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Extensions of the matrix-valued $q-$Sturm–Liouville operators

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Abstract: In this paper, we investigate the matrix-valued $q-$Sturm–Liouville problems. We establish an existence and uniqueness result. Later, we introduce the corresponding maximal and minimal operators for this system. Moreover, we give a criterion under which these operators are self-adjoint. Finally, we characterize extensions (maximal dissipative, maximal accumulative, and self-adjoint) of the minimal symmetric operator.

Key words: Boundary value space, boundary condition, dissipative extensions, accretive extensions, self-adjoint extensions

1. Introduction

This paper deals with the extension theory of symmetric operators. This topic is one of the main research areas of operator theory. This theory was developed originally by J. Von Neumann [44]. In [22], Calkin gave the description of self-adjoint extensions of a symmetric operator in terms of abstract boundary conditions. Extensions of a symmetric operator with aid of linear relations were given by Rofe-Beketov [39]. Later, in [21, 32], the notion of a space of boundary values was introduced. The readers may find some papers that are related to extension theory in [28, 38, 47]. In [34], the authors obtained a description of extensions of a second-order symmetric operator. In [27], the author obtained a description of self-adjoint extensions of Sturm–Liouville operators with an operator potential. In the case when the deficiency indices take indeterminate values, a description of extensions of differential operators was given in [6, 35–37].

On the other hand, the study of matrix-valued Sturm–Liouville equations has become an important area of research because such equations arise in a variety of physical problems (for example, see [15–17, 20, 23]). Although the matrix Sturm–Liouville equations is more difficult than the scalar Sturm–Liouville equations the matrix-valued Sturm–Liouville equations have intensively been investigated during the last two decades (see [9, 13, 18, 19, 24, 45, 46] and references therein).

Recently, $q-$difference equations have attracted tremendous interest since they have a lot of applications in sciences, e.g., quantum theory, orthogonal polynomials, hypergeometric functions (see [25] for more details). Specially, $q-$Sturm–Liouville problems were studied in [1–5, 8, 10–12, 14, 26, 31, 41–43]. The goal of our paper is to study the matrix-valued $q-$Sturm–Liouville operators. In the analysis that follows, we will largely follow a development of the theory in [6, 29, 33, 40, 47].

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This paper is organized as follows. In Section 2, fundamental concepts of quantum analysis are given. In Section 3, an existence and uniqueness theorem is proved for the matrix-valued $q-$Sturm–Liouville equation. Later, the corresponding maximal and minimal operators for this equation are constructed. In Section 4, a criterion under which the matrix-valued $q-$Sturm–Liouville operators are self-adjoint is given. In Section 5, maximal dissipative, maximal accumulative, and self-adjoint extensions of the minimal operators are studied.

2. Preliminaries

In this section, we recall some basic concepts and useful results of quantum calculus. We refer to [7, 25, 30] and some references cited therein. Let q be a positive number with $0 < q < 1$. A set $A \subset \mathbb{R}$ is called $q-$geometric if for every $x \in A$, $qx \in A$. Let y be a complex-valued function on A. Then, the $q-$*difference operator* D_q *is* defined by

$$D_q y(x) = [y(qx) - y(x)](qx - x)^{-1} \text{ for all } x \in A.$$

The $q-$derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} [y(xq^n) - y(0)] x^{-1} q^{-n} \quad (x \in A),$$

if the limit exists and does not depend on x (see [7]).

The *Jackson* $q-$*integration* is given by

$$\int_0^x f(t)\, d_q t = (1-q)\, x \sum_{n=0}^{\infty} f(q^n x)\, q^n \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t \quad (a, b \in A).$$

A function f which is defined on A, $0 \in A$, is said to be $q-$regular at zero if

$$\lim_{n\to\infty} f(xq^n) = f(0),$$

for every $x \in A$. Through the remainder of the paper, we deal only with $q-$regular functions at zero.

3. The matrix-valued $q-$Sturm–Liouville equation

Let us consider the following matrix-valued $q-$Sturm–Liouville equation

$$\Upsilon(y) := -\frac{1}{q} D_{q^{-1}} [P(x) D_q y(x)] + Q(x)\, y(x) = \lambda V(x)\, y(x), \; x \in (0, a), \tag{3.1}$$

where $P(x), V(x)$, and $Q(x)$ are $n \times n$ complex Hermitian matrix-valued functions, defined on $[0, q^{-1}a]$, continuous at zero, $q-$integrable over $[0, a]$, $detP(x) \neq 0$, $P^{-1}(x)$ is $q-$integrable over $[0, a]$, $V(x)$ is positive, and λ is a complex parameter.

Now, we can transform equation (3.1) into the Hamiltonian system. Let

$$\mathcal{Y}(x)=\begin{pmatrix} y(x) \\ P(x)D_q y(x)\end{pmatrix}, \quad \mathcal{Y}^{[q]}(x)=\begin{pmatrix} D_q y(x) \\ \frac{1}{q}D_{q^{-1}}\left(P(x)D_q y(x)\right)\end{pmatrix},$$

$$W_1(x)=\begin{pmatrix} V(x) & O_n \\ O_n & O_n\end{pmatrix}, \quad J=\begin{pmatrix} O_n & -I_n \\ I_n & O_n\end{pmatrix},$$

$$W_2(x)=\begin{pmatrix} -Q(x) & O_n \\ O_n & P^{-1}(x)\end{pmatrix},$$

where I_n is a unit matrix and O_n is a zero matrix. Then, the equation (3.1) becomes

$$\tau(\mathcal{Y}):=J\mathcal{Y}^{[q]}(x)-W_2(x)\mathcal{Y}(x)=\lambda W_1(x)\mathcal{Y}(x), \ x\in(0,a). \tag{3.2}$$

Let $L^2_{q,W_1}[(0,a);E]=\left\{\mathcal{Y}:\int_0^a (W_1\mathcal{Y},\mathcal{Y})_E\, d_qx=\int_0^a \mathcal{Y}^*W_1\mathcal{Y}d_qx<\infty\right\}$ with the inner product

$$(\mathcal{X},\mathcal{Y}):=\int_0^a (W_1\mathcal{X},\mathcal{Y})_E\, d_qx=\int_0^a \mathcal{Y}^*W_1\mathcal{Y}d_qx,$$

where $E=\mathbb{C}^{2n}$ is the $2n$-dimensional Euclidean space. For any function $\mathcal{Y}\in L^2_{q,W_1}[(0,a);E]$, $\mathcal{Y}(0)$ can be defined as

$$\mathcal{Y}(0):=\lim_{n\to\infty}\mathcal{Y}(q^n). \tag{3.3}$$

Since $\mathcal{Y}$ is $q-$regular at zero, the limit in (3.3) exists and is finite.

Let $C^2_q[(0,a);E]=\{\mathcal{Y}: \ \mathcal{Y}$ and $D_q\mathcal{Y}$ are $q-$regular at zero$\}$. It is clear that $C^2_q[(0,a);E]\subset L^2_{q,W_1}[(0,a);E]$.

Theorem 3.1 *For $K\in\mathbb{C}^{2n}$, the equation (3.2) with initial condition*

$$\mathcal{Y}(0,\lambda)=K \qquad (\lambda\in\mathbb{C}) \tag{3.4}$$

has a unique solution in $C^2_q[(0,a);E]$.

Proof An integration yields

$$\mathcal{Y}(x,\lambda)=K-q\int_0^x J\left[\lambda W_1(qt,\lambda)+W_2(qt,\lambda)\right]\mathcal{Y}(qt,\lambda)\,d_qt, \tag{3.5}$$

where $x\in(0,a)$. Conversely, every solution of equation (3.5) is also a solution of the equation (3.2).

Let us define the sequence $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ of successive approximations by

$$\mathcal{Y}_0(x,\lambda)=K,$$

$$\mathcal{Y}_{n+1}(x,\lambda)=$$

$$K-q\int_0^x J\left[\lambda W_1(qt,\lambda)+W_2(qt,\lambda)\right]\mathcal{Y}_n(qt,\lambda)\,d_qt, \tag{3.6}$$

where $n = 0, 1, 2, ...,$ and $x \in (0, a)$. Then, we shall prove that the sequence $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ converges to a function y uniformly on each compact subset of $(0, a)$. There exist positive numbers $\vartheta(\lambda)$ and $\varsigma(\lambda)$ such that

$$\|J[\lambda W_1(x,\lambda) + W_2(x,\lambda)]\| \leq \vartheta(\lambda),$$

$$\|\mathcal{Y}_1(x,\lambda)\| \leq \varsigma(\lambda), \; x \in (0, a).$$

Using mathematical induction, we get

$$\|\mathcal{Y}_{n+1}(x,\lambda) - \mathcal{Y}_n(x,\lambda)\| \leq \vartheta(\lambda)\, q^{\frac{n(n+1)}{2}} \frac{(\varsigma(\lambda)\, x\,(1-q))^n}{(q;q)_n} \quad (n \in \mathbb{N}).$$

An application of the Weierstrass M-test implies that the sequence $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ converges to a function y uniformly on each compact subset of $(0, a)$. One can prove that $\mathcal{Y}$ and $D_q\mathcal{Y}$ are continuous on $[0, a]$. It is clear that the function $\mathcal{Y}$ satisfies the condition (3.4).

Now, we show that the equation (3.2) has a unique solution, assume $\mathcal{Z}$ is another one. Then $\mathcal{Z}$ is continuous. Therefore, there exists a positive number $\mathcal{M}$ such that $\|\mathcal{Y} - \mathcal{Z}\| \leq \mathcal{M}$. Proceeding as above we conclude that

$$\|\mathcal{Y}(x,\lambda) - \mathcal{Z}(x,\lambda)\| \leq \mathcal{M}\vartheta(\lambda)\, q^{\frac{n(n+1)}{2}} \frac{(x\,(1-q))^n}{(q;q)_n} \qquad (n \in \mathbb{N})$$

Since

$$\lim_{n\to\infty} \mathcal{M}\vartheta(\lambda)\, q^{\frac{n(n+1)}{2}} \frac{(x\,(1-q))^n}{(q;q)_n} = 0,$$

we arrive at $\mathcal{Y} = \mathcal{Z}$ on $[0, a]$. □

Now, we will give the definition of maximal and minimal operators for the matrix-valued $q-$Sturm–Liouville equations.

Denote

$$\mathcal{D}_{\max} :=$$

$$\left\{ \begin{array}{c} \mathcal{Y} \in L^2_{q,W_1}[(0,a);E] : J\mathcal{Y}^{[q]}(x) - W_2(x)\,\mathcal{Y}(x) = W_1 F \text{ exists in } (0,a) \\ \text{and } F \in L^2_{q,W_1}[(0,a);E] \end{array} \right\},$$

$$\mathcal{D}_{\min} :=$$

$$\left\{ \begin{array}{c} \mathcal{Y} \in L^2_{q,W_1}[(0,a);E] : \mathcal{Y} \text{ and } PD_q\mathcal{Y} \text{ are } q-\text{regular at zero,} \\ J\mathcal{Y}^{[q]}(x) - W_2(x)\,\mathcal{Y}(x) = W_1 F \text{ exists in } (0,a), \\ F \in L^2_{q,W_1}[(0,a);E], \\ \text{and } \widehat{\mathcal{Y}}(0) = \widehat{\mathcal{Y}}(a) = 0. \end{array} \right\}, \tag{3.7}$$

where $\widehat{\mathcal{Y}}(x) = \begin{pmatrix} y(x) \\ P(x)\, D_{q^{-1}} y(x) \end{pmatrix}$.

The operator $T_{\min}$ defined by

$$T_{\min} : \mathcal{D}_{\min} \to L^2_{q,W_1}[(0,a);E],$$

$$\mathcal{Y} \to T_{\min}\mathcal{Y} = \tau(\mathcal{Y}).$$

is called the minimal operator generated by the matrix-valued $q-$Sturm–Liouville equation. Similarly, the operator $T_{\max}$ defined by

$$T_{\max} : \mathcal{D}_{\max} \to L^2_{q,W_1}\left[(0,a);E\right],$$
$$\mathcal{Y} \to T_{\max}\mathcal{Y} = \tau\left(\mathcal{Y}\right).$$

is called the maximal operator for the matrix-valued $q-$Sturm–Liouville equation.

Now, we establish the following Green's formula.

Theorem 3.2 *Let* $\mathcal{Y}, \mathcal{Z} \in \mathcal{D}_{\max}$. *Then we have*

$$\int_0^a \mathcal{Z}^*\left(x\right) J\mathcal{Y}^{[q]}(x) d_q x - \int_0^a \left\{ J\mathcal{Z}^{[q]}(x) \right\}^* \mathcal{Y}\left(x\right) d_q x$$

$$= \widehat{\mathcal{Z}}^*(a) J\widehat{\mathcal{Y}}(a) - \widehat{\mathcal{Z}}^*(0) J\widehat{\mathcal{Y}}(0),$$

where $x \in (0,a)$.

Proof

$$\int_0^a \mathcal{Z}^*\left(x\right) J\mathcal{Y}^{[q]}(x) d_q x - \int_0^a \left\{ J\mathcal{Z}^{[q]}(x) \right\}^* \mathcal{Y}\left(x\right) d_q x$$

$$= \int_0^a \begin{pmatrix} z\left(x\right) \\ P\left(x\right) D_q z\left(x\right) \end{pmatrix}^* \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix} \begin{pmatrix} D_q y\left(x\right) \\ \frac{1}{q} D_{q^{-1}}\left(P\left(x\right) D_q y\left(x\right)\right) \end{pmatrix} d_q x$$

$$- \int_0^a \left\{ \begin{pmatrix} D_q z\left(x\right) \\ \frac{1}{q} D_{q^{-1}}\left(P\left(x\right) D_q z\left(x\right)\right) \end{pmatrix} \right\}^* \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix} \begin{pmatrix} y\left(x\right) \\ P\left(x\right) D_q y\left(x\right) \end{pmatrix} d_q x$$

$$= \int_0^a \left[z^*\left(x\right) \left\{ \frac{1}{q} D_{q^{-1}}\left(P\left(x\right) D_q y\left(x\right)\right) \right\} + \left(P\left(x\right) D_q z\left(x\right)\right)^* D_q y\left(x\right) \right] d_q x$$

$$- \int_0^a \left[\left\{ -\frac{1}{q} D_{q^{-1}}\left(P\left(x\right) D_q z\left(x\right)\right) \right\}^* y\left(x\right) + \left(D_q z\left(x\right)\right)^* P\left(x\right) D_q y\left(x\right) \right] d_q x$$

$$= \int_0^a D_q \left\{ \left[\left(P D_q z\right)\left(q^{-1}x\right) \right]^* y\left(x\right) - z^*(x) \left(P D_q y\right)\left(q^{-1}x\right) \right\} d_q x$$

$$= \widehat{\mathcal{Z}}^*(a) J\widehat{\mathcal{Y}}(a) - \widehat{\mathcal{Z}}^*(0) J\widehat{\mathcal{Y}}(0).$$

$\square$

Theorem 3.3 (Green's formula)

$$(T_{\max}\mathcal{Y},\mathcal{Z})-(\mathcal{Y},T_{\max}\mathcal{Z})=[\mathcal{Y},\mathcal{Z}]_a-[\mathcal{Y},\mathcal{Z}]_0, \tag{3.8}$$

where $[\mathcal{Y},\mathcal{Z}]_x:=\widehat{\mathcal{Z}}^*(a)J\widehat{\mathcal{Y}}(x),\ x\in[0,a].$

Lemma 3.4 *The operator* $T_{\min}$ *is Hermitian.*

Proof For $\mathcal{Y},\mathcal{Z}\in\mathcal{D}_{\min}$, there exist $F,G\in L^2_{q,W_1}[(0,a);E]$ such that $\tau(\mathcal{Y})=W_1F$ and $\tau(\mathcal{Z})=W_1G$. From (3.7) and (3.8), we conclude that

$$(T_{\min}\mathcal{Y},\mathcal{Z})-(\mathcal{Y},T_{\min}\mathcal{Z})=(F,\mathcal{Z})-(\mathcal{Y},G)$$

$$=\int_0^a\left[\mathcal{Z}^*(t)\,W_1F-G^*(t)\,W_1\mathcal{Y}(t)\right]d_qt$$

$$=\int_0^a\left[\mathcal{Z}^*(t)\,\tau(\mathcal{Y})-\tau^*(\mathcal{Z})\,\mathcal{Y}(t)\right]d_qt$$

$$=[\mathcal{Y},\mathcal{Z}]_a-[\mathcal{Y},\mathcal{Z}]_0=0.$$

□

The following lemma has a similar proof to that of Lemma 3.4.

Lemma 3.5 *For all* $\mathcal{Y}\in\mathcal{D}_{\min}$ *and for all* $\mathcal{Z}\in\mathcal{D}_{\max}$, *we have*

$$(T_{\min}\mathcal{Y},\mathcal{Z})=(\mathcal{Y},T_{\max}\mathcal{Z}).$$

Lemma 3.6 *Let the null space and the range of an operator* T *be denoted by* $\mathcal{N}(T)$ *and* $\mathcal{R}(T)$, *respectively. Then we have*

$$\mathcal{R}(T_{\min})=\mathcal{N}(T_{\max})^{\perp},$$

where the superscript $\perp$ *denotes the orthogonal complement of a subspace.*

Proof Given any $\xi\in\mathcal{R}(T_{\min})$, there exists $\mathcal{Y}\in\mathcal{D}_{\min}$ such that $T_{\min}\mathcal{Y}=\xi$. It follows from Lemma 3.5 that

$$(\xi,\mathcal{Z})=(T_{\min}\mathcal{Y},\mathcal{Z})=(\mathcal{Y},T_{\max}\mathcal{Z})=0,$$

for each $\mathcal{Z}\in\mathcal{N}(T_{\max})$.

Now we prove that $\mathcal{N}(T_{\max})^{\perp}\subset\mathcal{R}(T_{\min})$. For any given $\xi\in\mathcal{N}(T_{\max})^{\perp}$ and for all $\mathcal{Z}\in\mathcal{N}(T_{\max})$, we have $(\xi,\mathcal{Z})=0$. Let us consider the following problem:

$$\begin{gathered}J\mathcal{Y}^{[q]}(x)-W_2(x)\,\mathcal{Y}(x)=W_1(x)\,\xi(x),\ x\in(0,a)\\ \mathcal{Y}(0,\lambda)=P(0)\,D_{q^{-1}}\mathcal{Y}(0,\lambda)=0.\end{gathered} \tag{3.9}$$

It follows from Theorem 3.1 that the problem (3.9) has a unique solution on $(0,a)$. Let $\Psi(x)=(\psi_1,\psi_2,...,\psi_n)$ be the fundamental solution of the system

$$J\mathcal{Y}^{[q]}(x)-W_2(x)\mathcal{Y}(x)=0, \Psi(a)=J,\ x\in(0,a).$$

It is clear that $\psi_i\in\mathcal{N}(T_{\max})$ for $1\leq i\leq n$. By Theorem 3.3, for $1\leq i\leq n$, we get

$$\begin{aligned}0=(\xi,\psi_i)&=\int_0^a\psi_i^*(t)W_1(x)\xi(t)d_qt=\int_0^a\psi_i^*(t)\tau(\mathcal{Y})(t)d_qt\\&=\int_0^a\psi_i^*(t)\tau(\mathcal{Y})(t)d_qt-\int_0^a\tau(\psi_i)^*(t)\mathcal{Y}(t)d_qt\\&=[\mathcal{Y},\psi_i]_a-[\mathcal{Y},\psi_i]_0=[\mathcal{Y},\psi_i]_a.\end{aligned}$$

Thus, we have $[\mathcal{Y},\psi_i]_a=\widehat{\Psi}^*(a)J\widehat{\mathcal{Y}}(a)=\widehat{\mathcal{Y}}(a)=0$, i.e., $\xi\in\mathcal{R}(T_{\min})$. □

Theorem 3.7 *The operator $T_{\min}$ is a symmetric operator and the operator $T_{\max}$ is a densely defined operator. Furthermore, $T_{\min}^*=T_{\max}$, where $T_{\min}^*$ denotes the adjoint operator of $T_{\min}$.*

Proof Firstly, we prove that $\mathcal{D}_{\min}^{\perp}=\{0\}$. Assume that $\xi\in\mathcal{D}_{\min}^{\perp}$. Then, for all $\mathcal{Z}\in\mathcal{D}_{\min}$, we have $(\xi,\mathcal{Z})=0$. Set $T_{\min}\mathcal{Z}(x)=\varphi(x)$. Let $y(.)$ be any solution of the system

$$J\mathcal{Y}^{[q]}(x)-W_2(x)\mathcal{Y}(x)=W_1(x)\xi(x),\ x\in(0,a).$$

It follows from Theorem 3.3 that

$$\begin{aligned}(\mathcal{Y},\varphi)-(\xi,\mathcal{Z})&=\int_0^a\varphi^*(t)W_1(t)\mathcal{Y}(t)d_qt-\int_0^a\mathcal{Z}^*(t)W_1(t)\xi(t)d_qt\\&=\int_0^a\tau(\mathcal{Z})^*(t)\mathcal{Y}(t)d_qt-\int_0^a\mathcal{Z}^*(t)\tau(\mathcal{Y})(t)d_qt\\&=-[\mathcal{Y},\mathcal{Z}]_a+[\mathcal{Y},\mathcal{Z}]_0=0,\end{aligned}$$

i.e. $(\mathcal{Y},\varphi)=(\xi,\mathcal{Z})=0$. From Lemma 3.6, we see that $\mathcal{Y}\in\mathcal{R}(T_{\min})=\mathcal{N}(T_{\max})^{\perp}$. Thus, $\xi=0$.

We will denote by $\mathcal{D}_{\min}^*$ the domain of the operator $T_{\min}^*$. Now, we prove that $\mathcal{D}_{\min}^*=\mathcal{D}_{\max}$, and $T_{\min}^*\mathcal{Y}=T_{\max}\mathcal{Y}$ for all $\mathcal{Y}\in\mathcal{D}_{\min}^*$. It follows from Lemma 3.5 that, for any given $\mathcal{Y}\in\mathcal{D}_{\max}$,

$$(\mathcal{Y},T_{\min}\mathcal{Z})=(T_{\max}\mathcal{Y},\mathcal{Z})\text{ for all }\mathcal{Z}\in\mathcal{D}_{\min}.$$

Consequently, the functional $(\mathcal{Y},T_{\min}(.))$ is continuous on $\mathcal{D}_{\min}$ and $\mathcal{Y}\in\mathcal{D}_{\min}^*$, i.e. $\mathcal{D}_{\max}\subset\mathcal{D}_{\min}^*$.

We prove the reverse conclusion, i.e. $\mathcal{D}_{\min}^*\subset\mathcal{D}_{\max}$. If $\mathcal{Y}\in\mathcal{D}_{\min}^*$, then $\mathcal{Y},\varphi\in L_{q,W_1}^2[(0,a);E]$, where $\varphi:=T_{\min}^*\mathcal{Y}$. Assume that $\mathcal{U}$ is a solution of the equation

$$J\mathcal{U}^{[q]}(x)-W_2(x)\mathcal{U}(x)=W_1(x)\varphi(x). \tag{3.10}$$

By Lemma 3.5, we see that

$$(\varphi, \mathcal{Z}) = (T_{\max}\mathcal{U}, \mathcal{Z}) = (\mathcal{U}, T_{\min}\mathcal{Z}).$$

Thus, we get

$$\begin{aligned}(\mathcal{Y} - \mathcal{U}, T_{\min}\mathcal{Z}) &= (\mathcal{Y}, T_{\min}\mathcal{Z}) - (\mathcal{U}, T_{\min}\mathcal{Z}) \\ &= (T_{\min}^{*}\mathcal{Y}, \mathcal{Z}) - (\varphi, \mathcal{Z}) = 0,\end{aligned}$$

i.e. $\mathcal{Y} - \mathcal{U} \in \mathcal{R}(T_{\min})^{\perp}$. It follows from Lemma 3.6 that $\mathcal{Y} - \mathcal{U} \in \mathcal{N}(T_{\max})$.

Using (3.10), we get

$$\begin{aligned}& J\mathcal{Y}^{[q]}(x) - W_2(x)\mathcal{Y}(x) \\ &= J\mathcal{U}^{[q]}(x) - W_2(x)\mathcal{U}(x) = W_1(x)\varphi(x), \ x \in (0, a).\end{aligned}$$

Since $\mathcal{Y}, \varphi \in L^2_{q,W_1}[(0,a); E]$, we have that $\mathcal{Y} \in \mathcal{D}_{\max}$ and

$$T_{\max}\mathcal{Y} = \varphi = T_{\min}^{*}\mathcal{Y}.$$

This completes the proof. □

4. A criterion of the self-adjoint matrix-valued $q-$Sturm–Liouville operators

In this section, we give a criterion under which the matrix-valued $q-$Sturm–Liouville operators are self-adjoint.

Let Σ and Λ be $m \times 2n$ matrices such that $rank(\Sigma : \Lambda) = m$. Then, we define the operator T by

$$T : \mathcal{D} \to L^2_{q,W_1}[(0,a); E], \tag{4.1}$$

where

$$\mathcal{D} := \left\{ \begin{array}{c} \mathcal{Y} \in L^2_{q,V}(0,a) : \mathcal{Y} \text{ and } PD_q\mathcal{Y} \text{ are } q-\text{regular at zero,} \\ J\mathcal{Y}^{[q]}(x) - W_2(x)\mathcal{Y}(x) = W_1(x)F(x) \text{ exists in } (0,a), \\ F \in L^2_{q,V}(0,a), \text{ and} \\ \Sigma\widehat{\mathcal{Y}}(0) + \Lambda\widehat{\mathcal{Y}}(a) = 0. \end{array} \right\}. \tag{4.2}$$

Let Ω and Γ be $(4n-m) \times 2n$ matrices, chosen so that $rank(\Omega : \Gamma) = 4n - m$ and $\begin{pmatrix} \Sigma & \Lambda \\ \Omega & \Gamma \end{pmatrix}$ is nonsingular. Let $\begin{pmatrix} \widetilde{\Sigma} & \widetilde{\Lambda} \\ \widetilde{\Omega} & \widetilde{\Gamma} \end{pmatrix}$ be chosen so that

$$\begin{pmatrix} \widetilde{\Sigma} & \widetilde{\Lambda} \\ \widetilde{\Omega} & \widetilde{\Gamma} \end{pmatrix}^{*} \begin{pmatrix} \Sigma & \Lambda \\ \Omega & \Gamma \end{pmatrix} = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}. \tag{4.3}$$

Theorem 4.1 *For $\mathcal{Y}, \mathcal{Z} \in \mathcal{D}_{\max}$, we have*

$$\begin{aligned}(T_{\max}\mathcal{Y}, \mathcal{Z}) - (\mathcal{Y}, T_{\max}\mathcal{Z}) &= \left[\widetilde{\Sigma}\widehat{\mathcal{Z}}(0) + \widetilde{\Lambda}\widehat{\mathcal{Z}}(a)\right]^{*}\left[\Sigma\widehat{\mathcal{Y}}(0) + \Lambda\widehat{\mathcal{Y}}(a)\right] \\ &+ \left[\widetilde{\Omega}\widehat{\mathcal{Z}}(0) + \widetilde{\Gamma}\widehat{\mathcal{Z}}(a)\right]^{*}\left[\Omega\widehat{\mathcal{Y}}(0) + \Gamma\widehat{\mathcal{Y}}(a)\right].\end{aligned}$$

Proof By virtue of (3.8) and (4.3), we get

$$(T_{\max}\mathcal{Y},\mathcal{Z})-(\mathcal{Y},T_{\max}\mathcal{Z})=[\mathcal{Y},\mathcal{Z}]_a-[\mathcal{Y},\mathcal{Z}]_0$$

$$=\begin{pmatrix}\widehat{\mathcal{Z}}^*(0) & \widehat{\mathcal{Z}}^*(a)\end{pmatrix}\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}\widehat{\mathcal{Y}}(0)\\ \widehat{\mathcal{Y}}(a)\end{pmatrix}$$

$$=\begin{pmatrix}\widehat{\mathcal{Z}}^*(0) & \widehat{\mathcal{Z}}^*(a)\end{pmatrix}\begin{pmatrix}\widetilde{\Sigma} & \widetilde{\Lambda}\\ \widetilde{\Omega} & \widetilde{\Gamma}\end{pmatrix}^*\begin{pmatrix}\Sigma & \Lambda\\ \Omega & \Gamma\end{pmatrix}\begin{pmatrix}\widehat{\mathcal{Y}}(0)\\ \widehat{\mathcal{Y}}(a)\end{pmatrix}$$

$$=\left[\begin{pmatrix}\widetilde{\Sigma} & \widetilde{\Lambda}\\ \widetilde{\Omega} & \widetilde{\Gamma}\end{pmatrix}\begin{pmatrix}\widehat{\mathcal{Z}}(0)\\ \widehat{\mathcal{Z}}(a)\end{pmatrix}\right]^*\left[\begin{pmatrix}\Sigma & \Lambda\\ \Omega & \Gamma\end{pmatrix}\begin{pmatrix}\widehat{\mathcal{Y}}(0)\\ \widehat{\mathcal{Y}}(a)\end{pmatrix}\right]$$

$$=\begin{pmatrix}\widetilde{\Sigma}\widehat{\mathcal{Z}}(0)+\widetilde{\Lambda}\widehat{\mathcal{Z}}(a)\\ \widetilde{\Omega}\widehat{\mathcal{Z}}(0)+\widetilde{\Gamma}\widehat{\mathcal{Z}}(a)\end{pmatrix}^*\begin{pmatrix}\Sigma\widehat{\mathcal{Y}}(0)+\Lambda\widehat{\mathcal{Y}}(a)\\ \Omega\widehat{\mathcal{Y}}(0)+\Gamma\widehat{\mathcal{Y}}(a)\end{pmatrix}.$$

□

Now, we describe the adjoint of the operator T. Denote

$$\mathcal{D}^*:=$$

$$\left\{\begin{array}{c}\mathcal{Z}\in L^2_{q,W_1}\left[(0,a);E\right]: J\mathcal{Z}^{[q]}-W_2\left(x\right)\mathcal{Z}\left(x\right)=W_1\left(x\right)F_1\left(x\right)\text{ exists in }(0,a),\\ F_1\in L^2_{q,W_1}\left[(0,a);E\right]\text{ and }\widetilde{\Omega}\widehat{\mathcal{Z}}(0)+\widetilde{\Gamma}\widehat{\mathcal{Z}}(a)=0.\end{array}\right\},$$

Theorem 4.2 *For $\mathcal{Z}\in\mathcal{D}^*$, $T^*\mathcal{Z}=\widehat{F_1}$ if and only if*

$$J\mathcal{Z}^{[q]}-W_2\left(x\right)\mathcal{Z}\left(x\right)=W_1\left(x\right)F_1\left(x\right).$$

Proof Since $T_{\min}\subset T\subset T_{\max}$, we have $T_{\min}\subset T^*\subset T_{\max}$. Let $\mathcal{Y}\in\mathcal{D}$ and $\mathcal{Z}\in\mathcal{D}^*$. It follows from Theorem 4.1 that

$$\begin{aligned}(T\mathcal{Y},\mathcal{Z})-(\mathcal{Y},T^*\mathcal{Z})&=\left[\widetilde{\Sigma}\widehat{\mathcal{Z}}(0)+\widetilde{\Lambda}\widehat{\mathcal{Z}}(a)\right]^*\left[\Sigma\widehat{\mathcal{Y}}(0)+\Lambda\widehat{\mathcal{Y}}(a)\right]\\ &\quad+\left[\widetilde{\Omega}\widehat{\mathcal{Z}}(0)+\widetilde{\Gamma}\widehat{\mathcal{Z}}(a)\right]^*\left[\Omega\widehat{\mathcal{Y}}\left(0\right)+\Gamma\widehat{\mathcal{Y}}(a)\right].\end{aligned}$$

Then we get

$$0=\left[\widetilde{\Omega}\widehat{\mathcal{Z}}(0)+\widetilde{\Gamma}\widehat{\mathcal{Z}}(a)\right]^*\left[\Omega\widehat{\mathcal{Y}}\left(0\right)+\Gamma\widehat{\mathcal{Y}}(a)\right].$$

Since $\Omega\widehat{\mathcal{Y}}\left(0\right)+\Gamma\widehat{\mathcal{Y}}(a)$ is arbitrary, we have $\widetilde{\Omega}\widehat{\mathcal{Z}}(0)+\widetilde{\Gamma}\widehat{\mathcal{Z}}(a)=0$.

Conversely, if $\mathcal{Z}$ satisfies the criteria listed above then $\mathcal{Z}\in\mathcal{D}^*$. □

Now, we find parametric boundary conditions for $\mathcal{D}$ and $\mathcal{D}^*$. Recall that

$$\Sigma\widehat{\mathcal{Y}}(0)+\Lambda\widehat{\mathcal{Y}}(a)=0,\Omega\widehat{\mathcal{Y}}(0)+\Gamma\widehat{\mathcal{Y}}(a)=F_2, \tag{4.4}$$

where F_2 is arbitrary. Hence, we get

$$\begin{pmatrix} \Sigma & \Lambda \\ \Omega & \Gamma \end{pmatrix}\begin{pmatrix} \widehat{\mathcal{Y}}(0) \\ \widehat{\mathcal{Y}}(a) \end{pmatrix}=\begin{pmatrix} 0 \\ F_2 \end{pmatrix}. \tag{4.5}$$

If we multiply both sides of (4.5) by

$$\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} \widetilde{\Sigma} & \widetilde{\Lambda} \\ \widetilde{\Omega} & \widetilde{\Gamma} \end{pmatrix}^*$$

we conclude that

$$\begin{pmatrix} \widehat{\mathcal{Y}}(0) \\ \widehat{\mathcal{Y}}(a) \end{pmatrix}=\begin{pmatrix} J\widetilde{\Omega}^*F_2 \\ -J\widetilde{\Gamma}^*F_2 \end{pmatrix}. \tag{4.6}$$

Similarly, we can find parametric boundary conditions for $\mathcal{D}^*$. Since

$$\widetilde{\Sigma}\widehat{\mathcal{Z}}(0)+\widetilde{\Lambda}\widehat{\mathcal{Z}}(a)=F_3,\ \widetilde{\Omega}\widehat{\mathcal{Z}}(0)+\widetilde{\Gamma}\widehat{\mathcal{Z}}(a)=0,$$

where F_3 is arbitrary, we obtain

$$\begin{pmatrix} \widehat{\mathcal{Z}}^*(0) & \widehat{\mathcal{Z}}^*(a) \end{pmatrix}\begin{pmatrix} \widetilde{\Sigma} & \widetilde{\Lambda} \\ \widetilde{\Omega} & \widetilde{\Gamma} \end{pmatrix}^*=\begin{pmatrix} F_3^* & 0 \end{pmatrix}. \tag{4.7}$$

Multiplying both sides of (4.7) by

$$\begin{pmatrix} \Sigma & \Lambda \\ \Omega & \Gamma \end{pmatrix}\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}$$

it follows that

$$\widehat{\mathcal{Z}}(0)=-J\Sigma^*F_3,\ \widehat{\mathcal{Z}}(a)=J\Lambda^*F_3. \tag{4.8}$$

Now, we have the following theorem.

Theorem 4.3 *The operator T is self-adjoint if and only if $m=2n$ and $\Sigma J\Sigma^*=\Lambda J\Lambda^*$.*

Proof Let T be a self-adjoint operator. Then $\mathcal{Z}$ satisfies the boundary conditions for $\mathcal{D}$, that is

$$\Sigma\widehat{\mathcal{Z}}(0)+\Lambda\widehat{\mathcal{Z}}(a)=0.$$

It follows from (4.8) that

$$\Sigma(-J\Sigma^*F_3)+\Lambda(J\Lambda^*F_3)=0$$
$$[\Sigma J\Sigma^*-\Lambda J\Lambda^*]F_3=0.$$

Since F_3 is arbitrary, we see that

$$\Sigma J\Sigma^*=\Lambda J\Lambda^*.$$

Conversely, if $\Sigma J\Sigma^* = \Lambda J\Lambda^*$, then we get

$$\begin{pmatrix} -\Sigma J & \Lambda J \end{pmatrix} \begin{pmatrix} \Sigma^* \\ \Lambda^* \end{pmatrix} = 0,$$

i.e. the columns of $\begin{pmatrix} \Sigma^* \\ \Lambda^* \end{pmatrix}$ for $2n$ independent solutions to the equation

$$\begin{pmatrix} -\Sigma J & \Lambda J \end{pmatrix} X = 0.$$

By virtue of (4.4) and (4.6), we deduce that

$$\begin{pmatrix} -\Sigma J & \Lambda J \end{pmatrix} \begin{pmatrix} \widetilde{\Omega}^* \\ \widetilde{\Gamma}^* \end{pmatrix} = 0.$$

Thus, there must be a constant, nonsingular matrix K such that

$$\begin{pmatrix} \widetilde{\Omega}^* \\ \widetilde{\Gamma}^* \end{pmatrix} K^* = \begin{pmatrix} \Sigma^* \\ \Lambda^* \end{pmatrix}.$$

or

$$\begin{pmatrix} \Sigma & \Lambda \end{pmatrix} = K \begin{pmatrix} \widetilde{\Omega} & \widetilde{\Gamma} \end{pmatrix}.$$

Clearly, the conditions $\Sigma\widehat{\mathcal{Y}}(0) + \Lambda\widehat{\mathcal{Y}}(a) = 0$ and $\Omega\widehat{\mathcal{Y}}(0) + \Gamma\widehat{\mathcal{Y}}(a) = 0$ are equivalent. Since the forms of T and T^* are the same, this gives $T = T^*$. □

5. Extensions of the matrix-valued $q-$Sturm–Liouville operators

In this section, we shall describe all the self-adjoint, dissipative, and accumulative extensions of the corresponding minimal operator $T_{\min}$.

We begin this section with a definition (see [21, 29, 32]).

Definition 5.1 *Let $\mathbb{H}$ be a Hilbert space; let Π_1 and Π_2 be linear mappings of $\mathcal{D}(\mathcal{B}^*)$ into $\mathbb{H}$, where $\mathcal{B}$ is a closed symmetric operator acting in a Hilbert space $\mathcal{H}$ with equal (finite or infinite) deficiency indices. Then the triplet $(\mathbb{H}, \Pi_1, \Pi_2)$ is called a space of boundary values of the operator $\mathcal{B}$ if*

(i) $(\mathcal{B}^* h, g)_{\mathcal{H}} - (h, \mathcal{B}^* g)_{\mathcal{H}} = (\Pi_1 h, \Pi_2 g)_{\mathbb{H}} - (\Pi_2 h, \Pi_1 g)_{\mathbb{H}}$, $\forall h, g \in \mathcal{D}((\mathcal{B}^*)$, and

(ii) for every $G_1, G_2 \in \mathbb{H}$, there exists a vector $g \in \mathcal{D}(\mathcal{B}^*)$ such that $\Pi_1 g = G_1$ and $\Pi_2 g = G_2$.

In the next results, we use the following notation:

$\Pi_1,\ \Pi_2 : \mathcal{D}_{\max} \to E \oplus E$, where $E = \mathbb{C}^n$,

$$\Pi_1\mathcal{Y} = \begin{pmatrix} -\mathcal{Y}(0) \\ \mathcal{Y}(a) \end{pmatrix},\ \Pi_2\mathcal{Y} = \begin{pmatrix} (PD_{q^{-1}}\mathcal{Y})(0) \\ (PD_{q^{-1}}\mathcal{Y})(a) \end{pmatrix}, \tag{5.1}$$

and $y \in \mathcal{D}_{\max}$.

Theorem 5.2 *The triplet* $(E \oplus E, \Pi_1, \Pi_2)$ *defined by* (5.1) *is a space of boundary values of the symmetric operator* $T_{\min}$.

Proof From (5.1) and (3.8), we conclude that

$$(\Pi_1 \mathcal{Y}, \Pi_2 \mathcal{Z})_{E\oplus E} - (\Pi_2 \mathcal{Y}, \Pi_1 \mathcal{Z})_{E\oplus E} = -\left(\mathcal{Y}(0), (PD_{q^{-1}}\mathcal{Z})(0)\right)_E$$

$$+\left(\mathcal{Y}(a), (PD_{q^{-1}}\mathcal{Z})(a)\right)_E + \left((PD_{q^{-1}}\mathcal{Y})(0), \mathcal{Z}(0)\right)_E$$

$$-\left((PD_{q^{-1}}\mathcal{Y})(a), \mathcal{Z}(a)\right)_E = [\mathcal{Y}, \mathcal{Z}](a) - [\mathcal{Y}, \mathcal{Z}](0)$$

$$= (T_{\max}\mathcal{Y}, \mathcal{Z}) - (\mathcal{Y}, T_{\max}\mathcal{Z}),$$

where $\mathcal{Y}, \mathcal{Z} \in \mathcal{D}_{\max}$.

Now, we show the second assumption of the definition of space of boundary values.

Let $\Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}, \Gamma = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \in E \oplus E$. Then the vector-valued function

$$\mathcal{Y}(t) = \alpha_1(t)\, o\Lambda_1 + \alpha_2(t)\, o\Gamma_1 + \beta_1(t)\, o\Lambda_2 + \beta_2(t)\, o\Gamma_2,$$

where o is a symbol of the Hadamard product of vectors and

$$\alpha_1(t) = \begin{pmatrix} \alpha_{11}(t) \\ \ldots \\ \alpha_{1n}(t) \end{pmatrix}, \ \alpha_2(t) = \begin{pmatrix} \alpha_{21}(t) \\ \ldots \\ \alpha_{2n}(t) \end{pmatrix} \in E,$$

$$\beta_1(t) = \begin{pmatrix} \beta_{11}(t) \\ \ldots \\ \beta_{1n}(t) \end{pmatrix}, \ \beta_2(t) = \begin{pmatrix} \beta_{21}(t) \\ \ldots \\ \beta_{2n}(t) \end{pmatrix} \in E,$$

satisfy the conditions

$$\alpha_1(0) = \begin{pmatrix} \alpha_{11}(0) \\ \ldots \\ \alpha_{1n}(0) \end{pmatrix} = \begin{pmatrix} -1 \\ \ldots \\ -1 \end{pmatrix},$$

$$\alpha_1(a) = \begin{pmatrix} \alpha_{11}(a) \\ \ldots \\ \alpha_{1n}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 0 \end{pmatrix},$$

$$D_{q^{-1}}\alpha_1(0) = \begin{pmatrix} D_{q^{-1}}\alpha_{11}(0) \\ \ldots \\ D_{q^{-1}}\alpha_{1n}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 0 \end{pmatrix},$$

$$D_{q^{-1}}\alpha_1(a) = \begin{pmatrix} D_{q^{-1}}\alpha_{11}(a) \\ \ldots \\ D_{q^{-1}}\alpha_{1n}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 0 \end{pmatrix},$$

$$\alpha_2(0) = \begin{pmatrix} \alpha_{21}(0) \\ \dots \\ \alpha_{2n}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix},$$

$$\alpha_2(a) = \begin{pmatrix} \alpha_{21}(a) \\ \dots \\ \alpha_{2n}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix},$$

$$D_{q^{-1}}\alpha_2(0) = \begin{pmatrix} D_{q^{-1}}\alpha_{21}(0) \\ \dots \\ D_{q^{-1}}\alpha_{2n}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix},$$

$$D_{q^{-1}}\alpha_2(a) = \begin{pmatrix} D_{q^{-1}}\alpha_{21}(a) \\ \dots \\ D_{q^{-1}}\alpha_{2n}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix},$$

$$\beta_1(0) = \begin{pmatrix} \beta_{11}(0) \\ \dots \\ \beta_{1n}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix},$$

$$\beta_1(a) = \begin{pmatrix} \beta_{11}(a) \\ \dots \\ \beta_{1n}(a) \end{pmatrix} = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix},$$

$$D_{q^{-1}}\beta_1(0) = \begin{pmatrix} D_{q^{-1}}\beta_{11}(0) \\ \dots \\ D_{q^{-1}}\beta_{1n}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix},$$

$$D_{q^{-1}}\beta_1(a) = \begin{pmatrix} D_{q^{-1}}\beta_{11}(a) \\ \dots \\ D_{q^{-1}}\beta_{1n}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix},$$

$$\beta_2(0) = \begin{pmatrix} \beta_{21}(0) \\ \dots \\ \beta_{2n}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix},$$

$$\beta_2(a) = \begin{pmatrix} \beta_{21}(a) \\ \dots \\ \beta_{2n}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix},$$

$$D_{q^{-1}}\beta_2(0) = \begin{pmatrix} D_{q^{-1}}\beta_{21}(0) \\ \dots \\ D_{q^{-1}}\beta_{2n}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix},$$

$$D_{q^{-1}}\beta_2(a) = \begin{pmatrix} D_{q^{-1}}\beta_{21}(a) \\ \dots \\ D_{q^{-1}}\beta_{2n}(a) \end{pmatrix} = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix}.$$

belongs to the set $\mathcal{D}_{\max}$ and $\Pi_1\mathcal{Y} = \Lambda,\ \Pi_2\mathcal{Y} = \Gamma$. This completes the proof. □

Now, we give the following definition.

Definition 5.3 ([29]) *Let $\mathcal{L}$ be a linear operator with dense domain $\mathcal{D}(\mathcal{L})$ acting on some Hilbert space $\mathcal{H}$. The operator $\mathcal{L}$ is called dissipative if*

$$\operatorname{Im}(\mathcal{L}f, f) \geq 0$$

for all $f \in \mathcal{D}(\mathcal{L})$ and is called maximal dissipative if it does not have a proper dissipative extension. Similarly, The operator $\mathcal{L}$ is called accumulative

$$\operatorname{Im}(\mathcal{L}f, f) \leq 0$$

for all $f \in \mathcal{D}(\mathcal{L})$ and is called maximal accumulative if it does not have a proper accumulative extension.

Let

$$D_1 = \{\xi \in \mathcal{D}_{\max} : (M - I)\Pi_1\xi + i(M + I)\Pi_2\xi = 0\}, \tag{5.2}$$

$$D_2 = \{\xi \in \mathcal{D}_{\max} : (M - I)\Pi_1\xi + i(M + I)\Pi_2\xi = 0\}, \tag{5.3}$$

where M is a contraction operator *in* $E \oplus E$.

Then by Theorem 5.2, the following theorem is obtained [29].

Theorem 5.4 *The restriction of the operator $T_{\max}$ to the set D_1 is a maximal dissipative extension of the symmetric operator $T_{\min}$. Conversely, any maximal dissipative extensions of $T_{\min}$ is the restriction of $T_{\max}$ to a set D_1. Similarly, the restriction of the operator $T_{\max}$ to the set D_2 is a maximal accumulative extension of the symmetric operator $T_{\min}$. Conversely, any maximal accumulative extensions of $T_{\min}$ is the restriction of $T_{\max}$ to a set D_2. Here, the contraction M is uniquely determined by the extension. If the operator M is unitary, these conditions define a self-adjoint extension of $T_{\min}$.*

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