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Repdigits as sums of two generalized Lucas numbers

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Abstract: A generalization of the well–known Lucas sequence is the k-Lucas sequence with some fixed integer $k \geq 2$. The first k terms of this sequence are $0, \ldots, 0, 2, 1$, and each term afterwards is the sum of the preceding k terms. In this paper, we determine all repdigits, which are expressible as sums of two k-Lucas numbers. This work generalizes a prior result of Şiar and Keskin who dealt with the above problem for the particular case of Lucas numbers and a result of Bravo and Luca who searched for repdigits that are k-Lucas numbers.

Key words: Generalized Lucas number, repdigit, linear form in logarithms, reduction method

1. Introduction

For an integer $k \geq 2$, let the k-generalized Fibonacci sequence (or simply, the k-Fibonacci sequence) $F^{(k)} := (F_n^{(k)})_{n \geq 2-k}$ be defined by the linear recurrence sequence of order k

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$

with initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. The above sequence is one among the several generalizations of the Fibonacci sequence $(F_n)_{n \geq 0}$. For example, when $k = 2$, this coincides with the Fibonacci sequence, while, when $k = 3$, this sequence is also known as the Tribonacci sequence.

Let us now consider the k-generalized Lucas sequence (or simply, the k-Lucas sequence) $L^{(k)} := (L_n^{(k)})_{n \geq 2-k}$ whose terms satisfy the recurrence relation of order k

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \cdots + L_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$

with initial conditions $L_{-(k-2)}^{(k)} = L_{-(k-3)}^{(k)} = \cdots = L_{-1}^{(k)} = 0$, $L_0^{(k)} = 2$ and $L_1^{(k)} = 1$. The expression $L_n^{(k)}$ denotes the n^{th} term of the k-Lucas sequence. The usual Lucas sequence $(L_n)_{n \geq 0}$ is obtained when $k = 2$.

Repdigits are natural numbers having a single digit in their decimal expansion, i.e., numbers of the form

$$a\left(\frac{10^\ell - 1}{9}\right) \text{ for some } \ell \geq 1 \text{ where } a \in \{1, 2, \cdots, 9\}.$$

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During the past decade, there has been a flurry of activity regarding finding all members of certain classical recurrence sequences which are repdigits. For example, repdigits in Fibonacci, Lucas, Pell, Pell–Lucas, balancing and Lucas-balancing numbers have been explored in [7, 9, 14]. A similar study has been carried out by replacing Fibonacci, Lucas, balancing and Lucas-balancing numbers by their respective consecutive products (see [8, 10, 14]). In [15], Rayaguru and Panda extended their previous work [14] by exploring the existence of repdigits that are products of balancing and Lucas-balancing numbers with their indices in arithmetic progressions. Repdigits in k-Fibonacci and k-Lucas sequences have been studied in [2, 3]. An investigation for the repdigits that are sums of two k-Fibonacci numbers has been done in [4]. In a recent paper [17], Şiar and Keskin searched for the repdigits that are sums of two Lucas numbers.

As a generalization to the work of Şiar and Keskin [17] and to the work of Bravo and Luca [3], we address a similar problem with the k-Lucas sequence $L^{(k)}$ i.e., we determine all the solutions of the Diophantine equation

$$L_n^{(k)} + L_m^{(k)} = a\left(\frac{10^\ell - 1}{9}\right) \tag{1.1}$$

in nonnegative integers n, m, k, a and ℓ with $k \geq 2$, $n \geq m$, $1 \leq a \leq 9$ and $\ell \geq 2$. Our result is the following theorem.

Theorem 1.1 *All solutions of the Diophantine equation (1.1) in nonnegative integers n, m, k, a, and ℓ with $k \geq 2$, $n \geq m$, $1 \leq a \leq 9$, and $\ell \geq 2$ are*

$L_4 + L_3 = 11$	$L_9 + L_1 = 77$	$2L_5^{(4)} = 44$
$2L_5 = 22$	$L_{12} + L_5 = 333$	$L_7^{(3)} + L_0^{(3)} = 66$
$L_6 + L_3 = 22$	$L_4^{(3)} + L_1^{(3)} = 11$	$L_7^{(3)} + L_6^{(3)} = 99$
$L_7 + L_3 = 33$	$L_5^{(3)} + L_2^{(3)} = 22$	$L_7^{(k)} + L_2^{(k)} = 99, \forall\, k \geq 7$

Our proof combines linear forms in logarithms, reduction techniques, and some estimates from [3] to deal with large values of k. In this paper, we follow the approach and the presentation described in [4].

2. Auxiliary results

We start with some properties of $L^{(k)}$. First, it is known that the characteristic polynomial of the sequence $L^{(k)}$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one zero $\alpha := \alpha(k)$ outside the unit circle. To simplify the notation, we shall omit the dependence on k of α. The other zeros of $\Psi_k(x)$ are strictly inside the unit circle. Furthermore, $\alpha \in \big(2(1-2^{-k}), 2\big)$ (see, for example, [12, 13, 19]).

We now consider for an integer $s \geq 2$, the function

$$f_s(x) = \frac{x-1}{2+(s+1)(x-2)} \text{ for } x > 2(1-2^{-s}).$$

With the above notation, we have the following properties of $L^{(k)}$.

Lemma 2.1 [3, Lemma 2] *Let $k \geq 2$ be an integer. Then*

(a) $\alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n$ *for all* $n \geq 1$*;*

(b) $L^{(k)}$ *satisfies the following Binet - like formula:*

$$L_n^{(k)} = \sum_{i=1}^{k}(2\alpha_i - 1)f_k(\alpha_i)\alpha_i^{n-1},$$

where $\alpha = \alpha_1, \ldots, \alpha_k$ *are roots of* $\Psi_k(x)$*;*

(c) $|L_n^{(k)} - (2\alpha - 1)f_k(\alpha)\alpha^{n-1}| < 3/2$ *holds for all* $n \geq 2 - k$*;*

(d) *If* $2 \leq n \leq k$*, then* $L_n^{(k)} = 3 \cdot 2^{n-2}$*.*

Note that property (a) of Lemma 2.1 also holds for $n = 0$. Moreover, using the identity $L_n^{(k)} = 2L_{n-1}^{(k)} - L_{n-k-1}^{(k)}$, which holds for all $n \geq 3$, one can prove that

$$L_n^{(k)} < 3 \cdot 2^{n-2} \quad \text{holds for all} \quad n \geq k + 1.$$

To solve Diophantine equations involving repdigits and terms of linear recurrence sequences, many authors have used lower bounds for linear forms in logarithms of algebraic numbers. These bounds play an important role while solving such Diophantine equations. Before presenting the result that will be used here, it is worth recalling some important aspects from algebraic number theory.

Let η be an algebraic number of degree d having the minimal polynomial

$$f(x) = \sum_{j=0}^{d} a_j x^{d-j} = a_0 \prod_{j=1}^{d}(x - \eta^{(j)}) \in \mathbb{Z}[x],$$

where the a_j's are relatively prime integers with $a_0 > 0$ and the $\eta^{(j)}$'s are conjugates of η. The logarithmic height of η is given by

$$h(\eta) = \frac{1}{d}\left(\log\, a_0 + \sum_{j=1}^{d} \max\{0, \log|\eta^{(j)}|\}\right).$$

In particular, if $\eta = a/b$ is a rational number with $\gcd(a,b) = 1$ and $b > 1$, then $h(\eta) = \log(\max\{|a|, b\})$. The following properties [18, Property 3.3] of the logarithmic height will be used with or without further reference as and when needed.

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2,$$
$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma),$$
$$h(\eta^s) = |s|\, h(\eta), s \in \mathbb{Z}.$$

A modified version of a result of Matveev [11] appears in [5]. Let $\mathbb{L}$ be a real algebraic number field of degree D. Let $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real algebraic numbers in $\mathbb{L}$ and $b_1, b_2, \ldots, b_t$ be rational integers. Put

$$B = \max\{|b_1|, \ldots, |b_t|\} \quad \text{and} \quad \Lambda = \prod_{i=1}^{t} \gamma_i^{b_i} - 1.$$

Theorem 2.2 [5, Theorem 9.4] *If* $\Lambda \neq 0$*, then*

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1+\log D)(1+\log B)A_1 \cdots A_t),$$

where

$$A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\} \quad \text{for all } i = 1, \dots, t.$$

Using the above theorem and properties of the logarithmic height, we will obtain upper bounds for the index of k-Lucas numbers. We will also use the following estimate which can be derived from [3] and will be the key point in addressing the large values of k.

Lemma 2.3 *Let* $k \geq 12$ *and* r *be positive integers such that* $r-1 < 2^{k/2}$*. Then*

$$(2\alpha - 1)f_k(\alpha)\alpha^{r-1} = 3 \cdot 2^{r-2}(1 + \zeta(r,k)) \quad \textit{where} \quad |\zeta(r,k)| < \frac{4}{2^{k/2}}.$$

Proof It follows from [3, p. 150] that

$$(2\alpha - 1)f_k(\alpha)\alpha^{r-1} = 3 \cdot 2^{r-2} + 3 \cdot 2^{r-1}\eta + \frac{\delta}{2} + \eta\delta,$$

where η and δ are real numbers such that

$$|\eta| < \frac{2k}{2^k} \quad \text{and} \quad |\delta| < \frac{2^{r+2}}{2^{k/2}}.$$

Thus

$$(2\alpha - 1)f_k(\alpha)\alpha^{r-1} = 3 \cdot 2^{r-2}(1 + \zeta(r,k)),$$

where

$$\begin{aligned} |\zeta(r,k)| &= \left| 2\eta + \frac{\delta}{3 \cdot 2^{r-1}} + \frac{\eta\delta}{3 \cdot 2^{r-2}} \right| \\ &< \frac{4k}{2^k} + \frac{3}{2^{k/2}} + \frac{11k}{2^{3k/2}} \\ &< \frac{4}{2^{k/2}} \quad \text{for all } k \geq 12. \end{aligned}$$

□

Lemma 2.4 *For* $k \geq 2$*, let* α *be the dominant root of the characteristic polynomial* $\Psi_k(x)$ *of the* k*-Lucas sequence, and consider the function* $f_k(x)$ *defined in* (2)*. Then*

$$h(f_k(\alpha)) < \log(k+1) + \log 4 < 3\log k,$$

where $h(\cdot)$ *represents the logarithmic height function. Moreover, if* $r > 1$ *is an integer satisfying* $r - 1 < 2^{k/2}$*, then*

$$(2\alpha - 1)f_k(\alpha)\alpha^{r-1} = 3 \cdot 2^{r-2} + \frac{\delta}{2} + 3 \cdot 2^{r-1}\eta + \eta\delta,$$

where δ and η are real numbers such that

$$|\delta| < \frac{2^{r+2}}{2^{k/2}} \quad \text{and} \quad |\eta| < \frac{2k}{2^k}.$$

In [6], Dujella and Pethö gave a version of the reduction method based on the Baker–Davenport lemma. We will apply the following version from [1], which is an immediate variation of the result [6, Lemma 5(a)] again and again for the further reduction of the obtained upper bounds of k and the index of k-Lucas numbers so that in the remaining range the repdigits which are sum of two k-Lucas numbers can be verified with direct computation.

Lemma 2.5 *Let τ be an irrational number, and let A, B, and μ be real numbers with $A > 0$ and $B > 1$. Assume that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of τ such that $q > 6M$ and let*

$$\epsilon = ||\mu q|| - M||\tau q||,$$

where $||\cdot||$ denotes the distance to the nearest integer. If $\epsilon > 0$, then there is no solution of the inequality

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$\frac{\log(Aq/\epsilon)}{\log B} \le w \quad \text{and} \quad u \le M.$$

In this preliminary section, we focus on the first part of the sequence $L^{(k)}$, where we have powers of 2 with a multiple of 3 and present the following lemma, which gives a partial answer to (1.1).

Lemma 2.6 *The only integer solution (x, y, a, ℓ) of the Diophantine equation*

$$3(2^x + 2^y) = a\left(\frac{10^\ell - 1}{9}\right), \tag{2.1}$$

with $0 \le x \le y$, $1 \le a \le 9$ and $\ell \ge 2$ is $(x, y, a, \ell) = (0, 5, 9, 2)$.

Proof We first rewrite (2.1) as

$$3 \cdot 2^x(1 + 2^{y-x}) = a\left(\frac{10^\ell - 1}{9}\right). \tag{2.2}$$

Comparing the exponent of 2 in both sides of (2.2) for the case $x = y$, we obtain that $x \le 2$ since $(10^\ell - 1)/9$ is odd for all $\ell \ge 2$. But one can easily check that (2.2) has no solutions for $0 \le x = y \le 2$ and $\ell \ge 2$. The same argument as above for the case $x < y$ shows that $x \le 3$. One can easily check that (2.2) has no solutions for $0 \le x < y \le 2$ and $\ell \ge 2$. Thus, from now on, we assume that $y \ge 3$. Then,

$$3 \cdot 2^{\ell-1} < 10^{\ell-1} < 3(2^x + 2^y) \le 3 \cdot 2^{y+1}$$

yielding $\ell - 2 < y$. Now, we rewrite (2.2) as

$$27(2^x + 2^y) = a(10^\ell - 1) \Leftrightarrow 27 \cdot 2^x + a = 2^{\ell-2}(4a \cdot 5^\ell - 27 \cdot 2^{y-\ell+2})$$

and use $2^{\ell-2} \leq 27 \cdot 2^x + a \leq 225$ to obtain $\ell \leq 9$. So, the inequalities

$$3(1+2^y) \leq 3(2^x+2^y) = a\left(\frac{10^\ell - 1}{9}\right) \leq 10^\ell - 1 \leq 10^9 - 1$$

imply that $y \leq 28$. Finally, we compute the numbers of the form $3(2^x + 2^y)$ in the range $0 \leq x < y \leq 28$ with $x \leq 3$, and check that the only solution of (2.2) in this range is $(x, y, a, \ell) = (0, 5, 9, 2)$, which completes the proof. □

3. An inequality for n in terms of k

Throughout this paper we assume that (1.1) holds and that $k \geq 3$ as the case $k = 2$ has already been treated by Şiar and Keskin [17]. At first, observe that when $n = m$, (1.1) reduces to

$$L_n^{(k)} = \frac{a}{2}\left(\frac{10^\ell - 1}{9}\right).$$

Thus, a must be even. But in view of [3, Theorem 1] the only solution of the above equation is $(n, k, a, \ell) = (5, 4, 4, 2)$. So, from now on, we assume that $n > m$. If $2 \leq n \leq k$, then by Lemma 2.1 (d), $L_n^{(k)} = 3 \cdot 2^{n-2}$ and $L_m^{(k)} = 3 \cdot 2^{m-2}$ and hence in this case, Lemma 2.6 gives the solutions

$$L_7^{(k)} + L_2^{(k)} = 99, \quad \forall k \geq 7.$$

Let us now suppose that $n \geq k+1$. If $n \leq 15$, then we must have that $k \leq 14$. In this case, a brute force search with *Mathematica* in the range $0 \leq m < n \leq 15$ and $3 \leq k \leq 14$ gives the solutions shown in the statement of Theorem 1.1. Thus, for the rest of the paper, we assume that $n \geq 16$. Also, since the 16^{th} 3-Lucas number > 10000, it follows that $\ell \geq 4$.

In view of Lemma 2.1(a) and $10^{\ell-1} < L_n^{(k)} + L_m^{(k)} < 10^\ell$, one gets that

$$10^{\ell-1} < 2(\alpha^n + \alpha^m) = 2\alpha^n(1 + \alpha^{m-n}) \leq 2\alpha^n(1 + \alpha^{-1}) < 2\alpha^{n+1},$$

and

$$\alpha^{n-1} < L_n^{(k)} < L_n^{(k)} + L_m^{(k)} < 10^\ell.$$

Thus,

$$(n-1)\left(\frac{\log\alpha}{\log 10}\right) < \ell < (n+1)\left(\frac{\log\alpha}{\log 10}\right) + 2.61.$$

Using the fact that $7/4 < \alpha < 2$, we conclude that

$$\frac{n}{5} < \ell < \frac{3n}{2}, \tag{3.1}$$

which is an estimate on ℓ in terms of n.

Now using (1.1) and Lemma 2.1(c), we get that

$$\left|\frac{a10^\ell}{9} - (2\alpha - 1)f_k(\alpha)\alpha^{n-1}\right| < \frac{3}{2} + \frac{a}{9} + L_m^{(k)} \leq \frac{5}{2} + L_m^{(k)} \leq \frac{5}{2} + 2\alpha^m. \tag{3.2}$$

Dividing both sides of the above inequality by $(2\alpha-1)f_k(\alpha)\alpha^{n-1}$, which is positive, we obtain

$$\left|10^\ell \alpha^{-(n-1)}\frac{a}{9}(2\alpha-1)^{-1}(f_k(\alpha))^{-1}-1\right| < \frac{9}{\alpha^{n-m}}, \tag{3.3}$$

where we used the facts $f_k(\alpha)>1/2$ and $1/(2\alpha-1)<1/2$. With the notation of Theorem 2.2, we take $t=3$ and the parameters

$$\gamma_1=10,\ \gamma_2=\alpha,\ \gamma_3=\frac{a}{9}(2\alpha-1)^{-1}(f_k(\alpha))^{-1}.$$

We also take $b_1=\ell$, $b_2=-(n-1)$ and $b_3=1$. The real number field containing $\gamma_1,\gamma_2,\gamma_3$ is $\mathbb{L}=\mathbb{Q}(\alpha)$, so we can take $D=[\mathbb{L}:\mathbb{Q}]=k$. The left–hand side of (3.3) is nonzero. In fact, if it were zero, then

$$a10^\ell = 9(2\alpha-1)f_k(\alpha)\alpha^{n-1}.$$

Conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\Psi_k(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that for any $i\geq 2$,

$$a10^\ell = 9\left|(2\alpha_i-1)f_k(\alpha_i)\alpha_i^{n-1}\right|.$$

But the above equality is impossible since its left–hand side is ≥ 100, whereas its right–hand side is <27. Thus, the left–hand side of (3.3) is nonzero.

Since $\ell<3n/2$ by (3.1), we can choose $B:=3n/2$. The logarithmic heights of γ_1 and γ_2 are $\log 10$ and $(\log\alpha)/k$, respectively. Furthermore, from the properties of $h(\cdot)$ we get

$$h(\gamma_3)\leq \log\frac{a}{9}+h(2\alpha-1)+h(f_k(\alpha))\leq \log 27+3\log k\leq 6\log k$$

for all $k\geq 3$, where we used the estimates $h(2\alpha-1)<\log 3$ and $h(f_k(\alpha))<3\log k$ (see [3, p. 147]). Hence, we can take $A_1=2.31k$, $A_2=0.7$ and $A_3=6k\log k$. Now, using Theorem 2.2 to get a lower bound on the left–hand side of (3.3) we obtain

$$\exp(-C_1(k)\times(1+\log B)(2.31k)(0.7)(6k\log k)) < \frac{9}{\alpha^{n-m}},$$

where $C_1(k)=1.4\times 30^6\times 3^{4.5}\times k^2\times(1+\log k)<1.5\times 10^{11}k^2(1+\log k)$. Taking logarithms on both sides of the above inequality, together with a straightforward calculation, gives

$$(n-m)\log\alpha < 5.83\times 10^{12}k^4\log^2 k\log n, \tag{3.4}$$

where we used that $1+\log B<2\log n$ for all $n\geq 16$ and $1+\log k<2\log k$ for all $k\geq 3$.

Now, we will define a second linear form in logarithms. To this, we use (1.1) and Lemma 2.1(c) to get

$$\begin{aligned}
&\left|\frac{a10^\ell}{9}-(2\alpha-1)f_k(\alpha)\alpha^{n-1}(1+\alpha^{m-n})\right| \\
&\qquad = \left|(L_n^{(k)}-(2\alpha-1)f_k(\alpha)\alpha^{n-1})+(L_m^{(k)}-(2\alpha-1)f_k(\alpha)\alpha^{m-1})+\frac{a}{9}\right| \\
&\qquad \leq 3+\frac{a}{9}\leq 4,
\end{aligned} \tag{3.5}$$

and dividing it across by $(2\alpha-1)f_k(\alpha)\alpha^{n-1}(1+\alpha^{m-n})$, we obtain

$$\left|10^{\ell}\alpha^{-(n-1)}\frac{a}{9}(2\alpha-1)^{-1}(f_k(\alpha))^{-1}(1+\alpha^{m-n})^{-1}-1\right|<\frac{4}{\alpha^{n-1}}. \tag{3.6}$$

In a second application of Matveev's result Theorem 2.2, we take the same parameters as in the first application, except by γ_3 which in this case is given by

$$\gamma_3=\frac{a}{9}(2\alpha-1)^{-1}(f_k(\alpha))^{-1}(1+\alpha^{m-n})^{-1}.$$

As before $\mathbb{L}=\mathbb{Q}(\alpha)$ contains $\gamma_1,\gamma_2,\gamma_3$ and has $D=k$. The choices of B, A_1 and A_2 are also the same as before. To see why the left–hand side of (3.6) is nonzero, note that otherwise, we would get the relation

$$a10^{\ell}=9(2\alpha-1)f_k(\alpha)(\alpha^{n-1}+\alpha^{m-1}).$$

Now, the same argument used before gives us an absurdity. Indeed, conjugating the above relation by an automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ such that $\sigma(\alpha)=\alpha_i$ for some $i\geq 2$, and then taking absolute values, we have

$$100\leq a10^{\ell}=9|(2\alpha_i-1)f_k(\alpha_i)(\alpha_i^{n-1}+\alpha_i^{m-1})|<54,$$

which is impossible. Thus, the left–hand side of (3.6) is nonzero. Let us now estimate $h(\gamma_3)$. Applying the properties of $h(\cdot)$ and taking into account inequality (3.4), we get

$$h(\gamma_3)<6\log k+|m-n|\left(\frac{\log\alpha}{k}\right)+\log 2,$$

and so $kh(\gamma_3)<7k\log k+(n-m)\log\alpha<5.85\times10^{12}k^4\log^2 k\log n$. Therefore, we take $A_3=5.85\times10^{12}k^4\log^2 k\log n$. Then, Matveev's theorem applied to the left–hand side of the inequality (3.6) gives

$$\exp(-C_1(k)\times(1+\log B)(2.31k)(0.7)(5.85\times10^{12}k^4\log^2 k\log n)<\frac{4}{\alpha^{n-1}},$$

which leads to

$$n<1.14\times10^{25}k^7\ \log^3 k\ \log^2 n. \tag{3.7}$$

It is easy to check that for all $A\geq 100$ the inequality $x<A\ \log^2 x$ implies $x<4A\ \log^2 A$. Using this fact with $A=1.14\times10^{25}k^7\ \log^3 k$, inequality (3.7) yields

$$n<1.67\times10^{29}k^7\ \log^5 k.$$

Thus, by (3.1), we get $\ell<2.51\times10^{29}k^7\ \log^5 k$. We record this in the following lemma.

Lemma 3.1 *If (n,m,k,a,ℓ) is a solution of (1.1) with $n>m$, $n\geq 16$ and $k\geq 3$, then*

$$n<1.67\times10^{29}k^7\ \log^5 k\quad\text{and}\quad \ell<2.51\times10^{29}k^7\ \log^5 k.$$

4. The case of small k

In this section, we treat the cases when $k \in [3, 400]$. We need to find better bounds for n, m, and ℓ than those implied by Lemma 3.1 for these values of k. To do this, we first let

$$z_1 = \ell \log 10 - (n-1)\log\alpha + \log s(a,k), \tag{4.1}$$

where $s(a,k) = a(2\alpha-1)^{-1}(f_k(\alpha))^{-1}/9$. In view of (1.1) and Lemma 2.1(c), we have

$$(2\alpha-1)f_k(\alpha)\alpha^{n-1} < L_n^{(k)} + \frac{3}{2} < L_n^{(k)} + L_m^{(k)} < \frac{a10^\ell}{9},$$

and so

$$e^{z_1} - 1 = 10^\ell \alpha^{-(n-1)}\frac{a}{9}(2\alpha-1)^{-1}(f_k(\alpha))^{-1} - 1 > 0.$$

This, together with (3.3) gives

$$0 < z_1 \le e^{z_1} - 1 < \frac{9}{\alpha^{n-m}}.$$

Replacing z_1 in the above inequality by its formula (4.1) and dividing it across by $\log\alpha$, we get

$$0 < \ell\left(\frac{\log 10}{\log\alpha}\right) - n + \left(1 + \frac{\log\, s(a,k)}{\log\alpha}\right) < 18 \cdot \alpha^{-(n-m)}. \tag{4.2}$$

Putting

$$\tau_k := \frac{\log 10}{\log\alpha}, \quad \mu_k := \mu_k(a,k) = 1 + \frac{\log s(a,k)}{\log\alpha}, \quad A = 18, \quad B = \alpha,$$

inequality (4.2) implies

$$0 < \ell\tau_k - n + \mu_k < A \cdot B^{-(n-m)}. \tag{4.3}$$

Note that τ_k is an irrational number since α is a unit in $\mathcal{O}_{\mathbb{L}}$, the ring of integers of $\mathbb{L}$. We also put $M_k = \lfloor 2.51 \times 10^{29} k^7 \log^5 k \rfloor$, which is an upper bound on ℓ by Lemma 3.1. We now apply Lemma 2.5 to inequality (4.3) for each $k \in [3, 400]$ and all choices $a \in \{1, \ldots, 9\}$. A computer search in *Mathematica* revealed that the maximum value of $\log(Aq_k/\epsilon_k)/\log B$ is < 190, where $q_k > 6M_k$ is a denominator of a convergent of the continued fraction of τ_k such that $\epsilon_k = ||\mu_k q_k|| - M_k ||\tau_k q_k|| > 0$. Hence, we deduce that the possible solutions (n, m, k, a, ℓ) of (1.1) for which k is in the range $[3, 400]$ all have $n - m \in [1, 190]$.

Now, using (3.6) we will find a better upper bound on n. Let

$$z_2 = \ell \log 10 - (n-1)\log\alpha + \log s(a,k,n-m),$$

where $s(a,k,n-m) = a(2\alpha-1)^{-1}(f_k(\alpha))^{-1}(1+\alpha^{m-n})^{-1}/9$. Then, from estimate (3.6), we deduce that

$$|e^{z_2} - 1| < \frac{4}{\alpha^{n-1}}.$$

Note that $z_2 \neq 0$ and $|e^{z_2} - 1| < 4/\alpha^{n-1} < 1/2$ for all $n \ge 16$. We shall distinguish the following cases. If $z_2 > 0$, then $0 < z_2 \le e^{z_2} - 1 < 4/\alpha^{n-1}$. If, on the contrary, $z_2 < 0$, then $e^{|z_2|} < 2$ and therefore, $0 < |z_2| \le e^{|z_2|} - 1 = e^{|z_2|}|e^{z_2} - 1| < 8/\alpha^{n-1}$. In any case we have that

$$0 < |z_2| < \frac{8}{\alpha^{n-1}} \quad \text{holds for all} \quad n \ge 16.$$

Arguing as in (4.2), this time we arrive at

$$0 < \left|\ell\left(\frac{\log 10}{\log\alpha}\right) - n + \left(1 + \frac{\log s(a,k,n-m)}{\log\alpha}\right)\right| < 16\cdot\alpha^{-(n-1)}. \tag{4.4}$$

Here also we put $M_k = \lfloor 2.51\times 10^{29}k^7\,\log^5 k\rfloor$ and, as we explained before, we apply Lemma 2.5 to inequality (4.4) in order to obtain an upper bound on $n-1$. Indeed, with the help of *Mathematica* we find that if $k\in[3,400]$ and $n-m\in[1,190]$, then the maximum value of $\log(16q_k/\epsilon_k)/\log\alpha$ is <201 for all choices $a\in\{1,\ldots,9\}$. Thus, the possible solutions (n,m,k,a,ℓ) of (1.1) for which k is in the range $[3,400]$ all have $n\le 201$.

Finally, we checked that there are no solutions to equation (1.1) in the range

$$3\le k\le 400,\quad \max\{16,k+1\}\le n\le 201\quad\text{and}\quad 1\le m\le n-1.$$

This completes the analysis in the case $k\in[3,400]$.

5. The case of large k

We now suppose that $k>400$ and note that for such k we have

$$m<n<1.67\times 10^{29}k^7\,\log^5 k<2^{k/2}.$$

Here, it follows from Lemma 2.4 that

$$\left|3\cdot 2^{n-2}-\frac{a10^\ell}{9}\right| = \left|(2\alpha-1)f_k(\alpha)\alpha^{n-1}) - \frac{a10^\ell}{9} - 3\cdot 2^{n-2}\zeta(n,k)\right|,$$

where $\zeta(n,k)$ is a real number such that $|\zeta(n,k)|<4/2^{k/2}$. This, together with (3.2), gives

$$\left|3\cdot 2^{n-2}-\frac{a10^\ell}{9}\right| < \frac{5}{2}+3\cdot 2^{m-2}+\frac{12\cdot 2^{n-2}}{2^{k/2}}.$$

Dividing both sides of the above inequality by $3\cdot 2^{n-2}$ and using that $1/2^{n-2}<1/2^{k/2}$ because $n\ge k+1$, we get that

$$\left|1-\frac{a}{27}\cdot 10^\ell\cdot 2^{-(n-2)}\right| < \frac{5}{2^{k/2}}+\frac{1}{2^{n-m}}. \tag{5.1}$$

We next lower bound the left–hand side of (5.1) using again Theorem 2.2. We take the parameters $t=3$ and

$$\gamma_1=a/27,\ \gamma_2=10,\ \gamma_3=2\quad\text{with}\quad b_1=1,\ b_2=\ell,\ b_3=-(n-2).$$

First, notice that the left–hand side of (5.1) is nonzero, because if this is so, then $a\cdot 10^\ell = 27\cdot 2^{n-2}$ and since $\ell\ge 2$ and $n\ge 16$, we get $5\mid 27\cdot 2^{n-2}$, which is false. In this third application of Theorem 2.2, we take $\mathbb{L}:=\mathbb{Q}$, $D=1$, $A_1:=\log 27$, $A_2:=\log 10$, $A_3:=\log 2$ and $B:=3n/2$. We thus get

$$\exp(-C(1+\log B)(\log 27)(\log 10)(\log 2)) < \frac{6}{2^\Gamma},$$

where $C = 1.4 \times 30^6 \times 3^{4.5}$ and $\Gamma = \min\{k/2, n-m\}$. Taking logarithms in both sides of the above inequality, we obtain

$$\Gamma < 2.18 \times 10^{12} \log n.$$

In view of Lemma 3.1, we have $n < 1.67 \times 10^{29} k^7 \log^5 k$ and hence,

$$\Gamma < 4.36 \times 10^{13} \log k. \tag{5.2}$$

Now, if we put

$$z_3 = \ell \log 10 - (n-2)\log 2 + \log(a/27),$$

then (5.1) can be written as

$$|e^{z_3} - 1| < \frac{6}{2^{\Gamma}}.$$

Assuming $\Gamma \geq 4$, we get that the right–hand side above is less than $1/2$. Thus, similar arguments apply here to get that $0 < |z_3| < 12/2^{\Gamma}$, and therefore

$$0 < \left| \ell \left(\frac{\log 10}{\log 2} \right) - n + \left(2 + \frac{\log(a/27)}{\log 2} \right) \right| < 18 \cdot 2^{-\Gamma}. \tag{5.3}$$

Now, we will discuss the following two cases.

5.1. The case $\Gamma = k/2$

In this case, it follows from (5.2) that

$$k < 8.72 \times 10^{13} \log k,$$

yielding $k < 3.12 \times 10^{15}$. So, by Lemma 3.1, we obtain $n < 2.78 \times 10^{145}$ and $\ell < 4.18 \times 10^{145}$. We record this result in the following lemma.

Lemma 5.1 *If (n, m, k, a, ℓ) is a solution of (1.1) with $n > m$, $k > 400$ and $k/2 \leq n - m$, then*

$$n < 2.78 \times 10^{145}, \quad k < 3.12 \times 10^{15} \quad \text{and} \quad \ell < 4.18 \times 10^{145}.$$

Since these bounds are too large to handle, we wish to reduce our bounds further, by using again Lemma 2.5. To do this, we take $M = 4.18 \times 10^{145}$ (upper bound on ℓ from Lemma 5.1), and as we explained before, we apply Lemma 2.5 to inequality (5.3) for all $a \in \{1, \ldots, 9\}$. Using *Mathematica* we found that $k \leq 992$. This new upper bound on k implies, by Lemma 3.1, $\ell < 3.72 \times 10^{54}$. Taking now $M = 3.72 \times 10^{54}$, a new application of Lemma 2.5 on (5.3) gives $k < 387$ for all choices $a \in \{1, \ldots, 9\}$. This contradicts the fact that $k > 400$.

5.2. The case $\Gamma = n - m$

In this case, inequality (5.2) becomes

$$n - m < 4.36 \times 10^{13} \log k. \tag{5.4}$$

We now apply Lemma 2.4 with $r = n$ and $r = m$ to get

$$\left|3 \cdot 2^{n-2} + 3 \cdot 2^{m-2} - \frac{a10^{\ell}}{9}\right|$$
$$= \left|(2\alpha - 1)f_k(\alpha)(\alpha^{n-1} + \alpha^{m-1}) - \frac{a10^{\ell}}{9} - 3 \cdot 2^{n-2}\zeta_1(n,k) - 3 \cdot 2^{m-2}\zeta_2(m,k)\right|,$$

where $\zeta_1(n,k)$ and $\zeta_2(m,k)$ are real numbers such that $\max\{|\zeta_1(n,k)|, |\zeta_1(n,k)|\} < 4/2^{k/2}$. Combining this with (3.5) we obtain

$$\left|3 \cdot 2^{n-2} + 3 \cdot 2^{m-2} - \frac{a10^{\ell}}{9}\right| < 4 + \frac{4(3 \cdot 2^{n-2} + 3 \cdot 2^{m-2})}{2^{k/2}},$$

which implies

$$\left|1 - \frac{a}{27} \cdot 10^{\ell} \cdot 2^{-(n-2)}(1 + 2^{m-n})^{-1}\right| < \frac{6}{2^{k/2}} \tag{5.5}$$

because $1/2^{n-2} < 1/2^{k/2}$. Note that if the left–hand side of inequality (5.5) were zero, then we might have that $a \cdot 10^{\ell} = 27 \cdot 2^{m-2}(1 + 2^{n-m})$, and so

$$\ell \le \upsilon_5(a \cdot 10^{\ell}) = \upsilon_5(1 + 2^{n-m}), \tag{5.6}$$

where $\upsilon_p(t)$ denotes the exponent at which the prime p appears in the prime factorization of t.

Before proceeding further, an observation on the sequence $(u_s)_{s\ge 0} = (2^s - 1)_{s\ge 0}$ is worth making. It turns out that $(u_s)_{s\ge 0}$ can be seen as the Lucas sequence $u_s = 3u_{s-1} - 2u_{s-2}$ for all $s \ge 2$ with $u_0 = 0$ and $u_1 = 1$ as initial conditions. Hence, it follows from a result of Sanna [16, Theorem 1.5] that

$$\upsilon_5(u_s) = \upsilon_5(2^s - 1) = \begin{cases} 1 + \upsilon_5(s), & \text{if } s \equiv 0 \pmod 4, \\ 0, & \text{if } s \not\equiv 0 \pmod 4. \end{cases}$$

From this and using the identity $2^{2s} - 1 = (2^s - 1)(2^s + 1)$, it is a straightforward exercise to check that

$$\upsilon_5(2^s + 1) = \begin{cases} 0, & \text{if } s \equiv 0, 1, 3 \pmod 4, \\ 1 + \upsilon_5(s), & \text{if } s \equiv 2 \pmod 4. \end{cases}$$

In particular, the inequality

$$\upsilon_5(2^s + 1) \le 1 + \upsilon_5(s) \quad \text{holds for all} \quad s \ge 0.$$

By using this last fact and (5.6), as well as (3.1), we can assert that

$$\frac{n}{5} < \ell \le 1 + \upsilon_5(n - m) \le 1 + \frac{\log(n - m)}{\log 5} < 1 + \frac{\log n}{\log 5},$$

which implies that $n \le 12$. However, this contradicts our initial assumption that $n \ge 16$. Thus, the left–hand side of (5.5) is nonzero. With a view towards applying Matveev's theorem in a fourth time, we take the parameters $t = 4$ and

$$\gamma_1 = a/27, \quad \gamma_2 = 10, \quad \gamma_3 = 2 \quad \text{and} \quad \gamma_4 = 1 + 2^{m-n}.$$

We also take

$$b_1 = 1, \quad b_2 = \ell, \quad b_3 = -(n-2) \quad \text{and} \quad b_4 = -1.$$

Here, we have $\mathbb{L} = \mathbb{Q}$ for which $D = 1$. We take $A_1 = \log 27$, $A_2 = \log 10$, $A_3 = \log 2$ and $B = 3n/2$. Now, we will determine the value of A_4. Observe that

$$h(\gamma_4) = h\left(\frac{1+2^{n-m}}{2^{n-m}}\right) = \log(1+2^{n-m}) < \log 2^{n-m+1} = (n-m+1)\log 2.$$

So, we can take $A_4 = (n-m+1)\log 2$. Then, Matveev's theorem gives

$$\exp(-C(1+\log B)(\log 27)(\log 10)(\log 2)(n-m+1)\log 2) < \frac{6}{2^{k/2}},$$

where now $C = 1.4 \times 30^7 \times 4^{4.5}$. From this, we obtain, after some calculations, that

$$k < 3.3 \times 10^{14}(\log n)(n-m+1).$$

Using now the upper bounds for n and $n-m$ from Lemma 3.1 and (5.4) respectively, we conclude that

$$k < 2.88 \times 10^{29}\log^2 k,$$

yielding $k < 1.69 \times 10^{33}$. So, by Lemma 3.1 once again, we obtain $n < 3.53 \times 10^{269}$ and $\ell < 5.24 \times 10^{269}$. We record this in the following lemma.

Lemma 5.2 *If (n,m,k,a,ℓ) is a solution of (1.1) with $n > m$, $n \geq 16$, $k > 400$ and $n-m < k/2$, then*

$$n < 3.53 \times 10^{269}, \quad k < 1.69 \times 10^{33} \quad \text{and} \quad \ell < 5.24 \times 10^{269}.$$

We now proceed as we did in the case when $\Gamma = k/2$. Indeed, taking $M = 5.24 \times 10^{269}$, which is an upper bound on ℓ from Lemma 5.2, we apply Lemma 2.5 on (5.3) to find an upper bound on $n-m$. In this case, *Mathematica* gives $n-m \leq 1811$ for all choices $a \in \{1,\ldots,9\}$.

Finally, we let

$$z_4 = \ell \log 10 - (n-2)\log 2 + \log s(a, n-m),$$

where $s(a,n-m) = a(1+2^{m-n})^{-1}/27$, and observe that (5.5) can be rewritten as

$$|e^{z_4} - 1| < \frac{6}{2^{k/2}}.$$

The same argument that we have been using then shows that

$$0 < \left|\ell\left(\frac{\log 10}{\log 2}\right) - n + \left(2 + \frac{\log s(a,n-m)}{\log 2}\right)\right| < 18 \cdot (\sqrt{2})^{-k}. \tag{5.7}$$

We take $M = 5.24 \times 10^{269}$ (upper bound on ℓ) and apply Lemma 2.5 on (5.7) for each $n-m \in [1,1811]$ and for all choices of $a \in \{1,\ldots,9\}$. With the help of *Mathematica*, we get that $k \leq 1850$, and so, by Lemma 3.1, $\ell < 7.42 \times 10^{51}$. After taking $M = 7.42 \times 10^{51}$, a new application of Lemma 2.5 on (5.7) gives $k \leq 383$ for all choices of $n-m \in [1,1811]$ and $a \in \{1,\ldots,9\}$. This contradicts our assumption that $k > 400$. This completes the analysis of the case when $\Gamma = n-m$ and therefore the proof of Theorem 1.1.

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