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Existence results for a class of boundary value problems for fractional differential equations

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Abstract: By application of some fixed point theorems, that is, the Banach fixed point theorem, Schaefer's and the Leray-Schauder fixed point theorem, we establish new existence results of solutions to boundary value problems of fractional differential equations. This paper is motivated by Agarwal et al. (Georgian Math. J. 16 (2009) No.3, 401-411).

Key words: Fractional differential equations, boundary value problems, Caputo fractional derivative, uniqueness, existence, fixed point

1. Introduction

In this manuscript, we study the fractional differential equations

$$ {}^cD^{\nu}u(t) = f(t, u(t)), \qquad t \in \mathcal{J} = [0, b], \tag{1.1}$$

subject to boundary conditions

$$ u(0) - u'(0) = u_0, \qquad u(b) + u'(b) = u_b, \qquad u''(b) = u_b^*. \tag{1.2}$$

Here $2 < \nu \leq 3$, ${}^cD^{\nu}$ designates the Caputo fractional derivative of order ν, $f : \mathcal{J} \times \mathcal{R} \to \mathcal{R}$ is a continuous and u_0, u_b, u_b^* are real constants. We investigate the existence and uniqueness of solutions for boundary value problems (BVPs) for fractional differential equations.

Fractional differential equations have captivated much concentration since they can be practiced in diverse areas of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control theory, porous media, electromagnetism, etc. can be modelled as fractional differential eqautions. For structures, view the documents of Kilbas et al. [16], Kiryakova [17], Miller and Ross [20], Oldham and Spanier [21], Podlubny [22], and Samko et al.[23]. Some the latest improvements to the hypothesis of fractional differential equations may be observed in [1–14, 18, 19, 24–28].

In [5], Bai and Lü studied the BVP of fractional order

$$ D_{0+}^{\alpha}x(t) + f(t, x(t)) = 0, \qquad t \in (0, 1), \qquad \alpha \in (1, 2], \tag{1.3}$$

$$ x(0) = x(1) = 0. \tag{1.4}$$

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Here $D^{\alpha}_{0+}x(t)$ is the Caputo fractional derivative, and $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous.They found some existence and multiplicity results of positive solutions by employing some fixed point theorems .

In [26], Zhang investigated the existence and multiplicity of positive solutions for BVPs of nonlinear fractional differential equations

$$D^{\alpha}_{0+}x(t)=f(t,x(t)),\qquad t\in(0,1),\qquad \alpha\in(1,2], \tag{1.5}$$

$$x(0)+x'(0)=0,\qquad x(1)+x'(1)=0. \tag{1.6}$$

Here $D^{\alpha}_{0+}x(t)$ is the Caputo fractional derivative, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. He found some existence results of positive solutions by employing fixed point theorem on cones.

In [28], the authors established the existence of multiple positive solutions for the nonlinear fractional differential equation BVP

$$D^{\alpha}_{0+}x(t)+f(t,x(t))=0,\qquad t\in(0,1),\qquad \alpha\in(2,3], \tag{1.7}$$

$$x(0)=x'(0)=x'(1)=0, \tag{1.8}$$

where $D^{\alpha}_{0+}x(t)$ is the Caputo fractional derivative. They established some new existence criteria for singular and nonsingular fractional differential equation BVP by the characteristics of the Green function, the lower and upper solution method and fixed point theorem.

Inspired by the work of the above papers, in this paper we discuss the BVPs (1.1),(1.2). Using the Banach fixed point theorem, for the applications of Schaefer's and the Leray-Schauder fixed point theorem, we give some new existence results for BVPs (1.1),(1.2). Recently, Agarwal et al. examined the BVPs for fractional differential equations (1) and (2) in [1]. As far as we know, BVPs for fractional differential equations (1.1),(1.2) have not been studied. We will complete this opening in the written matter.

The plan of this manuscript is the following way. In Section 2, we recall several useful preliminaries that will be used to verify main results. Main results are given in Section 3. In Section 4, an illustration is displayed to clarify the principal conclusions.

2. Background

For the usefulness of the reader, one gives several notation, descriptions, preliminary facts, which can be found in [16, 22]. They will be used in the proofs of our results.

By $\mathcal{C}(\mathcal{J},\mathcal{R})$ one represents the Banach space of all continuous functions from $\mathcal{J}$ into $\mathcal{R}$ endowed with

$$||u||_{\infty}:=\sup\{|u(t)|:t\in\mathcal{J}\}.$$

Definition 2.1 The fractional integral of order $\nu\in\mathcal{R}_+$ for a function $y\in L^1([a,b],\mathcal{R}_+)$ is given as

$$I^{\nu}_{a}y(t)=\frac{1}{\Gamma(\nu)}\int_a^t(t-s)^{\nu-1}y(s)ds,$$

here Γ is the gamma function. If $a=0$, then one has $I^{\nu}y(t)=y(t)\star\varphi_{\nu}(t)$, where $\varphi_{\nu}(t)=\frac{t^{\nu-1}}{\Gamma(\nu)}$ for $t>0$, $\varphi_{\nu}(t)=0$ for $t\leq 0$, $\varphi_{\nu}\to\delta(t)$ as $\nu\to 0$, here δ is the delta function.

Definition 2.2 The Riemann–Liouville fractional derivative of order ν for a function y defined on $[a,b]$ can be described by

$$D^{\nu}_{a+}y(t)=\frac{1}{\Gamma(k-\nu)}\left(\frac{d}{dt}\right)^{k}\int_{a}^{t}\frac{y(s)}{(t-s)^{\nu-k+1}}ds,$$

where

$$k=\begin{cases}[\nu]+1 & \text{if } k\notin\{0,1,2,\ldots\}\\ \nu & \text{if } k\in\{0,1,2,\ldots\}\end{cases}$$

and $[\nu]$ denotes the integer part of ν.

Definition 2.3 Caputo's fractional derivative of order ν for a function y given on $[a,b]$ can be prescribed as

$${}^{c}D^{\nu}_{a+}y(t)=\frac{1}{\Gamma(k-\nu)}\int_{a}^{t}\frac{y^{(k)}(s)}{(t-s)^{\nu-k+1}}ds.$$

3. Main results

We shall begin with the following definition.

Definition 3.1 $u\in\mathcal{C}^{2}(\mathcal{J},\mathcal{R})$ with its ν-derivative occuring on $\mathcal{J}$ is called to be a solution of BVP (1.1) and (1.2) if u solves the equation ${}^{c}D^{\nu}u(t)=f(t,u(t))$ on $\mathcal{J}$ and boundary conditions $u(0)-u'(0)=u_0,\quad u(b)+u'(b)=u_b,\quad u''(b)=u_b^{\star}$.
Here the notation $\mathcal{C}^{2}(\mathcal{J},\mathcal{R})$ means continuously twice differentiable.

In order to establish the existence of solutions for the BVP (1.1) and (1.2), one has to use lemma as follows:

Lemma 3.2 *([26]). For $\nu>0$, the solution of the differential equation ${}^{c}D^{\nu}y(t)=0$ is presented as*

$$y(t)=c_0+c_1t+c_2t^2+\cdots c_{n-1}t^{n-1},\quad \textit{here}\quad c_j\in\mathcal{R},\quad j=0,1,2,\ldots,n-1$$

where

$$n=\begin{cases}[\nu]+1 & \textit{if } n\notin\{0,1,2,\ldots\}\\ \nu & \textit{if } n\in\{0,1,2,\ldots\}.\end{cases}$$

Taking into account Lemma 3.2, one has that

$$I^{\nu c}D^{\nu}y(t)=y(t)+c_0+c_1t+c_2t^2+\cdots c_{n-1}t^{n-1}$$

for $c_j\in\mathcal{R},\quad j=0,1,2,\ldots,n-1,$ where

$$n=\begin{cases}[\nu]+1 & \text{if } n\notin\{0,1,2,\ldots\}\\ \nu & \text{if } n\in\{0,1,2,\ldots\}.\end{cases}$$

By means of Lemma 3.2 one has the next result which is valuable.

Lemma 3.3 *Suppose that $y:\mathcal{J}\to\mathcal{R}$ is continuous,and $2<\nu\leq 3$. Let $d=b+2\neq 0$. Then u is a solution of the integral expression*

$$\begin{aligned}
u(t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-w)^{\nu-1}y(w)dw - \frac{1}{2\Gamma(\nu-2)}t^2\int_0^b (b-w)^{\nu-3}y(w)dw \\
&+\frac{b^2}{2d\Gamma(\nu-2)}t\int_0^b (b-w)^{\nu-3}y(w)dw - \frac{1}{d\Gamma(\nu)}t\int_0^b (b-w)^{\nu-1}y(w)dw \\
&+\frac{b}{d\Gamma(\nu-2)}t\int_0^b (b-w)^{\nu-3}y(w)dw - \frac{1}{d\Gamma(\nu-1)}t\int_0^b (b-w)^{\nu-2}y(w)dw \\
&+\frac{b^2}{2d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw - \frac{1}{d\Gamma(\nu)}\int_0^b (b-w)^{\nu-1}y(w)dw \\
&+\frac{b}{d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw - \frac{1}{d\Gamma(\nu-1)}\int_0^b (b-w)^{\nu-2}y(w)dw \\
&+\frac{u_b^*}{2}t^2+\frac{u_b}{d}t-\frac{u_0}{d}t-\frac{u_b^*b^2}{2d}t-\frac{u_b^*b}{d}t+u_0+\frac{u_b}{d}-\frac{u_0}{d}-\frac{u_b^*b^2}{2d}-\frac{u_b^*b}{d} \qquad (3.1)
\end{aligned}$$

$\Longleftrightarrow$ *u is a solution of the BVP*

$$ {}^cD^\nu u(t)=y(t), \qquad t\in\mathcal{J}, \qquad (3.2)$$

$$u(0)-u'(0)=u_0, \qquad u(b)+u'(b)=u_b, \qquad u''(b)=u_b^\star. \qquad (3.3)$$

Proof Express $u(t)$ using the RL integral as

$$u(t)=c_0+c_1t+c_2t^2+(I_a^\nu y)(t), \qquad (3.4)$$

and then point out that the Caputo derivative annihilates the polynomial terms in the front and acts as an inverse to the RL integral in this situation (see[[16], Lemma 2.21]).

Applying the boundary conditions for (3.3), we find that

$$\begin{aligned}
c_0 &= u_0+\frac{u_b}{d}-\frac{u_0}{d}-\frac{u_b^*}{2d}b^2+\frac{b^2}{2d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw \\
&-\frac{1}{d\Gamma(\nu)}\int_0^b (b-w)^{\nu-1}y(w)dw-\frac{u_b^*}{d}b+\frac{b}{d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw \\
&-\frac{b}{d\Gamma(\nu-1)}\int_0^b (b-w)^{\nu-2}y(w)dw,
\end{aligned}$$

$$\begin{aligned}
c_1 &= \frac{u_b}{d}-\frac{u_0}{d}-\frac{u_b^*}{2d}b^2+\frac{b^2}{2d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw \\
&-\frac{1}{d\Gamma(\nu)}\int_0^b (b-w)^{\nu-1}y(w)dw-\frac{u_b^*}{d}b+\frac{b}{d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw \\
&-\frac{b}{d\Gamma(\nu-1)}\int_0^b (b-w)^{\nu-2}y(w)dw,
\end{aligned}$$

and

$$c_2 = \frac{u_b^*}{2} - \frac{1}{2\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw.$$

If we substitute the values $c_0, c_1,\ c_2$ in (3.4), then we obtain the solution given by (3.1). Inversely, it is obvious that if u fulfills Eq. (3.1), then Eq. (3.2) and (3.3) are satisfied. □

By applying the Banach fixed point theorem, we give our first result.

Theorem 3.4 *Let $f : \mathcal{J}\times\mathcal{R}\to\mathcal{R}$ be a continuous function satisfying the Lipschitz conditions:*

(C1) $|f(t,w)-f(t,\overline{w})| \le N_1|w-\overline{w}|$, *for* $t\in\mathcal{J}$, $w,\overline{w}\in\mathcal{R}$, $N_1>0$.

Let

$$N_1 b^{\nu}\left[\frac{3}{\Gamma(\nu+1)}+\frac{2}{\Gamma(\nu)}+\frac{7}{2\Gamma(\nu-1)}\right] < 1. \tag{3.5}$$

There exists a unique solution for BVP (1.1) and (1.2) on $\mathcal{J}$.

Proof Convert BVP (1.1) and (1.2) into a fixed point problem. Define an operator $Q : \mathcal{C}(\mathcal{J},\mathcal{R})\to\mathcal{C}(\mathcal{J},\mathcal{R})$ as

$$\begin{aligned}
Qu(t) = {} & \frac{1}{\Gamma(\nu)}\int_0^t (t-\sigma)^{\nu-1}f(\sigma,u(\sigma))d\sigma - \frac{1}{2\Gamma(\nu-2)}t^2\int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma))d\sigma\\
& + \frac{b^2}{2d\Gamma(\nu-2)}t\int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma))d\sigma - \frac{1}{d\Gamma(\nu)}t\\
& \times\int_0^b (b-\sigma)^{\nu-1}f(\sigma,u(\sigma))d\sigma + \frac{b}{d\Gamma(\nu-2)}t\int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma))d\sigma\\
& - \frac{1}{d\Gamma(\nu-1)}t\int_0^b (b-\sigma)^{\nu-2}f(\sigma,u(\sigma))d\sigma + \frac{b^2}{2d\Gamma(\nu-2)}\\
& \times\int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma))d\sigma - \frac{1}{d\Gamma(\nu)}\int_0^b (b-\sigma)^{\nu-1}f(\sigma,u(\sigma))d\sigma\\
& + \frac{b}{d\Gamma(\nu-2)}\int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma))d\sigma - \frac{1}{d\Gamma(\nu-1)}\\
& \times\int_0^b (b-\sigma)^{\nu-2}f(\sigma,u(\sigma))d\sigma + \frac{u_b^*}{2}t^2 + \frac{u_b}{d}t - \frac{u_0}{d}t - \frac{u_b^*b^2}{2d}t - \frac{u_b^*b}{d}t\\
& + u_0 + \frac{u_b}{d} - \frac{u_0}{d} - \frac{u_b^*b^2}{2d} - \frac{u_b^*b}{d}.
\end{aligned}$$

Undoubtedly, the fixed points of the operator Q are solutions of the BVP (1.1) and (1.2). We will apply the Banach contraction principle to show that Q has a fixed point. We shall verify that Q is a contraction.

If $u_1, u_2\in\mathcal{C}(\mathcal{J},\mathcal{R})$, for $t\in\mathcal{J}$ we have

$$
\begin{aligned}
|(Qu_1)(t)-(Qu_2)(t)| \leq & \frac{1}{\Gamma(\nu)}\int_0^t (t-\varrho)^{\nu-1}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b^2}{2\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b^3}{2d\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b}{d\Gamma(\nu)}\int_0^b (b-\varrho)^{\nu-1}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b^2}{\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b}{d\Gamma(\nu-1)}\int_0^b (b-\varrho)^{\nu-2}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b^2}{2d\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{1}{d\Gamma(\nu)}\int_0^b (b-\varrho)^{\nu-1}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b}{d\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{1}{d\Gamma(\nu-1)}\int_0^b (b-\varrho)^{\nu-2}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
\leq & \frac{N_1||u_1-u_2||_\infty}{\Gamma(\nu)}\int_0^t (t-\rho)^{\nu-1}d\rho + \frac{b^2N_1||u_1-u_2||_\infty}{2\Gamma(\nu-2)} \\
& \times \int_0^b (b-\rho)^{\nu-3}d\rho + \frac{b^3N_1||u_1-u_2||_\infty}{2d\Gamma(\nu-2)}\int_0^b (b-\rho)^{\nu-3}d\rho \\
& + \frac{bN_1||u_1-u_2||_\infty}{d\Gamma(\nu)}\int_0^b (b-\rho)^{\nu-1}d\rho + \frac{b^2N_1||u_1-u_2||_\infty}{d\Gamma(\nu-2)} \\
& \times \int_0^b (b-\rho)^{\nu-3}d\rho + \frac{bN_1||u_1-u_2||_\infty}{d\Gamma(\nu-1)}\int_0^b (b-\rho)^{\nu-2}d\rho \\
& + \frac{b^2N_1||u_1-u_2||_\infty}{2d\Gamma(\nu-2)}\int_0^b (b-\rho)^{\nu-3}d\rho + \frac{N_1||u_1-u_2||_\infty}{(b+2)\Gamma(\nu)} \\
& \times \int_0^b (b-\rho)^{\nu-1}d\rho + \frac{bN_1||u_1-u_2||_\infty}{d\Gamma(\nu-2)}\int_0^b (b-\rho)^{\nu-3}d\rho \\
& + \frac{N_1||u_1-u_2||_\infty}{d\Gamma(\nu-1)}\int_0^b (b-\rho)^{\nu-2}d\rho \\
\leq & N_1b^\nu\left[\frac{1}{\Gamma(\nu+1)}+\frac{1}{2\Gamma(\nu-1)}+\frac{1}{2\Gamma(\nu-1)}+\frac{1}{\Gamma(\nu+1)}+\frac{1}{\Gamma(\nu-1)}\right. \\
& \left. + \frac{1}{\Gamma(\nu)}+\frac{1}{2\Gamma(\nu-1)}+\frac{1}{\Gamma(\nu+1)}+\frac{1}{\Gamma(\nu-1)}+\frac{1}{\Gamma(\nu)}\right]||u_1-u_2||_\infty.
\end{aligned}
$$

Consequently we obtain

$$
||Q(u_1)-Q(u_2)|| \leq N_1b^\nu\left[\frac{3}{\Gamma(\nu+1)}+\frac{2}{\Gamma(\nu)}+\frac{7}{2\Gamma(\nu-1)}\right]||u_1-u_2||_\infty,
$$

which, by (3.5), implies that Q is a contraction. We conclude that Q has a fixed point which is a solution of the BVP (1.1) and (1.2) by applying Banach fixed point theorem. □

Next, one will apply Schaefer's fixed point theorem to obtain the second result.

Theorem 3.5 *Suppose that*

(C2) $f : \mathcal{J} \times \mathcal{R} \to \mathcal{R}$ *is continuous;*

(C3) there exists a fixed $N_2 > 0$ *satisfying*

$$|f(t,z)| \leq N_2, \quad t \in \mathcal{J}, \quad z \in \mathcal{R}.$$

BVP (1.1) *and* (1.2) *has at least one solution on* $\mathcal{J}$.

Proof We need to prove that the operator Q has a fixed point. The proof can be broken down into steps as follows.

(1) Q **is continuous.**

If $\{u_n\}$ is a sequence such that $u_n \to u$ in $\mathcal{C}(\mathcal{J},\mathcal{R})$, then we have

$$\begin{aligned}
|(Qu_n)(t) - (Qu)(t)| \leq & \frac{1}{\Gamma(\nu)} \int_0^t (t-\vartheta)^{\nu-1} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b^2}{2\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b^3}{2d\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b}{d\Gamma(\nu)} \int_0^b (b-\vartheta)^{\nu-1} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b^2}{d\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b}{d\Gamma(\nu-1)} \int_0^b (b-\vartheta)^{\nu-2} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b^2}{2d\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{1}{d\Gamma(\nu)} \int_0^b (b-\vartheta)^{\nu-1} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b}{d\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{1}{d\Gamma(\nu-1)} \int_0^b (b-\vartheta)^{\nu-2} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta.
\end{aligned}$$

Because f is a continuous function, we have

$$||Q(u_n) - Q(u)||_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

(2) Q **maps bounded sets into bounded sets in** $\mathcal{C}(\mathcal{J},\mathcal{R})$.

In fact, it suffices to verify that for $\xi^* > 0$ there exists a fixed $R_1 > 0$ satisfying $u \in \mathcal{B}_{\xi^*} = \{u \in \mathcal{C}(\mathcal{J},\mathcal{R}) : ||u||_\infty \leq \xi^*\}$ we have $||Q(u)||_\infty \leq R_1$. From (C3), for $t \in \mathcal{J}$, we have

$$
\begin{aligned}
|Qu(t)| \leq & \frac{1}{\Gamma(\nu)} \int_0^t (t-\sigma)^{\nu-1} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b^2}{2\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b^3}{2d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b}{d\Gamma(\nu)} \int_0^b (b-\sigma)^{\nu-1} |(\sigma, u(\sigma))| d\sigma \\
& + \frac{b^2}{d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b}{d\Gamma(\nu-1)} \int_0^b (b-\sigma)^{\nu-2} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b^2}{2d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{1}{d\Gamma(\nu)} \int_0^b (b-\sigma)^{\nu-1} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b}{d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{1}{d\Gamma(\nu-1)} \int_0^b (b-\sigma)^{\nu-2} |f(\sigma, u(\sigma))| d\sigma \\
& + |u_0| \Big(\frac{b}{d} + 1 + \frac{1}{d} \Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \left[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d} \right] \\
\leq & \frac{N_2}{\Gamma(\nu)} \int_0^t (t-\sigma)^{\nu-1} d\sigma + \frac{b^2 N_2}{2\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma \\
& + \frac{b^3 N_2}{2d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma + \frac{bN_2}{d\Gamma(\nu)} \int_0^b (b-\sigma)^{\nu-1} d\sigma \\
& + \frac{b^2 N_2}{d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma + \frac{bN_2}{d\Gamma(\nu-1)} \int_0^b (b-\sigma)^{\nu-2} d\sigma \\
& + \frac{b^2 N_2}{2d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma + \frac{N_2}{d\Gamma(\nu)} \int_0^b (b-\sigma)^{\nu-1} d\sigma \\
& + \frac{bN_2}{d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma + \frac{N_2}{d\Gamma(\nu-1)} \int_0^b (b-\sigma)^{\nu-2} d\sigma \\
& + |u_0| \Big(\frac{b}{d} + 1 + \frac{1}{d} \Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \left[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d} \right] \\
\leq & \frac{N_2}{\Gamma(\nu+1)} b^\nu + \frac{N_2}{2\Gamma(\nu-1)} b^\nu + \frac{N_2}{2\Gamma(\nu-1)} b^\nu + \frac{N_2}{\Gamma(\nu+1)} b^\nu + \frac{N_2}{\Gamma(\nu-1)} b^\nu \\
& + \frac{N_2}{\Gamma(\nu)} b^\nu + \frac{N_2}{2\Gamma(\nu-1)} b^\nu + \frac{N_2}{\Gamma(\nu+1)} b^\nu + \frac{N_2}{\Gamma(\nu-1)} b^\nu + \frac{N_2}{\Gamma(\nu)} b^\nu \\
& + |u_0| \Big(\frac{b}{d} + 1 + \frac{1}{d} \Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \Big[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d} \Big],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
||Qu||_\infty \leq & \frac{3N_2}{\Gamma(\nu+1)} b^\nu + \frac{2N_2}{\Gamma(\nu)} b^\nu + \frac{7N_2}{2\Gamma(\nu-1)} b^\nu \\
& + \frac{|u_b^*|}{2d}(2b^3 + 5b^2 + 2b) + \frac{|u_b|}{d}(b+1) + \frac{|u_0|}{d}(2b+3) := R_1.
\end{aligned}
$$

(3) Q **maps bounded sets into equicontinuous sets of** $\mathcal{C}(\mathcal{J},\mathcal{R})$**.**

For $m_1, m_2 \in \mathcal{J}$, $m_1 < m_2$, suppose that $\mathcal{B}_{\xi^*}$ is a bounded set of $\mathcal{C}(\mathcal{J},\mathcal{R})$ like in (2). For any $u \in \mathcal{B}_{\xi^*}$, we have

$$
\begin{aligned}
|Qu(m_2)-Qu(m_1)| =& \Bigg| \frac{1}{\Gamma(\nu)} \int_0^{m_1} [(m_2-\varsigma)^{\nu-1} - (m_1-\varsigma)^{\nu-1}] f(\varsigma,u(\varsigma))d\varsigma \\
&+ \frac{(m_2-m_1)^2}{\Gamma(\nu-2)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma,u(\varsigma))d\varsigma + \frac{(m_2-m_1)}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma,u(\varsigma))d\varsigma + \frac{(m_2-m_1)}{\Gamma(\nu)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} f(\varsigma,u(\varsigma))d\varsigma + \frac{(m_2-m_1)}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma,u(\varsigma))d\varsigma + \frac{(m_2-m_1)}{\Gamma(\nu-1)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-2} f(\varsigma,u(\varsigma))d\varsigma + \frac{1}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma,u(\varsigma))d\varsigma + \frac{1}{\Gamma(\nu)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} f(\varsigma,u(\varsigma))d\varsigma + \frac{1}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma,u(\varsigma))d\varsigma + \frac{1}{\Gamma(\nu-1)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-2} f(\varsigma,u(\varsigma))d\varsigma \Bigg| + \frac{|u_b^*|}{2}(m_2-m_1)^2 \\
&+ \frac{|u_b|}{d}(m_2-m_1) + \frac{|u_0|}{d}(m_2-m_1) + \frac{|u_b^*|}{2d} b^2 (m_2-m_1) \\
&+ \frac{|u_b^*|}{d} b (m_2-m_1) + |u_0|\Big(1+\frac{1}{d}\Big) + \frac{|u_b|}{d} + |u_b^*|\Big(\frac{b^2}{2d}+\frac{b}{d}\Big) \\
\leq& \frac{N_2}{\Gamma(\nu)} \int_0^{m_1} [(m_1-\varsigma)^{\nu-1} - (m_2-\varsigma)^{\nu-1}] d\varsigma + \frac{N_2(m_2-m_1)^2}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2(m_2-m_1)}{\Gamma(\nu-2)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma \\
&+ \frac{N_2(m_2-m_1)}{\Gamma(\nu)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2(m_2-m_1)}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2(m_2-m_1)}{\Gamma(\nu-1)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma \\
&+ \frac{N_2}{\Gamma(\nu-2)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2}{\Gamma(\nu)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma \\
&+ \frac{N_2}{\Gamma(\nu-2)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2}{\Gamma(\nu-1)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma \\
&+ \frac{|u_b^*|}{2}(m_2-m)^2 + \frac{|u_b|}{d}(m_2-m_1) + \frac{|u_0|}{b+2}(m_2-m_1) \\
&+ \frac{|u_b^*|}{2d} b^2 (m_2-m_1) + \frac{|u_b^*|}{d} b (m_2-m_1) \\
&+ |u_0|\Big(1+\frac{1}{d}\Big) + \frac{|u_b|}{d} + |u_b^*|\Big(\frac{b^2}{2d}+\frac{b}{d}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{N_2}{\Gamma(\nu+1)}[(m_2-m_1)^\nu + m_1^\nu - m_2^\nu] + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu \\
& + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu + \frac{N_2}{\Gamma(\nu+1)}(m_2-m_1)^\nu \\
& + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu + \frac{N_2}{\Gamma(\nu)}(m_2-m_1)^\nu \\
& + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu + \frac{N_2}{\Gamma(\nu+1)}(m_2-m_1)^\nu \\
& + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu + \frac{N_2}{\Gamma(\nu)}(m_2-m_1)^\nu \\
& + \frac{|u_b^*|}{2}(m_2-m_1)^2 + \frac{|u_b|}{d}(m_2-m_1) + \frac{|u_0|}{d}(m_2-m_1) \\
& + \frac{|u_b^*|}{2d}b^2(m_2-m_1) + \frac{|u_b^*|}{d}b(m_2-m_1) \\
& + |u_0|\Big(1+\frac{1}{d}\Big) + \frac{|u_b|}{d} + |u_b^*|\Big(\frac{b^2}{2d}+\frac{b}{d}\Big) \to 0 \ \text{ as } \ m_1 \to m_2.
\end{aligned}
$$

In view of (1) to (3), it follows by Arzela-Ascoli theorem that $Q : \mathcal{C}(\mathcal{J},\mathcal{R}) \to \mathcal{C}(\mathcal{J},\mathcal{R})$ is completely continuous.

(4) A priori bounds.

Finally we will verify that the set

$$\mathbf{\Omega} = \{u \in \mathcal{C}(\mathcal{J},\mathcal{R}) \mid u = \lambda Q u, \quad \lambda \in (0,1)\}$$

is bounded.

Let $u \in \mathbf{\Omega},\ u = \lambda Q u$ for $\lambda \in (0,1)$. Therefore, for $t \in \mathcal{J},$ we have

$$
\begin{aligned}
u(t) \ = \ & \frac{\lambda}{\Gamma(\nu)}\int_0^t (t-\sigma)^{\nu-1} f(\sigma,u(\sigma))d\sigma - \frac{\lambda}{2\Gamma(\nu-2)}t^2\int_0^b (b-\sigma)^{\nu-3} f(\sigma,u(\sigma))d\sigma \\
& + \frac{\lambda b^2}{2d\Gamma(\nu-2)}t\int_0^b (b-\sigma)^{\nu-3} f(\sigma,u(\sigma))d\sigma - \frac{\lambda}{d\Gamma(\nu)}t \\
& \times \int_0^b (b-\sigma)^{\nu-1} f(\sigma,u(\sigma))d\sigma + \frac{\lambda b}{d\Gamma(\nu-2)}t\int_0^b (b-\sigma)^{\nu-3} f(\sigma,u(\sigma)d\sigma \\
& - \frac{\lambda}{d\Gamma(\nu-1)}t\int_0^b (b-\sigma)^{\nu-2} f(\sigma,u(\sigma))d\sigma + \frac{\lambda b^2}{2d\Gamma(\nu-2)} \\
& \times \int_0^b (b-\sigma)^{\nu-3} f(\sigma,u(\sigma))d\sigma - \frac{\lambda}{d\Gamma(\nu)}\int_0^b (b-\sigma)^{\nu-1} f(\sigma,u(\sigma))d\sigma \\
& + \frac{\lambda b}{d\Gamma(\nu-2)}\int_0^b (b-\sigma)^{\nu-3} f(\sigma,u(\sigma))d\sigma - \frac{\lambda}{d\Gamma(\nu-1)} \\
& \times \int_0^b (b-\sigma)^{\nu-2} f(\sigma,u(\sigma))d\sigma + \lambda\frac{u_b^*}{2}t^2 \\
& + \frac{\lambda t}{d}\Big[-u_0 + u_b - \frac{u_b^* b^2}{2} - u_b^* b\Big] + \lambda u_0\Big(1-\frac{1}{d}\Big) + \lambda\frac{u_b}{d} - \lambda u_b^*\Big(\frac{b^2}{2d}+\frac{b}{d}\Big).
\end{aligned}
$$

This indicates by (C3) that for $t \in \mathcal{J}$ we have

$$\begin{aligned}
|u(t)| \leq & \frac{N_2}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} d\xi + \frac{b^2 N_2}{2\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi \\
& + \frac{b^3 N_2}{2d\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi + \frac{bN_2}{d\Gamma(\nu)} \int_0^b (b-\xi)^{\nu-1} d\xi \\
& + \frac{b^2 N_2}{d\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi + \frac{bN_2}{d\Gamma(\nu-1)} \int_0^b (b-\xi)^{\nu-2} d\xi \\
& + \frac{b^2 N_2}{2d\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi + \frac{N_2}{d\Gamma(\nu)} \int_0^b (b-\xi)^{\nu-1} d\xi \\
& + \frac{bN_2}{d\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi + \frac{N_2}{d\Gamma(\nu-1)} \int_0^b (b-\xi)^{\nu-2} d\xi \\
& + |u_0|\Big(\frac{b}{d} + 1 + \frac{1}{d}\Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \left[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d}\right] \\
\leq & \frac{N_2}{\nu\Gamma(\nu)} b^\nu + \frac{N_2}{2(\nu-2)\Gamma(\nu-2)} b^\nu + \frac{N_2}{2(\nu-2)\Gamma(\nu-2)} b^\nu + \frac{N_2}{\nu\Gamma(\nu)} b^\nu \\
& + \frac{N_2}{(\nu-2)\Gamma(\nu-2)} b^\nu + \frac{N_2}{(\nu-1)\Gamma(\nu-1)} b^\nu + \frac{N_2}{2(\nu-2)\Gamma(\nu-2)} b^\nu \\
& + \frac{N_2}{\nu\Gamma(\nu)} b^\nu + \frac{N_2}{(\nu-2)\Gamma(\nu-2)} b^\nu + \frac{N_2}{(\nu-1)\Gamma(\nu-1)} b^\nu \\
& + |u_0|\Big(\frac{b}{d} + 1 + \frac{1}{d}\Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \left[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d}\right].
\end{aligned}$$

Hence for every $t \in \mathcal{J}$ we have

$$\begin{aligned}
||u||_\infty \leq & \frac{3N_2}{\Gamma(\nu+1)} b^\nu + \frac{2N_2}{\Gamma(\nu)} b^\nu + \frac{7N_2}{2\Gamma(\nu-1)} b^\nu \\
& + \frac{|u_b^*|}{2d}(2b^3 + 5b^2 + 2b) + \frac{|u_b|}{d}(b+1) + \frac{|u_0|}{d}(2b+3) := R_2.
\end{aligned}$$

We have shown that the set $\mathbf{\Omega}$ is bounded. We can complete that Q has a fixed point which is a solution of the BVP (1.1)-(1.2) by using Schaefer's fixed point theorem. □

Now we establish our existence results for the BVP (1.1) and (1.2) by employing the Leray-Schauder fixed point theorem, here the assumption (C3) is diminished.

Theorem 3.6 *Suppose that assumption (C2) holds. Define* $N_2 = D_1 + 1$, *where*

$$\begin{aligned}
D_1 = & ||I^q \varphi_f||_L^1 \psi(N_2) + \frac{b^2}{2}(I^{\nu-2}\varphi_f)(b)\psi(N_2) + \frac{b^3}{2d}(I^{\nu-2}\varphi_f)(b)\psi(N_2) \\
& + \frac{b}{d}(I^\nu \varphi_f)(b)\psi(N_2) + \frac{b^2}{d}(I^{\nu-2}\varphi_f)(b)\psi(N_2) + \frac{b}{d}(I^{\nu-1}\varphi_f)(b) \\
& \times \psi(N_2) + \frac{b^2}{2d}(I^{\nu-2}\varphi_f)(b)\psi(N_2) + \frac{1}{d}(I^\nu \varphi_f)(b)\psi(N_2) \\
& + \frac{b}{d}(I^{\nu-2}\varphi_f)(b)\psi(N_2) + \frac{1}{d}(I^{\nu-1}\varphi_f)(b)\psi(N_2) \\
& + |u_0|\Big(\frac{b}{d} + 1 + \frac{1}{d}\Big) + \frac{|u_b|}{d}(1+b) + |u_b^*|\left(\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d}\right).
\end{aligned}$$

Suppose that

(C4) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ *and a function* $\varphi_f \in L^1(\mathcal{J},\mathcal{R}^+)$ *satisfying*

$$|f(t,w)| \le \varphi_f(t)\psi(|w|), \qquad t \in \mathcal{J}, \qquad w \in \mathcal{R}.$$

Then the BVP (1.1) *and* (1.2) *has at least one solution on* $\mathcal{J}$.

Proof We recognize the operator Q described in Theorems 3.4, 3.5. It is easy to prove that Q is continuous, completely continuous. Let $\lambda \in [0,1]$. Suppose u is such that for $t \in \mathcal{J}$ we have $u(t) = \lambda(Qu)(t)$. By (C4) for $t \in \mathcal{J}$ we have

$$\begin{aligned}
|u(t)| \le & \frac{1}{\Gamma(\nu)}\int_0^t (t-\eta)^{\nu-1}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{b^2}{2\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta \\
& + \frac{b^3}{2d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{b}{d\Gamma(\nu)} \\
& \times \int_0^b (b-\eta)^{\nu-1}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{b^2}{d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta \\
& + \frac{b}{d\Gamma(\nu-1)}\int_0^b (b-\eta)^{\nu-2}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{b^2}{2d\Gamma(\nu-2)} \\
& \times \int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{1}{d\Gamma(\nu)}\int_0^b (b-\eta)^{\nu-1}\varphi_f(\eta)\psi(|u(\eta)|)d\eta \\
& + \frac{b}{d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{1}{d\Gamma(\nu-1)} \\
& \times \int_0^b (b-\eta)^{\nu-2}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + |u_0|\Big(\frac{b}{d}+1+\frac{1}{d}\Big) \\
& + \frac{|u_b|}{d}(1+b) + |u_b^*|\left(\frac{b^2}{2}+\frac{b^3}{2d}+\frac{b^2}{d}+\frac{b^2}{2d}+\frac{b}{d}\right) \\
\le & \psi(||u||_\infty)\frac{1}{\Gamma(\nu)}\int_0^t (t-\eta)^{\nu-1}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{b^2}{2\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta \\
& + \psi(||u||_\infty)\frac{b^3}{2d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{b}{d\Gamma(\nu)} \\
& \times \int_0^b (b-\eta)^{\nu-1}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{b^2}{d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta \\
& + \psi(||u||_\infty)\frac{b}{d\Gamma(\nu-1)}\int_0^b (b-\eta)^{\nu-2}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{b^2}{2d\Gamma(\nu-2)} \\
& \times \int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{1}{d\Gamma(\nu)}\int_0^b (b-\eta)^{\nu-1}\varphi_f(\eta)d\eta \\
& + \psi(||u||_\infty)\frac{b}{d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{1}{d\Gamma(\nu-1)} \\
& \times \int_0^b (b-\eta)^{\nu-2}\varphi_f(\eta)d\eta + |u_0|\Big(\frac{b}{d}+1+\frac{1}{d}\Big) \\
& + \frac{|u_b|}{d}(1+b) + |u_b^*|\left(\frac{b^2}{2}+\frac{b^3}{2d}+\frac{b^2}{d}+\frac{b^2}{2d}+\frac{b}{d}\right).
\end{aligned}$$

Therefore

$$\frac{||u||_\infty}{D_2} \leq 1,$$

where

$$\begin{aligned} D_2 &= \psi(||u||_\infty)||I^\nu \varphi_f||_L^1 + \frac{b^2}{2}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) + \frac{b^3}{2d}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) \\ &+\frac{b}{d}(I^\nu \varphi_f)(b)\psi(||u||_\infty) + \frac{b^2}{d}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) + \frac{b}{d}(I^{\nu-1}\varphi_f)(b) \\ &\times\psi(||u||_\infty) + \frac{b^2}{2d}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) + \frac{1}{d}(I^\nu \varphi_f)(b)\psi(||u||_\infty) \\ &+\frac{b}{d}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) + \frac{1}{d}(I^{\nu-1}\varphi_f)(b)\psi(||u||_\infty) \\ &+|u_0|\Big(\frac{b}{d}+1+\frac{1}{d}\Big) + \frac{|u_b|}{d}(1+b) + |u_b^*|\left(\frac{b^2}{2}+\frac{b^3}{2d}+\frac{b^2}{d}+\frac{b^2}{2d}+\frac{b}{d}\right). \end{aligned}$$

Put

$$W = \{u \in \mathcal{C}(\mathcal{J},\mathcal{R}) : ||u||_\infty < D_1 + 1\}.$$

Clearly, $Q : \overline{W} \to \mathcal{C}(\mathcal{J},\mathcal{R})$ is continuous, completely continuous. By the select of W, there is no $u \in \partial W$ satisfying $u = \lambda Q(u)$ for $0 < \lambda < 1$. We can complete that Q has a fixed point $u \in \overline{W}$, which is a solution of the BVP (1.1)-(1.2) by employing the Leray-Schauder fixed point theorem [15]. □

4. Applications

In this section, we present some examples to illustrate our results.

Example 4.1 *Examine the fractional BVP*

$$^cD^\nu u(t) = \frac{e^{-t}|u(t)|}{(9+e^t)(1+|u(t)|)}, \qquad t \in \mathcal{J} := [0,1], \qquad \nu \in (2,3], \tag{4.1}$$

$$u(0) - u'(0) = 0, \qquad u(1) + u'(1) = 1, \qquad u''(1) = 0. \tag{4.2}$$

Let

$$f(t,u) = \frac{e^{-t}u}{(9+e^t)(1+u)}, \quad (t,u) \in \mathcal{J} \times [0,\infty).$$

For $u_1, u_2 \in [0,\infty)$ *and* $t \in \mathcal{J}$, *one has*

$$\begin{aligned} |f(t,u_1)-f(t,u_2)| &= \frac{e^{-t}}{(9+e^t)}\left|\frac{u_1}{1+u_1}-\frac{u_2}{1+u_2}\right| \\ &= \frac{e^{-t}|u_1-u_2|}{(9+e^t)(1+u_1)(1+u_2)} \\ &\leq \frac{e^{-t}}{(9+e^t)}|u_1-u_2| \\ &\leq \frac{1}{10}|u_1-u_2|, \end{aligned}$$

where $N_1 = \frac{1}{10}$. *Thus, the assumption (C1) is satisfied. Choose* $b = 1$. *We will confirm that assumption* (3.5) *holds. In fact,*

$$N_1 b^{\nu}\left[\frac{3}{\Gamma(\nu+1)}+\frac{2}{\Gamma(\nu)}+\frac{7}{2\Gamma(\nu-1)}\right] < 1 \Leftrightarrow \frac{3}{\Gamma(\nu+1)}+\frac{2}{\Gamma(\nu)}+\frac{7}{2\Gamma(\nu-1)} < 10. \tag{4.3}$$

One has

$$\frac{1}{2} \leq \frac{3}{\Gamma(\nu+1)} < \frac{3}{2}, \qquad 1 \leq \frac{2}{\Gamma(\nu)} < 2, \tag{4.4}$$

and

$$\frac{7}{2} \leq \frac{7}{2\Gamma(\nu-1)} < x, \tag{4.5}$$

for an suitably selected constant x *that will be defined. From* (4.3)-(4.5), *we can write as*

$$\frac{3}{\Gamma(\nu+1)}+\frac{2}{\Gamma(\nu)}+\frac{7}{2\Gamma(\nu-1)} < \frac{3}{2}+2+x < 10. \tag{4.6}$$

From inequality (4.6), *it follows that*

$$x < \frac{13}{2}.$$

By (4.5) *we obtain*

$$\frac{7}{13} \approx 0.538462 < \Gamma(\nu-1), \tag{4.7}$$

which holds for some $2 < \nu \leq 3$. *Therefore, by Theorem 3.4, there exists a unique solution for the BVP* (4.1) *and* (4.2) *on* $[0,1]$. *It is easy to see that inequality* (4.7) *is satisfied for the values of* ν.

Example 4.2 *Consider the fractional BVP*

$${}^{c}D^{\frac{5}{2}}u(t) = \frac{\cos^2 t}{(e^{-t}+3)^2|u(t)|}, \qquad \text{for} \quad (t,u) \in ([0,1],\mathcal{R}_+), \tag{4.8}$$

$$u(0)-u'(0) = 0, \qquad u(1)+u'(1) = 1, \qquad u''(1) = 0, \tag{4.9}$$

where $\nu = \frac{5}{2}$, $b = 1$, *and* $f(t,u) := \dfrac{\cos^2 t}{(e^{-t}+3)^2|u|}$.

We have

$$|f(t,u)| = \Big|\frac{\cos^2 t}{(e^{-t}+3)^2|u|}\Big| \leq \frac{1}{9}.$$

Choosing $N_2 = \frac{1}{9}$, *then by Theorem 3.5, the BVP (4.8)-(4.9) has at least one solution on* $[0,1]$.

Example 4.3 *Consider the fractional BVP*

$$^cD^{\frac{5}{2}}u(t) = \Big(\frac{1}{5+3t+t^2}\Big)\Big(\frac{1}{3}\times\frac{|u|^7}{1+|u|^6}+4\Big), \qquad t\in[0,1], \tag{4.10}$$

$$u(0)-u'(0)=0, \qquad u(1)+u'(1)=1, \qquad u''(1)=0, \tag{4.11}$$

where $\nu = \frac{5}{2}$, $b = 1$, *and* $f(t,u) = \Big(\dfrac{1}{5+3t+t^2}\Big)\Big(\dfrac{1}{3}\times\dfrac{|u|^7}{1+|u|^6}+4\Big)$.

We have

$$|f(t,u)| \leq \frac{1}{5+3t+t^2}\Big(\frac{1}{3}|u|+4\Big).$$

Choosing $\varphi_f(t) = \dfrac{1}{5+3t+t^2}$ *and* $\psi(|u|) = \dfrac{1}{3}|u|+4$, *then by Theorem 3.6, the BVP (4.10)-(4.11) has at least one solution on* $[0,1]$.

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