use them.

As our results in the present paper take care of holomorphic pencils, the next step shall be the ultimate classification of genus-1 smooth Lefschetz pencils. Although we speculate that any genus-1 Lefschetz pencil is isomorphic to one of the holomorphic ones, we do not have the machinery to prove this. This leaves the following open question.

Question 1.3 *Is there a nonholomorphic genus-1 Lefschetz pencil? In other words, is there a k-holed torus relation that is not Hurwitz equivalent to any of the k-holed torus relations in Table 1?*

The paper is organized as follows. In Section 2, we briefly review basic properties of holomorphic Lefschetz pencils and monodromy factorizations. Section 3 is devoted to proving Theroem 1.1. In Section 4, we determine monodromy factorizations of the pencils f_n and f_s. We analyze combinatorial properties of the monodromy factorizations of f_n and f_s in Section 5, completing the proof of Theorem 1.2. Finally in Section 6, we discuss some additional topics related to the pencils and k-holed torus relations.

2. Preliminaries

Throughout this paper, we will assume that manifolds are smooth, connected, oriented, and closed unless otherwise noted. We denote the n-dimensional complex projective space by $\mathbb{P}^n$. Let X be a 4-manifold. A Lefschetz pencil on X is a smooth mapping $f : X \setminus B \to \mathbb{P}^1$ defined on the complement of a nonempty finite subset $B \subset X$ satisfying the following conditions:

- for any critical point $p \in X$ of f, there exists a complex coordinate neighborhood $(U, \varphi : U \to \mathbb{C}^2)$ (resp. $(V, \psi : V \to \mathbb{C})$) at p (resp. $f(p)$) compatible with the orientation such that $\psi \circ f \circ \varphi^{-1}(z, w)$ is equal to $z^2 + w^2$,

- for any $b \in B$, there exists a complex coordinate neighborhood (U, φ) of b compatible with the orientation and an orientation preserving diffeomorphism $\xi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\xi \circ f \circ \varphi^{-1}(z, w)$ is equal to $[z : w]$,

- the restriction $f|_{\mathrm{Crit}(f)}$ is injective.

The set B is called the base point set of f. In this paper we will use the dashed arrow $\dashrightarrow$ to represent Lefschetz pencils, e.g., $f : X \dashrightarrow \mathbb{P}^1$, when we do not need to represent the base point set explicitly (note that this symbol will be also used to represent meromorphic mappings). For a Lefschetz pencil f, the genus of the closure of a regular fiber is called the genus of f.

A Lefschetz pencil f is said to be relatively minimal if no fiber of f contains a (-1)-sphere. Let $f : X \dashrightarrow \mathbb{P}^1$ be a Lefschetz pencil, $\tilde{X}$ be a blow-up of X at a point and $\pi : \tilde{X} \to X$ be the blow-down mapping. One can construct a Lefschetz pencil $\tilde{f} : \tilde{X} \dashrightarrow \mathbb{P}^1$ so that $\tilde{f} = f \circ \pi$ on the complement of the exceptional sphere. Conversely, any relatively nonminimal Lefschetz pencil can be obtained from a relatively minimal one by this construction. In particular, relatively nonminimal Lefschetz pencils are inessential in the context of classification, and thus, we will assume that Lefschetz pencils are relatively minimal unless otherwise noted.

2.1. Holomorphic Lefschetz pencils

A Lefschetz pencil $f : X \dashrightarrow \mathbb{P}^1$ is said to be holomorphic if there exists a complex structure of X such that f is holomorphic and we can take biholomorphic φ, ψ, and ξ in the conditions in the definition above. A Lefschetz pencil on a complex surface S is said to be holomorphic if it is holomorphic with respect to the given complex structure. For a complex surface S, it is well-known that a divisor $D \in \mathrm{Div}(S)$ gives rise to a line bundle over S, which we denote by $[D]$ (see [8] for details).

Proposition 2.1 *Let S be a complex surface, $f : S \dashrightarrow \mathbb{P}^1$ be a holomorphic Lefschetz pencil, and $F \subset S$ be the closure of a fiber of f.*

1. *The genus of f is equal to $(2 + \mathcal{F}^2 + K_S(\mathcal{F}))/2$, where $K_S \in H^2(S; \mathbb{Z})$ is the canonical class of S and $\mathcal{F} \in H_2(S; \mathbb{Z})$ is the homology class represented by F.*

2. *There exist sections s_0, s_1 of the line bundle $[F]$ such that f is equal to $[s_0 : s_1] : S \dashrightarrow \mathbb{P}^1$.*

3. *Let $C \subset S$ be an irreducible curve. The intersection number $C \cdot F$ is greater than or equal to 0. Furthermore, it is equal to 0 if and only if C is a component of a fiber of f without base points.*

Proof   (1) is merely a consequence of the adjunction formula, and we can prove (2) in the same way as that for [11, Lemma 3.1]. In what follows we will prove (3). Let $\tilde{S}$ be the complex surface obtained by blowing-up S at all the base points of f and $\tilde{C} \subset \tilde{S}$ (resp. $\tilde{F} \subset \tilde{S}$) be the proper transform of C (resp. F). The pencil f induces a fibration $\tilde{f} : \tilde{S} \to \mathbb{P}^1$. Without loss of generality we can assume that $\tilde{F}$ does not contain any singular point of $\tilde{C}$ and any critical point of $\tilde{f}|_{\tilde{C}}$. Since $\tilde{F}$ is a fiber of $\tilde{f}$, the intersection number $\tilde{F} \cdot \tilde{C}$ is equal to $\sharp(\tilde{F} \cap \tilde{C})$. Hence, we obtain:

$$F \cdot C = \tilde{F} \cdot \tilde{C} + \sharp(C \cap B) = \sharp(\tilde{F} \cap \tilde{C}) + \sharp(C \cap B) \geq 0.$$

Moreover, the equality holds only if $C \cap B = \emptyset$ and $\tilde{f}|_{\tilde{C}}$ is a constant map. The latter condition implies that C is contained in a fiber of f. □

Remark 2.2 *For a line bundle L over a complex surface S with sections, we can define a meromorphic mapping $\varphi_L : S \dashrightarrow \mathbb{P}^{l-1}$ as follows:*

$$\varphi_L(x) = [s_1(x) : \cdots : s_l(x)],$$

where $s_1, \ldots, s_l$ is a basis of $H^0(S; L)$. The statement (2) of Proposition 2.1 implies that f is the composition of $\varphi_{[F]} : S \dashrightarrow \mathbb{P}^{m-1}$ (where $m = \dim H^0(S; [F])$) and a projection $\mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^1$. Note that the composition of φ_L and a projection $\mathbb{P}^l \dashrightarrow \mathbb{P}^1$ is not always a Lefschetz pencil. It is known, however, that the composition is a Lefschetz pencil provided that L is very ample and the projection is generic. Moreover, the smooth isomorphism class of the Lefschetz pencil does not depend on the choice of this projection (see [11, 32]).

2.2. Monodromy factorizations

For a compact oriented connected surface Σ (possibly with boundaries), we denote by $\mathrm{Diff}(\Sigma)$ the set of self-diffeomorphisms of Σ preserving the boundary pointwise, endowed with the Whitney C^∞-topology. Let $\mathrm{MCG}(\Sigma) = \pi_0(\mathrm{Diff}(\Sigma))$, which has the group structure defined by the composition of representatives.

Let $f : X \setminus B \to \mathbb{P}^1$ be a genus-g Lefschetz pencil with k base points and $Q = \{q_1, \ldots, q_m\} \subset \mathbb{P}^1$ be the set of critical values of f. We take a point $q_0 \in \mathbb{P}^1 \setminus Q$ and a path $\alpha_i \subset \mathbb{P}^1$ $(i = 1, \ldots, m)$ from q_0 to q_i satisfying the following conditions:

- $\alpha_1, \ldots, \alpha_m$ are mutually distinct except at the common initial point q_0,

- $\alpha_1, \ldots, \alpha_m$ appear in this order when we go around q_0 counterclockwise.

The system of paths $(\alpha_1, \ldots, \alpha_m)$ satisfying the conditions above is called a Hurwitz path system of f. Let $\gamma_i \subset \mathbb{P}^1$ be a based loop with the base point q_0 obtained by connecting q_0 with a small circle oriented counterclockwise by α_i. It is known that the monodromy along γ_i is the Dehn twist along some simple closed curve $c_i \subset \overline{f^{-1}(q_0)}$, called a vanishing cycle of f with respect to the path α_i. Furthermore, we can obtain the following relation in $\mathrm{MCG}(\overline{f^{-1}(q_0)} \setminus \nu B)$:

$$t_{c_m} \cdots t_{c_1} = t_{\delta_1} \cdots t_{\delta_k}, \tag{2.1}$$

where νB is a tubular neighborhood of $B \subset \overline{f^{-1}(q_0)}$ and $\delta_1, \ldots, \delta_k \subset \overline{f^{-1}(q_0)} \setminus \nu B$ are simple closed curves parallel to the boundary components. We call this relation a monodromy factorization of f. Conversely, let Σ_g^k be a genus-g compact surface with k boundary components, and $c_1, \ldots, c_m \subset \Sigma_g^k$ be simple closed curves satisfying the relation (2.1) in $\mathrm{MCG}(\Sigma_g^k)$. We can construct a genus-g Lefschetz pencil $f : X \setminus B \to \mathbb{P}^1$ with k base points and vanishing cycles $c_1, \ldots, c_m$, under some identification of the complement $\overline{f^{-1}(q_0)} \setminus \nu B$ of the closure of a regular fiber with Σ_g^k.

3. Complex surfaces admitting genus-1 Lefschetz pencils

This section is devoted to proving Theroem 1.1, which one can easily deduce from the following theorem.

Theorem 3.1 *Let S be a complex surface, $f : S \dashrightarrow \mathbb{P}^1$ be a genus-1 holomorphic Lefschetz pencil, and F be the closure of a fiber of f. Suppose that no fibers of f contain an embedded sphere. Then either of the following holds:*

- *the complex surface S can be obtained by blowing-up $\mathbb{P}^2$ at $l \leq 8$ points and F is linearly equivalent to $3H - \sum_{i=1}^{l} E_i$, where H is the total transform of a projective line H' in $\mathbb{P}^2$ and $E_1, \ldots, E_l$ are the exceptional spheres.*

- *the complex surface S is $\mathbb{P}^1 \times \mathbb{P}^1$ and F is linearly equivalent to $2F_1 + 2F_2$, where F_i is a fiber of the projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the i-th component.*

Remark 3.2 *This remark concerns the assumption that no fibers of a pencil contain an embedded sphere, which is required in not only the theorem above but also the main theorems in the paper. Even if a Lefschetz pencil is relatively minimal, a fiber of it might contain an embedded sphere. For example, let us consider the Lefschetz pencil $f_{g,k} : X_{g,k} \dashrightarrow \mathbb{P}^1$ with the following monodromy factorization:*

$$t_{\delta_1} \cdots t_{\delta_k} = t_{\delta_1} \cdots t_{\delta_k} \;\; in \;\; \mathrm{MCG}(\Sigma_g^k). \tag{3.1}$$

The total space $X_{g,k}$ is a ruled surface and the pencil $f_{g,k}$ has k critical (resp. base) points corresponding to the twists in the left-hand (resp. right-hand) side of (3.1). This pencil is relatively minimal but each singular fiber of it contains a sphere. Furthermore, there exist other types of such Lefschetz pencils with genus-0: the pencils of degree 1 and 2 curves in $\mathbb{P}^2$. The former (resp. the latter) gives rise to the trivial relation $1 = t_\delta$ in $\mathrm{MCG}(D^2)$ (resp. the lantern relation) as the monodromy factorization. Such pencils, however, are not important in the context of classification; if a (not necessarily holomorphic) relatively minimal Lefschetz pencil has an embedded sphere in a fiber, it is isomorphic to one of the examples given here. This follows from the observation in [26, Remark 2.4] for genus-0 and the lemma below for higher genera.

Lemma 3.3 *Let $f : X \dashrightarrow \mathbb{P}^1$ be a relatively minimal Lefschetz pencil with genus-$g \geq 1$. Suppose that there exists an embedded sphere in a fiber of f. Then a monodromy factorization of f is $t_{\delta_1} \cdots t_{\delta_k} = t_{\delta_1} \cdots t_{\delta_k}$.*

Proof [Proof of Lemma 3.3] Let m and k be the numbers of critical points and base points of f, respectively, and $t_{c_m} \cdots t_{c_1} = t_{\delta_1} \cdots t_{\delta_k}$ be a monodromy factorization of f, where $c_1, \ldots, c_m \subset \Sigma_g^k$ be simple closed curves in Σ_g^k. By capping the boundary of Σ_g^k by disks, we can regard Σ_g^k as a subsurface of the closed surface Σ_g. By the assumption, one of the vanishing cycles of f, say c_1, is not essential in Σ_g. Let S be the closure of the genus-0 component of the complement $\Sigma_g^k \setminus c_1$. Since f is relatively minimal, S contains a boundary component of Σ_g^k. By capping all the boundary components of Σ_g^k except for one in S, we obtain the following relation in $\mathrm{MCG}(\Sigma_g^1)$:

$$t_\delta \cdot t_{c_2} \cdots t_{c_m} = t_\delta \Rightarrow t_{c_2} \cdots t_{c_m} = 1$$

If one of the curves $c_2, \ldots, c_m$ is essential in Σ_g^1, the equality above implies that there exists a relatively minimal nontrivial genus-g Lefschetz fibration with a square-zero section, contradicting [29, Proposition 3.3] and [30, Lemma 2.1]. Thus, all the curves $c_1, \ldots, c_m$ bounds a genus-0 subsurface in Σ_g^k. We can also deduce from the observation above that the fundamental group of X is isomorphic to that of Σ_g.

Let S_i be the genus-0 component of $\Sigma_g^k \setminus c_i$. Suppose that S_i contains more than one component of $\partial \Sigma_g^k$ for some i. Then X has a symplectic structure such that there exists an embedded symplectic sphere C with positive square. Since X is not rational, one can verify in the same way as that in the proof of [17, Theorem 1.4 (ii)] that X is an irrational ruled surface, C is away from a maximal disjoint family of exceptional spheres, and after blow-down C becomes a fiber of a $\mathbb{P}^1$-bundle. However, this contradicts that C has positive square. We can eventually show that S_i contains only one component of $\partial \Sigma_g^k$ for each $i = 1, \ldots, m$, in particular each c_i is isotopic to some δ_j. The lemma then follows from the fact that the subgroup of $\mathrm{MCG}(\Sigma_g^k)$ generated by $t_{\delta_1}, \ldots, t_{\delta_k}$ is isomorphic to the free abelian group $\mathbb{Z}^k$. $\qquad\square$

Proof [Proof of Theorem 3.1] Let $f : S \dashrightarrow \mathbb{P}^1$ be a genus-1 holomorphic Lefschetz pencil and $\tilde{f} : \tilde{S} \to \mathbb{P}^1$ be the Lefschetz fibration obtained by blowing-up all the base points of f. By Lemma 3.3 and the assumption, $\tilde{f}$ is relatively minimal. We can deduce from the classification of genus-1 Lefschetz fibrations in the smooth category (given in [18]) that $\tilde{S}$ is diffeomorphic to $\mathbb{P}^2 \sharp 9\overline{\mathbb{P}^2}$. Thus, applying [6, Corollary 2], we can show that S is a rational surface, in particular S is either $\mathbb{P}^2$ or a blow-up of the Hirzebruch surface S_n for some $n \geq 0$ (for the definition of S_n, see [8, Chap. 4, §.3]). Suppose that S is the projective plane. Then the number of the

base points of f is equal to 9; thus, the self-intersection of F is also equal to 9. Since the line bundle $[F]$ has at least two linearly independent sections by (2) of Proposition 2.1, F is linearly equivalent to aH for some $a > 0$ (note that the linear equivalence class of a divisor of a simply connected Kähler manifold is uniquely determined by the corresponding second cohomology class). The self-intersection of aH is equal to a^2. Hence, we can conclude that F is linearly equivalent to $3H$.

In the rest of the proof, we assume that S can be obtained by blowing-up the Hirzebruch surface S_n ($n \geq 0$) l' times ($0 \leq l' \leq 7$). Let $E'_\infty \subset S_n$ be a section of S_n (as a $\mathbb{P}^1$-bundle) with self-intersection $-n$ (which is unique when $n > 0$), and $C' \subset S_n$ be a fiber of the same $\mathbb{P}^1$-bundle on S_n. We denote the total transforms of E'_∞ and C' by E_∞ and C, respectively. Let $\hat{E}_i \subset S$ be the total transform of the exceptional sphere appearing in the i-th blow-up of S_n. Since S is simply connected and Kähler, we can assume that F is linearly equivalent to the following divisor:

$$aE_\infty + bC - \sum_{i=1}^{l'} c_i \hat{E}_i \quad (a, b, c_i \in \mathbb{Z}).$$

All the components of E_∞, C and $\hat{E}_i$ are spheres. Since no fiber of f contains a sphere, we can deduce the following inequality from (3) of Proposition 2.1:

$$a > 0, \ b > na, \text{ and } c_i > 0 \ (i = 1, \ldots, l'). \tag{3.2}$$

Since the number of base points of f is equal to the self-intersection of F, we obtain the following equality:

$$8 - l' = -na^2 + 2ab - \sum_{i=1}^{l'} c_i^2. \tag{3.3}$$

The canonical class of S_n is represented by the divisor $-2E_\infty - (2+n)C$ (see [8, Chap. 4, §.3]). Thus, we can deduce the following equality from (1) of Proposition 2.1:

$$8 - l' + (n-2)a - 2b + \sum_{i=1}^{l'} c_i = 0 \Leftrightarrow b = \frac{1}{2}\left(8 - l' + (n-2)a + \sum_{i=1}^{l'} c_i\right). \tag{3.4}$$

Combining the equalities (3.3) and (3.4), we obtain:

$$-na^2 + a\left((n-2)a + 8 - l' + \sum_{i=1}^{l'} c_i\right) - \sum_{i=1}^{l'} c_i^2 - (8 - l') = 0$$

$$\Leftrightarrow -2a^2 + \left(8 - l' + \sum_{i=1}^{l'} c_i\right)a - \sum_{i=1}^{l'} c_i^2 - (8 - l') = 0. \tag{3.5}$$

We can regard (3.5) as a quadratic equation on a, whose discriminant must be nonnegative. Thus, the following inequality holds:

$$\left(8 - l' + \sum_{i=1}^{l'} c_i\right)^2 - 8\left(\sum_{i=1}^{l'} c_i^2 + (8 - l')\right) \geq 0$$

$$\Leftrightarrow \left(\sum_{i=1}^{l'} c_i\right)^2 + 2(8-l')\sum_{i=1}^{l'} c_i - 8\sum_{i=1}^{l'} c_i^2 - l(8-l') \geq 0. \tag{3.6}$$

Applying the Cauchy-Schwarz inequality to the vectors $\left(\sum_{i=1}^{l'} c_i, \ldots, \sum_{i=1}^{l'} c_i\right)$ and $(c_1, \ldots, c_{l'})$, we obtain the following inequality:

$$\left(\sum_{i=1}^{l'} c_i\right)^2 \leq \sqrt{l'}\left(\sum_{i=1}^{l'} c_i\right) \cdot \sqrt{\sum_{i=1}^{l'} c_i^2} \Rightarrow \sum_{i=1}^{l'} c_i \leq \sqrt{l'\sum_{i=1}^{l'} c_i^2}. \tag{3.7}$$

Combining the inequalities (3.6) and (3.7), we eventually obtain:

$$l'\sum_{i=1}^{l'} c_i^2 + 2\sqrt{l'}(8-l')\sqrt{\sum_{i=1}^{l'} c_i^2} - 8\sum_{i=1}^{l'} c_i^2 - l'(8-l') \geq 0$$

$$\Rightarrow -(8-l')\left(\sqrt{\sum_{i=1}^{l'} c_i^2} - \sqrt{l'}\right)^2 \geq 0.$$

Since l' is less than 8, we can deduce from this inequality that the sum $\sum_{i=1}^{l'} c_i^2$ is equal to l'. This equality together with the inequality in (3.2) implies that $c_1, \ldots, c_{l'}$ are all equal to 1. By substituting 1 for all the c_i's in (3.5), we obtain:

$$-2a^2 + 8a - 8 = 0 \Rightarrow a = 2.$$

We can further deduce from (3.4) that b is equal to $n+2$. Since b is greater than na, n is equal to 0 or 1.

If n is equal to 1, the complex surface S is a blow-up of S_1, which is a blow-up of $\mathbb{P}^2$ at a single point. In other words, there is a sequence of blow-up from $\mathbb{P}^2$ to S:

$$S^{(0)} := \mathbb{P}^2 \leftarrow S^{(1)} := S_1 \leftarrow S^{(2)} \leftarrow \cdots \leftarrow S^{(l)} =: S, \text{ where } l = l' + 1. \tag{3.8}$$

We denote the exceptional sphere in Σ_1 by $\hat{E}'_0$. The divisors E'_∞ and C' are respectively linearly equivalent to $\hat{E}'_0$ and $H' - \hat{E}'_0$. Let $\hat{E}_0 \subset S$ be the total transform of $\hat{E}'_0$. The closure of a fiber F of f is linearly equivalent to $aE_\infty + bC - \sum_{i=1}^{l'} \hat{E}_i = 3H - \sum_{i=1}^{l} E_i$, where $E_i = \hat{E}_{i-1}$. Suppose that the j-th blow-up in the sequence (3.8) is applied at a point on the exceptional sphere appearing in the i-th blow-up for some $i < j$. Then the divisor $E_i - E_j$ would be linearly equivalent to a positive linear combination of spheres. By (3) of Proposition 2.1, the self-intersection $(E_i - E_j) \cdot F$ would be positive, but this is not the case since F is linearly equivalent to $3H - \sum_{i=1}^{l} E_i$. We can eventually conclude that S is obtained from $\mathbb{P}^2$ by blowing-up l points.

Finally, suppose that n is equal to 0. The complex surface S is $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and E'_∞ and C' are respectively equal to F_1 and F_2. Since the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a single point is biholomorphic to the surface obtained by blowing-up $\mathbb{P}^2$ at two points, we can assume that l is equal to 0 without loss of generality. The closure of a fiber F is then linearly equivalent to $aF_1 + bF_2 = 2F_2 + 2F_2$. This completes the proof of Theorem 3.1. $\square$

Proof [Proof of Theroem 1.1] We first observe that the Veronese embedding v_3 and the composition $v_2 \circ \sigma$ are embeddings corresponding to the very ample line bundles $[3H]$ and $[2F_1 + 2F_2]$, respectively. Thus, according

to Remark 2.2, the corollary holds if S is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that S is obtained by blowing up $\mathbb{P}^2$ l times. We can regard the Lefschetz pencil f as a projective line in the complete linear system $\mathbb{P}(H^0(S; [F]))$. Let $E = \sum_{i=1}^{l} E_i$ be the exceptional divisor, $\pi : S \to \mathbb{P}^2$ be the blow-down mapping and $s \in H^0(S; [E])$ a nontrivial section. We can then define the following linear mapping:

$$\xi : H^0(S; [F]) \to H^0(S; [F] \otimes [E]) = H^0(S; \pi^*[3H]), \ \ \xi(\tau) = \tau \otimes s.$$

Since the blow-down mapping π is birational, we can identify $H^0(S; [F] \otimes [E])$ with $H^0(\mathbb{P}^2; [3H])$ via π. Under this identification, the image $\xi(f)$ is a genus-1 Lefschetz pencil defined on $\mathbb{P}^2$, which is smoothly isomorphic to f_n by Remark 2.2, and f can be obtained by blowing-up $\xi(f)$. $\qquad\square$

4. Vanishing cycles of genus-1 Lefschetz pencils

As we have shown, any genus-1 Lefschetz pencil can be obtained by blowing-up either of the pencils f_n or f_s in Theroem 1.1. In this section we will determine vanishing cycles of these pencils relying on the theory of braid monodromies due to Moishezon and Teicher. Throughout this section, we denote the projective varieties $v_3(\mathbb{P}^2)$ and $v_2 \circ \sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ by U_n and U_s, respectively.

4.1. Braid monodromy techniques

In this subsection, we will give a brief review on the theory of braid monodromies. We will first explain how the theory is related with vanishing cycles of Lefschetz pencils appearing as generic pencils of very ample line bundles, and then recall several facts we need to obtain monodromies of f_n and f_s. The reader can refer to [20–23] for more details on this subject.

Let $V \subset \mathbb{P}^n$ be a nonsingular projective surface. Restricting a generic projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^2$, we obtain a regular mapping $\pi : V \to \mathbb{P}^2$ whose critical value set C is a curve with nodes and cusps. We further take a generic projection $\pi' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with base point $p_0 \in \mathbb{P}^2$ so that the composition $f := \pi' \circ \pi : V \dashrightarrow \mathbb{P}^1$ is a Lefschetz pencil. The critical point set of f is equal to the set of critical points of π whose image by π is a branch point of the restriction $\pi'|_C$. We can obtain the monodromy (or equivalently, vanishing cycles) of f from the braid monodromy of C (around branch points of $\pi'|_C$) explained below.

Let $Q := \{q_1, \ldots, q_m\} \subset \mathbb{P}^1$ be the set of images (by π') of branch points of $\pi'|_C$. Take a reference point $q_0 \in \mathbb{P}^1 \setminus Q$. The closure of the preimage $\overline{\pi'^{-1}(q_0)}$ (which is equal to $\pi'^{-1}(q_0) \cup \{p_0\}$) is a line in $\mathbb{P}^2$ intersecting C at $d := \deg C$ points. We take $d+1$ points $v_0, v_1, \ldots, v_d \in S^2$ and fix an identification of the triple $(\overline{\pi'^{-1}(q_0)}, \overline{\pi'^{-1}(q_0)} \cap C, \{p_0\})$ with $(S^2, \{v_1, \ldots, v_d\}, \{v_0\})$. Note that we can also identify the restriction $\pi|_{\overline{f^{-1}(q_0)}} : \overline{f^{-1}(q_0)} \to \overline{\pi'^{-1}(q_0)}$ with a simple branched covering $\theta : \Sigma \to S^2$ branched at $v_1, \ldots, v_d$ (where a simple branched covering is a branched covering such that all the branched points have degree 2). We next take a Hurwitz path system $(\alpha_1, \ldots, \alpha_m)$ of f with the base point q_0, and the corresponding loops γ_i for $i = 1, \ldots, m$ as we took in Section 2.2. Taking the isotopy class of a parallel transport along γ_i preserving C, we obtain a sequence of elements $\tau_1, \ldots, \tau_m$ of the braid group B_d defined as follows:

$$B_d := \pi_0(\mathrm{Diff}(S^2, \{v_1, \ldots, v_d\}, v_0)),$$

where we denote by $\mathrm{Diff}(S^2, \{v_1, \ldots, v_d\}, v_0)$ the group of orientation-preserving self-diffeomorphisms of S^2 preserving v_0 and the set $\{v_1, \ldots, v_d\}$. It is easy to see that each element τ_i is a half twist along some

path $\beta_i \subset S^2$ between two points in $\{v_1, \ldots, v_d\}$. The path β_i is called a Lefschetz vanishing cycle of the corresponding branched point of $\pi'|_C$. The preimage $\theta^{-1}(\beta_i)$ has the unique circle component $c_i \subset \Sigma$, and this circle is a vanishing cycle of a Lefschetz singularity of f in $f^{-1}(q_i)$ with respect to the path α_i.

Remark 4.1 *In the series of papers of Moishzon–Teicher, a Lefschetz vanishing cycle and a braid monodromy are defined not only for branched points of the restriction of a projection on the critical value set, but also for multiple points and cusps of a general curve in $\mathbb{P}^2$. The reader can refer to [21], for example, for details of this subject. Note that we will deal with braid monodromies of multiple points (which is a Dehn twist along some simple closed curve in a punctured sphere) in order to determine vanishing cycles of f_n and f_s.*

Remark 4.2 *Although the product $t_{c_m} \cdots t_{c_1}$ in $\mathrm{MCG}(\overline{f^{-1}(q_0)})$ is equal to the unit, the product $\tau_m \cdots \tau_1$ is not equal to the unit since we do not consider braid monodromies of nodes and cusps of the critical value set C.*

In summary, we can get vanishing cycles of the Lefschetz pencils f_n and f_s in Theroem 1.1 once we obtain Lefschetz vanishing cycles of the branch points of the critical value sets of generic projections from U_n and U_s to $\mathbb{P}^2$ (and the monodromies of simple branched coverings defined over a line in $\mathbb{P}^2$, which can be obtained easily in our situations). Moishezon and Teicher [23] obtained the Lefschetz vanishing cycles for U_n by giving a projective degeneration of U_n to a union of planes, and then analyzing how Lefschetz vanishing cycles are changed in the regeneration (the opposite deformation of the degeneration). As we will observe below, the Lefschetz vanishing cycles for U_s can also be obtained in the same way. In what follows, we will review the definition of a projective degeneration and those for U_n and U_s given in [22] and [19], respectively.

An algebraic set $U_0 \subset \mathbb{P}^{n_0}$ is said to be equivalent to another algebraic set $U_1 \subset \mathbb{P}^{n_1}$ if there exist an algebraic set $W \subset \mathbb{P}^N$ and projections $\pi_0 : \mathbb{P}^N \dashrightarrow \mathbb{P}^{n_0}$ and $\pi_1 : \mathbb{P}^N \dashrightarrow \mathbb{P}^{n_1}$ such that the restriction $\pi_i|_W : W \to U_i$ is an isomorphism for $i = 0, 1$. An algebraic set $U' \subset \mathbb{P}^m$ is a projective degeneration of $U \subset \mathbb{P}^n$ if there exists an algebraic set $W \subset \mathbb{P}^N \times \mathbb{C}$ such that $W \cap (\mathbb{P}^N \times \{0\})$ is equivalent to U' and $W \cap (\mathbb{P}^N \times \{\varepsilon\})$ is equivalent to U for any ε with sufficiently small $|\varepsilon| > 0$. In this paper, we mean by $U \rightsquigarrow U'$ that U' is the result of a projective degenerations from U. Following the notations in [22], we will describe components of algebraic sets as follows:

- A surface equivalent to the image of the Veronese embedding of degree d on $\mathbb{P}^2$ is denoted by V_d, and described by a triangle in figures.

- A surface equivalent to the image of the embedding $\varphi_{[E_\infty + lC]}$ on S_1 ($l > 1$) is denoted by T_l, and described by a trapezoid in figures.

- A surface equivalent to the image of the embedding $\varphi_{[aF_1 + bF_2]}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ ($a, b > 0$) is denoted by $U_{a,b}$, and described by a square in figures.

Theorem 4.3 ([22]. A projective degeneration of U_n.) *There exists a sequence of projective degenerations $U_n =: Y^{(0)} \rightsquigarrow Y^{(1)} \rightsquigarrow \cdots \rightsquigarrow Y^{(5)}$ from U_n to a union of 9 planes $Y^{(5)}$. The intermediate algebraic sets are described in Figure 2.*
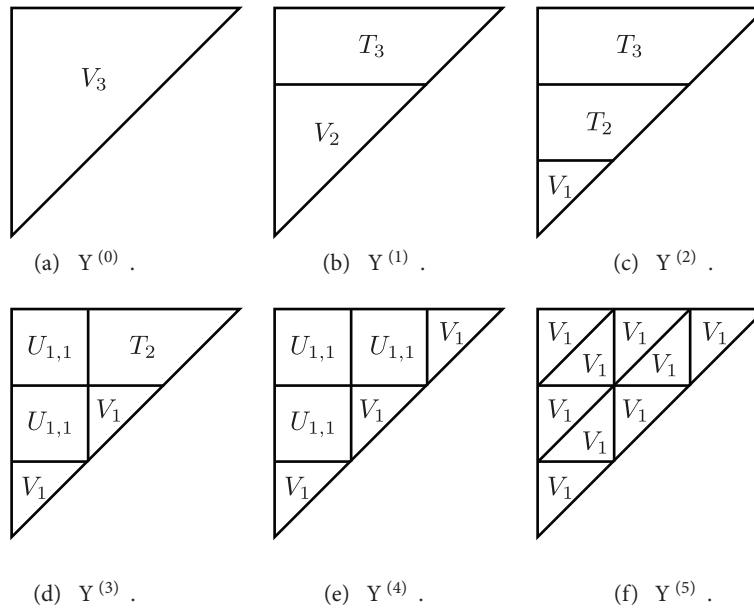
Figure 2. A sequence of projective degenerations of U_n.

Theorem 4.4 ([19]. A projective degeneration of U_s.) *There exists a sequence of projective degenerations $U_s =: Z^{(0)} \rightsquigarrow Z^{(1)} \rightsquigarrow Z^{(2)} \rightsquigarrow Z^{(3)}$ from U_s to a union of 8 planes $Z^{(3)}$. The intermediate algebraic sets are described in Figure 3.*
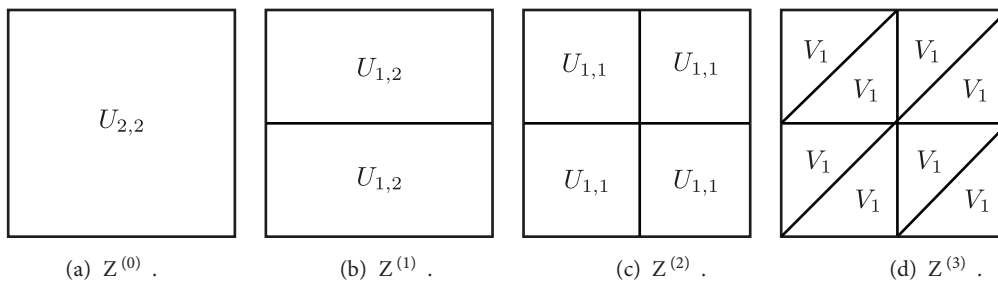

Figure 3. A sequence of projective degenerations of U_s.

Remark 4.5 *In each of the intermediate algebraic sets in Figure 2 and 3, any two components adjacent to each other intersect on a rational curve, and the configuration of these curves are the same as that of the segments between two regions in the figures. For example, in the algebraic set $Y^{(5)}$, there are 9 lines appearing as intersections of two adjacent components, and 7 multiple points in the line arrangement (see Figure 4).*

According to the observation in Remark 4.5, the sets of singular points of the algebraic sets $Y^{(5)}$ and $Z^{(3)}$ are unions of lines, and so are the images of them by projections to $\mathbb{P}^2$. The braid monodromies of these line arrangements in $\mathbb{P}^2$ are completely determined in [20, Theorem IX.2.1]. We will next review the relation between these braid monodromies and those for the original varieties U_n and U_s discussed in [21, 23].
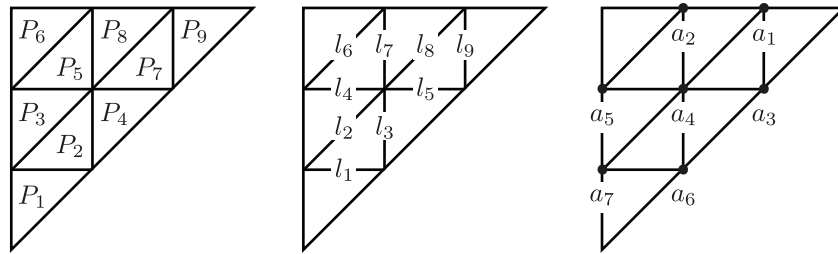
Figure 4. Planes, lines, and multiple points in $Y^{(5)}$.

In the sequences of regenerations given in Theorems 4.3 and 4.4, the line arrangement in $Y^{(5)}$ (resp. $Z^{(3)}$) is also regenerated to the critical value set of the restriction of a generic projection on U_n (resp. U_s). In this regeneration process, each line in the arrangement is "doubled" in the following sense: if some small disk D intersects a line in the arrangement at the center of D transversely, this disk intersects the critical value set of the restriction of a generic projection at two points (note that, without loss of generality, we can assume that the critical value set is sufficiently close to the line arrangement. See [19, §.1]). Furthermore, taking account of the plane arrangement and its regeneration, we can observe that each of the multiple points of the line arrangement has either of the following two properties:

- Three planes P_1, P_2, P_3 go through this point. Among the three planes, P_i and P_{i+1} $(i = 1, 2)$ intersect on a line, while P_1 and P_3 intersect only at the point, in particular two lines $P_1 \cap P_2$ and $P_2 \cap P_3$ go through the point. In the regeneration process the line $P_1 \cap P_2$ regenerates before the regeneration of $P_2 \cap P_3$

- Six planes $P_1, \ldots, P_6$ go through this point. Among the six planes, P_i and P_j intersect on a line if $|j - k| = 1$ modulo 6, or intersect only at the point otherwise. Among six lines $P_1 \cap P_2, \ldots, P_6 \cap P_1$, $P_1 \cap P_2$ and $P_4 \cap P_5$ regenerate first at the same time, $P_2 \cap P_3$ and $P_5 \cap P_6$ then regenerate at the same time, and lastly $P_3, \cap P_4$ and $P_6 \cap P_1$ regenerate at the same time.

In [19], the former multiple point is called a 2-point, while the latter one is called a type M 6-point. Following the rules below, we can determine the braid monodromies of branch points appearing around these points after the regeneration:

Theorem 4.6 ([23, Lemma 1]) *One branch point appears around a 2-point after the regeneration. Suppose that the Lefschetz vanishing cycle of the 2-point is a path β shown in Figure 5a, where v_i is the intersection of the reference fiber and the line $P_i \cap P_{i+1}$ $(i = 1, 2)$. Then the Lefschetz vanishing cycle of the branch point appearing after the regeneration is the path β' shown in Figure 5b.*
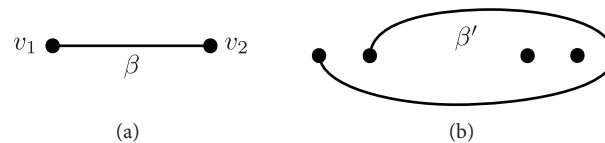

(a)   (b)

Figure 5. Lefschetz vanishing cycles of (a) a 2-point and (b) the branch point around a 2-point.

Theorem 4.7 *([23, Lemmas 5, 6, 7 and 8]) Six branch points appear around a type M 6-point after the regeneration. Suppose that the Lefschetz vanishing cycle of the type M 6-point is a system of paths shown in Figure 6a, where v_i is the intersection of the reference fiber and the line $P_i \cap P_{i+1}$ (taking indeces modulo 6). Then there exists a system of reference paths $(\alpha_1, \ldots, \alpha_6)$ for the six branch points, which appear in this order when we go around the reference point counterclockwise, such that the Lefschetz vanishing cycle associated with α_i is the path β_i, where β_1 and β_6 are shown in Figures 6b and 6c, while $\beta_2, \beta_3, \beta_4, \beta_5$ are defined as:*

$$\beta_2 = \beta, \ \ \beta_3 = \tau_{\gamma_3}^{-1}\tau_{\gamma_4}^{-1}(\beta), \ \ \beta_4 = \tau_{\gamma_1}^{-1}\tau_{\gamma_2}^{-1}(\beta), \ \ \beta_5 = \tau_{\gamma_1}^{-1}\tau_{\gamma_2}^{-1}\tau_{\gamma_3}^{-1}\tau_{\gamma_4}^{-1}(\beta).$$

(Here we denote the positive half twist along γ_i by τ_{γ_i}.)

(a) The Lefschetz vanishing cycle of a type M 6-point.

(b) The Lefschetz vanishing cycle β_1.

(c) The Lefschetz vanishing cycle β_6.

(d) The paths $\gamma_1, \ldots, \gamma_4$.
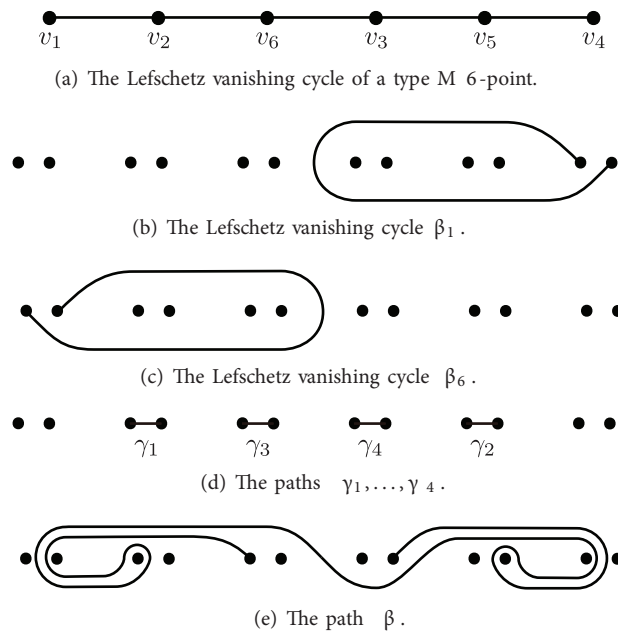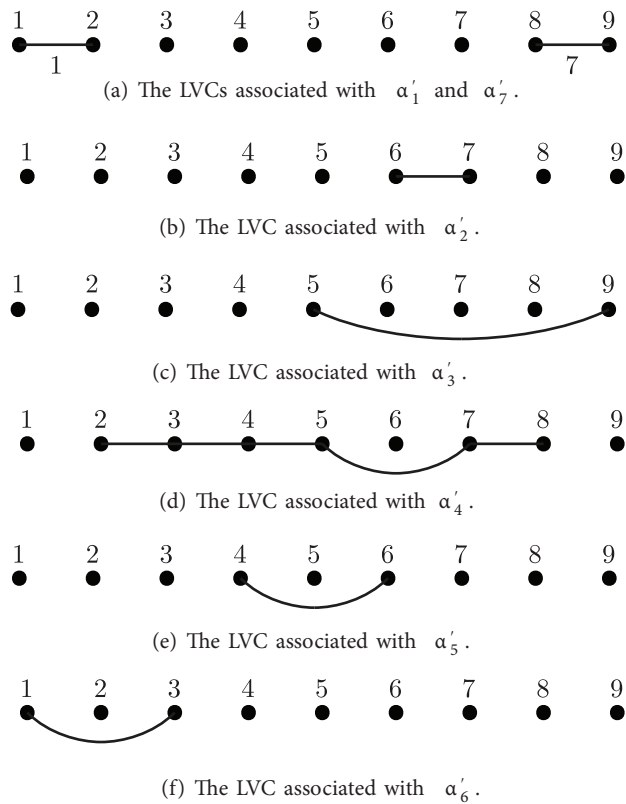
(e) The path β.

Figure 6. The paths around a type M 6-point.

Remark 4.8 *In general, a generic projection on a regenerated surface to $\mathbb{P}^2$ might have branch points which do not appear around multiple points of the original line arrangement. Such branch points are called extra branch points in [19, 27]. According to Proposition 3.3.4 in [27], there are no extra branch points in U_n, while there are two extra branch points in U_s (cf. [19, Proposition 5.2.4]). We will explain how to determine the braid monodromies of these branch points in Section 4.3.*

4.2. Vanishing cycles of a pencil of degree-3 curves in $\mathbb{P}^2$

We will first calculate vanishing cycles of the Lefschetz pencil $f_n : U_n \dashrightarrow \mathbb{P}^1$ given in Theroem 1.1. As shown in Theorem 4.3, we can take a sequence of projective degenerations from U_n to a union of 9 planes $Y^{(5)}$. Let C_n be the union of all the lines in $Y^{(5)}$ appearing as intersections of two planes in $Y^{(5)}$. We denote the planes in $Y^{(5)}$, the lines in C_n and the multiple points in C_n by $\{P_i\}_{i=1}^9$, $\{l_j\}_{j=1}^9$ and $\{a_k\}_{k=1}^7$, respectively, as shown in Figure 4. Note that all the multiple points in C_n are 2-points except for the unique type M 6-point a_4.
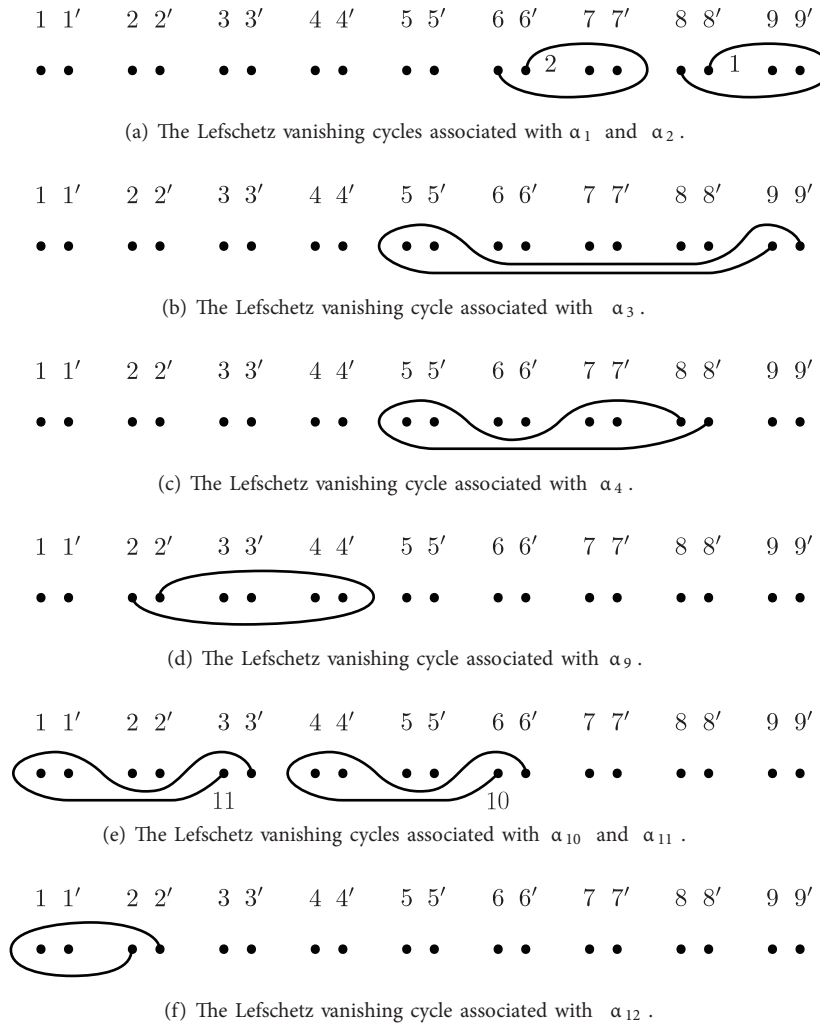
We can assume that $Y^{(5)}$ and $Y^{(0)} = U_n$ are both contained in $\mathbb{P}^N$ and these are sufficiently close (cf. [19, §.1]). Take generic projections $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^2$ and $\pi' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. Let $\tilde{\pi} : U_n \to \mathbb{P}^2$ be the restriction of π on U_n and $a'_i = \pi' \circ \pi(a_i)$. As observed in [23], we can regard C_n as a subarrangement of a line arrangement dual to generic introduced in [20, Section IX]. By [20, Theorem IX.2.1], we can take a point $a'_0 \in \mathbb{P}^1$ away from $a'_1, \ldots, a'_7$ and a simple path α'_i from a'_0 to a'_i so that α'_i's are mutually disjoint except at the common initial point a'_0, the paths $\alpha'_1, \ldots, \alpha'_7$ appear in this order when we go around a'_0 counterclockwise, and the Lefschetz vanishing cycles associated with the paths $\alpha'_1, \ldots, \alpha'_7$ are as shown in Figure 7, where the points labeled with i is the intersection between the fiber $\overline{\pi'^{-1}(a'_0)} \subset \mathbb{P}^2$ and the line $\pi(l_i)$ ($i = 1, \ldots, 9$). We next apply Theorems 4.6 and 4.7 in order to obtain the braid monodromies of the branch points of the restriction $\pi'|_{\widetilde{C_n}} \widetilde{C_n} \to \mathbb{P}^1$, where $\widetilde{C_n}$ is the critical value set of $\tilde{\pi} : U_n \to \mathbb{P}^2$. We eventually obtain a Hurwitz path system $(\alpha_1, \ldots, \alpha_{12})$ of f_n ($= \pi' \circ \tilde{\pi}$) such that the Lefschetz vanishing cycles of the branch points associated with $\alpha_1, \ldots, \alpha_4, \alpha_9, \ldots, \alpha_{12}$ are as shown in Figure 8, while those associated with $\alpha_5, \ldots, \alpha_8$ are respectively equal to $\beta, \tau_{\gamma_3}^{-1}\tau_{\gamma_4}^{-1}(\beta), \tau_{\gamma_1}^{-1}\tau_{\gamma_2}^{-1}(\beta), \tau_{\gamma_1}^{-1}\tau_{\gamma_2}^{-1}\tau_{\gamma_3}^{-1}\tau_{\gamma_4}^{-1}(\beta)$, where the paths $\beta, \gamma_1, \ldots, \gamma_4$ are given in Figure 9 and τ_{γ_i} is the half twist along γ_i.

(a) The LVCs associated with α'_1 and α'_7.

(b) The LVC associated with α'_2.

(c) The LVC associated with α'_3.

(d) The LVC associated with α'_4.

(e) The LVC associated with α'_5.

(f) The LVC associated with α'_6.

Figure 7. The Lefschetz vanishing cycles (LVC) of a line arrangement dual to points in general position.

In order to obtain vanishing cycles of f_n, we have to take the circle components of the preimages of the Lefschetz vanishing cycles under the branched covering

$$\tilde{\pi}|_{\overline{f_n^{-1}(a'_0)}} : \overline{f_n^{-1}(a'_0)} \to \overline{\pi'^{-1}(a'_0)} \tag{4.1}$$

(a) The Lefschetz vanishing cycles associated with α_1 and α_2.

(b) The Lefschetz vanishing cycle associated with α_3.

(c) The Lefschetz vanishing cycle associated with α_4.

(d) The Lefschetz vanishing cycle associated with α_9.

(e) The Lefschetz vanishing cycles associated with α_{10} and α_{11}.

(f) The Lefschetz vanishing cycle associated with α_{12}.

Figure 8. The Lefschetz vanishing cycles of branch points of the restriction $\pi'|_{\widetilde{C_n}} : \widetilde{C_n} \to \mathbb{P}^1$.

(a) The path β.

(b) The paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.
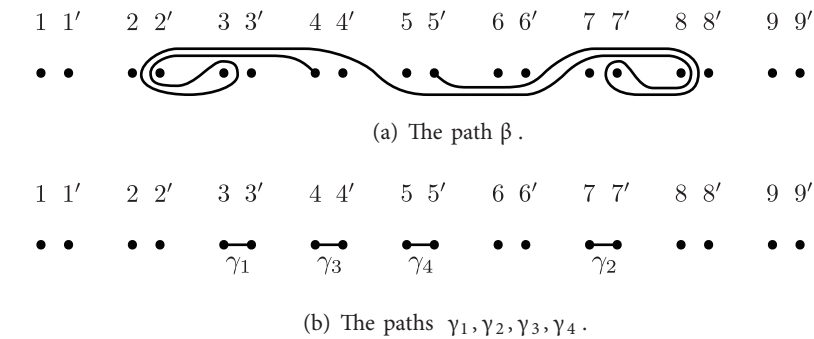
Figure 9. Paths in $\overline{\pi'^{-1}(a_0')}$.

branched at $\overline{\pi'^{-1}(a_0')} \cap \widetilde{C_n}$. We denote the closure $\overline{\pi'^{-1}(a_0')}$ by S, the intersection $\overline{\pi'^{-1}(a_0')} \cap \widetilde{C_n}$ by Q. We take a point $q_0 \in S \setminus Q$ and regard an element σ in the symmetry group $\mathfrak{S}_9$ as a self-bijections of $\tilde{\pi}^{-1}(q_0)$ sending the point in $\tilde{\pi}^{-1}(q_0)$ close to P_i to that close to $P_{\sigma(i)}$ for each $i = 1, \ldots, 9$ (note that we assumed that U_n is sufficiently close to $Y^{(5)}$). Let $\varrho : \pi_1(S \setminus Q, q_0) \to \mathfrak{S}_9$ be the monodromy representation of the branched covering (4.1). As shown in Figure 10, we take a system of oriented paths $\eta_1, \eta_1', \ldots, \eta_9, \eta_9'$ such that the common initial point of them is q_0 and the end point of η_i (resp. η_i') is the points labeled with i (resp. i').
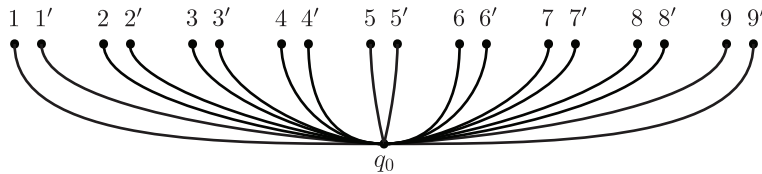
Figure 10. The paths $\eta_1, \eta_1', \ldots, \eta_9, \eta_9'$ in S. In this figure, these paths appear in this order when we go around q_0 clockwise.

Let $\widetilde{\eta_i}$ be a based loop in $S \setminus Q$ with the base point q_0 which can be obtained by connecting q_0 with a small clockwise circle around the point label with i using η_i. We also take a based loop $\widetilde{\eta_i'}$ in a similar manner. The images $\varrho([\widetilde{\eta_i}])$ and $\varrho([\widetilde{\eta_i'}])$ are easily calculated as follows:

$$\varrho([\widetilde{\eta_1}]) = \varrho([\widetilde{\eta_1'}]) = (12), \quad \varrho([\widetilde{\eta_2}]) = \varrho([\widetilde{\eta_2'}]) = (23), \quad \varrho([\widetilde{\eta_3}]) = \varrho([\widetilde{\eta_3'}]) = (24),$$

$$\varrho([\widetilde{\eta_4}]) = \varrho([\widetilde{\eta_4'}]) = (35), \quad \varrho([\widetilde{\eta_5}]) = \varrho([\widetilde{\eta_5'}]) = (47), \quad \varrho([\widetilde{\eta_6}]) = \varrho([\widetilde{\eta_6'}]) = (56),$$

$$\varrho([\widetilde{\eta_7}]) = \varrho([\widetilde{\eta_7'}]) = (58), \quad \varrho([\widetilde{\eta_8}]) = \varrho([\widetilde{\eta_8'}]) = (78), \quad \varrho([\widetilde{\eta_9}]) = \varrho([\widetilde{\eta_9'}]) = (79).$$

Note that all of these images are transpositions. We can thus describe the branched covering (4.1) as shown in Figure 11. In this figure, the red circles in the upper surface $\overline{f_n^{-1}(a_0')}$ are the circle components of the preimages of the red paths between branch points in the lower sphere $\overline{\pi'^{-1}(a_0')}$. The point represented by $\times$ in the lower sphere is the base point of π', while those in the upper surface are the preimages of it. As described in Figure 12, the complement of small disk neighborhoods of the base points of f_n in the closure $\overline{f_n^{-1}(a_0')}$ is a nine-holed torus. One can shrink the subsurfaces of the surface in Figure 12a labeled with $1, 6$, and 9 to obtain that in Figure 12b. Those in Figures 12b and 12c are homeomorphic to each other. Taking the preimages of the paths described in Figure 8 and 9 under the branched covering described in Figure 11, we can eventually obtain a monodromy factorization $t_{c_{12}} \circ \cdots \circ t_{c_1} = t_{\delta_1} \circ \cdots \circ t_{\delta_9}$ of f_n, where the simple closed curves $c_1, \ldots, c_5, c_9, \ldots, c_{12}$ are given in Figure 13, while c_6, c_7, c_8 are respectively equal to $t_{d_3}^{-1} t_{d_4}^{-1}(c_5), t_{d_1}^{-1} t_{d_2}^{-1}(c_5), t_{d_1}^{-1} t_{d_2}^{-1} t_{d_3}^{-1} t_{d_4}^{-1}(c_5)$, where the simple closed curves d_1, d_2, d_3 and d_4 are given in Figure 13d.

4.3. Vanishing cycles of a pencil of curves with bi-degree-$(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$

We will next calculate vanishing cycles of the Lefschetz pencil $f_s : U_s \dashrightarrow \mathbb{P}^1$. Again, let C_s be the union of all the lines in $Z^{(3)}$ appearing as intersections of two plane components, and denote the planes in $Z^{(3)}$, the lines in C_s and the multiple points in C_s by $\{P_s\}_{i=1}^8$, $\{l_j\}_{j=1}^8$ and $\{a_k\}_{k=2}^6$, respectively, as shown in Figure 14. We further take points a_1 and a_7 on l_8 and l_1, respectively. Suppose that $Z^{(3)}$ and $U_s = Z^{(0)}$ are both
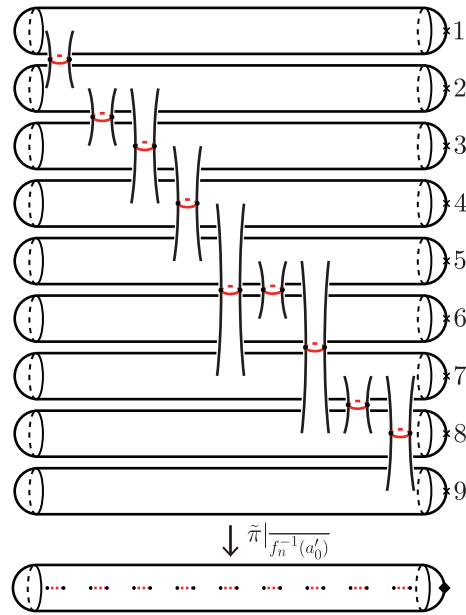
Figure 11. The branched covering given in Equation 4.1.

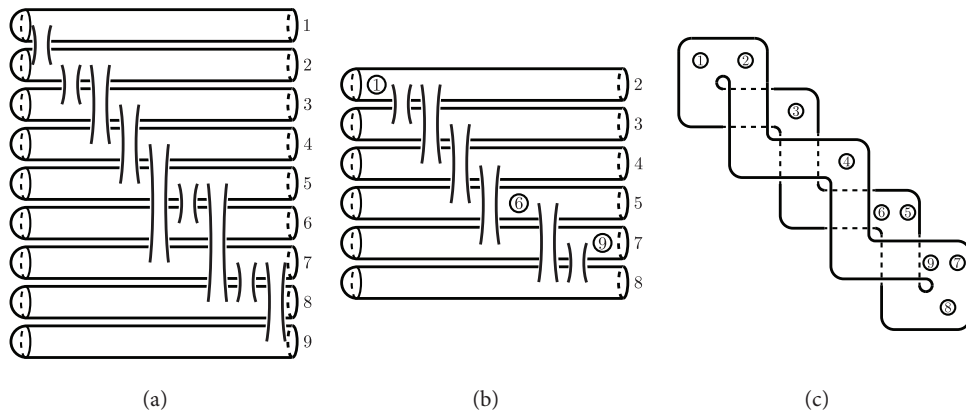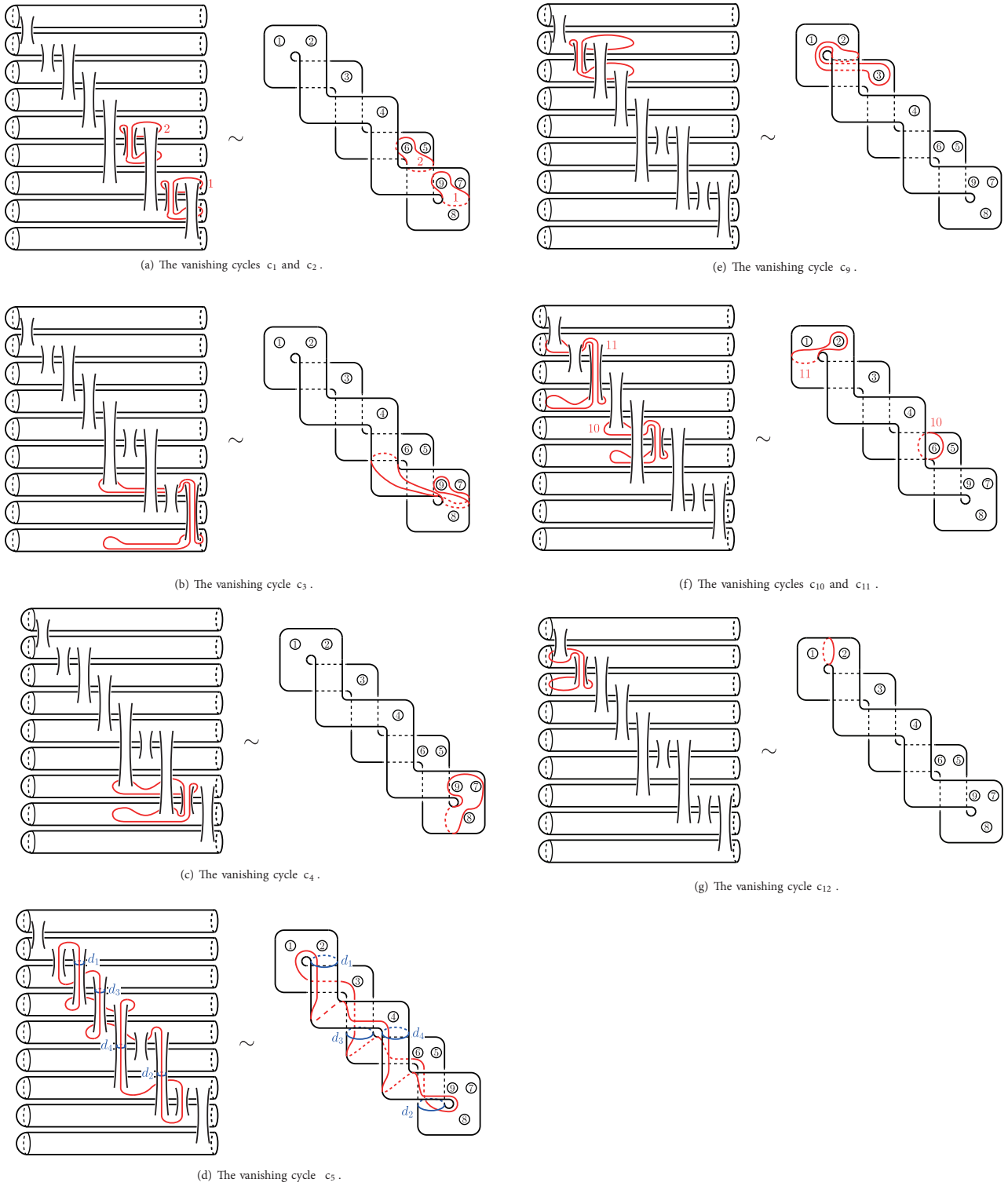(a)   (b)   (c)

Figure 12. The complement of neighborhoods of the base points in $\overline{f_n^{-1}(a_0')}$.

contained in $\mathbb{P}^N$ and these are sufficiently close. Moreover, without loss of generality, we can assume that the line arrangement C_s is the same as that given in [20, Theorem IX.2.1] and the order of the lines in C_s (given by indices) is the same as that in [20, Theorem IX.2.1] (meaning that the order of the vertices in C_s is opposite to that in [20, Theorem IX.2.1]). As in the previous subsection, let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^2$ and $\pi' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be generic projections, $\tilde{\pi} : U_s \to \mathbb{P}^2$ be the restriction of π, $\widetilde{C_s}$ be the critical value set of $\tilde{\pi}$ and $a_i' = \pi' \circ \pi(a_i)$. Applying [20, Theorem IX.2.1], we take a reference point $a_0' \in \mathbb{P}^1$ and reference paths α_i' from a_0' to a_i' $(i = 2, \ldots, 6)$ so that the the corresponding Lefschetz vanishing cycles are as shown in Figure 15.

As observed in Remark 4.8, there are branch points of $\tilde{\pi}|_{\widetilde{C_s}} : \widetilde{C_s} \to \mathbb{P}^1$ which are not close to multiple points of C_s. We take the regeneration from $Z^{(3)}$ to $Z^{(2)}$ so that the planes P_4 and P_7 (resp. P_2 and P_5) are

(a) The vanishing cycles c_1 and c_2.

(b) The vanishing cycle c_3.

(c) The vanishing cycle c_4.

(d) The vanishing cycle c_5.

(e) The vanishing cycle c_9.

(f) The vanishing cycles c_{10} and c_{11}.

(g) The vanishing cycle c_{12}.

Figure 13. Vanishing cycles of the Lefschetz pencil f_n.
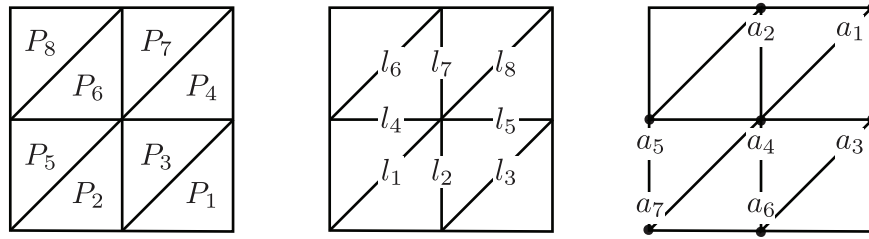
Figure 14. Planes, lines and multiple points in $Z^{(3)}$.

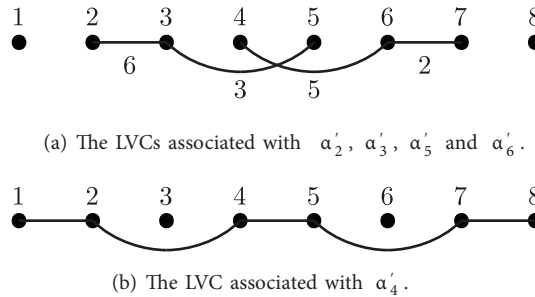(a) The LVCs associated with α'_2, α'_3, α'_5 and α'_6.

(b) The LVC associated with α'_4.

Figure 15. The Lefschetz vanishing cycles (LVC) of a line arrangement dual to points in general position.

regenerated to $U_{1,1}$ going through the points a_1, a_2, a_3, a_4 (resp. a_4, a_5, a_6, a_7). (See [22, §.3.5] for the detail of this regeneration.) Analyzing the model of such a regeneration given in the proof of [22, Proposition 14], we can verify that the two extra branch points of $\tilde{\pi}|_{\widetilde{C}_s}$ appear around a_1 and a_7. We can further show that, for suitable reference paths α_1 and α_{12} from a'_0 to the images of the branch points near a_1 and a_7, respectively, the Lefschetz vanishing cycles of the two extra branch points associated with α_1 and α_{12} are as shown in Figure 16 (cf. [27, §.3.3]). By applying Theorems 4.6 and 4.7, we can take reference paths α_i $(i = 2, \ldots, 11)$ so that $(\alpha_1, \ldots, \alpha_{12})$ is a Hurwitz path system of f_s, and the Lefschetz vanishing cycles of branch points of $\tilde{\pi}|_{\widetilde{C}_s}$ associated with $\alpha_2, \alpha_3, \alpha_4, \alpha_9, \alpha_{10}, \alpha_{11}$ are as shown in Figure 17, while those associated with $\alpha_5, \ldots, \alpha_8$ are respectively equal to $\beta, \tau_{\gamma_3}^{-1}\tau_{\gamma_4}^{-1}(\beta), \tau_{\gamma_1}^{-1}\tau_{\gamma_2}^{-1}(\beta), \tau_{\gamma_1}^{-1}\tau_{\gamma_2}^{-1}\tau_{\gamma_3}^{-1}\tau_{\gamma_4}^{-1}(\beta)$, where the paths $\beta, \gamma_1, \ldots, \gamma_4$ are given in Figures 17f and 17g. As in the previous subsection, we next consider the following branched covering:
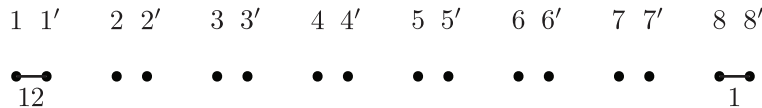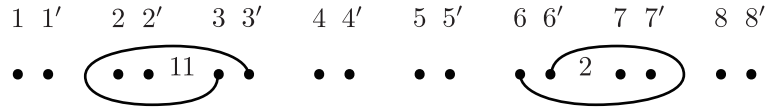

Figure 16. The Lefschetz vanishing cycles associated with α_1 and α_{12}.

$$\tilde{\pi}\Big|_{\overline{f_s^{-1}(a'_0)}} : \overline{f_n^{-1}(a'_0)} \to \overline{\pi'^{-1}(a'_0)}. \tag{4.2}$$

By calculating the monodromy representation of this covering, we can describe this branched covering as shown in Figures 18 and 19. Taking the preimages of the paths described in Figure 17 under the branched covering described in Figure 18, we can obtain a monodromy factorization $t_{c_{12}} \circ \cdots \circ t_{c_1} = t_{\delta_1} \circ \cdots \circ t_{\delta_8}$ of f_s, where
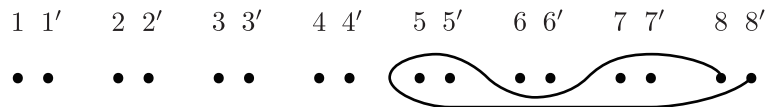
the simple closed curves $c_1, \ldots, c_5, c_9, \ldots, c_{12}$ are given in Figure 20, while c_6, c_7, c_8 are respectively equal to $t_{d_3}^{-1} t_{d_4}^{-1}(c_5), t_{d_1}^{-1} t_{d_2}^{-1}(c_5), t_{d_1}^{-1} t_{d_2}^{-1} t_{d_3}^{-1} t_{d_4}^{-1}(c_5)$, where the simple closed curves d_1, d_2, d_3, and d_4 are given in Figure 20e.
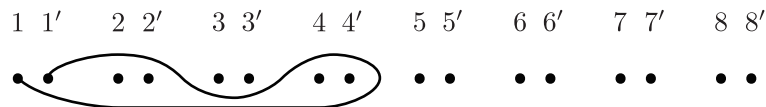

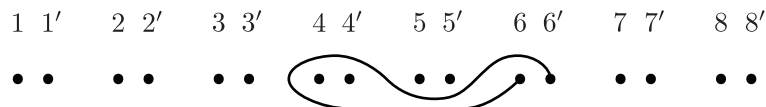(a) The Lefschetz vanishing cycles associated with α_2 and α_{11}.
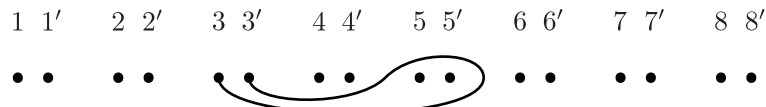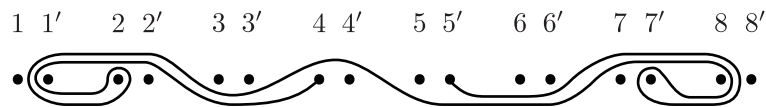

(b) The Lefschetz vanishing cycle associated with α_3.

(c) The Lefschetz vanishing cycle associated with α_4.

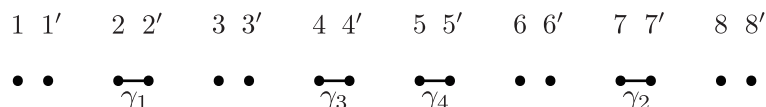(d) The Lefschetz vanishing cycle associated with α_9.

(e) The Lefschetz vanishing cycle associated with α_{10}.

(f) The path β.

(g) The paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

Figure 17. The Lefschetz vanishing cycles of branch points of the restriction $\pi'|_{\widetilde{C_s}} : \widetilde{C_s} \to \mathbb{P}^1$.
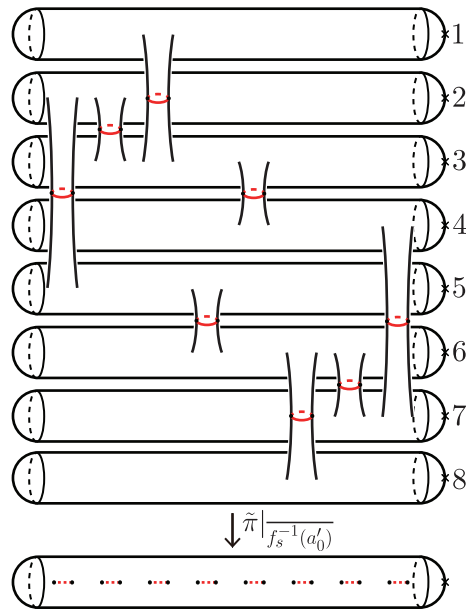
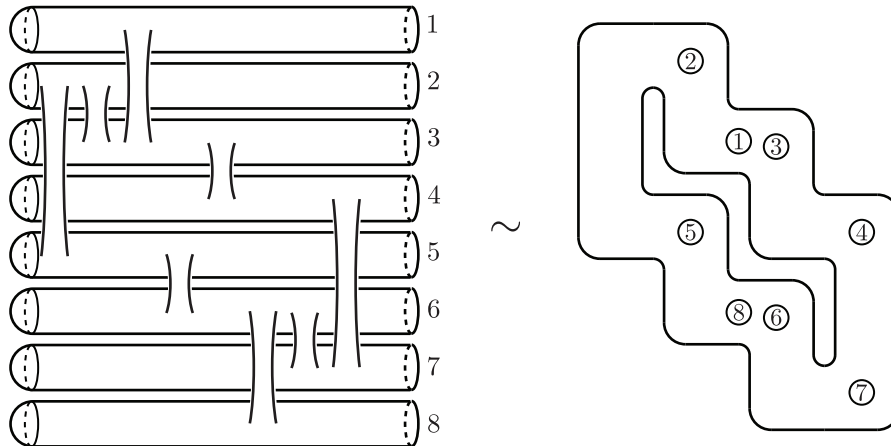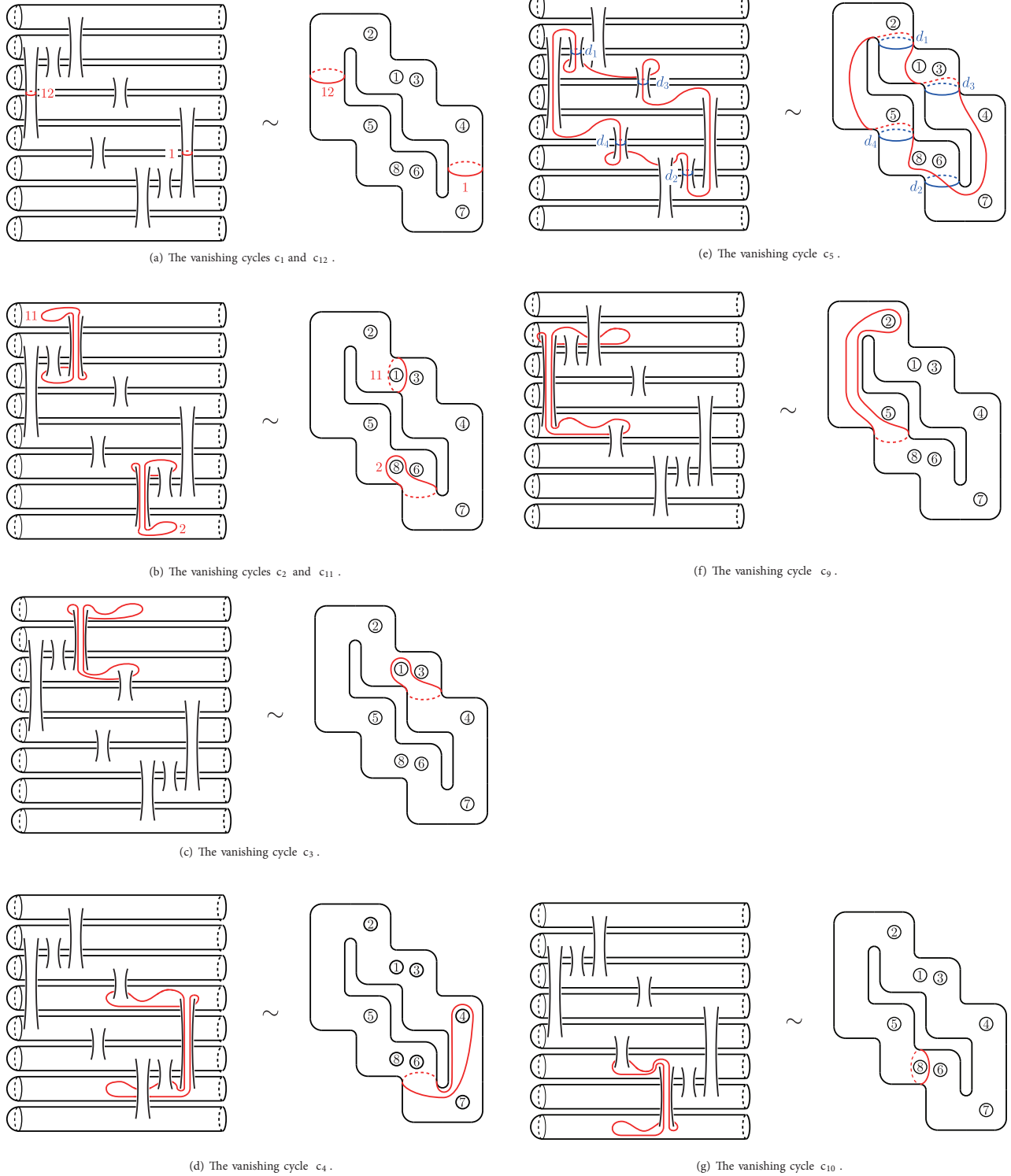**Figure 18**. The branched covering given in Equation 4.2.

Figure 19. The complement of neighborhoods of the base points in $\overline{f_s^{-1}(a_0')}$.

5. Combinatorial structures of genus-1 pencils

In this section we study the combinatorial structures of the monodromy factorizations associated with the genus-1 holomorphic Lefschetz pencils. We will simplify those factorizations and show that they are Hurwitz equivalent to the known k-holed torus relations, which were combinatorially constructed by Korkmaz-Ozbagci [14] and Tanaka [31]. In particular, we will see that a genus-1 holomorphic Lefschetz pencil obtained by blowing-up another holomorphic pencil is uniquely determined by the number of the blown-up points and independent of particular choices of such points. Thus, we complete the classification of genus-1 holomorphic Lefschetz pencils in the smooth category.

(a) The vanishing cycles c_1 and c_{12}.

(b) The vanishing cycles c_2 and c_{11}.

(c) The vanishing cycle c_3.

(d) The vanishing cycle c_4.

(e) The vanishing cycle c_5.

(f) The vanishing cycle c_9.

(g) The vanishing cycle c_{10}.

Figure 20. Vanishing cycles of the Lefschetz pencil f_s.

In the remainder of the paper, we simplify the notations regarding Dehn twists as follows. We will denote the right-handed Dehn twist along a curve α also by α, and its inverse, i.e. the left-handed Dehn twist along α, by $\bar{\alpha}$. We continue to use the functional notation for multiplication; $\beta\alpha$ means we first apply α and then β. In addition, we denote the conjugation $\alpha\beta\bar{\alpha}$ by ${}_\alpha(\beta)$, which is the Dehn twist along the curve $t_\alpha(\beta)$. Finally, we use the symbol ∂_k to denote the boundary multitwist $\delta_1\delta_2\cdots\delta_k$.

## 5.1. Monodromies of the minimal pencils

We first deal with the minimal holomorphic Lefschetz pencils f_n and f_s as they are the base cases in the sense that the other holomorphic pencils are obtained by blowing-up those two pencils.

5.1.1. Monodromy of f_n

In Section 4.2 we obtained a monodromy factorization of f_n,

$$c_{12}c_{11}c_{10}c_9c_8c_7c_6c_5c_4c_3c_2c_1 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9$$

with the vanishing cycles computed in Figure 13 where $c_6 = {}_{\bar{d}_3\bar{d}_4}(c_5)$, $c_7 = {}_{\bar{d}_1\bar{d}_2}(c_5)$, and $c_8 = {}_{\bar{d}_1\bar{d}_2\bar{d}_3\bar{d}_4}(c_5)$. The curves are redrawn on a standardly positioned torus in Figure 21. We further reposition the surface by pushing
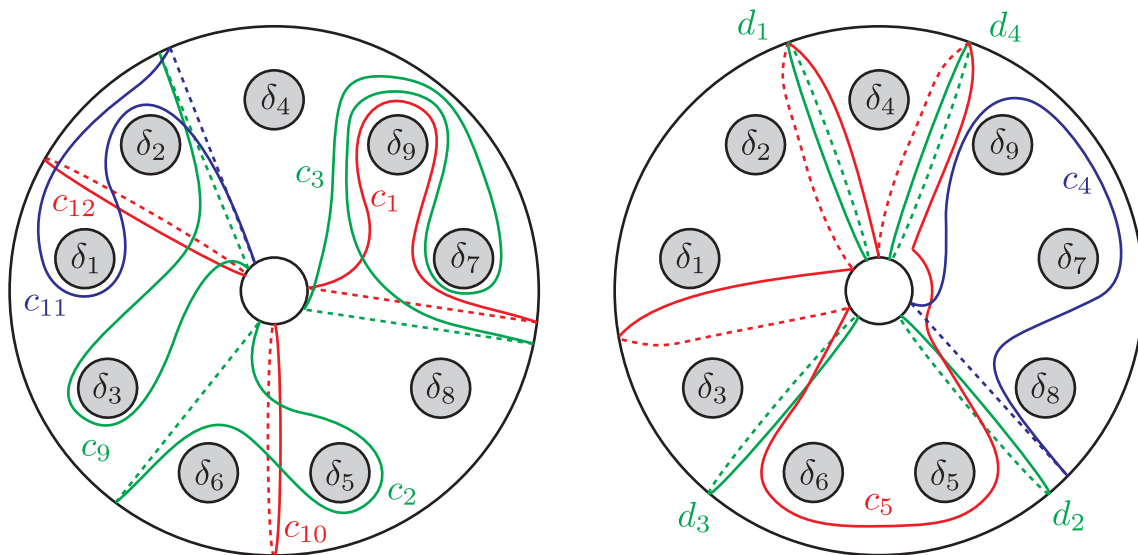
Figure 21. Redrawing of the vanishing cycles of f_n on a standard 9-holed torus Σ_1^9.

the boundary components as indicated in Figure 22; we first swap δ_1 and δ_3, also δ_7 and δ_9, then push the boundary components except for δ_2 and δ_3 along the meridian in the indicated directions. Accordingly, the vanishing cycles are now configured as in Figure 23. We further modify the factorization by Hurwitz moves.
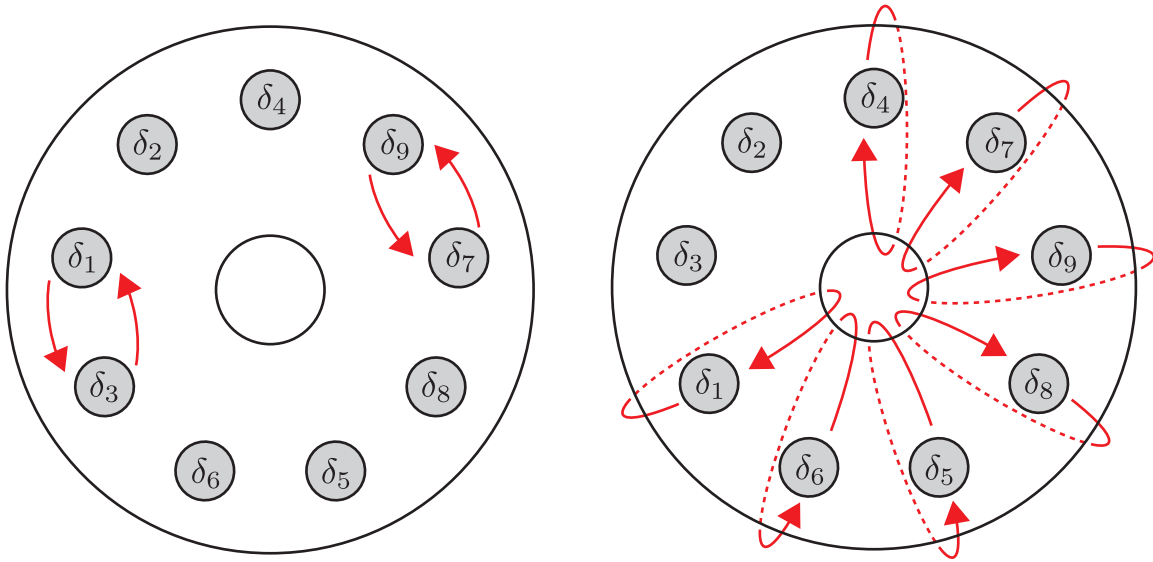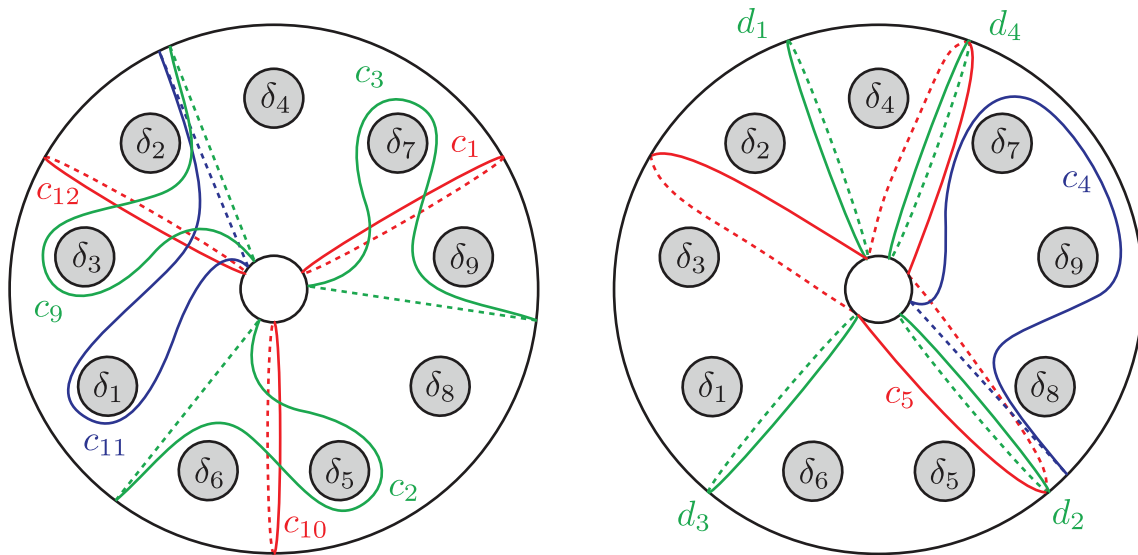
Figure 22. Pushing of boundary components.

Figure 23. Vanishing cycles after repositioning the surface Σ_1^9.

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9 = c_{12}c_{11}c_{10}c_9c_8c_7c_6c_5c_4c_3c_2c_1$$

$$\sim c_{11}\underline{c_9c_8}\ \underline{c_7c_6c_5c_{12}}c_4c_3c_1c_2c_{10}$$

$$\sim c_{11}c_8'c_9c_{12\bar{c}_{12}}(c_7)_{\bar{c}_{12}}(c_6)\underline{c_5'}c_4c_3c_1c_2c_{10}$$

$$\sim c_{11}c_8'c_9c_{12\bar{c}_{12}}(c_7)_{\underline{\bar{c}_{12}}}(c_6)\underline{c_4'}c_5'c_3c_1c_2c_{10}$$

$$\sim c_{11}c_8'c_9c_{12\bar{c}_{12}}(c_7)c_{4\underline{\bar{c}_4'\bar{c}_{12}}}(c_6)c_5'c_3c_1c_2c_{10}$$

$$\sim \underline{c_{11}c_8'}c_9c_{12\bar{c}_{12}}(c_7)c_4'c_3c_{1\underline{\bar{c}_1\bar{c}_3\bar{c}_4'\bar{c}_{12}}}(c_6)c_5''c_2c_{10}$$

$$\sim c_8'\underline{c_{11}'}c_9c_{12\bar{c}_{12}}(c_7)\underline{c_4'c_3}c_1c_5''\underline{c_6'c_2}c_{10}$$

$$\sim c_{8c_{11}'}'(c_9)\underline{c_{11}'c_{12}}c_{12\bar{c}_{12}}(c_7)_{c_4'}(c_3)\underline{c_4'c_1}c_{5c_6'}''(c_2)\underline{c_6'c_{10}}$$

$$\sim c_{8c_{11}'}'(c_9)c_{12\bar{c}_{12}}(c_{11}')_{\bar{c}_{12}}(c_7)_{c_4'}(c_3)c_{1\bar{c}_1}(c_4')c_{5c_6'}''(c_2)c_{10\bar{c}_{10}}(c_6')$$

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where $c_8' = {}_{c_9}(c_8)$, $c_5' = {}_{\bar{c}_{12}}(c_5)$, $c_4' = {}_{c_5'}(c_4)$, $c_5'' = {}_{\bar{c}_1\bar{c}_3}(c_5')$, $c_{11}' = {}_{\bar{c}_8'}(c_{11})$, $c_6' = {}_{\bar{c}_5''\bar{c}_1\bar{c}_3\bar{c}_4'\bar{c}_{12}}(c_6)$. It is routine to observe that the resulting curves are as depicted in Figure 24a and the last expression is $a_1b_1b_2b_3a_4b_4b_5b_6a_7b_7b_8b_9$ up to labeling and a permutation. Thus, we obtain the simpler monodromy factorization of f_n:
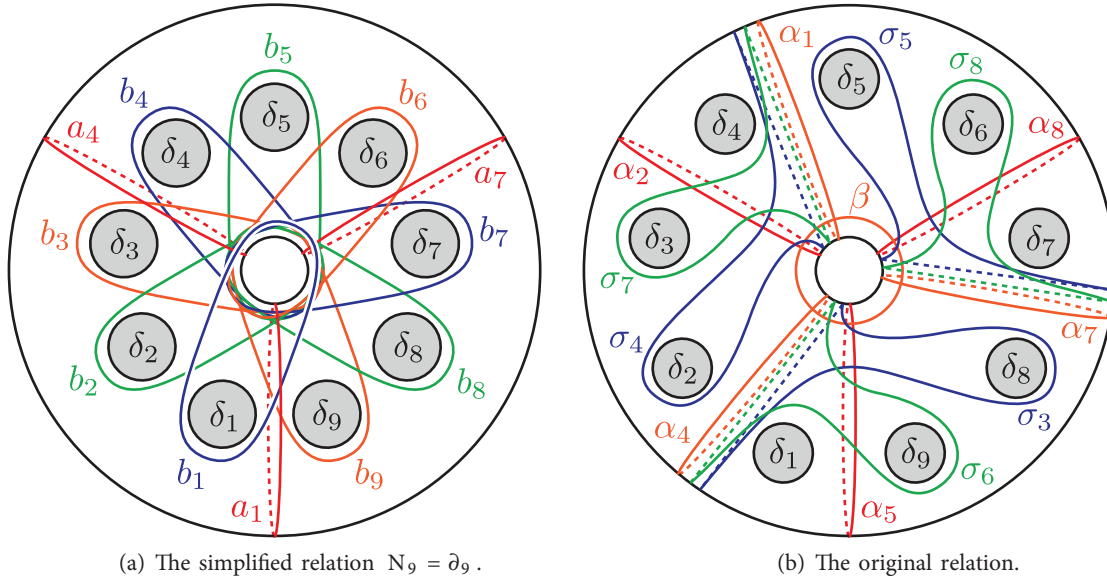

(a) The simplified relation $N_9 = \partial_9$.

(b) The original relation.

Figure 24. The curves for Korkmaz-Ozbagci's 9-holed torus relation.

$$a_1b_1b_2b_3a_4b_4b_5b_6a_7b_7b_8b_9 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9, \tag{5.1}$$

We refer to this relation as $N_9 = \partial_9$.

We are now ready for proving the following:

Theorem 5.1 *The monodromy factorization $N_9 = \partial_9$ is Hurwitz equivalent to Korkmaz-Ozbagci's 9-holed torus relation given in [14].*

Proof With the curves shown in Figure 24b, Korkmaz and Ozbagci [14] gave the 9-holed torus relation

$$\beta_4\sigma_3\sigma_6\alpha_5\beta_1\sigma_4\sigma_7\alpha_2\beta_7\sigma_5\sigma_8\alpha_8 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9, \tag{5.2}$$

where $\beta_4 = {}_{\alpha_4}(\beta)$, $\beta_1 = {}_{\alpha_1}(\beta)$ and $\beta_7 = {}_{\alpha_7}(\beta)$. We modify this relation as follows:

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9 = \underline{\beta_4\sigma_3\sigma_6}\alpha_5\underline{\beta_1\sigma_4\sigma_7}\alpha_2\underline{\beta_7\sigma_5\sigma_8}\alpha_8$$

$$\sim {}_{\beta_4}(\sigma_3){}_{\beta_4}(\sigma_6)\underline{\beta_4\alpha_5}\beta_1({}_{\beta_1}(\sigma_4){}_{\beta_1}(\sigma_7)\underline{\beta_1\alpha_2}\beta_7({}_{\beta_7}(\sigma_5){}_{\beta_7}(\sigma_8)\underline{\beta_7\alpha_8}$$

$$\sim {}_{\beta_4}(\sigma_3){}_{\beta_4}(\sigma_6)\alpha_{5\,\bar{\alpha}_5}(\beta_4){}_{\beta_1}(\sigma_4){}_{\beta_1}(\sigma_7)\alpha_{2\,\bar{\alpha}_2}(\beta_1){}_{\beta_7}(\sigma_5){}_{\beta_7}(\sigma_8)\alpha_{8\,\bar{\alpha}_8}(\beta_7)$$

$$\sim \alpha_{5\,\bar{\alpha}_5}(\beta_4){}_{\beta_1}(\sigma_4){}_{\beta_1}(\sigma_7)\alpha_{2\,\bar{\alpha}_2}(\beta_1){}_{\beta_7}(\sigma_5){}_{\beta_7}(\sigma_8)\alpha_{8\,\bar{\alpha}_8}(\beta_7){}_{\beta_4}(\sigma_3){}_{\beta_4}(\sigma_6).$$

It is straightforward to see that the last expression coincides with the factorization N_9. $\qquad \square$

5.1.2. Monodromy of f_s

A monodromy factorization of f_s was computed in Section 4.3 as

$$c_{12}c_{11}c_{10}c_9c_8c_7c_6c_5c_4c_3c_2c_1 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8$$

with the vanishing cycles found in Figure 20 where $c_6 = {}_{\bar{d}_3\bar{d}_4}(c_5)$, $c_7 = {}_{\bar{d}_1\bar{d}_2}(c_5)$, and $c_8 = {}_{\bar{d}_1\bar{d}_2\bar{d}_3\bar{d}_4}(c_5)$. The curves are redrawn on a standardly positioned torus in Figure 25. We perform the global conjugation by $\bar{d}_1\bar{d}_4d_2$ to put the vanishing cycles as in Figure 26. For simplicity, we keep using the same labeling c_i for the resulting curves. We then transform the factorization as follows.
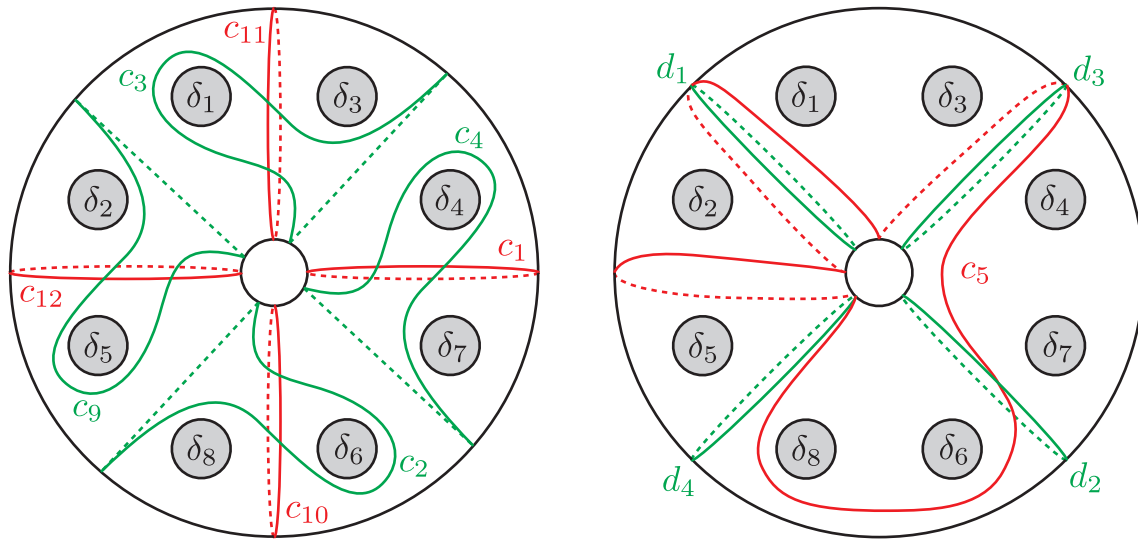

Figure 25. Redrawing of the vanishing cycles of f_s on a standard 8-holed torus Σ_1^8.
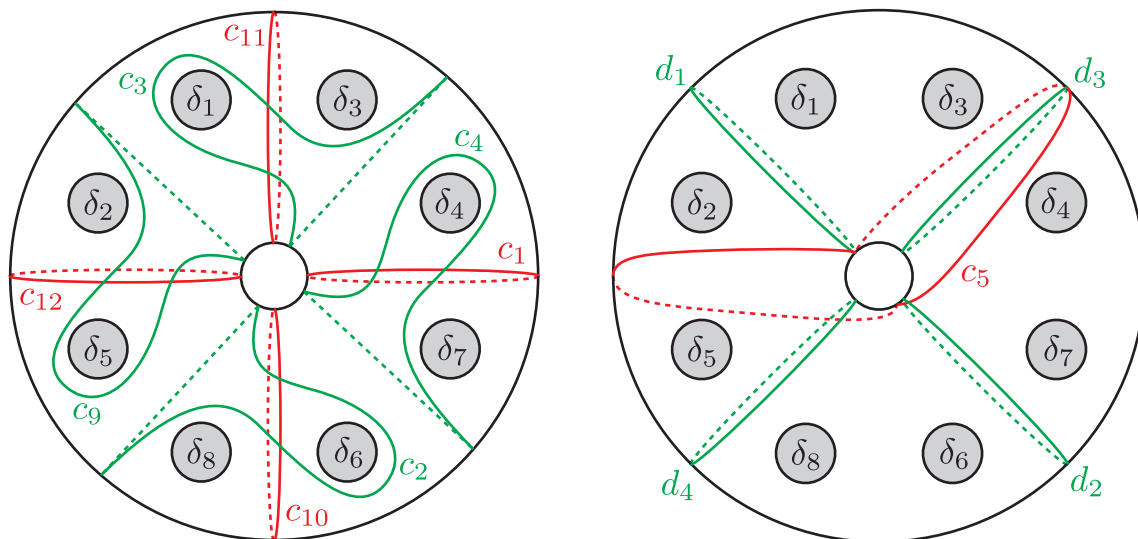

Figure 26. Vanishing cycles after the global conjugation by $\bar{d}_1\bar{d}_4d_2$.

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 = c_{12}c_{11}c_{10}c_9c_8c_7c_6c_5c_4c_3c_2c_1$$

$$\sim c_4c_1c_2\underline{c_{10}c_9c_8c_7}c_6\underline{c_5}c_{12}c_3c_{11}$$

$$\sim c_4\underline{c_1c_2c'_{8\,c_{10}c_9}(c_7)}c_{10}c_9c_6c_{12}c_3c'_5c_{11}$$

$$\sim \underline{c_4c'_7c_1} \cdot \underline{c_2c'_8c_{10}} \cdot \underline{c_9c_6c_{12}} \cdot \underline{c_3c'_5c_{11}}$$

$$\sim c'_7c_{1\,\bar c_1\bar c'_7}(c_4) \cdot c'_8c_{10\,\bar c_{10}\bar c'_8}(c_2) \cdot c_6c_{12\,\bar c_{12}\bar c_6}(c_9) \cdot c'_5c_{11\,\bar c_{11}\bar c'_5}(c_3)$$

where $c'_5 = {}_{\bar c_3\bar c_{12}}(c_5)$, $c'_8 = {}_{c_{10}c_9}(c_8)$, $c'_7 = {}_{c_1c_2c'_8c_{10}c_9}(c_7)$. The resulting curves are as depicted in Figure 27a and the last expression is $a_1b_1b_2a_3b_3b_4a_5b_5b_6a_7b_7b_8$ up to labeling and a permutation. Thus, we obtain the simpler

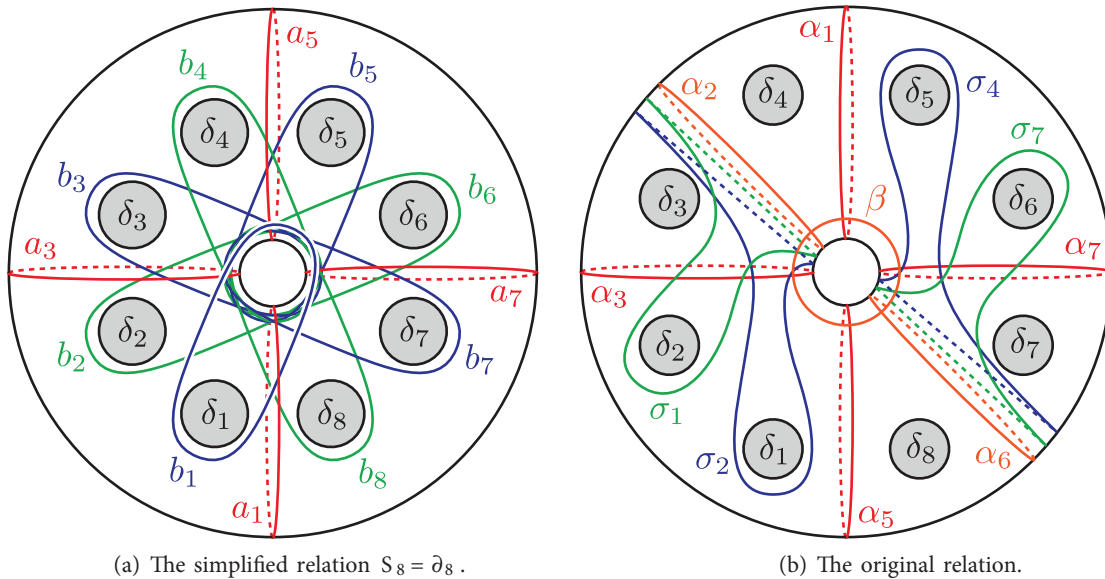(a) The simplified relation $S_8 = \partial_8$.   (b) The original relation.

Figure 27. The curves for Tanaka's 8-holed torus relation.

monodromy factorization of f_s:

$$a_1b_1b_2a_3b_3b_4a_5b_5b_6a_7b_7b_8 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8, \tag{5.3}$$

We refer to this relation as $S_8 = \partial_8$.

Theorem 5.2 *The monodromy factorization $S_8 = \partial_8$ is Hurwitz equivalent to Tanaka's 8-holed torus relation given in [31].*

Proof With the curves shown in Figure 27b, Tanaka [31] gave the 8-holed torus relation

$$\alpha_5\alpha_7\beta_{\bar 6}\beta_2\sigma_2\sigma_1\alpha_1\alpha_3\beta_{\bar 2}\beta_6\sigma_4\sigma_7 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8, \tag{5.4}$$

where $\beta_{\bar{6}} = {}_{\bar{\alpha}_6}(\beta)$, $\beta_2 = {}_{\alpha_2}(\beta)$, $\beta_{\bar{2}} = {}_{\bar{\alpha}_2}(\beta)$, and $\beta_6 = {}_{\alpha_6}(\beta)$. We modify this relation as follows:

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 = \underline{\alpha_5\alpha_7}\beta_{\bar{6}}\beta_2\sigma_2\sigma_1\underline{\alpha_1\alpha_3}\beta_{\bar{2}}\beta_6\sigma_4\sigma_7$$

$$\sim \alpha_7\underline{\alpha_5\beta_{\bar{6}}}\beta_2\sigma_2\sigma_1\alpha_3\underline{\alpha_1\beta_{\bar{2}}}\beta_6\sigma_4\sigma_7$$

$$\sim \alpha_{7\,\alpha_5}(\beta_{\bar{6}})\alpha_5\underline{\beta_2\sigma_2\sigma_1}\alpha_{3\,\alpha_1}(\beta_{\bar{2}})\alpha_1\underline{\beta_6\sigma_4\sigma_7}$$

$$\sim \alpha_{7\,\alpha_5}(\beta_{\bar{6}})\alpha_{5\,\beta_2}(\sigma_2)_{\beta_2}(\sigma_1)\beta_2\alpha_{3\,\alpha_1}(\beta_{\bar{2}})\alpha_{1\,\beta_6}(\sigma_4)_{\beta_6}(\sigma_7)\beta_6$$

$$\sim \alpha_{5\,\beta_2}(\sigma_2)_{\beta_2}(\sigma_1)\underline{\beta_2\alpha_3}\alpha_1(\beta_{\bar{2}})\alpha_{1\,\beta_6}(\sigma_4)_{\beta_6}(\sigma_7)\underline{\beta_6\alpha_7}\alpha_5(\beta_{\bar{6}})$$

$$\sim \alpha_{5\,\beta_2}(\sigma_2)_{\beta_2}(\sigma_1)\alpha_{3\,\bar{\alpha}_3}(\beta_2)_{\alpha_1}(\beta_{\bar{2}})\alpha_{1\,\beta_6}(\sigma_4)_{\beta_6}(\sigma_7)\alpha_{7\,\bar{\alpha}_7}(\beta_6)_{\alpha_5}(\beta_{\bar{6}}).$$

The last expression coincides with S_8. □

5.2. Monodromy and uniqueness of the nonminimal pencils

By blowing-up some of the base points of f_n or f_s we obtain a nonminimal holomorphic Lefschetz pencil. In terms of monodromy factorization this corresponds to capping boundary components of $N_9 = \partial_9$ or $S_8 = \partial_8$ with disks and obtaining a k-holed torus relation with smaller k. The question is whether the resulting pencil is (smoothly) determined only by the number of blow-ups and independent of a particular set of base points that we blow-up. We prove that the answer is affirmative by providing a "standard" k-holed torus relation $N_k = \partial_k$ for each $k \leq 8$ and showing the blow-up of any one base point of $N_k = \partial_k$, or additionally $S_8 = \partial_8$ when $k = 8$, is Hurwitz equivalent to N_{k-1}.

The next lemma summarizes the techniques that we will repeatedly use in the Hurwitz equivalence computations.

Lemma 5.3 *Consider the curves* a_i, b_i, b *in the k-holed torus Σ_1^k as in Figure 1. Then the following relations between Dehn twists in* $\mathrm{MCG}(\Sigma_1^k)$ *are achieved by Hurwitz moves.*

1. $bb_i \sim b_ib$, $a_ia_j \sim a_ja_i$.

2. $a_iba_i \sim ba_ib$.

3. $ba_ib_i \sim a_ib_ia_{i+1} \sim b_ia_{i+1}b \sim a_{i+1}ba_i$.

Here the indices are taken modulo k.

The verification is easy.

5.2.1. One-time blow-up of f_n and the 8-holed torus relation $N_8 = \partial_8$

We consider the 9-holed torus relation $N_9 = \partial_9$ with the curves in Figure 24a (or Figure 1) and cap one of the boundary components.

Case 1: Capping δ_9. This yields the relation

$$a_1b_1b_2b_3a_4b_4b_5b_6a_7b_7b_8b = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 \tag{5.5}$$

in $\mathrm{MCG}(\Sigma_1^8)$ where the curves are now understood to lie in Σ_1^8 as in Figure 1 with $k = 8$ (see also Figure 28). Notice that the curve b_9 becomes the central longitude b as the boundary δ_9 disappears. We modify the
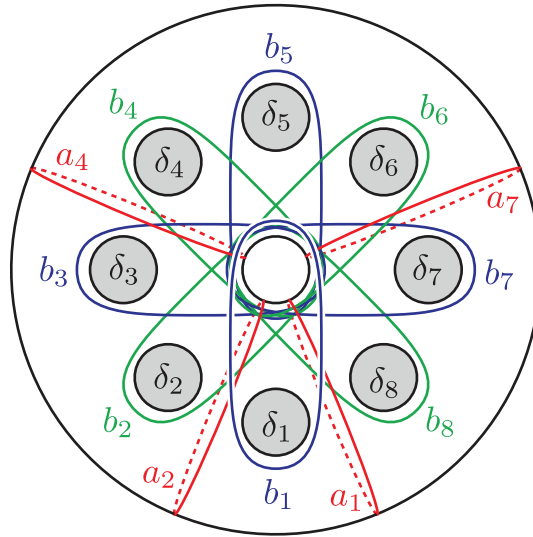
Figure 28. The relation $N_8 = \partial_8$.

relation as follows.

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 = a_1b_1b_2b_3a_4b_4b_5b_6a_7b_7b_8b$$

$$\sim \underline{ba_1}b_1b_2b_3a_4b_4b_5b_6a_7b_7b_8$$

$$\sim a_1b_1a_2b_2b_3a_4b_4b_5b_6a_7b_7b_8.$$

Thus, we have the 8-holed torus relation

$$a_1b_1a_2b_2b_3a_4b_4b_5b_6a_7b_7b_8 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8, \tag{5.6}$$

to which we refer as $N_8 = \partial_8$.

Case 2: Capping δ_8 or δ_7. Instead of δ_9, we now cap δ_8 or δ_7 of $N_9 = \partial_9$. Then, after relabeling the curves so that they match the curves in Figure 1 with $k = 8$, we have

$$a_1b_1b_2b_3a_4b_4b_5b_6a_7b_7\underline{bb_8} = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8,$$

$$a_1b_1b_2b_3a_4b_4b_5b_6a_7\underline{bb_7b_8} = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8.$$

Both of them are clearly equivalent to the relation (5.5), and hence to $N_8 = \partial_8$, since b commutes with b_7 and b_8.

Case 3: The other boundary components $\delta_1, \delta_2, \cdots, \delta_6$. We can take advantage of the symmetry that the relation $N_9 = \partial_9$ possesses and reduce to the cases we have already discussed. In Figure 24a, consider the clockwise rotation r by $2\pi/3$ about the axis perpendicular to the page and through the center of the figure. This diffeomorphism maps (a_i, b_i, δ_i) to $(a_{i+3}, b_{i+3}, \delta_{i+3})$, where the indices are taken modulo 9. Then, via the rotation r the relation $N_9 = \partial_9$ becomes

$$\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9\delta_1\delta_2\delta_3 = a_4b_4b_5b_6a_7b_7b_8b_9a_1b_1b_2b_3,$$

which is just a permutation of $N_9 = \partial_9$. Therefore, capping δ_6 of $N_9 = \partial_9$ is the same as capping δ_9 of $N_9 = \partial_9$ after applying r; hence, it results in the relation $N_8 = \partial_8$. In the same way, capping δ_4 or δ_5 reduces to capping δ_7 or δ_8, respectively. If we consider the counterclockwise rotation r^{-1} we can reduce the cases of δ_1, δ_2, and δ_3 to the cases of δ_7, δ_8, or δ_9, respectively.

5.2.2. Two-time blow-up of f_n and the 7-holed torus relation $N_7 = \partial_7$

We take the 8-holed torus relations $N_8 = \partial_8$ and $S_8 = \partial_8$ and cap each one of the boundary components.

Case 1: Capping δ_8 or δ_7 of $N_8 = \partial_8$. They give

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2b_2b_3a_4b_4b_5b_6a_7\underline{b_7b},$$

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2b_2b_3a_4b_4b_5b_6a_7\underline{bb_7},$$

which are clearly equivalent as b commutes with b_7. Then, from the second

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2b_2b_3a_4b_4b_5\underline{b_6a_7}bb_7$$

$$\sim a_1b_1a_2b_2b_3a_4b_4b_5a_6b_6a_7b_7$$

$$\sim a_7b_7a_1b_1a_2b_2b_3a_4b_4b_5a_6b_6.$$

Then perform the clockwise rotation by $2\pi/7$, which shifts all the indices by 1. This results in the following 7-holed torus relation $N_7 = \partial_7$:

$$a_1b_1a_2b_2a_3b_3b_4a_5b_5b_6a_7b_7 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7. \tag{5.7}$$

Case 2: Capping δ_6, δ_5, or δ_4 of $N_8 = \partial_8$. They respectively give

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2b_2b_3a_4\underline{b_4b_5}ba_6b_6b_7,$$

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2b_2b_3a_4\underline{b_4bb_5}a_6b_6b_7,$$

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2b_2b_3a_4\underline{bb_4b_5}a_6b_6b_7,$$

which are equivalent to each other. From the first,

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2b_2b_3a_4b_4b_5\underline{ba_6}b_6b_7$$

$$\sim a_1b_1a_2b_2b_3a_4b_4b_5a_6b_6a_7b_7,$$

which is the same as the one in Case 1 right before applying the rotation.

Case 3: Capping δ_2 or δ_3 of $N_8 = \partial_8$. They give the equivalent relations

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2\underline{b_2}ba_3b_3b_4b_5a_6b_6b_7,$$

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2\underline{bb_2}a_3b_3b_4b_5a_6b_6b_7.$$

From the first,

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1a_2b_2\underline{ba_3b_3}b_4b_5a_6b_6b_7$$

$$\sim a_1b_1a_2b_2a_3b_3a_4b_4b_5a_6b_6b_7.$$

By the counterclockwise rotation by $2\pi/7$ we can shift the indices by -1, which results in $N_7 = \partial_7$ up to a permutation.

Case 4: Capping δ_1 of $N_8 = \partial_8$. This yields

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = \underline{a_1ba_1}b_1b_2a_3b_3b_4b_5a_6b_6b_7$$

$$\sim \underline{ba_1\underline{bb_1}b_2}a_3b_3b_4b_5a_6\underline{b_6}b_7$$

$$\sim a_1b_1b_2\underline{ba_3b_3}b_4\underline{b_5}a_6bb_6b_7$$

$$\sim a_1b_1b_2a_3b_3a_4b_4a_5b_5a_6b_6b_7.$$

Shifting the indices by -3 (or $+4$) by rotation we see that this is equivalent to $N_7 = \partial_7$.

Case 5: Capping δ_8 or δ_7 of $S_8 = \partial_8$. They yield the equivalent relations

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1b_2a_3b_3b_4a_5b_5b_6a_7\underline{b_7b},$$

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = a_1b_1b_2a_3b_3b_4a_5b_5b_6a_7\underline{bb_7}.$$

From the first,

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = \underline{ba_1b_1}b_2a_3b_3b_4a_5b_5b_6a_7b_7$$

$$\sim a_1b_1a_2b_2a_3b_3b_4a_5b_5b_6a_7b_7,$$

which is exactly the expression N_7.

Case 6: The other boundary components $\delta_1, \delta_2, \cdots, \delta_6$ of $S_8 = \partial_8$. Observe that the relation $S_8 = \partial_8$ is symmetric with respect to the rotation by $2\pi/4$. Therefore, in the similar way as Case 3 in Section 5.2.1, we can reduce to the cases of capping δ_8 or δ_7.

Note that from the argument so far we deduce that the blow-up of any two base points of f_n and the blow-up of any one base point of f_s are isomorphic.

5.2.3. Three-time blow-up of f_n and the 6-holed torus relation $N_6 = \partial_6$

We cap each one of the boundary components of $N_7 = \partial_7$.

Case 1: Capping δ_7. We get

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 = a_1b_1a_2b_2a_3b_3b_4a_5b_5\underline{b_6a_1b}$$

$$\sim a_1b_1a_2b_2a_3b_3b_4a_5b_5\underline{a_6b_6a_1}$$

$$\sim a_1b_1a_2b_2a_3b_3b_4a_5\underline{b_5}ba_6b_6$$

$$\sim a_1b_1a_2b_2a_3b_3\underline{b_4a_5}bb_5a_6b_6$$

$$\sim a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5a_6b_6.$$

Thus, we obtain the following 6-holed torus relation $N_6 = \partial_6$:

$$a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5a_6b_6 = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6. \tag{5.8}$$

Case 2: Capping δ_6 or δ_5. They give the equivalent relations

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 = a_1b_1a_2b_2a_3b_3b_4a_5\underline{b_5}ba_6b_6,$$

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 = a_1b_1a_2b_2a_3b_3b_4a_5\underline{bb_5}a_6b_6.$$

From the second,

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 = a_1b_1a_2b_2a_3b_3\underline{b_4a_5}bb_5a_6b_6$$

$$\sim a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5a_6b_6 = N_6.$$

Case 3: Capping δ_4 or δ_3. They give the equivalent relations

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 = a_1b_1a_2b_2a_3\underline{b_3}ba_4b_4b_5a_6b_6,$$

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 = a_1b_1a_2b_2a_3\underline{bb_3}a_4b_4b_5a_6b_6.$$

From the first,

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 = a_1b_1a_2b_2a_3b_3\underline{ba_4b_4}b_5a_6b_6,$$

$$\sim a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5a_6b_6 = N_6.$$

Case 4: Capping δ_2. We get

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 = a_1b_1\underline{a_2ba_2}b_2b_3a_4b_4b_5a_6b_6$$

$$\sim a_1b_1ba_2\underline{bb_2}b_3\underline{a_4b_4}b_5a_6b_6$$

$$\sim a_1b_1\underline{ba_2b_2}b_3a_4b_4a_5b_5a_6b_6$$

$$\sim a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5a_6b_6 = N_6.$$

Case 5: Capping δ_1. We get

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 = \underline{a_1ba_1}b_1a_2b_2b_3a_4b_4b_5a_6b_6$$

$$\sim ba_1bb_1a_2b_2b_3a_4b_4b_5a_6b_6$$

$$\sim a_1b_1\underline{ba_2b_2}b_3a_4b_4\underline{b_5a_6}bb_6$$

$$\sim a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5a_6b_6 = N_6.$$

5.2.4. Four-time blow-up of f_n and the 5-holed torus relation $N_5 = \partial_5$

We cap each one of the boundary components of $N_6 = \partial_6$.
Case 1: Capping δ_6. We get

$$\delta_1\delta_2\delta_3\delta_4\delta_5 = a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5a_1b$$

$$\sim a_1\underline{ba_1b_1}a_2b_2a_3b_3a_4b_4a_5b_5$$

$$\sim a_1a_1b_1a_2a_2b_2a_3b_3a_4b_4a_5b_5.$$

We denote the resulting 5-holed torus relation by $N_5 = \partial_5$:

$$a_1a_1b_1a_2a_2b_2a_3b_3a_4b_4a_5b_5 = \delta_1\delta_2\delta_3\delta_4\delta_5. \tag{5.9}$$

Case 2: The other boundary components $\delta_1, \delta_2, \cdots, \delta_5$. Observe that the relation $N_6 = \partial_6$ is symmetric with respect to the rotation by $2\pi/6$. Therefore, we can reduce all the other cases to Case 1.

5.2.5. Five-time blow-up of f_n and the 4-holed torus relation $N_4 = \partial_4$

We cap each one of the boundary components of $N_5 = \partial_5$.
Case 1: Capping δ_5. We get

$$\delta_1\delta_2\delta_3\delta_4 = a_1a_1b_1a_2a_2b_2a_3b_3a_4b_4a_1b$$

$$\sim a_1ba_1a_1b_1a_2a_2b_2a_3b_3a_4b_4$$

$$\sim \underline{b}a_1ba_1b_1a_2a_2b_2a_3\underline{b_3a_4b_4}$$

$$\sim \underline{a_1}ba_1b_1a_2a_2b_2a_3a_3b_3a_4\underline{b_4}$$

$$\sim a_1a_1b_1a_2a_2b_2a_3a_3b_3a_4a_4b_4.$$

We take the last expression as the 4-holed torus relation $N_4 = \partial_4$:

$$a_1 a_1 b_1 a_2 a_2 b_2 a_3 a_3 b_3 a_4 a_4 b_4 = \delta_1 \delta_2 \delta_3 \delta_4. \tag{5.10}$$

Case 2: Capping δ_4. We get

$$\delta_1 \delta_2 \delta_3 \delta_4 = a_1 a_1 b_1 a_2 a_2 b_2 a_3 \underline{b_3 a_4} b a_4 b_4$$

$$\sim a_1 a_1 b_1 a_2 a_2 b_2 a_3 a_3 b_3 a_4 a_4 b_4 = N_4.$$

Case 3: Capping δ_3. We get

$$\delta_1 \delta_2 \delta_3 \delta_4 = a_1 a_1 b_1 a_2 a_2 b_2 a_3 \underline{b a_3} b_3 a_4 b_4$$

$$\sim a_1 a_1 b_1 a_2 a_2 b_2 a_3 a_3 b_3 a_4 a_4 b_4 = N_4.$$

Case 4: Capping δ_2. We get

$$\delta_1 \delta_2 \delta_3 \delta_4 = a_1 a_1 b_1 a_2 \underline{a_2 b a_2} b_2 a_3 b_3 a_4 b_4$$

$$\sim a_1 a_1 b_1 a_2 b a_2 \underline{b b_2 a_3 b_3} a_4 b_4$$

$$\sim a_1 a_1 b_1 a_2 \underline{b a_2 b_2} a_3 b_3 a_4 a_4 b_4$$

$$\sim a_1 a_1 b_1 a_2 a_2 b_2 a_3 a_3 b_3 a_4 a_4 b_4 = N_4.$$

Case 5: Capping δ_1. We get

$$\delta_1 \delta_2 \delta_3 \delta_4 = a_1 \underline{a_1 b a_1} a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4$$

$$\sim a_1 b \underline{a_1 b a_1} b_1 a_2 b_2 a_3 b_3 a_4 b_4$$

$$\sim \underline{a_1 b b} a_1 \underline{b b_1 a_2 b_2} a_3 b_3 a_4 \underline{b_4}$$

$$\sim a_1 \underline{b a_1 b_1} a_2 b_2 a_3 a_3 b_3 a_4 a_4 b_4$$

$$\sim a_1 a_1 b_1 a_2 a_2 b_2 a_3 a_3 b_3 a_4 a_4 b_4 = N_4.$$

Remark 5.4 *The 4-holed torus relation $N_4 = \partial_4$ has a different but equally symmetric expression, which is given in [14]. We can relate the two relations as follows.*

$$\delta_1 \delta_2 \delta_3 \delta_4 = N_4 = a_1 a_1 b_1 a_2 a_2 b_2 a_3 a_3 b_3 a_4 a_4 b_4$$

$$\sim \underline{a_1 b_1 a_2}\ \underline{a_2 b_2 a_3}\ \underline{a_3 b_3 a_4}\ \underline{a_4 b_4 a_1}$$

$$\sim \underline{b a_1 b_1}\ \underline{b a_2 b_2}\ \underline{b a_3 b_3}\ \underline{b a_4 b_4}$$

$$\sim a_2 b a_1 a_3 b a_2 a_4 b a_3 a_1 b a_4$$

$$\sim (a_1 a_3 b a_2 a_4 b)^2.$$

The last expression is Korkmaz-Ozbagci's 4-holed torus relation.

5.2.6. Six-time blow-up of f_n and the 3-holed torus relation $N_3 = \partial_3$

We need to cap each one of the boundary components of $N_4 = \partial_4$. However, noticing that $N_4 = \partial_4$ is symmetric with respect to the rotation by $2\pi/4$, it is clear that any capping gives an equivalent 3-holed torus relation.

For reference, we give a symmetric expression. By capping δ_4, we get

$$\delta_1\delta_2\delta_3 = a_1a_1b_1a_2a_2b_2a_3a_3b_3a_1a_1b$$

$$\sim a_1\underline{a_1ba_1}a_1b_1a_2a_2b_2a_3a_3b_3$$

$$\sim \underline{a_1ba_1}\,\underline{ba_1b_1}a_2a_2b_2a_3a_3b_3\underline{b_3}$$

$$\sim a_1a_1a_1b_1a_2a_2a_2b_2a_3a_3a_3b_3.$$

We take the last expression as the 3-holed torus relation $N_3 = \partial_3$:

$$a_1a_1a_1b_1a_2a_2a_2b_2a_3a_3a_3b_3 = \delta_1\delta_2\delta_3. \tag{5.11}$$

Remark 5.5 *The* 3*-holed torus relation* $N_3 = \partial_3$ *also has an alternative nice expression, which is called the star relation [7]. Here we show the equivalence explicitly.*

$$\delta_1\delta_2\delta_3 = N_3 = a_1a_1a_1b_1a_2a_2a_2b_2a_3a_3a_3b_3$$

$$\sim a_1\underline{a_1b_1a_2}a_2\underline{a_2b_2a_3}a_3\underline{a_3b_3a_1}$$

$$\sim a_1a_2ba_1a_2a_3ba_2a_3a_1ba_3$$

$$\sim (a_1a_2a_3b)^3.$$

The last expression gives nothing but the star relation.

5.2.7. Seven-time blow-up of f_n and the 2-holed torus relation $N_2 = \partial_2$

Since $N_3 = \partial_3$ is symmetric with respect to the rotation by $2\pi/3$ it is obvious that capping any one boundary component of $N_3 = \partial_3$ yields an equivalent 2-holed torus relation.

Capping δ_3 of $N_3 = \partial_3$ gives

$$\delta_1\delta_2 = a_1a_1\underline{a_1b_1a_2}a_2\underline{a_2b_2a_1}a_1a_1b$$

$$\sim \underline{a_1}a_1b_1a_2ba_2a_1ba_2a_1\underline{a_1b}$$

$$\sim \underline{ba_1b_1}a_2ba_2a_1ba_2a_1ba_1$$

$$\sim a_2ba_1a_2ba_2a_1ba_2a_1ba_1$$

$$\sim (a_1ba_2)^4.$$

Thus, we get the 2-holed torus relation $N_2 = \partial_2$:

$$(a_1ba_2)^4 = \delta_1\delta_2, \tag{5.12}$$

which is also known as the 3-chain relation.

5.2.8. Eight-time blow-up of f_n and the 1-holed torus relation $N_1 = \partial_1$

Capping either δ_2 or δ_1 of $N_2 = \partial_2$ gives

$$\delta_1 = (a_1ba_1)^4$$

$$= a_1ba_1\underline{a_1ba_1}a_1ba_1\underline{a_1ba_1}$$

$$\sim a_1ba_1ba_1ba_1ba_1ba_1b = (a_1b)^6.$$

Writing $a = a_1$, we get the 1-holed torus relation $N_1 = \partial_1$:

$$(ab)^6 = \delta_1, \tag{5.13}$$

which is also known as the 2-chain relation.

Remark 5.6 *Our nonspin k-holed torus relations $N_k = \partial_k$ are all Hurwitz equivalent to Korkmaz-Ozbagci's k-holed torus relations. The latter were constructed in the way that the 9-holed torus relation is a lift of the smaller k-holed torus relations; hence, conversely, they can be obtained by capping boundary components of the 9-holed torus relation.*

6. Final remarks

In this section, we discuss two additional topics related to the genus-1 holomorphic Lefschetz pencils and the k-holed torus relations. We first present handle diagrams associated with the genus-1 pencils using the simplified k-holed torus relations. We then summarize the contact topological aspect of the k-holed torus relations.

6.1. Handle diagrams

As we have shown, the relations $N_9 = \partial_9$ and $S_8 = \partial_8$ correspond to the minimal holomorphic Lefschetz pencils f_n on $\mathbb{P}^2$ and f_s on $\mathbb{P}^1 \times \mathbb{P}^1$, respectively (while the others $N_k = \partial_k$ ($k < 9$) are just blow-ups of them). In Figure 29, we draw two handle diagrams of the elliptic Lefschetz fibration $E(1) = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2 \to \mathbb{P}^1$ and locate the (-1)-sections corresponding to $N_9 = \partial_9$ and $S_8 = \partial_8$. In each of the diagrams (with $k = 9$ or 8), we
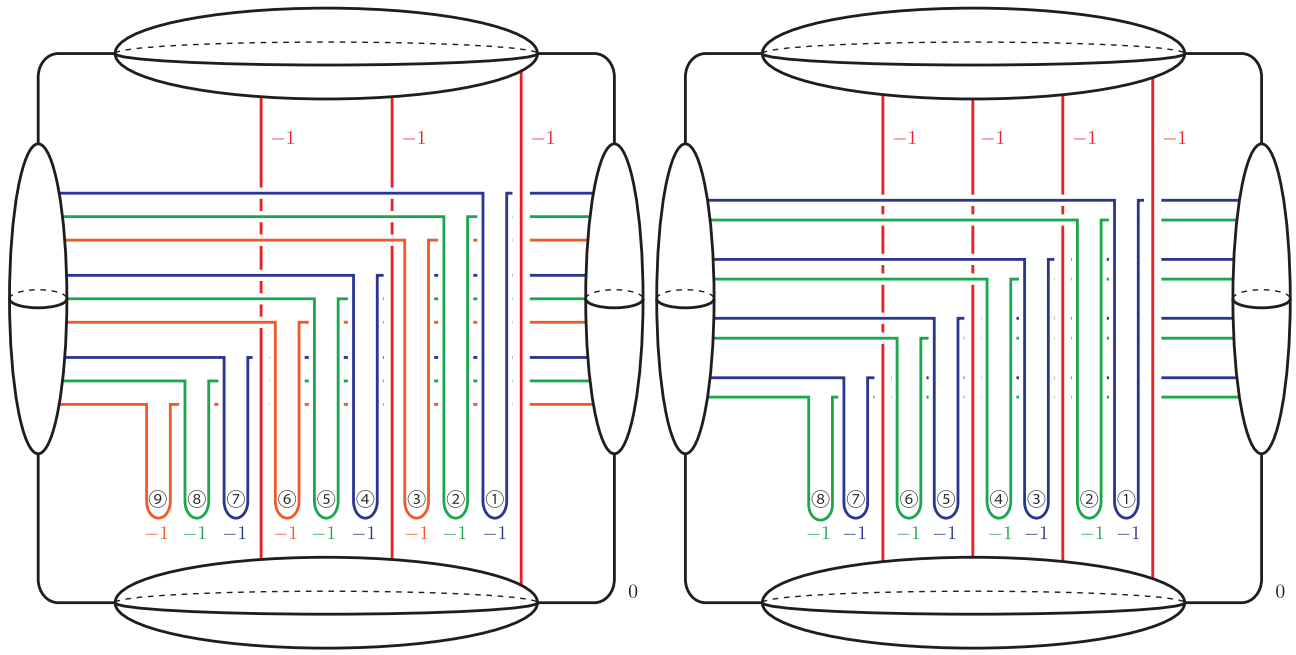
(a) The nine (-1)-sections corresponding to $N_9 = \partial_9$. (b) The eight (-1)-sections corresponding to $S_8 = \partial_8$.

Figure 29. Handle diagrams of the elliptic Lefschetz fibration $E(1) = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2 \to \mathbb{P}^1$ with configurations of (-1)-sections. The circled numbers indicate the location of the intersection points of the regular fiber and sections.

first construct the trivial T^2-fibration over D^2 by attaching two 1-handles to a 0-handle and then attaching a 2-handle with framing 0. We fix k points on the fiber that correspond to the boundary components of the k-holed torus. On this fiber with k fixed points, we attach twelve more 2-handles with framing -1 along the vanishing cycles of the respective monodromy factorization, which describes a Lefschetz fibration over a disk D_+^2. If we ignore the fixed points we can attach 2-, 3-, and 4-handles to build up the elliptic fibration $E(1)$. The meridians around the 2-handle with framing 0 that pass through the k fixed points are sections of the fibration restricted over $S^1 = \partial D_+^2$. The sections can be extended to sections of the entire fibration over S^2. Blowing-down those sections must yield the 4-manifolds $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, respectively, and the exceptional spheres become the base points of the Lefschetz pencils f_n and f_s.

6.2. Stein fillings and k-holed torus relations

Positive Dehn twist factorizations (with homologically nontrivial curves) of elements in mapping class groups of holed surfaces also provide positive allowable Lefschetz fibrations over D^2, which in turn represent Stein fillings of contact 3-manifolds. There is another elegant interpretation of the k-holed torus relations in this point of view. In this subsection, we summarize the role of the k-holed torus relations in contact topology as the intersection of various fields of study. More detailed discussion and relevant references can be found in [25].

A simple elliptic singularity of degree k is an isolated singularity such that the exceptional divisor of its minimal resolution consists of a nonsingular elliptic curve of self-intersection number $-k$. Let (M_k, ξ_k) be the link of the simple elliptic singularity of degree k with the contact structure given by the maximal complex tangencies. The contact 3-manifold (M_k, ξ_k) can also be viewed as the unit circle bundle of a complex hermitian line bundle $L_k \to T^2$ over a 2-torus with $c_1(L_k) = -k$ where the contact structure is the horizontal distribution given by a connection on the line bundle.

The contact 3-manifold (M_k, ξ_k) is Stein fillable and we present some natural examples of a Stein filling of it as follows.

- The minimal resolution of the simple elliptic singularity of degree k provides a Stein filling of (M_k, ξ_k). This filling can also be described as the unit disk bundle of the line bundle $L_k \to T^2$. Let D_k denote this (equivalent) Stein filling.

- For $0 < k \le 9$, the singularity of degree k has a smoothing that is simply connected. Let A_k denote this Stein filling.

- When $k = 8$, the singularity has another smoothing that is not simply connected. Let B_8 denote this Stein filling.

In fact, the above examples exhaust all the minimal Stein fillings of (M_k, ξ_k) up to diffeomorphism as the strong symplectic fillings (underlying structures of Stein fillings) of the link of the simple elliptic singularity are classified by Ohta and Ono [24].

Theorem 6.1 ([24]) *Suppose that X is a minimal Stein filling of (M_k, ξ_k).*

1. *If $k \ge 10$ then X is diffeomorphic to D_k.*

2. *If $k \le 9$ but $k \ne 8$ then X is diffeomorphic to either D_k or A_k.*

3. If $k = 8$ then X is diffeomorphic to either one of D_8, A_8, or B_8.

In [24], the diffeomorphism types of the smoothings are also described as follows;

- Consider the blow-up of $\mathbb{P}^2$ at $(9-k)$-points on a nonsingular cubic curve. The filling A_k is diffeomorphic to the complement of the proper transform of the cubic curve in $\mathbb{P}^2 \sharp (9-k)\overline{\mathbb{P}}^2$.

- The filling B_8 is diffeomorphic to the complement of a holomorphically embedded 2-torus in $\mathbb{P}^1 \times \mathbb{P}^1$ which is linearly equivalent to $2F_1 + 2F_2$, where F_i is a fiber of the projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the i-th component.

Therefore, we can view the smoothings of the simple elliptic singularities as the complements of a regular fiber in the genus-1 pencils.

- A_k is diffeomorphic to the complement of a regular fiber of $f_n \sharp (9-k)\overline{\mathbb{P}}^2$ in $\mathbb{P}^2 \sharp (9-k)\overline{\mathbb{P}}^2$.

- B_8 is diffeomorphic to the complement of a regular fiber of f_s in $\mathbb{P}^1 \times \mathbb{P}^1$.

Now we turn to the description of (M_k, ξ_k) and its Stein fillings in terms of an open book and positive allowable Lefschetz fibrations. The contact 3-manifold (M_k, ξ_k) indeed has the open book decomposition whose page is a k-holed torus and monodromy is the boundary multitwist $\psi_k = t_{\delta_1} \cdots t_{\delta_k}$ in $\mathrm{MCG}(\Sigma_1^k)$. Then, the positive allowable Lefschetz fibration over D^2 associated with the obvious Dehn twist factorization $\psi_k = t_{\delta_1} \cdots t_{\delta_k}$ prescribes a Stein filling of (M_k, ξ_k). This filling is nothing but the disk bundle D_k. If the open book monodromy ψ_k has another factorization (i.e. a k-holed torus relation) it also gives a Stein filling of (M_k, ξ_k). The relationship between those Stein fillings and the positive allowable Lefschetz fibrations associated with the k-holed torus relations is observed in [25] as follows (see also Table 2).

Table 2. Classification of the Stein fillings of the link of the simple elliptic singularity of degree k and corresponding genus-1 positive allowable Lefschetz fibrations.

Degree of singularity	Monodromy factorization	Stein filling	
$k > 1$	$\psi_k = t_{\delta_1} \cdots t_{\delta_k}$	D_k	Minimal resolution
9	$\psi_9 = t_{a_1} t_{b_1} t_{b_2} t_{b_3} t_{a_4} t_{b_4} t_{b_5} t_{b_6} t_{a_7} t_{b_7} t_{b_8} t_{b_9}$	A_9	Smoothing ($\pi_1 = 1$)
8	$\psi_8 = t_{a_1} t_{b_1} t_{b_2} t_{a_3} t_{b_3} t_{b_4} t_{a_5} t_{b_5} t_{b_6} t_{a_7} t_{b_7} t_{b_8}$	B_8	Smoothing ($\pi_1 \neq 1$)
8	$\psi_8 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{b_3} t_{a_4} t_{b_4} t_{b_5} t_{b_6} t_{a_7} t_{b_7} t_{b_8}$	A_8	Smoothing ($\pi_1 = 1$)
7	$\psi_7 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{b_4} t_{a_5} t_{b_5} t_{b_6} t_{a_7} t_{b_7}$	A_7	Smoothing ($\pi_1 = 1$)
6	$\psi_6 = t_{a_1} t_{b_1} t_{a_2} t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4} t_{a_5} t_{b_5} t_{a_6} t_{b_6}$	A_6	Smoothing ($\pi_1 = 1$)
5	$\psi_5 = t_{a_1}^2 t_{b_1} t_{a_2}^2 t_{b_2} t_{a_3} t_{b_3} t_{a_4} t_{b_4} t_{a_5} t_{b_5}$	A_5	Smoothing ($\pi_1 = 1$)
4	$\psi_4 = t_{a_1}^2 t_{b_1} t_{a_2}^2 t_{b_2} t_{a_3}^2 t_{b_3} t_{a_4}^2 t_{b_4}$ $\sim (t_{a_1} t_{a_3} t_b t_{a_2} t_{a_4} t_b)^2$	A_4	Smoothing ($\pi_1 = 1$)
3	$\psi_3 = t_{a_1}^3 t_{b_1} t_{a_2}^3 t_{b_2} t_{a_3}^3 t_{b_3}$ $\sim (t_{a_1} t_{a_2} t_{a_3} t_b)^3$	A_3	Smoothing ($\pi_1 = 1$)
2	$\psi_2 = (t_{a_1} t_b t_{a_2})^4$	A_2	Smoothing ($\pi_1 = 1$)
1	$\psi_1 = (t_{a_1} t_b)^6$	A_1	Smoothing ($\pi_1 = 1$)

Theorem 6.2 ([25]) *The Stein fillings of the link of the simple elliptic singularity of degree k are realized as genus-1 positive allowable Lefschetz fibrations except for D_1. More explicitly,*

1. *For $1 < k$, the positive allowable Lefschetz fibration associated with monodromy factorization $\psi_k = t_{\delta_1} \cdots t_{\delta_k}$ prescribes the Stein filling D_k.*

2. *For $0 < k \leq 9$, the positive allowable Lefschetz fibration associated with monodromy factorization $\psi_k = N_k$ prescribes the Stein filling A_k.*

3. *For $k = 8$, the positive allowable Lefschetz fibration associated with monodromy factorization $\psi_8 = S_8$ prescribes the Stein filling B_8.*

Note that the factorization $\psi_1 = t_{\delta_1}$ does not provide an allowable Lefschetz fibration since the vanishing cycle δ_1 is homologically trivial on the fiber. This is why the filling D_1 is excluded in Theorem 6.2. The fact that (M_k, ξ_k) has a unique Stein filling for $k \geq 10$ reflects that there is no k-holed torus relation for $k \geq 10$. This can also be seen from the fact that $E(1) = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$ can admit no more than nine (-1)-sections. The distinction between N_8 and S_8 up to Hurwitz equivalence is once again verified in this Theorem. This can be independently confirmed by computing the first homology groups of the positive allowable Lefschetz fibrations associated with N_8 and S_8 as demonstrated in [25] (Ozbagci used Korkmaz-Ozbagci's and Tanaka's 8-holed torus relations but the computation would be more straightforward if one uses our simplified expressions).

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