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# Yau-type ternary Hom-Lie bialgebras

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**Abstract:** The purpose of this paper is to introduce and study 3-Hom-Lie bialgebras, which are a ternary version of Hom-Lie bialgebras introduced by Yau (2015). We provide their properties, some key constructions and their 3-dimensional classification. Moreover we discuss their representation theory and their generalized derivations and coderivations. Furthermore, a more generalized notion called generalized 3-Hom-Lie bialgebra is also considered.

**Key words:** 3-Hom-Lie bialgebra, multiplicative 3-Hom-Lie bialgebra, extension, classification, representation, generalized derivation

## 1. Introduction

The notion of $n$-Lie algebra was introduced by Filippov in 1985 ([24]). The $n$-Lie algebras are a kind of multiple algebraic systems appearing in many fields in mathematics and mathematical physics. The first instances of 3-Lie algebras appeared in Nambu mechanics [39], for which an algebraic formulation was provided by Takhtajan [45]. Moreover, they were used in string theory and M-branes. The structure of 3-Lie algebras is applied to the study of the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes. Bagger and Lambert found in [8] a maximally supersymmetric three dimensional conformal field theory as a candidate description of the low energy world volume theory of multiple coincident M2-branes. In the study of supergravity solutions describing M2-branes ending on M5-branes, Basu and Harvey [9] suggested lifted Nahm equations based on 3-Lie algebras. The structure of $n$-Lie algebras was also explored from the algebraic point of view in various papers, see [6, 12, 32]. The $\mathbb{Z}_2$-graded case, motivated by supersymmetry in physics, was considered in [1–3].

Hom-type generalizations of $n$-Lie algebras, called $n$-Hom-Lie algebras, were introduced by Ataguema, Makhlouf, and Silvestrov in [7]. Each $n$-Hom-Lie algebra has $n-1$ linear twisting maps, which appear in a twisted generalization of the $n$-Jacobi identity called the $n$-Hom-Jacobi identity. If the twisting maps are all equal to the identity, one recovers $n$-Lie algebras. The twisting maps provide a substantial amount of freedom in manipulating Lie algebras.

The first instances of Lie bialgebras appeared first as the classical limit of mathematical structures underlying the quantum inverse scattering method developed by Faddeev [23] when studying quantum integrable

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systems. The theory of Lie bialgebras dates back to the early 80th and is mostly due to Drinfeld [21] and Semenov-Tian-Shansky [42], who introduced the Poisson-Lie groups and discussed the relationships with the concept of a classical $r$-matrix, which were introduced earlier by Sklyanin [**?** ], see also for concepts of Lie coalgebras and Lie bialgebras ([20, 38]). The generalization to twisted situation was considered in several papers, among them the study of purely Hom-Lie bialgebra by Cai and Sheng in [18] and the approach provided by Yau in [46] which we aim to extend to ternary algebras.

The class of 3-Lie bialgebras were studied first in [10]. In [22] the authors study local cocycle 3-Lie bialgebras and double construction 3-Lie bialgebras which were introduced in the study of the classical Yang–Baxter equation and Manin triples for 3-Lie algebras. Two types of 3-Lie bialgebras whose compatibility conditions between the multiplication and comultiplication are given by local cocycles and double constructions are also considered in [13]. One may mention also the work about 3-Leibniz bialgebras in [41]. An attempt to define 3-Hom-Lie bialgebras, in terms of Manin triple of 3-Hom-Lie algebras, was given recently in [29].

Derivations of different algebraic structures are an important subject of study in algebra and other areas. They appeared in many fields of mathematics and physics. In particular, they appear in representation theory and cohomology theory among other areas. They have various applications such as relating algebra to geometry and also allow the construction of new algebraic structures. There are many generalizations of derivations (for example, Leibniz derivations [33]). The notion of $\delta$-derivations appeared in the paper of Filippov [25], he studied $\delta$-derivations of prime Lie and Malcev algebras [26, 27]. The notion of generalized derivation is a generalization of $\delta$-derivation. The most important and systematic research on the generalized derivations algebras of a Lie algebra and their subalgebras was due to Leger and Luks [34]. In their article, they studied properties of generalized derivation algebras and their subalgebras, for example, the quasi-derivation algebras. They have determined the structure of algebras of by quasi-derivations and generalized derivations. They proved that the quasi-derivations algebra of a Lie algebra can be embedded into the derivations algebra of a larger Lie algebra. Derivations and generalized derivations of $n$-ary algebras were considered in [40, 48] and some other papers. For example, Williams proved that, unlike the case of binary algebras, for any $n$ there exists a non-nilpotent $n$-Lie algebra with invertible derivation, [48]. Generalized derivations for $n$-ary algebras and their Hom-type were considered in [15, 17, 31].

The aim of this paper is to introduce and study 3-Hom-Lie bialgebras, generalizing the Hom-Lie bialgebras introduced by Yau. In Section 2, we recall the basics about Hom-Lie bialgebras and then, in Section 3, we introduce and study various types of 3-Hom-Lie bialgebras: 3-Hom-Lie algebras, multiplicative 3-Hom-Lie algebras and generalized multiplicative 3-Hom-Lie algebras. We describe a twist construction and discuss their dualization. In Section 4, we provide a 3-dimensional classification of multiplicative 3-Hom-Lie bialgebras and point out the ones coming from 3-Lie bialgebras. Section 5 deals with a construction by an extension of a Hom-Lie bialgebras with a symmetric invariant nondegenerate bilinear form. In Section 6, we establish the representation theory of multiplicative 3-Hom-Lie bialgebras and in Section 7 we deal with their generalized derivations, coderivations and biderivations.

## 2. Hom-Lie bialgebras

In this section we recall some basic definitions and properties concerning Hom-Lie bialgebras introduced in [46]. Let $\mathbb{K}$ be a commutative field of characteristics zero.

**Definition 2.1** *A Hom-Lie algebra is a triple* $(\mathfrak{g},[\cdot,\cdot],\alpha)$ *consisting of a* $\mathbb{K}$*-vector space* $\mathfrak{g}$*, a bilinear map* $[\cdot,\cdot]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ *and a linear map* $\alpha:\mathfrak{g}\to\mathfrak{g}$ *satisfying :*

$$[x,y]=-[y,x] \quad (\textit{skew symmetric}), \tag{2.1}$$

$$\circlearrowleft_{x,y,z}[\alpha(x),[y,z]]=0, \quad \forall \ x,y,z\in\mathfrak{g} \quad (\textit{Hom-Jacobi identity}), \tag{2.2}$$

*where* $\circlearrowleft_{x,y,z}$ *denotes summation over the cyclic permutation on* $x,y,z$*. A Hom-Lie algebra is called multiplicative if* $\alpha\circ[\cdot,\cdot]=[\cdot,\cdot]\circ(\alpha\otimes\alpha)$*.*

Let $(\mathfrak{g},[\cdot,\cdot],\alpha)$ and $(\mathfrak{g}',[\cdot,\cdot]',\alpha')$ be two Hom-Lie algebras. A homomorphism $f:\mathfrak{g}\longrightarrow\mathfrak{g}'$ is said to be a Hom-Lie algebra morphism if

$$f([x,y])=[f(x),f(y)]', \quad \forall \ x,y\in\mathfrak{g}, \text{ and } f\circ\alpha=\alpha'\circ f.$$

**Definition 2.2** *A Hom-Lie coalgebra is a triple* $(\mathfrak{g},\triangle,\alpha)$ *consisting of a* $\mathbb{K}$*-vector space* $\mathfrak{g}$*, a linear map* $\Delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ *and a linear map* $\alpha:\mathfrak{g}\to\mathfrak{g}$ *satisfying*

$$Im(\Delta)\subset\mathfrak{g}\wedge\mathfrak{g}, \tag{2.3}$$

$$(id+\xi+\xi^2)\circ(\alpha\otimes\Delta)\circ\Delta=0, \tag{2.4}$$

*where* $id,\xi:\mathfrak{g}^{\otimes 3}\to\mathfrak{g}^{\otimes 3}$*, are defined for all* $x,y,z\in\mathfrak{g}$ *by* $id(x\otimes y\otimes z)=x\otimes y\otimes z$*,* $\xi(x\otimes y\otimes z)=y\otimes z\otimes x$ *and* $\xi^2=\xi\circ\xi$*.*
*A Hom-Lie coalgebra is called multiplicative if* $\Delta\circ\alpha=\alpha^{\otimes 2}\circ\Delta$*.*

Let $(\mathfrak{g},\Delta,\alpha)$ and $(\mathfrak{g}',\Delta',\alpha')$ be two Hom-Lie coalgebras. A homomorphism $f:\mathfrak{g}\longrightarrow\mathfrak{g}'$ is said to be a Hom-Lie coalgebra morphism if

$$\Delta'\circ f=f^{\otimes 2}\circ\Delta \text{ and } f\circ\alpha=\alpha'\circ f.$$

**Definition 2.3** *A Hom-Lie bialgebra is a quadruple* $(\mathfrak{g},[\cdot,\cdot],\Delta,\alpha)$ *such that:*

(1) $(\mathfrak{g},[\cdot,\cdot],\alpha)$ *is a Hom-Lie algebra,*

(2) $(\mathfrak{g},\Delta,\alpha)$ *is a Hom-Lie coalgebra,*

(3) $\Delta$ *and* $[\cdot,\cdot]$ *satisfy, for all* $x,y\in\mathfrak{g}$*, the condition :*

$$\Delta[x,y]=[\alpha(x),\Delta y]-[\alpha(y),\Delta x] \ (\textit{compatibility condition}), \tag{2.5}$$

*where* $[x,\Delta y]=(ad_x\otimes\alpha+\alpha\otimes ad_x)\circ\Delta y$ *and* $ad_x:\mathfrak{g}\to\mathfrak{g}$ *is the adjoint map defined by* $ad_x(y)=[x,y]$*.*

Let $(\mathfrak{g},[\cdot,\cdot],\Delta,\alpha)$ and $(\mathfrak{g}',[\cdot,\cdot]',\Delta',\alpha')$ be two Hom-Lie bialgebras. A homomorphism $f:\mathfrak{g}\longrightarrow\mathfrak{g}'$ is said to be a Hom-Lie bialgebra morphism if it is both a Hom-Lie algebra morphism and a Hom-Lie coalgebra morphism.

The Hom-Lie bialgebra is said multiplicative if it is multiplicative with respect to the bracket and the cobracket.

## 3. 3-Hom-Lie bialgebras

In this section, we introduce various type of 3-Hom-Lie bialgebras : 3-Hom-Lie algebras, multiplicative 3-Hom-Lie algebras and generalized multiplicative 3-Hom-Lie algebras. We aim to generalize 3-Lie bialgebras studied in [10] and follow the approach given by Yau for binary Hom-Lie bialgebras [46]. We provide some properties, key constructions and study their dualization.

**Definition 3.1 ([7])** *A 3-Hom-Lie algebra is a quadruple $(L, \mu, \alpha, \beta)$ consisting of a $\mathbb{K}$-vector space $L$, two linear maps $\alpha, \beta : L \to L$ and a trilinear skew-symmetric multiplication $\mu : L \times L \times L \to L$ satisfying the 3-Hom-Jacobi identity: for $x, y, z, u, v$ in $L$*

$$\begin{aligned} \mu(\alpha(u), \beta(v), \mu(x, y, z)) &= \mu(\mu(u, v, x), \alpha(y), \beta(z)) + \mu(\alpha(x), \mu(u, v, y), \beta(z)) \\ &+ \mu(\alpha(x), \beta(y), \mu(u, v, z)). \end{aligned} \tag{3.1}$$

**Remark 3.2**

1. *A 3-Hom-Lie algebra $(L, \mu, \alpha, \beta)$ is called multiplicative if $\alpha = \beta$ and $\alpha \circ \mu = \mu \circ \alpha^{\otimes 3}$.*
2. *A multiplicative 3-Hom-Lie algebra $(L, \mu, \alpha)$ is called regular if $\alpha$ is bijective.*
3. *We recover the classical 3-Lie algebra when $\alpha = \beta = id$.*

Let $(L, \mu, \alpha, \beta)$ and $(L', \mu', \alpha', \beta')$ be two 3-Hom-Lie algebras. A homomorphism $f : L \longrightarrow L'$ is said to be a 3-Hom-Lie algebra morphism if

$$f \circ \alpha = \alpha' \circ f, \quad f \circ \beta = \beta' \circ f \text{ and } f \circ \mu = \mu' \circ f^{\otimes 3}.$$

It is called 3-Hom-Lie algebra isomorphism if $f$ is bijective.

**Proposition 3.3** *Let $(L, \mu)$ be a 3-Lie algebra and $\alpha : L \to L$ be an algebra morphism. Then $(L, \mu_\alpha, \alpha)$, where $\mu_\alpha = \alpha \circ \mu$, is a multiplicative 3-Hom-Lie algebra.*

**Remark 3.4** *More generally, if $(L, \mu, \alpha, \beta)$ is a 3-Hom-Lie algebra and $\gamma : L \to L$ is an algebra morphism, then $(L, \mu_\gamma, \gamma \circ \alpha, \gamma \circ \beta)$, where $\mu_\gamma = \gamma \circ \mu$, is a 3-Hom-Lie algebra.*

**Remark 3.5** *In order to dualize the structure 3-Hom-Lie algebra, we consider its following equivalent description. Given a permutation $\sigma \in S_3$, we consider the map $\mathrm{p}_\sigma : L^{\otimes 3} \to L^{\otimes 3}$, defined for all $x_1, x_2, x_3 \in L$ by $\mathrm{p}_\sigma(x_1, x_2, x_3) = sign(\sigma)\ x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}$. We denote by $id_s : L^{\otimes s} \to L^{\otimes s}$ the identity map. We drop the subscript $s$ when there is no ambiguity. In the sequel $\tau_{ij}$ stands for the transposition in $S_5$ flipping the element at positions $i$ and $j$.*

*A 3-Hom-Lie algebra $(L, \mu, \alpha, \beta)$ is a $\mathbb{K}$-vector space $L$ with three linear maps $\mu : L^{\otimes 3} \to L$ and $\alpha, \beta : L \to L$ satisfying*

$$\mu(id_3 - \mathrm{p}_\sigma) = 0 \quad \forall \sigma \in S_3, \tag{3.2}$$

$$\mu \circ (\alpha \otimes \beta \otimes \mu) \circ (id_5 - \tau_{13} \circ \tau_{24} \circ (id_5 + \xi' + \tau_{34} \circ \xi'^2)) = 0, \tag{3.3}$$

*where $\xi' : L^{\otimes 5} \to L^{\otimes 5}$, defined as $\xi' = id \otimes id \otimes \xi$ with $\xi$ defined above. We have $\xi'^2 = \xi' \circ \xi' = id \otimes id \otimes \xi^2$.*

*In the case of multiplicative 3-Hom-Lie algebra $(L,\mu,\alpha)$ condition (3.3) becomes*

$$\mu\circ(\alpha\otimes\alpha\otimes\mu)\circ(id_5-\tau_{13}\circ\tau_{24}\circ(id_5+\xi'+\xi'^2))=0. \tag{3.4}$$

**Definition 3.6** *A 3-Hom-Lie coalgebra is a quadruple $(L,\Delta,\alpha,\beta)$ consisting of a $\mathbb{K}$-vector space $L$ with two endomorphisms $\alpha,\beta:L\to L$ and a linear map $\Delta:L\to L\otimes L\otimes L$ satisfying*

$$Im(\Delta)\subset L\wedge L\wedge L, \tag{3.5}$$

$$(id-\tau_{13}\circ\tau_{24}\circ(id+\xi'+\tau_{34}\circ\xi'^2))\circ(\alpha\otimes\beta\otimes\Delta)\circ\Delta=0 \quad \text{(3-Hom-co-Jacobi identity)}. \tag{3.6}$$

*A 3-Hom-Lie coalgebra $(L,\Delta,\alpha,\beta)$ is called multiplicative if $\alpha=\beta$ and $\Delta\circ\alpha=\alpha^{\otimes 3}\circ\Delta$. In this case the 3-Hom-co-Jacobi identity reduces to*

$$(id-\tau_{13}\circ\tau_{24}\circ(id+\xi'+\xi'^2))\circ(\alpha\otimes\alpha\otimes\Delta)\circ\Delta=0.$$

*Let $(L,\Delta,\alpha,\beta)$ and $(L',\Delta',\alpha',\beta')$ be two 3-Hom-Lie coalgebras. A homomorphism $f:L\longrightarrow L'$ is said to be a 3-Hom-Lie coalgebra morphism if*

$$f\circ\alpha=\alpha'\circ f,\quad f\circ\beta=\beta'\circ f \text{ and } f^{\otimes^3}\circ\Delta=\Delta'\circ f.$$

*It is called 3-Hom-Lie coalgebra isomorphism if $f$ is bijective.*

**Remark 3.7**

1. *We recover 3-Lie coalgebras when $\alpha=\beta=id$.*
2. *A multiplicative 3-Hom-Lie coalgebra $(L,\Delta,\alpha)$ is called regular if $\alpha$ is bijective.*

**Definition 3.8** *A 3-Hom-Lie bialgebra is a 5-tuple $(L,\mu,\Delta,\alpha,\beta)$ consisting of a $\mathbb{K}$-vector space $L$ such that*

1) *$(L,\mu,\alpha,\beta)$ is a 3-Hom-Lie algebra,*

2) *$(L,\Delta,\alpha,\beta)$ is a 3-Hom-Lie coalgebra,*

3) *$\Delta$ and $\mu$ satisfy the following compatibility condition:*

$$\Delta\circ\mu(x,y,z)=ad_\mu^{(3)}(\alpha(x),\beta(y))\bullet_{\alpha,\beta}\Delta(z)+ad_\mu^{(3)}(\alpha(y),\beta(z))\bullet_{\alpha,\beta}\Delta(x)-ad_\mu^{(3)}(\alpha(x),\beta(z))\bullet_{\alpha,\beta}\Delta(y), \tag{3.7}$$

*where*

$$\begin{aligned} ad_\mu^{(3)}(x,y)\bullet_{\alpha,\beta}(u\otimes v\otimes w) &= (ad_\mu(x,y)\otimes\alpha\otimes\beta)(u\otimes v\otimes w)+(\alpha\otimes ad_\mu(x,y)\otimes\beta)(u\otimes v\otimes w)\\ &+ (\alpha\otimes\beta\otimes ad_\mu(x,y))(u\otimes v\otimes w)\\ &= \mu(x,y,u)\otimes\alpha(v)\otimes\beta(w)+\alpha(u)\otimes\mu(x,y,v)\otimes\beta(w)\\ &+ \alpha(u)\otimes\beta(v)\otimes\mu(x,y,w) \end{aligned} \tag{3.8}$$

*and $ad_\mu:L\wedge L\to \mathrm{End}(L):z\mapsto ad_\mu(x,y)(z)=\mu(x,y,z)$, for $x,y,z,u,v,w\in L$.*

A 3-Hom-Lie bialgebra $(L,\mu,\Delta,\alpha,\beta)$ is called multiplicative if $\alpha=\beta$, $\alpha\circ\mu=\mu\circ\alpha^{\otimes 3}$ and $\alpha^{\otimes 3}\circ\Delta=\Delta\circ\alpha$. In this case the compatibility condition, Eq. (3.7), reduces to

$$\begin{aligned}\Delta\circ\mu(x,y,z) &= ad_\mu^{(3)}(\alpha(x),\alpha(y))\bullet_\alpha \Delta(z)+ad_\mu^{(3)}(\alpha(y),\alpha(z))\bullet_\alpha \Delta(x)+ad_\mu^{(3)}(\alpha(z),\alpha(x))\bullet_\alpha \Delta(y)\\ &= \circlearrowleft_{x,y,z} ad_\mu^{(3)}(\alpha(x),\alpha(y))\bullet_\alpha \Delta(z),\end{aligned}\tag{3.9}$$

where $x,y,z\in L$ and $\bullet_\alpha := \bullet_{\alpha,\alpha}$.

A multiplicative 3-Hom-Lie bialgebra $(L,\mu,\Delta,\alpha)$ is called regular if $\alpha$ is bijective.

**Remark 3.9** *We say that the 3-Hom-Lie bialgebras and multiplicative 3-Hom-Lie bialgebras defined above are of Yau-type in contrast with 3-Hom-Lie bialgebras defined using Manin triples, see [13, 29]. The two approaches are not equivalent.*

**Definition 3.10** *Let $(L,\mu,\Delta,\alpha,\beta)$ and $(L',\mu',\Delta',\alpha',\beta')$ be two 3-Hom-Lie bialgebras. A homomorphism $f:L\to L'$ is said to be a 3-Hom-Lie bialgebra morphism if it both a 3-Hom-Lie algebra and 3-Hom-Lie coalgebra morphism. It is a 3-Hom-Lie bialgebra isomorphism if in addition $f$ is bijective.*

**Remark 3.11** *We recover 3-Lie bialgebras when $\alpha=\beta=id$ (see [10]).*

In the following, we define Yau-type generalized multiplicative 3-Hom-Lie bialgebras that generalize slightly the concept of multiplicative 3-Hom-Lie bialgebra and allow to consider two twist endomorphisms.

**Definition 3.12** *A generalized multiplicative 3-Hom-Lie bialgebra is a tuple $(L,\mu,\alpha_1,\Delta,\alpha_2)$ such that*

1) *$(L,\mu,\alpha_1)$ is a multiplicative 3-Hom-Lie algebra,*

2) *$(L,\Delta,\alpha_2)$ is a multiplicative 3-Hom-Lie coalgebra,*

3) *the following compatibility conditions hold:*

$$\alpha_1^{\otimes 3}\circ\Delta=\Delta\circ\alpha_1,\quad \alpha_2\circ\mu=\mu\circ\alpha_2^{\otimes 3},\quad \alpha_1\circ\alpha_2=\alpha_2\circ\alpha_1,$$
$$\Delta\mu(x,y,z)=\circlearrowleft_{x,y,z} ad_\mu^{(3)}(\alpha_2(x),\alpha_2(y))\bullet_{\alpha_1}\Delta(z)$$

*for all $x,y,z\in L$ , where*

$$ad_\mu^{(3)}(x,y)\bullet_{\alpha_1}\Delta=(ad_\mu(x,y)\otimes\alpha_1\otimes\alpha_1+\alpha_1\otimes ad_\mu(x,y)\otimes\alpha_1+\alpha_1\otimes\alpha_1\otimes ad_\mu(x,y))\circ\Delta.$$

**Remark 3.13** *When $\alpha_1=\alpha_2$, we recover Yau-type multiplicative 3-Hom-Lie bialgebras.*

### 3.1. Twist constructions

In this section, we describe a way to construct a 3-Hom-Lie algebra along algebra homomorphisms.

**Proposition 3.14** *Let $(L,\Delta,\alpha,\beta)$ be a 3-Hom-Lie coalgebra and $\gamma:L\to L$ be a 3-Hom-Lie coalgebra morphism. Then $L_\gamma=(L,\Delta_\gamma,\gamma\circ\alpha,\gamma\circ\beta)$ is a 3-Hom-Lie coalgebra, where $\Delta_\gamma=\Delta\circ\gamma=\gamma^{\otimes 3}\circ\Delta$.*

**Proof** Using Definition 3.6, we have $Im(\Delta_\gamma)\subset Im\Delta\subset L\wedge L\wedge L$.

$$\begin{aligned}&(id-\tau_{13}\circ\tau_{24}\circ(id+\xi'+\tau_{34}\xi'^2))\circ((\gamma\circ\alpha)\otimes(\gamma\circ\beta)\otimes\Delta_\gamma)\circ\Delta_\gamma\\&=(id-\tau_{13}\circ\tau_{24}\circ(id+\xi'+\tau_{34}\xi'^2))((\gamma\circ\alpha)\otimes(\gamma\circ\beta)\otimes(\gamma^{\otimes^3}\circ\Delta))\circ(\gamma^{\otimes^3}\circ\Delta)\\&=(id-\tau_{13}\circ\tau_{24}\circ(id+\xi'+\tau_{34}\xi'^2))((\gamma\circ\alpha)\circ\gamma\otimes(\gamma\circ\beta)\circ\gamma\otimes\gamma^{\otimes^3}\circ\Delta\circ\gamma)\circ\Delta)\\&=(id-\tau_{13}\circ\tau_{24}\circ(id+\xi'+\tau_{34}\xi'^2))(\gamma^2\circ\alpha\otimes\gamma^2\circ\beta\otimes(\gamma^2\circ)^{\otimes3}\circ\Delta)\circ\Delta\\&=(\gamma^2)^{\otimes5}\circ(id-\tau_{13}\circ\tau_{24}\circ(id+\xi'+\tau_{34}\xi'^2))(\alpha\otimes\beta\otimes\Delta)\circ\Delta\\&=0.\end{aligned}$$

Then $(L,\Delta_\gamma,\gamma\circ\alpha,\gamma\circ\beta)$ is a 3-Hom-Lie coalgebra. □

**Theorem 3.15** *Let $(L,\mu,\Delta,\alpha,\beta)$ be a 3-Hom-Lie bialgebra and $\gamma:L\to L$ be a 3-Hom-Lie bialgebra morphism. Then $L_\gamma=(L,\mu_\gamma,\Delta_\gamma,\gamma\circ\alpha,\gamma\circ\beta)$ is a 3-Hom-Lie bialgebra, where $\Delta_\gamma=\Delta\circ\gamma$ and $\mu_\gamma=\gamma\circ\mu$.*

**Proof** According to Propositions 3.3 and 3.14, one needs only to show that the compatibility condition (3.7) holds. Indeed, we have

$$\Delta_\gamma\circ\mu_\gamma(x,y,z)=\Delta\circ\gamma^2\circ\mu(x,y,z)=(\gamma^2)^{\otimes3}\circ\Delta\circ\mu(x,y,z),$$

where $\gamma^2=\gamma\circ\gamma$.

Now for $x,y,z\in L$, we have

$$\begin{aligned}&ad^{(3)}_{\mu_\gamma}(\gamma\circ\alpha(x),\gamma\circ\beta(y))\bullet_{\gamma\circ\alpha,\gamma\circ\beta}\Delta_\gamma(z)=\mu_\gamma(\gamma\circ\alpha(x),\gamma\circ\beta(y),\gamma(z^{(1)}))\otimes\gamma\circ\alpha\circ\gamma(z^{(2)})\otimes\gamma\circ\beta\circ\gamma(z^{(3)})\\&\quad+\gamma\circ\alpha\circ\gamma(z^{(1)})\otimes\mu_\gamma(\gamma\circ\alpha(x),\gamma\circ\beta(y),\gamma(z^{(2)}))\otimes\gamma\circ\beta\circ\gamma(z^{(3)})\\&\quad+\gamma\circ\alpha\circ\gamma(z^{(1)})\otimes\gamma\circ\beta\circ\gamma(z^{(2)})\otimes\mu_\gamma(\gamma\circ\alpha(x),\gamma\circ\beta(y),\gamma(z^{(3)}))\\&=\mu(\gamma^2\circ\alpha(x),\gamma^2\circ\beta(y),\gamma^2(z^{(1)}))\otimes\gamma^2\circ\alpha(z^{(2)})\otimes\gamma^2\circ\beta(z^{(3)})\\&\quad+\gamma^2\circ\alpha(z^{(1)})\otimes\mu(\gamma^2\circ\alpha(x),\gamma^2\circ\beta(y),\gamma^2(z^{(2)}))\otimes\gamma^2\circ\beta(z^{(3)})\\&\quad+\gamma^2\circ\alpha(z^{(1)})\otimes\gamma^2\circ\beta(z^{(2)})\otimes\mu(\gamma^2\circ\alpha(x),\gamma^2\circ\beta(y),\gamma^2(z^{(3)}))\\&=(\gamma^2)^{\otimes3}(\mu(\alpha(x),\beta(y),(z^{(1)}))\otimes\alpha(z^{(2)})\otimes\beta(z^{(3)})\\&\quad+\alpha(z^{(1)})\otimes\mu(\alpha(x),\beta(y),(z^{(2)}))\otimes\beta(z^{(3)})+\alpha(z^{(1)})\otimes\beta(z^{(2)})\otimes\mu(\alpha(x),\beta(y),z^{(3)}))\\&=(\gamma^2)^{\otimes3}\circ ad^3_{\mu(\alpha(x),\beta(y))}\bullet_{\alpha,\beta}\Delta(z).\end{aligned}$$

Then

$$\begin{aligned}\Delta_\gamma\circ\mu_\gamma(x,y,z)&=ad^{(3)}_{\mu_\gamma}(\gamma\circ\alpha(x),\gamma\circ\beta(y))\bullet_{\gamma\circ\alpha,\gamma\circ\beta}\Delta_\gamma(z)-ad^{(3)}_{\mu_\gamma}(\gamma\circ\alpha(x),\gamma\circ\beta(z))\bullet_{\gamma\circ\alpha,\gamma\circ\beta}\Delta_\gamma(y)\\&+ad^{(3)}_{\mu_\gamma}(\gamma\circ\alpha(y),\gamma\circ\beta(z))\bullet_{\gamma\circ\alpha,\gamma\circ\beta}\Delta_\gamma(x)\\&=(\gamma^2)^{\otimes3}\circ ad^3_{\mu(\alpha(x),\beta(y))}\bullet_{\alpha,\beta}\Delta(z)-(\gamma^2)^{\otimes3}\circ ad^3_{\mu(\alpha(x),\beta(z))}\bullet_{\alpha,\beta}\Delta(y)\\&+(\gamma^2)^{\otimes3}\circ ad^3_{\mu(\alpha(y),\beta(z))}\bullet_{\alpha,\beta}\Delta(x)\\&=(\gamma^2)^{\otimes3}\circ\Delta\circ\mu(x,y,z).\end{aligned}$$

□

**Definition 3.16** *Let* $(L,\mu,\Delta,\alpha)$ *be a multiplicative* 3*-Hom-Lie bialgebra. If there exists a* 3*-Lie bialgebra* $(L,\mu',\Delta')$ *such that* $\mu=\alpha\circ\mu'$ *and* $\Delta=\alpha^{\otimes 3}\circ\Delta'$*, then* $(L,\mu,\Delta,\alpha)$ *is said to be of* 3*-Lie-type and* $(L,\mu',\Delta')$ *is called the induced* 3*-Lie bialgebra of* $(L,\mu,\Delta,\alpha)$.

**Remark 3.17** *Let* $(L,\mu,\Delta,\alpha)$ *be a regular multiplicative* 3*-Hom-Lie bialgebra. Then* $(L,\mu,\Delta,\alpha)$ *is of* 3*-Lie-type with the induced* 3*-Lie bialgebra* $(L,\mu',\Delta')$*, where* $\mu'=\alpha^{-1}\circ\mu$ *and* $\Delta'=\Delta\circ\alpha^{-1}$.

**Example 3.18** *Let* $(L,\mu,\Delta)$ *be a* 3*-dimensional* 3*-Lie bialgebra, where the multiplication and comultiplication are defined with respect to a basis* $\{e_1,e_2,e_3\}$ *by* $\mu(e_1,e_2,e_3)=e_1$ *and* $\Delta(e_1)=e_1\wedge e_2\wedge e_3$.
*The linear map* $\alpha:L\longrightarrow L$ *defined by* $\alpha(e_1)=-e_1,\ \alpha(e_2)=b_1e_1+b_2e_3,\ \alpha(e_3)=c_1e_1-\frac{1}{b_2}e_2+c_2e_3,\quad b_2\neq 0.$
*is a* 3*-Lie bialgebra morphism.*
*Using Theorem 3.15, the quadruple* $(L,\mu_\alpha,\Delta_\alpha,\alpha)$ *is a multiplicative* 3*-Hom-Lie bialgebra.*
*We have* $\mu_\alpha(e_1,e_2,e_3)=-e_1$ *and* $\Delta_\alpha(e_1)=-e_1\wedge e_2\wedge e_3$.

**Example 3.19** *Let* $(L,\mu,\Delta)$ *be a* 4*-dimensional* 3*-Lie bialgebra, where the multiplication and comultiplication are defined with respect to a basis* $\{e_1,e_2,e_3,e_4\}$ *by* $\mu(e_1,e_3,e_4)=e_1,\quad \mu(e_2,e_3,e_4)=e_2,\ \Delta(e_1)=e_3\wedge e_2\wedge e_4,\quad \Delta(e_3)=e_1\wedge e_2\wedge e_4$*. The linear map* $\alpha:L\longrightarrow L$ *defined by* $\alpha(e_1)=a_1e_1+a_2e_2,\ \alpha(e_2)=a_1e_2,\ \alpha(e_3)=a_3e_2-a_1e_3+a_4e_4,\quad \alpha(e_4)=a_5e_2-\frac{1}{a_1}e_4,\quad a_1\neq 0$*, is a* 3*-Lie bialgebra morphism.*
*Using Theorem 3.15, the quadruple* $(L,\mu_\alpha,\Delta_\alpha,\alpha)$ *is a multiplicative* 3*-Hom-Lie bialgebra. We have*

$$\mu_\alpha(e_1,e_3,e_4)=a_1e_1+a_2e_2,\quad \mu_\alpha(e_2,e_3,e_4)=a_1e_2,$$
$$\Delta_\alpha(e_1)=a_1e_3\wedge e_2\wedge e_4,\quad \Delta_\alpha(e_3)=-a_1e_1\wedge e_2\wedge e_4.$$

**Proposition 3.20** *Let* $(L,\mu,\Delta)$ *be a* 3*-Lie bialgebra and* $\alpha_1,\alpha_2:L\to L$ *be two commuting* 3*-Lie bialgebra morphisms. Then* $(L,\mu_{\alpha_1},\alpha_1,\Delta_{\alpha_2},\alpha_2)$ *is a generalized multiplicative* 3*-Hom-Lie bialgebra, where* $\mu_{\alpha_1}=\alpha_1\circ\mu$*, and* $\Delta_{\alpha_2}=\Delta\circ\alpha_2$.

**Proof** Using Propositions 3.3 and 3.14, we have that $(L,\mu,\alpha_1)$ is a 3-Hom-Lie algebra and $(L,\Delta,\alpha_2)$ and $(L,\Delta,\alpha_2)$ is a 3-Hom-Lie algebra, respectively.
Now, let $x,y,z\in L$,

$$\begin{aligned}
&ad^{(3)}_{\mu_{\alpha_1}}(\alpha_2(x),\alpha_2(y))\bullet_{\alpha_1}\Delta_{\alpha_2}(z)+ad^{(3)}_{\mu_{\alpha_1}}(\alpha_2(x),\alpha_2(z))\bullet_{\alpha_1}\Delta_{\alpha_2}(y)+ad^{(3)}_{\mu_{\alpha_1}}(\alpha_2(y),\alpha_2(z))\bullet_{\alpha_1}\Delta_{\alpha_2}(x)\\
&=(ad_{\mu_{\alpha_1}}(\alpha_2(x),\alpha_2(y))\otimes\alpha_1\otimes\alpha_1+\alpha_1\otimes ad_{\mu_{\alpha_1}}(\alpha_2(x),\alpha_2(y))\otimes\alpha_1+\alpha_1\otimes\alpha_1\otimes ad_{\mu_{\alpha_1}}(\alpha_2(x),\alpha_2(y)))\circ\Delta_{\alpha_2}(z)\\
&+(ad_{\mu_{\alpha_1}}(\alpha_2(y),\alpha_2(z))\otimes\alpha_1\otimes\alpha_1+\alpha_1\otimes ad_{\mu_{\alpha_1}}(\alpha_2(y),\alpha_2(z))\otimes\alpha_1+\alpha_1\otimes\alpha_1\otimes ad_{\mu_{\alpha_1}}(\alpha_2(y),\alpha_2(z)))\circ\Delta_{\alpha_2}(x)\\
&+(ad_{\mu_{\alpha_1}}(\alpha_2(z),\alpha_2(x))\otimes\alpha_1\otimes\alpha_1+\alpha_1\otimes ad_{\mu_{\alpha_1}}(\alpha_2(z),\alpha_2(x))\otimes\alpha_1+\alpha_1\otimes\alpha_1\otimes ad_{\mu_{\alpha_1}}(\alpha_2(z),\alpha_2(x)))\circ\Delta_{\alpha_2}(y)\\
&=(\alpha_1\alpha_2\otimes\alpha_1\alpha_2\otimes\alpha_1\alpha_2)(ad_\mu(x,y)\bullet\Delta(z)+ad_\mu(y,z)\bullet\Delta(x)+ad_\mu(z,x)\bullet\Delta(y))\\
&=(\alpha_1\alpha_2\otimes\alpha_1\alpha_2\otimes\alpha_1\alpha_2)\Delta\mu(x,y,z)\\
&=\Delta_{\alpha_2}\mu_{\alpha_1}(x,y,z).
\end{aligned}$$

□

**Example 3.21** *Let* $(L, \mu, \Delta)$ *be a* 4*-dimensional* 3*-Lie bialgebra, where the multiplication and comultiplication are defined with respect to a basis* $\{e_1, e_2, e_3, e_4\}$ *by* $\mu(e_1, e_3, e_4) = a_1e_1 + a_2e_2, \quad \mu(e_2, e_3, e_4) = a_1e_2, \quad a_1 \neq 0,$ *and* $\Delta(e_1) = e_3 \wedge e_2 \wedge e_4, \quad \Delta(e_3) = e_1 \wedge e_2 \wedge e_4$*.*
*We consider the linear map* $\alpha : L \longrightarrow L$ *defined by* $\alpha_1(e_1) = a_1e_1 + a_2e_2, \ \alpha_1(e_2) = a_1e_2, \ \alpha_1(e_3) = a_3e_2 - a_1e_3 + a_4e_4, \ \alpha_1(e_4) = a_5e_2 - \frac{1}{a_1}e_4, \quad a_1 \neq 0.$
*Then* $(L, \mu_{\alpha_1} = \alpha_1 \circ \mu, \alpha_1, \Delta, id)$ *is a generalized multiplicative* 3*-Hom-Lie bialgebra.*

**Remark 3.22** *Similarly to Proposition 3.15, one may construct a new generalized* 3*-Hom-Lie bialgebra starting with a given generalized* 3*-Hom-Lie bialgebra and a pair of commuting generalized* 3*-Hom-bialgebra morphisms.*

### 3.2. Dualization of multiplicative 3-Hom-Lie bialgebras

In this section, we give the relationships between multiplicative 3-Hom-Lie bialgebras and their dual. In the following, we consider $L$ to be a finite dimensional vector space.

**Proposition 3.23** *A triple* $(L, \Delta, \alpha)$ *defines a multiplicative* 3*-Hom-Lie coalgebra if and only if* $(L^*, \Delta^*, \alpha^*)$ *is a multiplicative* 3*-Hom-Lie algebra, where* $L^*$ *is the dual space of* $L$ *and* $\Delta^*$ *and* $\alpha^*$ *are the dual mapping of* $\mu$ *and* $\alpha$ *respectively (i.e.* $< \Delta^*(\chi \otimes \eta \otimes \zeta), x >=< \chi \otimes \eta \otimes \zeta, \Delta(x) >$ *and* $< \alpha^*(\chi), x >=< \chi, \alpha(x) >$*) for all* $\chi, \eta, \zeta \in L^*$ *and* $x \in L$*.*

**Proof** Let $f, g, \chi, \eta, \zeta \in L^*$ and $x \in L$. Then we have

$$\begin{aligned}
&< \Delta^* \circ (\alpha^* \otimes \alpha^* \otimes \Delta^*) \circ (id_5 - \tau_{13} \circ \tau_{24} \circ (id_5 + \xi' + \xi'^2))(f \otimes g \otimes \chi \otimes \eta \otimes \zeta), x > \\
&=< (\alpha^* \otimes \alpha^* \otimes \Delta^*) \circ (id_5 - \tau_{13} \circ \tau_{24} \circ (id_5 + \xi' + \xi'^2))(f \otimes g \otimes \chi \otimes \eta \otimes \zeta), \Delta(x) > \\
&=< f \otimes g \otimes \chi \otimes \eta \otimes \zeta, (id_5 - \tau_{13} \circ \tau_{24} \circ (id_5 + \xi' + \xi'^2)) \circ (\alpha \otimes \alpha \otimes \Delta) \circ \Delta(x) > .
\end{aligned}$$

□

**Proposition 3.24** *Let* $(L_1, \Delta_1, \alpha_1)$ *and* $(L_2, \Delta_2, \alpha_2)$ *be two multiplicative* 3*-Hom-Lie coalgebras. Then* $\varphi : L_1 \to L_2$ *is a* 3*-Hom-Lie coalgebra isomorphism from* $(L_1, \Delta_1, \alpha_1)$ *to* $(L_2, \Delta_2, \alpha_2)$ *if and only if the dual mapping* $\varphi^* : L_2^* \to L_1^*$ *is a* 3*-Hom-Lie algebra isomorphism from* $(L_2^*, \Delta_2^*, \alpha_2^*)$ *to* $(L_1^*, \Delta_1^*, \alpha_1^*)$*, where* $< \varphi^*(\chi), v >=< \chi, \varphi(v) >$ *for every* $\chi \in L_2^*$ *and* $v \in L_1$*.*

**Proof**
Let $(L_1, \Delta_1, \alpha_1)$ and $(L_2, \Delta_2, \alpha_2)$ be two multiplicative 3-Hom-Lie coalgebras. Then, using Proposition 3.23, $(L_1^*, \Delta_1^*, \alpha_1^*)$ and $(L_2^*, \Delta_2^*, \alpha_2^*)$ are two multiplicative 3-Hom-Lie algebras. Let $\varphi : L_1 \to L_2$ be a multiplicative 3-Hom-Lie coalgebra isomorphism from $(L_1, \Delta_1, \alpha_1)$ and $(L_2, \Delta_2, \alpha_2)$. Then, the dual mapping $\varphi^* : L_2^* \to L_1^*$ is a bijective linear map. For every $\chi, \eta, \zeta \in L_2^*$ and $x \in L_1$, we have

$$< \varphi^* \circ \alpha_2^*(\chi), x >=< \chi, \alpha_2 \circ \varphi(x) >=< \chi, \varphi \circ \alpha_1(x) >=< \alpha_1^* \circ \varphi^*(\chi), x >,$$

and

$$\begin{aligned}
< \varphi^* \circ \Delta_2^*(\chi, \eta, \zeta), x > \quad &= \quad < \chi \otimes \eta \otimes \zeta, \Delta_2(\varphi(x)) >=< \chi \otimes \eta \otimes \zeta, (\varphi \otimes \varphi \otimes \varphi) \circ \Delta_1(x) > \\
&= \quad < \varphi^*(\chi) \otimes \varphi^*(\eta) \otimes \varphi^*(\zeta), \Delta_1(x) >=< \Delta_1^*(\varphi^*(\chi), \varphi^*(\eta), \varphi^*(\zeta)), x > .
\end{aligned}$$

Then $\varphi^* \circ \Delta_2^* = \Delta_1^* \circ (\varphi^* \otimes \varphi^* \otimes \varphi^*)$, that is $\varphi^*$ is a 3-Hom-Lie algebra isomorphism. □

**Theorem 3.25** *Let $(L,\mu,\Delta,\alpha)$ be a multiplicative 3-Hom-Lie bialgebra. Then $(L^*,\Delta^*,\mu^*,\alpha^*)$ is a multiplicative 3-Hom-Lie bialgebra, which is called the dual multiplicative 3-Hom-Lie bialgebra of $(L,\mu,\Delta,\alpha)$.*

**Proof** As $(L,\mu,\Delta,\alpha)$ is a multiplicative 3-Hom-Lie bialgebra, by Proposition 3.23, $(L^*,\Delta^*,\alpha^*)$ is a multiplicative 3-Hom-Lie algebra, and $(L^*,\mu^*,\alpha^*)$ is a multiplicative 3-Hom-Lie coalgebra. Now, we prove that $(\Delta^*,\mu^*)$ satisfy the compatibility condition (3.9), that is the following identity holds for every $\chi,\eta,\zeta\in L^*$,

$$\mu^*(\Delta^*(\chi,\eta,\zeta))=ad^{(3)}_{\Delta^*}(\alpha^*(\chi),\alpha^*(\eta))\bullet_{\alpha*}\mu^*(\zeta)+ad^{(3)}_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))\bullet_{\alpha*}\mu^*(\chi)+ad^{(3)}_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))\bullet_{\alpha*}\mu^*(\eta). \tag{3.10}$$

Let $x,y,z\in L$ and $\chi,\eta,\zeta\in L^*$, we have

$$\begin{aligned}&<\mu^*\circ\Delta^*(\chi,\eta,\zeta),x\otimes y\otimes z>=<\chi\otimes\eta\otimes\zeta,\Delta\mu(x,y,z)>\\&=<\chi\otimes\eta\otimes\zeta,ad^{(3)}_\mu(\alpha(x),\alpha(y))\bullet_\alpha\Delta(z)>+<\chi\otimes\eta\otimes\zeta,ad^{(3)}_\mu(\alpha(z),\alpha(x))\bullet_\alpha\Delta(y)>\\&+<\chi\otimes\eta\otimes\zeta,ad^{(3)}_\mu(\alpha(y),\alpha(z))\bullet_\alpha\Delta(x)>.\end{aligned}$$

Without loss of generality, suppose that $\Delta(z)=\sum_i a_i\otimes b_i\otimes c_i$, where $a_i,b_i,c_i\in L$. Then,

$$\begin{aligned}&<\chi\otimes\eta\otimes\zeta,ad^{(3)}_\mu(\alpha(x),\alpha(y))\bullet_\alpha\Delta(z)>=<\chi\otimes\eta\otimes\zeta,ad^{(3)}_\mu(\alpha(x),\alpha(y))(\sum_i a_i\otimes b_i\otimes c_i)>\\&=<\chi\otimes\eta\otimes\zeta,\sum_i ad_\mu(\alpha(x),\alpha(y))(a_i)\otimes\alpha(b_i)\otimes\alpha(c_i)\\&\quad+\alpha(a_i)\otimes ad_\mu(\alpha(x),\alpha(y))(b_i)\otimes\alpha(c_i))+\alpha(a_i)\otimes\alpha(b_i))\otimes ad_\mu(\alpha(x),\alpha(y))(c_i)>\\&=-\sum_i<ad^*_\mu(\alpha(x),\alpha(y))\chi\otimes\alpha^*(\eta)\otimes\alpha^*(\zeta)+\alpha^*(\chi)\otimes ad^*_\mu(\alpha(x),\alpha(y))(\eta)\otimes\alpha^*(\zeta)\\&\quad+\alpha^*(\chi)\otimes\alpha^*(\eta)\otimes ad^*_\mu(\alpha(x),\alpha(y))\zeta,a_i\otimes b_i\otimes c_i>\\&=-<ad^*_\mu(\alpha(x),\alpha(y))(\chi)\otimes\alpha^*(\eta)\otimes\alpha^*(\zeta)+\alpha^*(\chi)\otimes ad^*_\mu(\alpha(x),\alpha(y))(\eta)\otimes\alpha^*(\zeta)\\&\quad+\alpha^*(\chi)\otimes\alpha^*(\eta)\otimes ad^*_\mu(\alpha(x),\alpha(y))(\zeta),\Delta(z)>\\&=-<\Delta^*(ad^*_\mu(\alpha(x),\alpha(y))(\chi),\alpha^*(\eta),\alpha^*(\zeta))+\Delta^*(\alpha^*(\chi),ad^*_\mu(\alpha(x),\alpha(y))(\eta),\alpha^*(\zeta))\\&\quad+\Delta^*(\alpha^*(\chi),\alpha^*(\eta),ad^*_\mu(\alpha(x),\alpha(y))(\zeta),z>\\&=-<ad_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))(ad^*_\mu(\alpha(x),\alpha(y))(\chi))+ad_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))(ad^*_\mu(\alpha(x),\alpha(y))(\eta))\\&\quad+ad_{\Delta^*}(\alpha^*(\chi),\alpha^*(\eta))(ad^*_\mu(\alpha(x),\alpha(y))(\zeta)),z>.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&<\chi\otimes\eta\otimes\zeta,ad^{(3)}_\mu(\alpha(z),\alpha(x))\bullet_\alpha\Delta(y)>=-<ad_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))(ad^*_\mu(\alpha(z),\alpha(x))(\chi))\\&+ad_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))(ad^*_\mu(\alpha(z),\alpha(x))(\eta))+ad_{\Delta^*}(\alpha^*(\chi),\alpha^*(\eta))(ad^*_\mu(\alpha(z),\alpha(x))(\zeta)),y>,\end{aligned}$$

and

$$\begin{aligned}&<\chi\otimes\eta\otimes\zeta,ad^{(3)}_\mu(\alpha(y),\alpha(z))\bullet_\alpha\Delta(x)>=-<ad_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))(ad^*_\mu(\alpha(y),\alpha(z))(\chi))\\&+ad_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))(ad^*_\mu(\alpha(y),\alpha(z))(\eta))+ad_{\Delta^*}(\alpha^*(\chi),\alpha^*(\eta))(ad^*_\mu(\alpha(y),\alpha(z))(\zeta)),x>.\end{aligned}$$

As well as

$$-<ad_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))(ad^*_\mu(\alpha(x),\alpha(y))(\chi)),z>-<ad_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))(ad^*_\mu(\alpha(y),\alpha(z))(\chi)),x>$$
$$-<ad_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))(ad^*_\mu(\alpha(z),\alpha(x))(\chi)),y>=-<\chi,\mu(\alpha(x),\alpha(y),ad^*_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))z)>$$
$$-<\chi,\mu(ad^*_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))x,\alpha(y),\alpha(z))>-<\chi,\mu(\alpha(x),ad^*_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))y,\alpha(z))>$$
$$=-<\mu^*(\chi),ad^{*(3)}_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))(x\otimes y\otimes z)>=<ad^{(3)}_{\Delta^*}(\alpha^*(\eta),\alpha^*(\zeta))\bullet_{\alpha*}\mu^*(\chi),x\otimes y\otimes z>,$$

$$-<ad_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))(ad^*_\mu(\alpha(x),\alpha(y))(\eta)),z>-<ad_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))(ad^*_\mu(\alpha(y),\alpha(z))(\eta)),x>$$
$$-<ad_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))(ad^*_\mu(\alpha(z),\alpha(x))(\eta)),y>=-<\mu^*(\eta),ad^{*(3)}_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))(x\otimes y\otimes z)>$$
$$=<ad^{(3)}_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))\bullet_{\alpha*}\mu^*(\eta),x\otimes y\otimes z>,$$

$$-<ad_{\Delta^*}(\alpha^*(\chi),\alpha^*(\eta))(ad^*_\mu(\alpha(x),\alpha(y))(\zeta)),z>-<ad_{\Delta^*}(\alpha^*(\chi),\alpha^*(\eta))(ad^*_\mu(\alpha(y),\alpha(z))(\zeta)),x>$$
$$-<ad_{\Delta^*}(\alpha^*(\chi),\alpha^*(\eta))(ad^*_\mu(\alpha(z),\alpha(x))(\zeta)),y>=-<\mu^*(\eta),ad^{*(3)}_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))(x\otimes y\otimes z)>$$
$$=<ad^{(3)}_{\Delta^*}(\alpha^*(\zeta),\alpha^*(\chi))\bullet_{\alpha*}\mu^*(\eta),x\otimes y\otimes z>.$$

Therefore, the identity (3.10) holds, which completes the proof. □

Similarly to Theorem 3.25, we get the following duality Theorem for generalized multiplicative 3-Hom-Lie bialgebras.

**Theorem 3.26** *Let $(L,\mu,\alpha_1,\Delta,\alpha_2)$ be a generalized multiplicative 3-Hom-Lie bialgebra. Then $(L^*,\Delta^*,\alpha_2^*,\mu^*,\alpha_1^*)$ is a generalized multiplicative 3-Hom-Lie bialgebra and it is called the dual generalized multiplicative 3-Hom-Lie bialgebra of $(L,\mu,\alpha_1,\Delta,\alpha_2)$.*

## 4. Classification of 3-dimensional multiplicative 3-Hom-Lie bialgebras

In this section, we provide the classification up to isomorphism of 3-dimensional multiplicative 3-Hom-Lie bialgebras over complex numbers. Moreover, we point out multiplicative 3-Hom-Lie bialgebras which are of 3-Lie-type.

Let $L$ be a 3-dimensional vector space generated by $\{e_1,e_2,e_3\}$. In the sequel, we will consider the structure linear map under the diagonal and Jordan forms with respect to a suitable basis. First, we deal with diagonal case and then with Jordan case. Notice that the Jordan case splits into the case where there is two eigenvalues and the case with only one eigenvalue.

The structure of the bracket $[\cdot,\cdot,\cdot]$ is given by

$$[e_1,e_2,e_3]=b_1e_1+b_2e_2+b_3e_3,$$

where $b_1,b_2,b_3$ are parameters in $\mathbb{K}$. The structure of the cobracket $\Delta$ is given by

$$\Delta(e_1)=c_1e_1\wedge e_2\wedge e_3,\quad \Delta(e_2)=c_2e_1\wedge e_2\wedge e_3,\quad \Delta(e_3)=c_3e_1\wedge e_2\wedge e_3,$$

where $c_1,c_2,c_3$ are parameters in $\mathbb{K}$.

**1) Diagonal case** : We consider the linear map $\alpha$, corresponding with respect to a suitable basis to the matrix of the form: $\alpha = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$.

Table 1 provides a classification of 3-Hom-Lie bialgebras up to isomorphism where the structure map has the previous form with respect to the given basis.

**Table 1**. Classification of 3-Hom-Lie bialgebras- Diagonal case.

| | Bracket | Cobracket | Linear map |
|---|---|---|---|
| (1) | $[e_1, e_2, e_3] = e_1 + e_2$ | $\Delta(e_1) = e_1 \wedge e_2 \wedge e_3,\ \Delta(e_2) = c_2 e_1 \wedge e_2 \wedge e_3,$ $\Delta(e_3) = 0,\ c_2 \neq 0.$ | $\alpha(e_1) = ae_1,\ \ \alpha(e_2) = ae_2,$ $\alpha(e_3) = \frac{1}{a} e_3,\ a \neq 0.$ |
| (2) | $[e_1, e_2, e_3] = e_1 + e_2$ | $\Delta(e_1) = 0,\ \Delta(e_2) = c_2 e_1 \wedge e_2 \wedge e_3,\ \Delta(e_3) = 0,$ $c_2 \neq 0.$ | $\alpha(e_1) = ae_1,\ \ \alpha(e_2) = ae_2,$ $\alpha(e_3) = \frac{1}{a} e_3,\ a \neq 0.$ |
| (3) | $[e_1, e_2, e_3] = e_1$ | $\Delta(e_1) = 0,\ \Delta(e_2) = e_1 \wedge e_2 \wedge e_3,\ \Delta(e_3) = 0,$ | $\alpha(e_1) = ae_1,\ \ \alpha(e_2) = ae_2,$ $\alpha(e_3) = \frac{1}{a} e_3,\ a \neq 0.$ |
| (4) | $[e_1, e_2, e_3] = e_1$ | $\Delta(e_1) = c_1 e_1 \wedge e_2 \wedge e_3,\ ,\Delta(e_2) = \Delta(e_3) = 0$ $c_1 \neq 0.$ | $\alpha(e_1) = ae_1,\ a \neq 0,$ $\alpha(e_2) = b,\ \ \alpha(e_3) = \frac{1}{b} e_3,$ $b \neq 0.$ |
| (5) | $[e_1, e_2, e_3] = e_1 + e_2$ | $\Delta(e_1) = c_1 e_1 \wedge e_2 \wedge e_3,\ \Delta(e_2) = -c_1 e_1 \wedge e_2 \wedge e_3,$ $\Delta(e_3) = 0,\ c_1 \neq 0.$ | $\alpha(e_1) = \alpha(e_2) = 0,$ $\alpha(e_3) = ce_3, c \neq 0.$ |
| (6) | $[e_1, e_2, e_3] = e_1$ | $\Delta(e_1) = \Delta(e_3) = 0,\ \Delta(e_2) = e_1 \wedge e_2 \wedge e_3,$ | $\alpha(e_1) = \alpha(e_2) = 0,$ $\alpha(e_3) = ce_3,\ \ c \neq 0.$ |
| (7) | $[e_1, e_2, e_3] = e_1$ | $\Delta(e_1) = c_1 e_1 \wedge e_2 \wedge e_3,\ \Delta(e_2) = 0,\ \Delta(e_3) = 0,$ $c_1 \neq 0.$ | $\alpha(e_1) = 0,\ \ \alpha(e_2) = be_2,$ $\alpha(e_3) = \frac{-1}{b} e_3,\ b \neq 0.$ |
| (8) | $[e_1, e_2, e_3] = e_3$ | $\Delta(e_1) = \Delta(e_2) = 0,\ \Delta(e_3) = c_3 e_1 \wedge e_2 \wedge e_3,$ $c_3 \neq 0.$ | $\alpha(e_1) = ae_1,\ \ \alpha(e_2) = \frac{1}{a} e_2,$ $\alpha(e_3) = 0,\ a \neq 0.$ |

**Remark 4.1** *The* 3*-Hom-Lie bialgebras* $(1) - (4)$ *are of* 3*-Lie-type.*

**1) Jordan case with two eigenvalues**: Now, we consider the linear map $\alpha$ corresponding with respect to a suitable basis to the matrix of the form $\alpha = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$.

We obtain the following classification of 3-Hom-Lie bialgebras up to isomorphism where the structure map has the previous form with respect to the given basis.

**Remark 4.2** *In this case the multiplicative* 3*-Hom-Lie bialgebras* (7) *and* (8) *are of* 3*-Lie-type,* (9) *is of* 3*-Lie-type if* $a_2 = a_1$ *and* (10) *is of* 3*-Lie-type if* $a_2 = -a_1$*.*

**1) Jordan case with one eigenvalue**: Now, we consider the linear map $\alpha$ where the corresponding matrix is of the form $\alpha = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$.

**Table 2**. Classification of 3-Hom-Lie bialgebras- Jordan form 1 case.

| | Bracket | Cobracket | Linear map |
|---|---|---|---|
| (1) | $[e_1, e_2, e_3] = e_1$. | $\Delta(e_1) = \Delta(e_3) = 0,\ \Delta(e_2) = c_2 e_1 \wedge e_2 \wedge e_3$, $c_2 \neq 0$. | $\alpha(e_1) = 0,\ \ \alpha(e_2) = e_1$, $\alpha(e_3) = be_3$ |
| (2) | $[e_1, e_2, e_3] = e_1$. | $\Delta(e_1) = 0,\ \Delta(e_2) = e_1 \wedge e_2 \wedge e_3$, $\Delta(e_3) = e_1 \wedge e_2 \wedge e_3$. | $\alpha(e_1) = 0,\ \ \alpha(e_2) = e_1$, $\alpha(e_3) = 0$. |
| (3) | $[e_1, e_2, e_3] = e_1 - e_2 + e_3$. | $\Delta(e_1) = \Delta(e_2) = 0$, $\Delta(e_3) = c_3 e_1 \wedge e_2 \wedge e_3,\ c_3 \neq 0, 1$. | $\alpha(e_1) = e_1$, $\alpha(e_2) = e_1 + e_2$, $\alpha(e_3) = 0$. |
| (4) | $[e_1, e_2, e_3] = e_1 - ie_2 - e_3$ | $\Delta(e_1) = \Delta(e_2) = 0$, $\Delta(e_3) = c_3 e_1 \wedge e_2 \wedge e_3,\ c_3 \neq 0$. | $\alpha(e_1) = ie_1$, $\alpha(e_2) = e_1 + ie_2$, $\alpha(e_3) = 0$. |
| (5) | $[e_1, e_2, e_3] = e_1 + ie_2 - e_3$. | $\Delta(e_1) = \Delta(e_2) = 0,\ \Delta(e_3) = c_3 e_1 \wedge e_2 \wedge e_3$, $c_3 \neq 0$. | $\alpha(e_1) = -ie_1$ $, \alpha(e_2) = e_1 - ie_2$, $\alpha(e_3) = 0$. |
| (6) | $[e_1, e_2, e_3] = e_1 + e_2 + e_3$. | $\Delta(e_1) = \Delta(e_2) = 0,\ \Delta(e_3) = c_3 e_1 \wedge e_2 \wedge e_3$, $c_3 \neq 0, 1$. | $\alpha(e_1) = -e_1$, $\alpha(e_2) = e_1 - e_2$, $\alpha(e_3) = 0$. |
| (7) | $[e_1, e_2, e_3] = a_1 e_1$. | $\Delta(e_1) = 0,\ \Delta(e_2) = \Delta(e_3) = e_1 \wedge e_2 \wedge e_3$. | $\alpha(e_1) = -e_1$, $\alpha(e_2) = e_1 - e_2$, $\alpha(e_3) = -e_3$. |
| (8) | $[e_1, e_2, e_3] = a_1 e_1$. | $\Delta(e_1) = 0,\ \Delta(e_2) = \Delta(e_3) = e_1 \wedge e_2 \wedge e_3$. | $\alpha(e_1) = e_1$, $\alpha(e_2) = e_1 + e_2$, $\alpha(e_3) = e_3$. |
| (9) | $[e_1, e_2, e_3] = a_1 e_1 + a_2 e_2 + \frac{a_2^2}{a_1} e_3$, $a_1.a_2 \neq 0$. | $\Delta(e_1) = \Delta(e_2) = 0,\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_3$. | $\alpha(e_1) = -e_1$, $\alpha(e_2) = e_1 - e_2$, $\alpha(e_3) = (\frac{a_2}{a_1} - 1)e_3$. |
| (10) | $[e_1, e_2, e_3] = a_1 e_1 + a_2 e_2 + \frac{a_2^2}{a_1} e_3$, $a_1.a_2 \neq 0$. | $\Delta(e_1) = \Delta(e_2) = 0,\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_3$. | $\alpha(e_1) = e_1$, $\alpha(e_2) = e_1 + e_2$, $\alpha(e_3) = (\frac{a_2}{a_1} + 1)e_3$. |

Table 3 provides a classification of 3-Hom-Lie bialgebras up to isomorphism where the structure map has the previous form with respect to the given basis.

**Remark 4.3** *In this case all multiplicative* $3$*-Hom-Lie bialgebras are of* $3$*-Lie-type.*

## 5. Construction of $3$-Hom-Lie bialgebras by two-dimensional extensions

In this section, we construct 3-Hom-Lie bialgebras by two dimensional extensions from a given quadratic multiplicative Hom-Lie bialgebras, that is multiplicative Hom-Lie bialgebras with symmetric invariant nondegenerate bilinear forms, see [14] for the classical case.

Let $(L, \mu, \alpha)$ be a multiplicative Hom-Lie algebra with a basis $\{x_1, \cdots, x_m\}$ and $B : L \otimes L \to \mathbb{K}$ be a

**Table 3**. Classification of 3-Hom-Lie bialgebras- Jordan form 2 case.

| | Bracket | Cobracket | Linear map |
|---|---|---|---|
| (1) | $[e_1, e_2, e_3] = e_1 + e_3$. | $\Delta(e_1) = \Delta(e_2) = 0,\ \Delta(e_3) = c_3 e_1 \wedge e_2 \wedge e_3$, $c_3 \neq 0$. | $\alpha(e_1) = -e_1,\ \ \alpha(e_2) = e_1 - e_2$, $\alpha(e_3) = e_2 - e_3$. |
| (2) | $[e_1, e_2, e_3] = e_1 + e_3$. | $\Delta(e_1) = \Delta(e_2) = 0,\ \Delta(e_3) = c_3 e_1 \wedge e_2 \wedge e_3$, $c_3 \neq 0$. | $\alpha(e_1) = e_1,\ \ \alpha(e_2) = e_1 + e_2$, $\alpha(e_3) = e_2 + e_3$. |
| (3) | $[e_1, e_2, e_3] = e_1$. | $\Delta(e_1) = \Delta(e_2) = 0,\ \Delta(e_3) = c_3 e_1 \wedge e_2 \wedge e_3$ $c_3 \neq 0$. | $\alpha(e_1) = e_1,\ \ \alpha(e_2) = e_1 + e_2$, $\alpha(e_3) = e_2 + e_3$. |
| (4) | $[e_1, e_2, e_3] = e_1$. | $\Delta(e_1) = \Delta(e_2) = 0,\ \Delta(e_3) = c_3 e_1 \wedge e_2 \wedge e_3$, $c_3 \neq 0$. | $\alpha(e_1) = -e_1,\ \ \alpha(e_2) = e_1 - e_2$, $\alpha(e_3) = e_2 - e_3$. |

nondegenerate bilinear symmetric form which is $ad_\mu$-invariant, that is for all $x, y, z \in L$

$$B(\mu(y,x), z) + B(x, \mu(y,z)) = 0, \tag{5.1}$$

and $\alpha$-symmetric, that is for all $x, y \in L$

$$B(\alpha(x), y) = B(x, \alpha(y)). \tag{5.2}$$

The quadruple $(L, \mu, \alpha, B)$ is called a quadratic multiplicative Hom-Lie algebra, see [16].

A 3-Hom-Lie algebra $(L, \mu, \alpha)$ is called quadratic, see [4], if there exists a nondegenerate bilinear symmetric form which is $ad_\mu$-invariant, that is for all $x, y, z \in L$

$$B(\mu(x,y,z), t) + B(z, \mu(x,y,t)) = 0,$$

and $\alpha$-symmetric, that is for all $x, y \in L$

$$B(\alpha(x), y) = B(x, \alpha(y)).$$

Fix $x_0, x_{-1}$ to be elements not contained in $L$. Set $\bar{L} = L \oplus \mathbb{K}x_0 \oplus \mathbb{K}x_{-1}$ (direct sum of vector spaces). With the above notations, we have the following result.

**Theorem 5.1** *Let $(L, \mu, \alpha, B)$ be an $m$-dimensional quadratic multiplicative Hom-Lie algebra and let $\{x_1, \cdots, x_m\}$ be with a basis of $L$ and $\lambda_1 \in \mathbb{K}$. Then the vector space $\bar{L} = L \oplus \mathbb{K}x_0 \oplus \mathbb{K}x_{-1}$ with the following skew-symmetric multiplication $\bar{\mu}$ and linear map $\bar{\alpha}$, defined with respect to the basis $\{x_{-1}, x_0, x_1, \cdots, x_m\}$ by :*

$$\bar{\mu}(x_0, x_i, x_j) = \mu(x_i, x_j),\ 1 \leq i, j \leq m, \tag{5.3}$$

$$\bar{\mu}(x_{-1}, x_i, x_j) = 0, \quad 0 \leq i, j \leq m, \tag{5.4}$$

$$\bar{\mu}(x_i, x_j, x_k) = B(\mu(x_i, x_j), x_k)x_{-1},\ 1 \leq i, j, k \leq m. \tag{5.5}$$

$$\bar{\alpha}(x) = \alpha(x),\ if\ x \in L, \quad \bar{\alpha}(x_0) = x_0 + \lambda_1 x_{-1}, \quad \bar{\alpha}(x_{-1}) = x_{-1}, \tag{5.6}$$

*is a 3-Hom-Lie algebra.*

**Proof** Since the multiplication $\mu$ of the Hom-Lie algebra $L$ is skew-symmetric and the nondegenerate bilinear form $B$ satisfies identity (5.1), the multiplication $\bar{\mu}$ on $\bar{L}$ is well defined and is skew-symmetric. Now we prove that it satisfies the 3-Hom-Jacobi identity (3.1). For arbitrary $x_{i_1}, x_{i_2}, x_{i_3}, x_{j_1}, x_{j_2} \in L$ , $1 \leq i_1, i_2, i_3, j_1, j_2 \leq m$, by identities (5.4) and (5.5), we have

$$\bar{\mu}(\bar{\mu}(x_{i_1}, x_{i_2}, x_{i_3}), \bar{\alpha}(x_{j_1}), \bar{\alpha}(x_{j_2})) = \bar{\mu}(B(\mu(x_{i_1}, x_{i_2}), x_{i_3})x_{-1}, \alpha(x_{j_1}), \alpha(x_{j_2})) = 0,$$

$$\bar{\mu}(\bar{\mu}(x_{i_1}, x_{j_1}, x_{j_2})\bar{\alpha}(x_{i_2}), \bar{\alpha}(x_{i_3})) + \bar{\mu}(\bar{\alpha}(x_{i_1}), \bar{\mu}(x_{i_2}, x_{j_1}, x_{j_2}), \bar{\alpha}(x_{i_3})) + \bar{\mu}(\bar{\alpha}(x_{i_1}), \bar{\alpha}(x_{i_2}), \bar{\mu}(x_{i_3}, x_{j_1}, x_{j_2})) = 0.$$

Using identities (5.3) and (2.2), we have

$$\bar{\mu}(\bar{\mu}(x_0, x_{i_1}, x_{i_2}), \bar{\alpha}(x_0), \bar{\alpha}(x_{j_1})) = \bar{\mu}(\mu(x_{i_1}, x_{i_2}), x_0 + \lambda_1 x_{-1}, \alpha(x_{j_1}))$$

$$= \bar{\mu}(\mu(x_{i_1}, x_{i_2}), x_0, \bar{\alpha}(x_{j_1})) = -\mu(\mu(x_{i_1}, x_{i_2}), \alpha(x_{j_1}))$$

$$= -\mu(\mu(x_{i_1}, x_{j_1}), \alpha(x_{i_2})) - \mu(\alpha(x_{i_1}), \mu(x_{i_2}, x_{j_1}))$$

$$= \bar{\mu}(\bar{\alpha}(x_0), \bar{\mu}(x_0, x_{j_1}, x_{i_1}), \bar{\alpha}(x_{i_2})) + \bar{\mu}(\bar{\alpha}(x_0), \bar{\alpha}(x_{i_1}), \bar{\mu}(x_0, x_{j_1}, x_{i_2}), \bar{\alpha}(x_{i_2}))$$

$$+\bar{\mu}(\bar{\mu}(x_0, x_{j_1}, x_0), \bar{\alpha}(x_{i_1}), \bar{\alpha}(x_{i_2})).$$

By identities (5.1) and (5.5), we get

$$\bar{\mu}(\bar{\mu}(x_0, x_{i_1}, x_{i_2}), \bar{\alpha}(x_{j_1}), \bar{\alpha}(x_{j_2})) = \bar{\mu}(\mu(x_{i_1}, x_{i_2}), \alpha(x_{j_1}), \alpha(x_{j_2})) = \bar{\mu}(\alpha(x_{j_1}), \bar{\alpha}(x_{j_2}), \mu(x_{i_1}, x_{i_2}))$$

$$= B(\mu(\alpha(x_{j_1}), \alpha(x_{j_2})), \mu(x_{i_1}, x_{i_2}))x_{-1} = B(\alpha \circ \mu(x_{j_1}, x_{j_2}), \mu(x_{i_1}, x_{i_2}))x_{-1},$$

$$\bar{\mu}(\bar{\mu}(x_0, x_{j_1}, x_{j_2}), \bar{\alpha}(x_{i_1}), \bar{\alpha}(x_{i_2})) + \bar{\mu}(\bar{\alpha}(x_0), \bar{\mu}(x_{i_1}, x_{j_1}, x_{j_2}), \bar{\alpha}(x_{i_2})) + \bar{\mu}(\bar{\alpha}(x_0), \bar{\alpha}(x_{i_1}), \bar{\mu}(x_{i_2}, x_{j_1}, x_{j_2}))$$

$$= \bar{\mu}(\bar{\mu}(x_0, x_{j_1}, x_{j_2}), \bar{\alpha}(x_{i_1}), \bar{\alpha}(x_{i_2})) + \bar{\mu}(\bar{\alpha}(x_0), B(\mu(x_{i_1}, x_{j_1}), x_{j_2})x_{-1}, \bar{\alpha}(x_{i_2}))$$

$$+\bar{\mu}(\bar{\alpha}(x_0), \bar{\alpha}(x_{i_1}), B(\mu(x_{i_2}, x_{j_1}), x_{j_2})x_{-1})$$

$$= \bar{\mu}(\bar{\mu}(x_0, x_{j_1}, x_{j_2}), \bar{\alpha}(x_{i_1}), \bar{\alpha}(x_{i_2})) = \bar{\mu}(\bar{\alpha}(x_{i_1}), \bar{\alpha}(x_{i_2}), \bar{\mu}(x_0, x_{j_1}, x_{j_2}))$$

$$= \bar{\mu}(\alpha(x_{i_1}), \alpha(x_{i_2}), \mu(x_{j_1}, x_{j_2})) = B(\mu(\alpha(x_{i_1}), \alpha(x_{i_2})), \mu(x_{j_1}, x_{j_2}))x_{-1}$$

$$= B(\alpha(\mu(x_{i_1}, x_{i_2})), \mu(x_{j_1}, x_{j_2}))x_{-1} = B(\mu(x_{i_1}, x_{i_2}), \alpha \circ \mu(x_{j_1}, x_{j_2}))x_{-1}$$

$$= B(\alpha \circ \mu(x_{j_1}, x_{j_2}), \mu(x_{i_1}, x_{i_2}))x_{-1}.$$

Therefore, the multiplication $\bar{\mu}$ satisfies identity (2.2). The proof is completed. □

**Corollary 5.2** *Let $(L, \mu, \alpha, B)$ be a quadratic Hom-Lie algebra and $\alpha$ be an involution, i.e. $\alpha^2 = id$. Then the 3-Hom-Lie algebra $(\bar{L}, \bar{\mu}, \bar{\alpha})$ is multiplicative.*

**Proof** One has to show that $\bar{\alpha}$ is an algebra morphism. Indeed, let $x_0, x_{-1}, x_{i_1}, x_{i_2}, x_{i_3} \in \bar{L}$, $1 \leq i_1, i_2, i_3 \leq m$,

$$\begin{aligned}\bar{\alpha} \circ \bar{\mu}(x_0, x_{i_1}, x_{i_2}) &= \bar{\alpha} \circ \mu(x_{i_1}, x_{i_2}) = \alpha \circ \mu(x_{i_1}, x_{i_2}) = \mu(\alpha(x_{i_1}), \alpha(x_{i_2}))) \\ &= \bar{\mu}(x_0, \alpha(x_{i_1}), \alpha(x_{i_2})) + \bar{\mu}(\lambda_1 x_{-1}, \alpha(x_{i_1}), \alpha(x_{i_2})) \\ &= \bar{\mu}(\bar{\alpha}(x_0), \bar{\alpha}(x_{i_1}), \bar{\alpha}(x_{i_2})).\end{aligned}$$

$$\bar{\alpha} \circ \bar{\mu}(x_{-1}, x_{i_1}, x_{i_2}) = \bar{\alpha}(0) = 0 = \bar{\mu}(x_{-1}, \alpha(x_{i_1}), \bar{\alpha}(x_{i_2})) = \bar{\mu}(\bar{\alpha}(x_{-1}), \bar{\alpha}(x_{i_1}), \bar{\alpha}(x_{i_2})).$$

$$\begin{aligned}&\bar{\mu}(\bar{\alpha}(x_{i_1}), \bar{\alpha}(x_{i_2}), \bar{\alpha}(x_{i_3})) = \bar{\mu}(\alpha(x_{i_1}), \alpha(x_{i_2}), \alpha(x_{i_3})) = B(\mu(\alpha(x_{i_1}), \alpha(x_{i_2})), \alpha(x_{i_3}))x_{-1} \\ &= B(\alpha \circ \mu(x_{i_1}, x_{i_2}), \alpha(x_{i_3}))x_{-1} = B(\mu(x_{i_1}, x_{i_2}), \alpha^2(x_{i_3}))x_{-1} = B(\mu(x_{i_1}, x_{i_2}), x_{i_3})\bar{\alpha}(x_{-1}) \\ &= \bar{\alpha} \circ \bar{\mu}(x_{i_1}, x_{i_2}, x_{i_3}).\end{aligned}$$

□

Define the symmetric bilinear form, $\bar{B} : \bar{L} \otimes \bar{L} \to \mathbb{K}$, by: $\bar{B}(x, y) = B(x, y)$, $\bar{B}(x_0, x_0) = 1$, $\bar{B}(x_{-1}, x_0) = -1$, $\bar{B}(x_{-1}, y) = \bar{B}(x_{-1}, x_{-1}) = \bar{B}(x_0, y) = 0$, for all $x, y \in L$.

**Proposition 5.3** *With the above notations $\bar{B}$ is an invariant, $\bar{\alpha}$-symmetric bilinear form on the 3-Hom-Lie algebra $(\bar{L}, \bar{\mu}, \bar{\alpha})$.*

**Proof** We only need to prove $\bar{B}$ satisfying Eqs. (5.1) and (5.2), for all $x_{i_1}, x_{i_2}, x_{i_3} \in L$.

$$\bar{B}(\bar{\mu}(x_{i_1}, x_{i_2}, x_{i_3}), x_{-1}) = \bar{B}(B(\mu(x_{i_1}, x_{i_2}), x_{i_3})x_{-1}, x_{-1}) = 0$$

$$\begin{aligned}&\bar{B}(\bar{\mu}(x_{i_1}, x_{i_2}, x_{i_3}), x_0) = \bar{B}(B(\mu(x_{i_1}, x_{i_2}), x_{i_3})x_{-1}, x_0) = \bar{B}(x_{-1}, x_0)\bar{B}(\mu(x_{i_1}, x_{i_2}), x_{i_3}) \\ &= -\bar{B}(\bar{\mu}(x_0, x_{i_1}, x_{i_2}), x_{i_3}) = -\bar{B}(\bar{\mu}(x_{i_1}, x_{i_2}, x_0), x_{i_3}) = -\bar{B}(x_{i_3}, \bar{\mu}(x_{i_1}, x_{i_2}, x_0)).\end{aligned}$$

$$\begin{aligned}&\bar{B}(\bar{\mu}(x_{i_1}, x_{i_2}, x_{i_3}), x_k) = \bar{B}(B(\mu(x_{i_1}, x_{i_2}), x_{i_3})x_{-1}, x_k) = B(\mu(x_{i_1}, x_{i_2}), x_{i_3})\bar{B}(x_{-1}, x_k) = 0 \\ &= -\bar{B}(x_{i_3}, x_{-1})B(\mu(x_{i_1}, x_{i_2}), x_k) = -\bar{B}(x_{i_3}, B(\mu(x_{i_1}, x_{i_2}), x_k)x_{-1}) = -\bar{B}(x_{i_3}, \bar{\mu}(x_{i_1}, x_{i_2}, x_k)).\end{aligned}$$

We check the second condition for different pairs.
(1) $\bar{B}(\bar{\alpha}(x), y) = B(\alpha(x), y) = B(x, \alpha(y)) = \bar{B}(x, \bar{\alpha}(y))$,
(2) $\bar{B}(\bar{\alpha}(x_0), x_0) = \bar{B}(x_0, \bar{\alpha}(x_0))$,
(3) $\bar{B}(\bar{\alpha}(x_{-1}), x_0) = \bar{B}(x_{-1}, x_0) = \bar{B}(x_{-1}, x_0) + \bar{B}(x_{-1}, \lambda_1 x_{-1}) = \bar{B}(x_{-1}, \bar{\alpha}(x_0)) = \bar{B}(x_{-1}, \bar{\alpha}(x_0))$,
(4) $\bar{B}(\bar{\alpha}(x_{-1}), y) = \bar{B}(x_{-1}, y) = 0 = \bar{B}(x_{-1}, \alpha(y)) = \bar{B}(x_{-1}, \bar{\alpha}(y))$,
(5) $\bar{B}(\bar{\alpha}(x_{-1}), x_{-1}) = \bar{B}(x_{-1}, \alpha(x_{-1}))$,
(6) $\bar{B}(\bar{\alpha}(x_0), y) = \bar{B}(x_0 + \lambda_1 x_{-1}, y) = \bar{B}(x_0, y) + \lambda_1 \bar{B}(x_{-1}, y) = 0 = \bar{B}(x_0, \alpha(y))$.

□

**Corollary 5.4** *Let $(L, \mu, \alpha)$ be an $m$-dimensional Hom-Lie algebra with a basis $\{x_1, \cdots, x_m\}$. Then $(\bar{L} = L \oplus \mathbb{K}x_0 \oplus \mathbb{K}x_{-1}, \bar{\mu}, \bar{\alpha})$ where $\bar{\mu}$ is defined in Eqs.* (5.3) *and* (5.4)*, $\bar{\alpha}$ is defined in Eq.*(5.6) *and satisfying*

$$\bar{\mu}(x_i, x_j, x_k) = 0, \ 1 \leq i, j, k \leq m, \tag{5.7}$$

*is a 3-Hom-Lie algebra.*

**Theorem 5.5** *Let $(L,\mu,\Delta,\alpha)$ be an $m$-dimensional Hom-Lie bialgebra. Let $x_0, x_{-1}$ be elements not contained in $L$. Let $B : L\otimes L \to \mathbb{K}$ be a bilinear symmetric, $\alpha$-symmetric and $ad_\mu$-invariant form. Then $(\bar{L},\bar{\mu},\bar{\Delta},\bar{\alpha})$ with $\bar{L} = L\oplus \mathbb{K}x_0 \oplus \mathbb{K}x_{-1}$, $\bar{\mu}$ defined by Eqs. (5.3)–(5.5), $\bar{\alpha}$ defined by Eq. (5.6) and $\bar{\Delta}$ defined as*

$$\bar{\Delta}(x_0) = \bar{\Delta}(x_{-1}) = 0, \quad \bar{\Delta}(x_k) = x_{-1}\wedge \Delta(x_k), \tag{5.8}$$

*is an $(m+2)$-dimensional 3-Hom-Lie bialgebra.*

**Proof** Let $\{x_{-1}, x_0, x_1, \cdots, x_m\}$ be a basis of $\bar{L}$ and $\{x^{-1}, x^0, x^1, \cdots, x^m\}$ be the dual basis of $(\bar{L})^*$. By Lemma 5.4 and by means of structure constants, we can suppose

$$\bar{\Delta}(x_k) = \sum_{j_1,j_2} b_k^{j_1j_2} x_{-1}\wedge x_{j_1}\wedge x_{j_2}, 1\le j_1 < j_2 \le m, \quad \mu(x_{j_2}, x_{j_3}) = \Big(\sum_{t=1}^{m} c_{j_2j_3}^{t} x_t\Big).$$

Then, we need to prove that $\bar{\mu}$ and $\bar{\Delta}$ satisfy Eq. (3.9). Therefore we divide the computations into four cases.

**(1)** Set $j_1 = -1$ and $1\le j_2, j_3 \le m$. By Theorem 5.1, we have $\bar{\Delta}\bar{\mu}(x_{-1}, x_{j_2}, x_{j_3}) = \bar{\Delta}(0) = 0$.

$$\begin{aligned}
&ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_{-1}),\bar{\alpha}(x_{j_2}))\bullet_\alpha \bar{\Delta}(x_{j_3}) - ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_{-1}),\bar{\alpha}(x_{j_3}))\bullet_\alpha \bar{\Delta}(x_{j_2}) + ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_{j_2}),\bar{\alpha}(x_{j_3}))\bullet_\alpha \bar{\Delta}(x_{-1})\\
&= ad_{\bar{\mu}}^{(3)}(x_{-1},\alpha(x_{j_2}))\bullet_\alpha \bar{\Delta}(x_{j_3}) - ad_{\bar{\mu}}^{(3)}(x_{-1},\alpha(x_{j_3}))\bullet_\alpha \bar{\Delta}(x_{j_2})\\
&= ad_{\bar{\mu}}^{(3)}(x_{-1},\alpha(x_{j_2}))(\sum_{l,k} b_{j_3}^{lk} x_{-1}\wedge x_l\wedge x_k) - ad_{\bar{\mu}}^{(3)}(x_{-1},\alpha(x_{j_3}))(\sum_{l,k} b_{j_2}^{lk} x_{-1}\wedge x_l\wedge x_k).
\end{aligned}$$

Since for all $-1\le t_1, t_2, t_3 \le m$,

$$ad_{\bar{\mu}}^{(3)}(x_{-1},\alpha(x_{j_2}))(x_{t_1}\wedge x_{t_2}\wedge x_{t_3}) = ad_{\bar{\mu}}^{(3)}(x_{-1},\alpha(x_{j_3}))(x_{t_1}\wedge x_{t_2}\wedge x_{t_3}) = 0,$$

we have

$$\circlearrowleft_{-1,j_1,j_2} ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_{-1}),\bar{\alpha}(x_{j_2}))\bullet \bar{\Delta}(x_{j_3}) = \bar{\Delta}\bar{\mu}(x_{-1}, x_{j_2}, x_{j_3}) = 0.$$

**(2)** Set $j_1 = -1$, $j_2 = 0$ and $1\le j_3\le m$. By Eq.(5.4), $\bar{\Delta}(x_{-1}) = \bar{\Delta}(x_0) = 0$. Then we have

$$\begin{aligned}
&ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_{-1}),\bar{\alpha}(x_0))\bullet_\alpha \bar{\Delta}(x_{j_3}) - ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_{-1}),\bar{\alpha}(x_{j_3}))\bullet_\alpha \bar{\Delta}(x_0) + ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_0),\bar{\alpha}(x_{j_3}))\bullet_\alpha \bar{\Delta}(x_{-1})\\
&= ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_{-1}),\bar{\alpha}(x_0))\bullet_\alpha \bar{\Delta}(x_{j_3})\\
&= ad_{\bar{\mu}}^{(3)}(x_{-1}, x_0 + \lambda_1 x_{-1})\bullet_\alpha \bar{\Delta}(x_{j_3})\\
&= ad_{\bar{\mu}}^{(3)}(x_{-1}, x_0)\bullet_\alpha \bar{\Delta}(x_{j_3}) + \lambda_1 ad_{\bar{\mu}}^{(3)}(x_{-1}, x_{-1})\bullet_\alpha \bar{\Delta}(x_{j_3})\\
&= 0 = \bar{\Delta}\bar{\mu}(x_{-1}, x_{j_2}, x_{j_3}).
\end{aligned}$$

**(3)** Set $j_1 = 0$, and $1\le j_2, j_3\le m$. Thanks to Eqs. (5.3) and (5.8), we get

$$\bar{\Delta}\bar{\mu}(x_0, x_{j_2}, x_{j_3}) = \bar{\Delta}\mu(x_{j_2}, x_{j_3}) = \bar{\Delta}\Big(\sum_{t=1}^{m} c_{j_2j_3}^t x_t\Big) = \sum_{t=1}^{m} c_{j_2j_3}^t \bar{\Delta}(x_t) = \sum_{l,k}\sum_{t=1}^{m} b_t^{lk} c_{j_2j_3}^t x_{-1}\wedge x_l \wedge x_k$$
$$= x_{-1}\wedge \Delta\mu(x_{j_2}, x_{j_3}).$$

$$\begin{aligned}
&ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_0), \bar{\alpha}(x_{j_2})) \bullet_\alpha \bar{\Delta}(x_{j_3}) - ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_0), \bar{\alpha}(x_{j_3})) \bullet_\alpha \bar{\Delta}(x_{j_2}) + ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_{j_2}), \bar{\alpha}(x_{j_3})) \bullet_\alpha \bar{\Delta}(x_0)\\
&= ad_{\bar{\mu}}^{(3)}(x_0+\lambda_1 x_{-1}, \bar{\alpha}(x_{j_2}))(\sum_{l,k} b_{j_3}^{lk} x_{-1}\wedge x_l\wedge x_k) - ad_{\bar{\mu}}^{(3)}(x_0+\lambda_1 x_{-1}, \bar{\alpha}(x_{j_3}))(\sum_{l,k} b_{j_2}^{lk} x_{-1}\wedge x_l\wedge x_k)\\
&= \sum_{l,k} b_{j_3}^{lk}(\bar{\alpha}(x_{-1})\wedge \bar{\mu}(x_0+\lambda_1 x_{-1}, \bar{\alpha}(x_{j_2}), x_l)\wedge \bar{\alpha}(x_k) + \bar{\alpha}(x_{-1})\wedge\bar{\alpha}(x_l)\wedge \bar{\mu}(x_0+\lambda_1 x_{-1}, \bar{\alpha}(x_{j_2}), x_k))\\
&- \sum_{l,k} b_{j_2}^{lk}(\bar{\alpha}(x_{-1})\wedge \bar{\mu}(x_0+\lambda_1 x_{-1}, \bar{\alpha}(x_{j_3}), x_l)\wedge \bar{\alpha}(x_k) + \bar{\alpha}(x_{-1})\wedge\bar{\alpha}(x_l)\wedge \bar{\mu}(x_0+\lambda_1 x_{-1}, \bar{\alpha}(x_{j_3}), x_k))\\
&= \sum_{l,k} b_{j_3}^{lk} x_{-1}\wedge (ad_\mu(\alpha(x_{j_2}))\otimes\alpha + \alpha\otimes ad_\mu(\alpha(x_{j_2})))(x_l\wedge x_k)\\
&- \sum_{l,k} b_{j_2}^{lk} x_{-1}\wedge (ad_\mu(\alpha(x_{j_3}))\otimes\alpha + \alpha\otimes ad_\mu(\alpha(x_{j_3})))(x_l\wedge x_k)\\
&= x_{-1}\wedge((ad_\mu(\alpha(x_{j_2}))\otimes\alpha+\alpha\otimes ad_\mu(\alpha(x_{j_2})))\Delta(x_{j_3}) - (ad_\mu(\alpha(x_{j_3}))\otimes\alpha+\alpha\otimes ad_\mu(\alpha(x_{j_3})))\Delta(x_{j_2}))\\
&= x_{-1}\wedge\Delta\mu(x_{j_2}, x_{j_3}) = \bar{\Delta}\bar{\mu}(x_0, x_{j_2}, x_{j_3}).
\end{aligned}$$

Therefore,

$$\bar{\Delta}\bar{\mu}(x_0, x_{j_2}, x_{j_3}) = ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_0), \bar{\alpha}(x_{j_2})) \bullet_\alpha \bar{\Delta}(x_{j_3}) - ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_0), \bar{\alpha}(x_{j_3})) \bullet_\alpha \bar{\Delta}(x_{j_2}) + ad_{\bar{\mu}}^{(3)}(\bar{\alpha}(x_{j_2}), \bar{\alpha}(x_{j_3})) \bullet_\alpha \bar{\Delta}(x_0).$$

**(4)** For all $1\le j_1, j_2, j_3\le m,$ by Eq. (5.5), we have

$$\bar{\Delta}\bar{\mu}(x_{j_1},x_{j_2},x_{j_3}) = B\big(\mu(x_{j_1},x_{j_2}),x_{j_3}\big)\bar{\Delta}(x_{-1}) = 0.$$

$$\begin{aligned}
&\circlearrowleft_{j_1,j_2,j_3} ad^{(3)}_{\bar{\mu}}(\bar{\alpha}(x_{j_1}),\bar{\alpha}(x_{j_2}))\bullet\bar{\Delta}(x_{j_3})\\
&=\circlearrowleft_{j_1,j_2,j_3} ad^{(3)}_{\bar{\mu}}(\bar{\alpha}(x_{j_1}),\bar{\alpha}(x_{j_2}))(\sum_{l,k} b^{lk}_{j_3}x_{-1}\wedge x_l\wedge x_k)\\
&=\sum_{l,k} b^{lk}_{j_3}(\bar{\alpha}(x_{-1})\wedge\bar{\mu}(\bar{\alpha}(x_{j_1}),\bar{\alpha}(x_{j_2}),x_l)\wedge\bar{\alpha}(x_k)+\bar{\alpha}(x_{-1})\wedge\bar{\alpha}(x_l)\wedge\bar{\mu}(\bar{\alpha}(x_{j_1}),\bar{\alpha}(x_{j_2}),x_k)))\\
&+\sum_{l,k} b^{lk}_{j_1}(\bar{\alpha}(x_{-1})\wedge\bar{\mu}(\bar{\alpha}(x_{j_2}),\bar{\alpha}(x_{j_3}),x_l)\wedge\bar{\alpha}(x_k)+\bar{\alpha}(x_{-1})\wedge\bar{\alpha}(x_l)\wedge\bar{\mu}(\bar{\alpha}(x_{j_2}),\bar{\alpha}(x_{j_3}),x_k)))\\
&+\sum_{l,k} b^{lk}_{j_2}(\bar{\alpha}(x_{-1})\wedge\bar{\mu}(\bar{\alpha}(x_{j_3}),\bar{\alpha}(x_{j_1}),x_l)\wedge\bar{\alpha}(x_k)+\bar{\alpha}(x_{-1})\wedge\bar{\alpha}(x_l)\wedge\bar{\mu}(\bar{\alpha}(x_{j_3}),\bar{\alpha}(x_{j_1}),x_k)))\\
&=\sum_{l,k} b^{lk}_{j_3}B(\mu(\alpha(x_{j_1}),\alpha(x_{j_2})),x_l)x_{-1}\wedge x_{-1}\wedge\alpha(x_k)+x_{-1}\wedge\bar{\alpha}(x_l)\wedge B(\mu(\bar{\alpha}(x_{j_2}),\bar{\alpha}(x_{j_3})),x_k)x_{-1}\\
&+\sum_{l,k} b^{lk}_{j_1}B(\mu(\alpha(x_{j_2}),\alpha(x_{j_3})),x_l)x_{-1}\wedge x_{-1}\wedge\bar{\alpha}(x_k)+x_{-1}\wedge\bar{\alpha}(x_l)\wedge B(\mu(\bar{\alpha}(x_{j_2}),\bar{\alpha}(x_{j_3})),x_k)x_{-1}\\
&+\sum_{l,k} b^{lk}_{j_2}B(\mu(\alpha(x_{j_3}),\alpha(x_{j_1})),x_l)x_{-1}\wedge x_{-1}\wedge\bar{\alpha}(x_k)+x_{-1}\wedge\bar{\alpha}(x_l)\wedge B(\mu(\bar{\alpha}(x_{j_3}),\bar{\alpha}(x_{j_1})),x_k)x_{-1}\\
&=0=\bar{\Delta}\bar{\mu}(x_{j_1},x_{j_2},x_{j_3}).
\end{aligned}$$

Therefore, $(\bar{L},\bar{\mu},\bar{\Delta},\bar{\alpha})$ is a multiplicative 3-Hom-Lie bialgebra. The proof is completed. □

## 6. Representations of multiplicative 3-Hom-Lie bialgebras

In this section, we recall the definition of representation of 3-Hom-Lie algebras introduced in [35] and extend it to Yau-type 3-Hom-Lie bialgebras.

**Definition 6.1** *A representation of a multiplicative 3-Hom-Lie algebra $(L,\mu,\alpha)$ with respect to $\alpha_V\in gl(V)$ is a skew-symmetric linear map $\rho: L\wedge L\to End(V)$ such that*

$$\rho(\alpha(x_1),\alpha(x_2))\circ\alpha_V=\alpha_V\circ\rho(x_1,x_2), \tag{6.1}$$

$$\rho(\alpha(x_1),\alpha(x_2))\rho(x_3,x_4)-\rho(\alpha(x_3),\alpha(x_4))\rho(x_1,x_2)=\big(\rho(\mu(x_1,x_2,x_3),\alpha(x_4))-\rho(\mu(x_1,x_2,x_4),\alpha(x_3))\big)\circ\alpha_V, \tag{6.2}$$

$$\rho(\mu(x_1,x_2,x_3),\alpha(x_4))\circ\alpha_V-\rho(\alpha(x_2),\alpha(x_3))\rho(x_1,x_4)=\rho(\alpha(x_3),\alpha(x_1))\rho(x_2,x_4)+\rho(\alpha(x_1),\alpha(x_2))\rho(x_3,x_4), \tag{6.3}$$

*for all $x_1,x_2,x_3$ and $x_4$ in $L$.*

**Remark 6.2** *A representation $(V,\rho,\alpha_V)$ of a multiplicative 3-Hom-Lie algebra $(L,\mu,\alpha)$ is an $L$-module (or module for short) via $\mu_V: L\wedge L\wedge V\to V$ defined by $\mu_V(x,y,a)=\rho(x,y)(a)$ for all $x,y\in L$ and $a\in V$. It is referred by a triple $(V,\mu_V,\alpha_V)$*

Notice that we recover representations of 3-Lie algebras when $\alpha=id_L$ and $\alpha_V=id_V$.

**Example 6.3** *Let $(L,\mu,\alpha)$ be the 3-dimensional multiplicative 3-Hom-Lie algebra defined, with respect to a basis $\{e_1,e_2,e_3\}$, by*

$$\mu(e_1,e_2,e_3)=a_1e_1,\quad \alpha(e_1)=a_1e_1,\quad \alpha(e_2)=a_1e_2,\quad \alpha(e_3)=\frac{1}{a_1}e_3.$$

*Let $V$ be a 2-dimensional vector space, $\{v_1, v_2\}$ its basis and $\alpha_V \in gl(V)$ defined by:*

$$\alpha_V(v_1) = a_1 v_1, \quad \alpha_V(v_2) = a_2 v_2,$$

*where $a_1, a_2$ are parameters in $\mathbb{K}$, such that $a_1 a_2 \neq 0, a_1 \neq a_2$.*
*We consider the module $(V, \mu_V, \alpha_V)$ of $L$, where $\mu_V$ is defined with respect to the basis by*

$$\mu_V(e_1, e_2, v_1) = 0,\ \mu_V(e_1, e_2, v_2) = 0,\ \mu_V(e_1, e_3, v_1) = 0,$$
$$\mu_V(e_1, e_3, v_2) = 0,\ \mu_V(e_2, e_3, v_1) = a_1 v_1,\ \mu_V(e_2, e_3, v_2) = a_1 v_2,$$

We have the following straightforward Yau twist construction and semidirect product characterization of a representation of a 3-Hom-Lie algebra.

**Theorem 6.4** *Let $(L, \mu)$ be a 3-Lie algebra and $(V, \rho)$ be a representation of $L$. Let $\alpha : L \to L$ be an algebra morphism on $L$ and $\alpha_V : V \to V$ be a linear map such that $\alpha_V \circ \rho(x_1, x_2)(a) = \rho(\alpha(x_1), \alpha(x_2)) \circ \alpha_V(a)$. Then $(V, \widetilde{\rho}, \alpha_V)$ is a representation of the multiplicative 3-Hom-Lie algebra $(L, \mu_\alpha, \alpha)$, where*

$$\mu_\alpha = \mu \circ \alpha^{\otimes 3} \quad \text{and} \quad \widetilde{\rho}(x_1, x_2)(a) = \alpha_V \circ \rho(x_1, x_2)(a).$$

**Theorem 6.5** *Let $(L, \mu, \alpha)$ be a multiplicative 3-Hom-Lie algebra, $V$ a vector space, $\alpha_V \in gl(V)$ and $\mu_V : L \otimes L \otimes V \longrightarrow V$ a skew-symmetric linear map. Then $(V, \mu_V, \alpha_V)$ is an $L$-module of $L$ if and only if there is a multiplicative 3-Hom-Lie algebra structure (called the semidirect product) on the direct sum $L \oplus V$ of vector spaces, defined by*

$$\mu_{L\oplus V}(x + a, y + b, z + c) = \mu(x, y, z) + \mu_V(x, y, c) + \mu_V(y, z, a) - \mu_V(x, z, b)$$

*and*

$$\alpha_{L\oplus V}(x + a) = \alpha(x) + \alpha_V(a)$$

*for all $x, y, z \in L$ and $a, b, c \in V$.*

Now, we define a comodule structure on a multiplicative 3-Hom-Lie coalgebra.

**Definition 6.6** *Let $(L, \Delta, \alpha)$ be a multiplicative 3-Hom-Lie coalgebra. Let $V$ be a vector space and $\alpha_V \in gl(V)$. A triple $(V, \Delta_V, \alpha_V)$ is called a comodule of $L$ if the linear map $\Delta_V : V \to L \wedge L \wedge V$ satisfies the following conditions*

$$(\alpha^{\otimes 2} \otimes \alpha_V) \circ \Delta_V = \Delta_V \circ \alpha_V, \tag{6.4}$$

$$(\alpha^{\otimes 2} \otimes \Delta_V) \circ \Delta_V - \omega'_1 \circ (\alpha^{\otimes 2} \otimes \Delta_V) \circ \Delta_V = (\Delta \otimes \alpha \otimes \alpha_V) \circ \Delta_V - \tau_{34}(\Delta \otimes \alpha \otimes \alpha_V) \circ \Delta_V, \tag{6.5}$$

$$(\Delta \otimes \alpha \otimes \alpha_V) \circ \Delta_V - (\alpha^{\otimes 2} \otimes \Delta_V) \circ \Delta_V = \omega'_2 \circ (\alpha^{\otimes 2} \otimes \Delta_V) \circ \Delta_V + \omega'_3 \circ (\alpha^{\otimes 2} \otimes \Delta_V) \circ \Delta_V, \tag{6.6}$$

*where for $i = 1, 2, 3$ and $j = 1, \cdots, 5$ and $x_j \in L$, we have $\omega'_i : L \otimes L \otimes L \otimes L \otimes V \longrightarrow L \otimes L \otimes L \otimes L \otimes V$*

$$\omega'_1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_3 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_5, \tag{6.7}$$

$$\omega'_2(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_3 \otimes x_1 \otimes x_2 \otimes x_4 \otimes x_5, \tag{6.8}$$

$$\omega'_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_2 \otimes x_3 \otimes x_1 \otimes x_4 \otimes x_5). \tag{6.9}$$

**Remark 6.7** *If $\alpha = id_L$ and $\alpha_V = id_V$, $(V, \Delta_V)$, we recover the comodule structure of a 3-Lie coalgebra.*

**Theorem 6.8** *Let $L$ and $V$ be two finite dimensional vector spaces. A triple $(V, \Delta_V, \alpha_V)$ is a comodule of a multiplicative 3-Hom-Lie coalgebra $(L, \Delta, \alpha)$ if and only if $(V^*, \Delta_V^*, \alpha_V^*)$ is a module of the multiplicative 3-Hom-Lie algebra $(L^*, \Delta^*, \alpha^*)$.*

**Proof** Let $x, y \in L$, $a \in V$, $x^*, y^*, z^*, t^* \in L^*$ and $a^* \in V^*$

$$
\begin{aligned}
&< (\alpha^{\otimes 2} \otimes \alpha_V) \circ \Delta_V(a) - \Delta_V \circ \alpha_V(a), x^* \otimes y^* \otimes a^* > \\
&=< (\alpha^{\otimes 2} \otimes \alpha_V) \circ \Delta_V(a), x^* \otimes y^* \otimes a^* > - < \Delta_V \circ \alpha_V(a), x^* \otimes y^* \otimes a^* > \\
&=< a, \Delta_V^* \circ ((\alpha^*)^{\otimes 2} \otimes \alpha_V^*)(x^* \otimes y^* \otimes a^*) > - < a, \alpha_V^* \circ \Delta_V^*(x^* \otimes y^* \otimes a^*) > .
\end{aligned}
$$

Then $\Delta_V^* \circ ((\alpha^*)^{\otimes 2} \otimes \alpha_V^*) = \alpha_V^* \circ \Delta_V^*$,

$$
\begin{aligned}
&< ((\alpha^{\otimes 2} \otimes \Delta_V) - \omega_1' \circ (\alpha^{\otimes 2} \otimes \Delta_V)) \circ \Delta_V(a) - (\Delta \otimes \alpha \otimes \alpha_V) \circ \Delta_V(a) \\
&- \tau_{34}(\Delta \otimes \alpha \otimes \alpha_V) \circ \Delta_V(a), x^* \otimes y^* \otimes z^* \otimes t^* \otimes a^* > \\
&=< a, \Delta_V^* \circ ((\alpha^*)^{\otimes 2} \otimes \Delta_V^*) - \Delta_V^* \circ ((\alpha^*)^{\otimes 2} \otimes \Delta_V^*) \circ \omega_1'(x^* \otimes y^* \otimes z^* \otimes t^* \otimes a^*) > \\
&- < a, (\Delta_V^* \circ (\Delta^* \otimes \alpha^* \otimes \alpha_V^*) - \Delta_V^* \circ (\Delta^* \otimes \alpha^* \otimes \alpha_V^*) \circ \tau_{34})(x^* \otimes y^* \otimes z^* \otimes t^* \otimes a^*) > .
\end{aligned}
$$

Then, we have

$$\Delta_V^* \circ ((\alpha^*)^{\otimes 2} \otimes \Delta_V^*) - \Delta_V^* \circ ((\alpha^*)^{\otimes 2} \otimes \Delta_V^*) \circ \omega_1' = \Delta_V^* \circ (\Delta^* \otimes \alpha^* \otimes \alpha_V^*) - \Delta_V^* \circ (\Delta^* \otimes \alpha^* \otimes \alpha_V^*) \circ \tau_{34}.$$

Similarly, we find

$$\Delta_V^* \circ (\Delta^* \otimes \alpha^* \otimes \alpha_V^*) - \Delta_V^* \circ ((\alpha^*)^{\otimes 2} \otimes \Delta_V^*) = \Delta_V^* \circ ((\alpha^*)^{\otimes 2} \otimes \Delta_V^*) \circ \omega_2' + \Delta_V^* \circ ((\alpha^*)^{\otimes 2} \otimes \Delta_V^*) \circ \omega_3'.$$

□

**Proposition 6.9** *Let $(L, \Delta)$ be a 3-Lie coalgebra and $(V, \Delta_V)$ be a comodule of $L$. Let $\alpha : V \to V$ be a colagebra morphism and $\alpha_V : V \to V$ be a linear map satisfying $\Delta_V \circ \alpha_V = (\alpha \otimes \alpha \otimes \alpha_V) \circ \Delta_V$. Then $(V, \widetilde{\Delta_V}, \alpha_V)$, where $\widetilde{\Delta_V} = \Delta_V \circ \alpha_V$, is a comodule of the multiplicative 3-Hom-Lie coalgebra $(L, \Delta_\alpha = \Delta \circ \alpha, \alpha)$.*

**Proof** Let $a \in L$, then we have

$$
\begin{aligned}
&(\Delta_\alpha \otimes \alpha \otimes \alpha_V) \circ \widetilde{\Delta_V}(a) - (\alpha^{\otimes 2} \otimes \widetilde{\Delta_V}) \circ \widetilde{\Delta_V}(a) - \omega_2' \circ (\alpha^{\otimes 2} \otimes \widetilde{\Delta_V}) \circ \widetilde{\Delta_V}(a) - \omega_3' \circ (\alpha^{\otimes 2} \otimes \widetilde{\Delta_V}) \circ \widetilde{\Delta_V}(a) \\
&= (\Delta \otimes Id \otimes Id_V) \circ \Delta_V \circ \alpha_V^2(a) - (Id^{\otimes 2} \otimes \Delta_V) \circ \Delta_V \circ \alpha_V^2(a) \\
&- \omega_2' \circ (Id^{\otimes 2} \otimes \Delta_V) \circ \Delta_V \circ \alpha_V^2(a) - \omega_3' \circ (Id^{\otimes 2} \otimes \Delta_V) \circ \Delta_V \circ \alpha_V^2(a) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
&(\alpha^{\otimes 2} \otimes \widetilde{\Delta_V}) \circ \widetilde{\Delta_V}(a) - \omega_1' \circ (\alpha^{\otimes 2} \otimes \widetilde{\Delta_V}) \circ \widetilde{\Delta_V}(a) - (\Delta_\alpha \otimes \alpha \otimes \alpha_V) \circ \widetilde{\Delta_V} + \tau_{34}(\Delta_\alpha \otimes \alpha \otimes \alpha_V) \circ \widetilde{\Delta_V}(a) \\
&= (Id^{\otimes 2} \otimes \Delta_V) \circ \Delta_V \circ \alpha_V^2(a) - \omega_1' \circ (Id^{\otimes 2} \otimes \Delta_V) \circ \Delta_V \circ \alpha_V^2(a) \\
&- (\Delta \otimes Id \otimes Id_V) \circ \Delta_V \circ \alpha_V^2 + \tau_{34}(\Delta \otimes Id \otimes Id_V) \circ \Delta_V \circ \alpha_V^2(a) \\
&= 0.
\end{aligned}
$$

□

**Theorem 6.10** *Let $(L,\Delta,\alpha)$ be a multiplicative 3-Hom-Lie coalgebra and $(V,\Delta_V,\alpha_V)$ be a comodule of the 3-Hom-Lie coalgebra of $L$. Then there exists a 3-Hom-Lie coalgebra structure on the direct sum $L\oplus V$ (called the semidirect coproduct) defined by*

$$\Delta_{L\oplus V}(x+a)=\Delta(x)+\Delta_V(a) \text{ and } \alpha_{L\oplus V}(x+a)=\alpha(x)+\alpha_V(a),$$

*for all $x\in L$ and $a\in V$.*

**Proof** We use identities (6.5) and co-Jacobi identity. Let $x+a\in L\oplus V$, we have

$$\begin{aligned}
&(\alpha\otimes\alpha\otimes\Delta_{L\oplus V})\circ\Delta_{L\oplus V}(x+a)-\omega_1'\circ(\alpha\otimes\alpha\otimes\Delta_{L\oplus V})\circ\Delta_{L\oplus V}(x+a)\\
&-(\Delta\otimes\alpha\otimes\alpha_{L\oplus V})\circ\Delta_{L\oplus V}(x+a)+\tau_{34}(\Delta\otimes\alpha\otimes\alpha_{L\oplus V})\circ\Delta_{L\oplus V}(x+a)\\
&=(\alpha\otimes\alpha\otimes\Delta)\circ\Delta(x)+(\alpha\otimes\alpha\otimes\Delta_V)\circ\Delta_V(a)-\omega_1'\circ(\alpha\otimes\alpha\otimes\Delta)\circ\Delta(x)-\omega_1'\circ(\alpha\otimes\alpha\otimes\Delta_V)\circ\Delta_V(a)\\
&-(\Delta\otimes\alpha\otimes\alpha)\circ\Delta(x)-(\Delta\otimes\alpha\otimes\alpha_V)\circ\Delta_V(a)+\tau_{34}(\Delta\otimes\alpha\otimes\alpha)\circ\Delta(x)+\tau_{34}(\Delta\otimes\alpha\otimes\alpha_V)\circ\Delta_V(a)=0.
\end{aligned}$$

□

Now, we define module structure for multiplicative 3-Hom-Lie bialgebras.

**Definition 6.11** *A tuple $(V,\mu_V,\Delta_V,\alpha_V)$ is said to be a module of a multiplicative 3-Hom-Lie bialgebra $(L,\mu,\Delta,\alpha)$ if*

*(i) $(V,\mu_V,\alpha_V)$ is a module of the multiplicative 3-Hom-Lie algebra $(L,\mu,\alpha)$,*

*(ii) $(V,\Delta_V,\alpha_V)$ is a comodule of the multiplicative 3-Hom-Lie coalgebra $(L,\Delta,\alpha)$,*

*(iii) the following compatibility condition is satisfied*

$$\begin{aligned}
&\Delta_V\circ\mu_V(x,y,a)\\
&=(ad_\mu(\alpha(x),\alpha(y))\otimes\alpha\otimes\alpha_V+\alpha\otimes ad_\mu(\alpha(x),\alpha(y))\otimes\alpha_V+\alpha\otimes\alpha\otimes\mu_V(\alpha(x),\alpha(y),\cdot))\circ\Delta_V(a)\\
&+(\mu_V(\alpha(\cdot),\alpha(y),a)\otimes\alpha\otimes\alpha+\alpha\otimes\mu_V(\alpha(\cdot),\alpha(y),a)\otimes\alpha+\alpha\otimes\alpha\otimes\mu_V(\alpha(\cdot),\alpha(y),a))\circ\Delta(x)\\
&+(\mu_V(\alpha(x),\alpha(\cdot),a)\otimes\alpha\otimes\alpha+\alpha\otimes\mu_V(\alpha(x),\alpha(\cdot),a)\otimes\alpha+\alpha\otimes\alpha\otimes\mu_V(\alpha(x),\alpha(\cdot),a))\circ\Delta(y),
\end{aligned}\tag{6.10}$$

*for all $x,y\in L$ and $a\in V$, where*

$$(\mu_V(\alpha(\cdot),\alpha(y),a)\otimes\alpha\otimes\alpha)\circ(x^{(1)}\otimes x^{(2)}\otimes x^{(3)})=\mu_V(\alpha(x^{(1)}),\alpha(y),a)\wedge\alpha(x^{(2)})\wedge\alpha(x^{(3)})\tag{6.11}$$

$$(\alpha\otimes\alpha\otimes\mu_V(\alpha(x),\alpha(y),\cdot))\circ(a^{(1)}\otimes a^{(2)}\otimes a_v^{(3)})=\alpha(a^{(1)})\wedge\alpha(a^{(2)})\wedge\mu_V(\alpha(x),\alpha(y),a_v^{(3)})\tag{6.12}$$

$$(\alpha\otimes\mu_V(\alpha(x),\alpha(\cdot),a)\otimes\alpha)(y^{(1)}\otimes y^{(2)}\otimes y^{(3)})=\alpha(y^{(1)})\wedge\mu_V(\alpha(x),\alpha(y^{(2)}),a)\wedge\alpha(y^{(3)}).\tag{6.13}$$

**Example 6.12** *Let $(L,\mu,\Delta,\alpha)$ be a multiplicative 3-Hom-Lie bialgebra. Then we have,* *(1) A multiplicative* $3$*-Hom-Lie bialgebra* $(L,\mu,\Delta,\alpha)$ *is a module over itself.*

*(2) If* $(V,\mu_V,\alpha_V)$ *is a module of a multiplicative* $3$*-Hom-Lie algebra* $(L,\mu,\alpha)$*. Then* $(V,\mu_V,0,\alpha_V)$ *is a module of a multiplicative* $3$*-Hom-Lie bialgebra* $(L,\mu,\Delta,\alpha)$*.*

*(3) If* $(V,\Delta_V,\alpha_V)$ *is a comodule of a multiplicative* $3$*-Hom-Lie coalgebra* $(L,\Delta,\alpha)$*. Then* $(V,0,\Delta_V,\alpha_V)$ *is a module of a multiplicative* $3$*-Hom-Lie bialgebra* $(L,\mu,\Delta,\alpha)$*.*

**Theorem 6.13** *Let* $(L,\mu,\Delta,\alpha)$ *be a multiplicative* $3$*-Hom-Lie bialgebra,* $V$ *be a vector space and* $\alpha_V\in End(V)$ *such that*

*(i)* $(V,\mu_V,\alpha_V)$ *is a module of the multiplicative* $3$*-Hom-Lie algebra* $(L,\mu,\alpha)$*,*

*(ii)* $(V,\Delta_V,\alpha_V)$ *is a comodule of the multiplicative* $3$*-Hom-Lie coalgebra* $(L,\Delta,\alpha)$*.*

*If* $(L\oplus V,\mu_{L\oplus V},\Delta_{L\oplus V},\alpha_{L\oplus V}=\alpha+\alpha_V)$ *is a multiplicative* $3$*-Hom-Lie bialgebra, Then* $(V,\mu_V,\Delta_V,\alpha_V)$ *is a module on the multiplicative* $3$*-Hom-Lie bialgebra* $(L,\mu,\Delta,\alpha)$*.*

**Proof** Let $x,y,z\in L$ and $a,b,c\in V$. We have

$$\Delta_{L\oplus V}\mu_{L\oplus V}(x+a,y+b,z+c)=\circlearrowleft_{x+a,y+b,z+c} ad^{(3)}_{L\oplus V}(x+a,y+b)\bullet_{\alpha_{L\oplus V}}\Delta_{L\oplus V}(z+c),$$

$$\begin{aligned}ad^{(3)}_{L\oplus V}(x+a,y+b)\bullet_{\alpha_{L\oplus V}}\Delta_{L\oplus V}(z+c)&=ad^{(3)}_{\mu}(\alpha(x),\alpha(y))\bullet_{\alpha}\Delta(z)+ad'^{(3)}_{L}(\alpha(x),\alpha(y))\bullet_{\alpha,\alpha_V}\Delta_V(c)\\&+\mu^{(3)}_V(\alpha(x),\alpha_V(b))\bullet_{\alpha}\Delta(z)+\mu'^{(3)}_V(\alpha(x),\alpha_V(b))\bullet_{\alpha,\alpha_V}\Delta_V(c)\\&-\mu^{(3)}_V(\alpha(y),\alpha_V(a))\bullet_{\alpha}\Delta(z)-\mu'^{(3)}_V(\alpha(y),\alpha_V(a))\bullet_{\alpha,\alpha_V}\Delta_V(c),\end{aligned}$$

where, for $x\in L,\ a\in V$, $\Delta(x)=x^{(1)}\otimes x^{(2)}\otimes x^{(3)}$ and $\Delta_V(a)=a^{(1)}\otimes a^{(2)}\otimes a^{(3)}_V$, such that $a^{(3)}_V\in V$ and $a^{(i)}\in L,$ for $i=1,2$.

$$\mu^{(3)}_V(x,b):L\otimes L\otimes L\longrightarrow L\otimes L\otimes V\quad and\quad \mu'^{(3)}_V(y,a):L\otimes L\otimes V\longrightarrow L\otimes V\otimes V,\tag{6.14}$$

are trilinear maps respectively satisfying

$$ad'^{(3)}_L(x,y)\bullet_{\alpha,\alpha_V}\Delta_V(c)=(ad_\mu(x,y)\otimes\alpha\otimes\alpha_V+\alpha\otimes ad_\mu(x,y)\otimes\alpha_V+\alpha\otimes\alpha\otimes\mu_V(x,y,\cdot))\Delta_V(c),$$

$$\mu^{(3)}_V(x,b)\bullet_{\alpha,\alpha_V}\Delta(z)=(\alpha\otimes\alpha\otimes\mu_V(\cdot,x,b)+\alpha\otimes\alpha\otimes\mu_V(\cdot,x,b)+\alpha\otimes\alpha\otimes\mu_V(\cdot,x,b))\Delta(z),$$

$$\mu'^{(3)}_V(y,a)\bullet_{\alpha,\alpha_V}\Delta_V(c)=(\alpha\otimes\mu_V(\cdot,y,a)\otimes\alpha_V+\alpha\otimes\alpha_V\otimes\mu_V(\cdot,y,a))\Delta(c).$$

Since $(L\oplus V,\mu_{L\oplus V},\Delta_{L\oplus V},\alpha_{L\oplus V})$ is a multiplicative 3-Hom-Lie bialgebra, then we have

$$\begin{aligned}
0 =&\Delta_{L\oplus V}\mu_{L\oplus V}(x+a,y+b,z+c)\\
=&\Delta\mu(x,y,z)+\Delta_V\mu_V(x,y,c)+\Delta_V\mu_V(y,z,a)-\Delta_V\mu_V(x,z,b)\\
&+\mu_V(z,a,b^{(1)})\otimes\alpha(b^{(2)})\otimes\alpha_V(b_V^{(3)})+\alpha(b^{(1)})\otimes\mu_V(z,a,b^{(2)})\otimes\alpha_V(b_V^{(3)})\\
&-\mu_V(x,c,b^{(1))}\otimes\alpha(b^{(2)})\otimes\alpha_V(b_V^{(3)})-\alpha(b^{(1)})\otimes\mu_V(x,c,b^{(2)})\otimes\alpha_V(b_V^{(3)})\\
&+\mu_V(y,c,a^{(1)})\otimes\alpha(a^{(2)})\otimes\alpha_V(a_V^{(3)})+\alpha(a^{(1)})\otimes\mu_V(y,c,a^{(2)})\otimes\alpha_V(a_V^{(3)})\\
&-\mu_V(z,b,a^{(1)})\otimes\alpha(a^{(2)})\otimes\alpha_V(a_V^{(3)})-\alpha(a^{(1)})\otimes\mu_V(z,b,a^{(2)})\otimes\alpha_V(a_V^{(3)})\\
&+\mu_V(x,b,c^{(1)})\otimes\alpha(c^{(2)})\otimes\alpha_V(c_V^{(3)})+\alpha(c^{(1)})\otimes\mu_V(x,b,c^{(2)})\otimes\alpha_V(c_V^{(3)})\\
&-\mu_V(y,a,c^{(1)})\otimes\alpha(c^{(2)})\otimes\alpha_V(c_V^{(3)})-\alpha(c^{(1)})\otimes\mu_V(y,a,c^{(2)})\otimes\alpha_V(c_V^{(3)}).
\end{aligned}$$

Thus, $\Delta_V$ and $\mu_V(x,z,b)=0$ are compatible. □

## 7. Generalized derivations of 3-Hom-Lie bialgebras

In the following, we recall some definitions and properties of derivations, quasi-derivations and generalized derivations of multiplicative 3-Hom-Lie algebras. We extend and discuss these notions in the case of multiplicative 3-Hom-Lie bialgebras. We define and show some properties of $\alpha^k$-derivations, $\alpha^k$-quasi-derivations, $\alpha^k$-coderivations, $\alpha^k$-centroids, $\alpha^k$-cocentroids, $\alpha^k$-quasi-derivations, $\alpha^k$-biderivations and their generalizations.

Let $(L,\mu,\alpha)$ be a multiplicative 3-Hom-Lie algebra.

**Definition 7.1** *A linear map $D:L\to L$ is said to be a derivation on $(L,\mu,\alpha)$ if $\alpha D=D\alpha$ and satisfies, for all $x,y,z\in L$, the following condition:*

$$D(\mu(x,y,z))=\mu(D(x),y,z)+\mu(x,D(y),z)+\mu(x,y,D(z)). \tag{7.1}$$

**Definition 7.2** *A linear map $D:L\to L$ is said to be an $\alpha^k$-derivation on $(L,\mu,\alpha)$ if $\alpha D=D\alpha$ and satisfies, for all $x,y,z\in L$, the following condition*

$$D(\mu(x,y,z))=\mu(D(x),\alpha^k(y),\alpha^k(z))+\mu(\alpha^k(x),D(y),\alpha^k(z))+\mu(\alpha^k(x),\alpha^k(y),D(z)). \tag{7.2}$$

We denote the set of all $\alpha^k$-derivations by $Der_{\alpha^k}(L)$. Set $Der(L)=\oplus_{k\geq 0}Der_{\alpha^k}(L)$, we can easily show that $Der(L)$ is equipped with a Lie algebra structure.

**Example 7.3** *Let $(L,[\cdot,\cdot,\cdot],\alpha)$ be a multiplicative 3-Hom-Lie algebra generated by $\{e_1,e_2,e_3\}$, given by*

$$[e_1,e_2,e_3]=e_3,\quad \alpha(e_1)=ae_1,\quad \alpha(e_2)=\frac{1}{a}e_2,\quad \alpha(e_3)=0.$$

*Define the linear map $D\in End(L)$ by $D(e_1)=b_1e_1$, $D(e_2)=b_2e_2$, $D(e_3)=b_3e_3$, where $a,b_1,b_2,b_3$ are parameters. Then $D$ is an $\alpha^3$-derivation on $(L,\mu,\alpha)$.*

**Definition 7.4** *A linear map $D \in End(L)$ is said to be an $\alpha^k$-generalized derivation on $(L, \mu, \alpha)$ if there exist linear maps $D', D'', D''' \in End(L)$ such that $\alpha$ commutes with $D, D', D'', D'''$ and holds the following condition*

$$D'''(\mu(x, y, z)) = \mu(D(x), \alpha^k(y), \alpha^k(z)) + \mu(\alpha^k(x), D'(y), \alpha^k(z)) + \mu(\alpha^k(x), \alpha^k(y), D''(z)), \quad (7.3)$$

*for all $x, y, z \in L$. We say that $D$ is associated with $(D', D'', D''')$. If $D = D' = D''$, then $D$ is called $\alpha^k$-quasi-derivation associated with $D'''$.*

The sets of $\alpha^k$-quasi-derivations and $\alpha^k$-generalized derivations will be denoted respectively by $QDer_{\alpha^k}(L)$ and $GDer_{\alpha^k}(L)$. Set $QDer(L) = \oplus_{k\geq 0} QDer_{\alpha^k}(L)$ and $GDer(L) = \oplus_{k\geq 0} GDer_{\alpha^k}(L)$. We have

$$Der(L) \subset QDer(L) \subset GDer(L).$$

**Example 7.5** *Let $(L, \mu, \alpha)$ be a 3-dimensional multiplicative 3-Hom-Lie algebra, where the multiplication and the structure map are defined with respect to a basis $\{e_1, e_2, e_3\}$ by*

$$\mu(e_1, e_2, e_3) = ce_1, \quad \alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2, \quad \alpha(e_3) = e_3.$$

*Let $D$ and $D'$ be linear maps defined as*

$$D(e_1) = b_1 e_1, \quad D(e_2) = b_2 e_2 + b_1 e_2 + b_3 e_3, \quad D(e_3) = b_4 e_1 + b_5 e_3,$$

$$D'(e_1) = (2b_1 - b_2 - 2b_5)e_1, \quad D'(e_2) = a_2 e_1 + (2b_1 - b_2 - 2b_5)e_2 + a_3 e_3, \quad D'(e_3) = a_4 e_1 + a_5 e_3.$$

*Then $D$ is an $\alpha^3$-quasi-derivation on $(L, \mu, \alpha)$ associated with $D'$.*

**Example 7.6** *Let $(L, \mu, \alpha)$ be a 3-dimensional multiplicative 3-Hom-Lie algebra, where the multiplication and the structure map are defined with respect to a basis $\{e_1, e_2, e_3\}$ by*

$$\mu(e_1, e_2, e_3) = e_1, \quad \alpha(e_1) = -e_1, \quad \alpha(e_2) = e_1 - e_2, \quad \alpha(e_3) = e_2 - e_3.$$

*Let $D, D', D'', D'''$ be linear maps defined as*

$$D(e_1) = a_1 e_1, \quad D(e_2) = a_2 e_1 + a_1 e_2, D(e_3) = a_3 e_1 + a_2 e_2 + a_1 e_3,$$

$$D'(e_1) = b_1 e_1, \quad D'(e_2) = b_2 e_1 + b_1 e_2, D'(e_3) = b_3 e_1 + b_2 e_2 + b_1 e_3,$$

$$D''(e_1) = c_1 e_1, \quad D''(e_2) = c_2 e_1 + c_1 e_2, D''(e_3) = c_3 e_1 + c_2 e_2 + c_1 e_3,$$

$$D'''(e_1) = (a_1 + b_1 + c_1)e_1, \quad D'''(e_2) = d_2 e_1 + (a_1 + b_1 + c_1)e_2, D'''(e_3) = d_3 e_1 + d_2 e_2 + (a_1 + b_1 + c_1)e_3.$$

*Then $D$ is an $\alpha^5$-generalized derivation on $(L, \mu, \alpha)$ associated with $(D', D'', D''')$.*

**Proposition 7.7** *Let $(L, \mu, \alpha)$ be a multiplicative 3-Hom-Lie algebra and a linear map $D$ associated with $(D', D'', D''')$ be an $\alpha^k$-generalized derivation on $L$. Then $\alpha^p D$ associated with $(\alpha^p D', \alpha^p D'', \alpha^p D''')$ is an $\alpha^{p+k}$-generalized derivation on $L$.*

**Proof** Let $D$ be an $\alpha^k$-generalized derivation on $L$ associated with $(D', D'', D''')$. Then, for all $x, y, z \in L$, we have

$$\begin{aligned}
&\alpha^p D'''(\mu(x,y,z)) \\
&= \alpha^p(\mu(D(x), \alpha^k(y), \alpha^k(z))) + \alpha^p(\mu(\alpha^k(x), D'(y), \alpha^k(z))) + \alpha^p(\mu(\alpha^k(x), \alpha^k(y), D''(z))) \\
&= \mu(\alpha^p D(x), \alpha^{p+k}(y), \alpha^{p+k}(z)) + \mu(\alpha^{p+k}(x), \alpha^p D'(y), \alpha^{p+k}(z)) + \mu(\alpha^{p+k}(x), \alpha^{p+k}(y), \alpha^p D''(z)).
\end{aligned}$$

□

**Proposition 7.8** *Let $(L, \mu)$ be a 3-Lie algebra, $D$ associated with $(D', D'', D''')$ be a generalized derivation on $L$ and $\alpha$ be a morphism on $L$ commuting wit $D, D', D''$ and $D'''$. Then $\alpha^k D$ associated with $(\alpha^k D', \alpha^k D'', \alpha^k D''')$ is an $\alpha^k$-generalized derivation on $(L, \mu_\alpha, \alpha)$.*

**Proof** Straightforward. □

**Definition 7.9** *A linear map $\theta \in End(L)$ is said to be an $\alpha^k$-centroid of a 3-Hom-Lie algebra $(L, \mu, \alpha)$ if $\theta$ and $\alpha$ commute and satisfy, for all $x, y, z \in L$, the following condition*

$$\theta(\mu(x,y,z)) = \mu(\theta(x), \alpha^k(y), \alpha^k(z)) = \mu(\alpha^k(x), \theta(y), \alpha^k(z)) = \mu(\alpha^k(x), \alpha^k(y), \theta(z)). \tag{7.4}$$

The set of $\alpha^k$-centroids of $(L, \mu, \alpha)$ is denoted by $C_{\alpha^k}(L)$.

**Definition 7.10** *A linear map $\theta \in End(L)$ is said to be an $\alpha^k$-quasi-centroid of a 3-Hom-Lie algebra $(L, \mu, \alpha)$ if $\theta$ and $\alpha$ commute and satisfy the following condition*

$$\mu(\theta(x), \alpha^k(y), \alpha^k(z)) = \mu(\alpha^k(x), \theta(y), \alpha^k(z)) = \mu(\alpha^k(x), \alpha^k(y), \theta(z)), \quad \forall \ \ x, y, z \in L. \tag{7.5}$$

The set of $\alpha^k$-quasi-centroids of $L$ is denoted by $QC_{\alpha^k}(L)$.
Set $C(L) = \oplus_{k \geq 0} C_{\alpha^k}(L)$ and $QC(L) = \oplus_{k \geq 0} QC_{\alpha^k}(L)$.

**Proposition 7.11** *With the above notations we have*

1. $C(L) \subseteq QC(L)$.
2. $C(L) \subseteq QDer(L)$.

**Proof** Straightforward. □

In the following, we introduce the notion of generalized $\alpha^k$-coderivation, $\alpha^k$-cocentroids and $\alpha^k$-quasi-cocentroid of 3-Hom-Lie coalgebras. Let $(L, \Delta, \alpha)$ be a multiplicative 3-Hom-Lie coalgebra.

**Definition 7.12** *A linear map $\Psi : L \to L$ is said to be a coderivation on $(L, \Delta, \alpha)$, if $\Psi \circ \alpha = \alpha \circ \Psi$ and satisfies the following condition*

$$\Delta \circ \Psi = (\Psi \otimes Id \otimes Id) \circ \Delta + (Id \otimes \Psi \otimes Id) \circ \Delta + (Id \otimes Id \otimes \Psi) \circ \Delta. \tag{7.6}$$

**Definition 7.13** *A linear map $\Psi : L \to L$ is said to be an $\alpha^k$-coderivation on $(L, \Delta, \alpha)$, if $\alpha\Psi = \Psi\alpha$ and satisfies the following condition*

$$\Delta \circ \Psi = (\Psi \otimes \alpha^k \otimes \alpha^k) \circ \Delta + (\alpha^k \otimes \Psi \otimes \alpha^k) \circ \Delta + (\alpha^k \otimes \alpha^k \otimes \Psi) \circ \Delta. \tag{7.7}$$

We denote the set of all $\alpha^k$-coderivations by $Der^{\Delta}_{\alpha^k}(L)$ and set $Der^{\Delta}(L) = \oplus_{k\geq 0} Der^{\Delta}_{\alpha^k}(L)$.

**Proposition 7.14** *The set $Der^{\Delta}(L)$ equipped with the commutator give a Lie algebra structure.*

**Proof** Let $D \in Der^{\Delta}_{\alpha^k}(L)$ and $D' \in (Der_{\alpha^{k'}}$, we have

$$\begin{aligned}
&\Delta \circ [D, D'] = \Delta \circ (D \circ D' - D' \circ D)\\
&= \Delta \circ D \circ D' - \Delta \circ D' \circ D\\
&= (\Delta \circ D) \circ D' - (\Delta \circ D') \circ D\\
&= (D' \otimes \alpha^{k'} \otimes \alpha^{k'}) \circ (\Delta \circ D) + (\alpha^{k'} \otimes D' \otimes \alpha^{k'}) \circ (\Delta \circ D) + (\alpha^{k'} \otimes \alpha^{k'} \otimes D') \circ (\Delta \circ D)\\
&- (D \otimes \alpha^{k} \otimes \alpha^{k}) \circ (\Delta \circ D') + (\alpha^{k} \otimes D \otimes \alpha^{k}) \circ (\Delta \circ D') + (\alpha^{k} \otimes \alpha^{k} \otimes D) \circ (\Delta \circ D')\\
&= (D' \circ D \otimes \alpha^{k+k'} \otimes \alpha^{k+k'}) \circ \Delta + (\alpha^{k+k'} \otimes D' \circ D \otimes \alpha^{k+k'}) \circ \Delta + (\alpha^{k+k'} \otimes \alpha^{k+k'} \otimes D' \circ D) \circ \Delta\\
&- (D \circ D' \otimes \alpha^{k+k'} \otimes \alpha^{k+k'}) \circ \Delta + (\alpha^{k+k'} \otimes D \circ D' \otimes \alpha^{k+k'}) \circ \Delta + (\alpha^{k+k'} \otimes \alpha^{k+k'} \otimes D \circ D') \circ \Delta\\
&= ([D, D'] \otimes \alpha^{k+k'} \otimes \alpha^{k+k'}) \circ \Delta + (\alpha^{k+k'} \otimes [D, D'] \otimes \alpha^{k+k'}) \circ \Delta + (\alpha^{k+k'} \otimes \alpha^{k+k'} \otimes [D, D']) \circ \Delta.
\end{aligned}$$

Thus $[D, D']$ is an $\alpha^{k+k'}$-coderivation. □

**Definition 7.15** *A linear map $\Psi \in End(L)$ is said to be an $\alpha^k$-generalized coderivation on $(L, \Delta, \alpha)$ if there exist linear maps $\Psi', \Psi'', \Psi''' \in End(L)$ such that $\alpha$ commutes with $\Psi, \Psi', \Psi'', \Psi'''$ and satisfies the following condition*

$$\Delta \circ \Psi''' = (\Psi \otimes \alpha^k \otimes \alpha^k) \circ \Delta + (\alpha^k \otimes \Psi' \otimes \alpha^k) \circ \Delta + (\alpha^k \otimes \alpha^k \otimes \Psi'') \circ \Delta. \tag{7.8}$$

*We also say that $\Psi$ is associated with $(\Psi', \Psi'', \Psi''')$. If $\Psi = \Psi' = \Psi''$, then $\Psi$ is called $\alpha^k$-quasi-coderivation.*

The sets of $\alpha^k$-quasi-coderivations and $\alpha^k$-generalized coderivations are denoted respectively by $QDer^{\Delta}_{\alpha^k}(L)$ and $GDer^{\Delta}_{\alpha^k}(L)$. Set $QDer^{\Delta}(L) = \oplus_{k\geq 0} QDer^{\Delta}_{\alpha^k}(L)$ and $GDer^{\Delta}(L) = \oplus_{k\geq 0} GDer^{\Delta}_{\alpha^k}(L)$.

**Example 7.16** *Let $L$ be a 3-dimensional multiplicative 3-Hom-Lie coalgebra. Let $\{e_1, e_2, e_3\}$ be a basis on which the comultiplication $\Delta$ is defined by $\Delta(e_1) = ce_1 \wedge e_2 \wedge e_3$, $\Delta(e_2) = \Delta(e_3) = 0$, and the morphism $\alpha$ by $\alpha(e_1) = ae_1$, $\alpha(e_2) = be_2$, $\alpha(e_3) = \frac{1}{b}e_3$, where $a, b$ are parameters such that $a, b \neq 0, 1$.*

*We consider linear maps $\Psi, \Psi', \Psi'', \Psi''' \in End(L)$ defined by*

$$\Psi(e_1) = a_1e_1, \quad \Psi(e_2) = a_2e_2, \quad \Psi(e_3) = a_3e_3,$$

$$\Psi'(e_1) = b_1e_1, \quad \Psi'(e_2) = b_2e_2, \quad \Psi'(e_3) = b_3e_3,$$

$$\Psi''(e_1) = c_1e_1, \quad \Psi''(e_2) = c_2e_2, \quad \Psi''(e_3) = c_3e_3,$$

$$\Psi'''(e_1) = (a_1 + b_2 + \frac{a^k}{b^k} + a^kb^kc_3)e_1, \quad \Psi'''(e_2) = d_1e_2, \quad \Psi'''(e_3) = d_2e_3,$$

*for* $a_i, b_i, c_i, d_j \in \mathbb{K}$, $1 \le i \le 3$ *and* $1 \le j \le 2$. *Then* $\Psi$ *is an* $\alpha^k$*-coderivation associated with* $(\Psi', \Psi'', \Psi''')$.

**Proposition 7.17** *Let* $(L, \Delta)$ *be a* 3*-Lie algebra,* $\Psi$ *associated with* $(\Psi', \Psi'', \Psi''')$ *be a generalized coderivation on* $L$ *and* $\alpha$ *be a morphism of* $L$ *commuting with* $\Psi, \Psi', \Psi'', \Psi'''$. *Then* $\alpha^k\Psi$ *associated with* $(\alpha^k\Psi', \alpha^k\Psi'', \alpha^k\Psi''')$ *are* $\alpha^k$*-generalized coderivations on* $(L, \Delta_\alpha, \alpha)$.

**Definition 7.18** *A linear map* $\Theta \in End(L)$ *is said to be an* $\alpha^k$*-cocentroid on* $(L, \Delta, \alpha)$ *if* $\alpha \circ \Theta = \Theta \circ \alpha$ *and holds the following condition*

$$\Delta \circ \Theta = (\Theta \otimes \alpha^k \otimes \alpha^k) \circ \Delta = (\alpha^k \otimes \Theta \otimes \alpha^k) \circ \Delta = (\alpha^k \otimes \alpha^k \otimes \Theta) \circ \Delta. \tag{7.9}$$

**Example 7.19** *Let* $L$ *be a* 3*-dimensional multiplicative* 3*-Hom-Lie coalgebra and* $\{e_1, e_2, e_3\}$ *be a basis on which the comultiplication* $\Delta$ *is defined by* $\Delta(e_1) = \Delta(e_2) = 0, \Delta(e_3) = ce_1 \wedge e_2 \wedge e_3$, *and the morphism* $\alpha$ *by* $\alpha(e_1) = -e_1, \quad \alpha(e_2) = e1 - e_2, \quad \alpha(e_3) = e_2 - e_3$.

*We consider a linear map* $\Theta \in End(L)$ *defined by*

$$\Theta(e_1) = 0, \quad \Theta(e_2) = a_1 e_2, \quad \Theta(e_3) = a_1 e_1 + a_2 e_2, \quad for\ a_1, a_2 \in \mathbb{K}.$$

*Then,* $\Theta$ *is an* $\alpha^2$*-cocentroid of* $(L, \Delta, \alpha)$.

**Definition 7.20** *A linear map* $\Theta \in End(L)$ *is said to be an* $\alpha^k$*-quasi-cocentroid on* $(L, \Delta, \alpha)$ *if* $\Theta \circ \alpha = \alpha \circ \Theta$ *and satisfies the following condition*

$$(\Theta \otimes \alpha^k \otimes \alpha^k) \circ \Delta = (\alpha^k \otimes \Theta \otimes \alpha^k) \circ \Delta = (\alpha^k \otimes \alpha^k \otimes \Theta) \circ \Delta. \tag{7.10}$$

The set of $\alpha^k$-cocentroids and $\alpha^k$-quasi-cocentroids of $L$ are denoted respectively by $C^{\Delta}_{\alpha^k}(L)$ and $QC^{\Delta}_{\alpha^k}(L)$. Set $C^{\Delta}(L) = \oplus_{k\ge0} C^{\Delta}_{\alpha^k}(L)$ and $QC^{\Delta}(L) = \oplus_{k\ge0} QC^{\Delta}_{\alpha^k}(L)$.

**Proposition 7.21** *With the above notations, we have*

1. $C^{\Delta}(L) \subset QC^{\Delta}(L)$.
2. $C^{\Delta}(L) \subseteq QDer^{\Delta}(L)$.

**Proof** Straightforward. □

Next, we discuss biderivations and their generalizations. Let $(L, \mu, \Delta, \alpha)$ be a multiplicative 3-Hom-Lie bialgebra.

**Definition 7.22** *A pair* $(D, \Psi)$ *of endomorphisms on* $L$ *is called an* $\alpha^k$*-biderivation of multiplicative* 3*-Hom-Lie bialgebra if* $D$ *is a* $\alpha^k$*-derivation of* $(L, \mu, \alpha)$ *and* $\Psi$ *is a* $\alpha^k$*-coderivation of* $(L, \Delta, \alpha)$.
*We denote the set of all* $\alpha^k$*-biderivations by* $Der^B_{\alpha^k}(L)$ *and set* $Der^B(L) = \oplus_{k\ge0} Der^B_{\alpha^k}(L)$.

*A pair* $(D, \Psi)$ *of endomorphisms on* $L$ *is called an* $\alpha^k$*-quasi-biderivation of* $(L, \mu, \Delta, \alpha)$, *if* $D$ *is an* $\alpha^k$*-quasi-derivation of* $(L, \mu, \alpha)$ *and* $\Psi$ *is an* $\alpha^k$*-quasi-coderivation of* $(L, \Delta, \alpha)$.
*We denote the set of all* $\alpha^k$*-quasi-biderivations by* $QDer^B_{\alpha^k}(L)$ *and set* $QDer^B(L) = \oplus_{k\ge0} QDer^B_{\alpha^k}(L)$.

*A pair $(D,\Psi)$ of endomorphisms on $L$ is called a generalized $\alpha^k$-biderivation of $(L,\mu,\Delta,\alpha)$, if $D$ is a generalized $\alpha^k$-derivation of $(L,\mu,\alpha)$ and $\Psi$ is a generalized $\alpha^k$-coderivation of $(L,\Delta,\alpha)$.*
*We denote the set of all generalized $\alpha^k$-biderivations by $GDer^B_{\alpha^k}(L)$ and set $GDer^B(L)=\oplus_{k\geq 0}GDer^B_{\alpha^k}(L)$.*

It is easy to see that $Der^B(L)\subset QDer^B(L)\subset GDer^B(L)$.

**Example 7.23** *Let $L$ be a 3-dimensional multiplicative 3-Hom-Lie bialgebra with a basis $\{e_1,e_2,e_3\}$ on which the multiplication and the comultiplication $\mu,\Delta$ are defined by $\mu(e_1,e_2,e_3)=e_1$, $\Delta(e_2)=e_1\wedge e_2\wedge e_3$, $\Delta(e_1)=\Delta(e_3)=0$, and the morphism $\alpha$ by $\alpha(e_1)=\alpha(e_2)=0$, $\alpha(e_3)=ce_3$, for $c\neq 0$.*
*We consider linear maps $D,D',D'',D''',\Psi,\Psi',\Psi'',\Psi''',\in End(L)$ defined by*

$$
\begin{aligned}
&D(e_1)=a_1e_1+e_2a_2,\quad D(e_2)=a_3e_1+a_4e_2,\quad D(e_3)=a_5e_3,\\
&D'(e_1)=a_6e_1+a_7e_2,\quad D'(e_2)=a_8e_1+a_9e_2,\quad D'(e_3)=a_{10}e_3,\\
&D''(e_1)=a_{11}e_1+a_{12}e_2,\quad D''(e_2)=a_{13}e_1+a_{14}e_2,\quad D''(e_3)=a_{15}e_3,\\
&D'''(e_1)=0,\quad D'''(e_2)=0,\quad D'''(e_3)=0,\\
&\Psi(e_1)=b_1e_1+b_2e_2,\quad \Psi(e_2)=b_3e_1+b_4e_2,\quad \Psi(e_3)=b_5e_3,\\
&\Psi'(e_1)=c_1e_1+c_2e_2,\quad \Psi'(e_2)=c_3e_1+c_4e_4,\quad \Psi'(e_3)=c_5e_3,\\
&\Psi''(e_1)=d_1e_1+d_2e_2,\quad \Psi''(e_2)=d_3e_1+d_4e_2,\quad \Psi''(e_3)=d_5e_3,\\
&\Psi'''(e_1)=d_6e_1,\quad \Psi'''(e_2)=d_7e_1,\quad \Psi'''(e_3)=d_8e_3,
\end{aligned}
$$

*where $a_i,b_l,c_k,d_j\in\mathbb{K}$, $1\leq i\leq 15$, $1\leq j\leq 8$ and $1\leq k,l\leq 5$.*
*Then $(D,\Psi)$ is an $\alpha^k$-biderivation associated with $(D',D'',D''',\Psi',\Psi'',\Psi''')$.*

**Corollary 7.24** *Let $(L,\mu,\Delta,\alpha)$ be a multiplicative 3-Hom-Lie bialgebra and $(D,\Psi)$ be a generalized $\alpha^k$-biderivation on $L$. Then $(\alpha^p\circ D,\Psi\circ\alpha^p)$ is a generalized $\alpha^{p+k}$-biderivation on $L$.*

**Definition 7.25** *A pair $(\theta,\Theta)$ is said to be an $\alpha^k$-(quasi)-bicentroid on $(L,\mu,\Delta,\alpha)$ if $\theta$ is an $\alpha^k$-(quasi)-centroid of $(L,\mu,\alpha)$ and $\Theta$ is an $\alpha^k$-(quasi)-cocentroid of $(L,\Delta,\alpha)$.*

The set of $\alpha^k$-bicentroids and $\alpha^k$-quasi-bicentroids of $(L,\mu,\Delta,\alpha)$ are denoted respectively by $C^B_{\alpha^{k'}}(L)$ and $QC^B_{\alpha^{k'}}(L)$. We set $C^B(L)=\oplus_{k\geq 0}C^B_{\alpha^k}(L)$ and $DC^B(L)=\oplus_{k\geq 0}QC^B_{\alpha^k}(L)$.

It is obvious that $C^B(L)\subset QC^B(L)$.

**Proposition 7.26** *Let $(D,\Psi)\in Der^B(L)$ and $(\theta,\Theta)\in C^B(L)$. Then $([D,\theta],[\Psi,\Theta])\in C^B(L)$.*

**Proof** Assume that $D\in Der_{\alpha^k}(L)$, $\theta\in C_{\alpha^{k'}}(L)$. For arbitrary $x,y,z\in L$, we have

$$
\begin{aligned}
D\theta(\mu(x,y,z))&=D(\mu(\theta(x),\alpha^{k'}(y),\alpha^{k'}(z)))\\
&=\mu(D\theta(x),\alpha^{k+k'}(y),\alpha^{k+k'}(z))+\mu(\theta\alpha^k(x),D\alpha^{k'}(y),\alpha^{k+k'}(z))+\mu(\theta(x),\alpha^{k+k'}(y),D\alpha^{k'}(z)).
\end{aligned}
\tag{7.11}
$$

and

$$\theta D(\mu(x,y,z)) = \theta(\mu(D(x),\alpha^k(y),\alpha^k(z)) + \mu(\alpha^k(x),D(y),\alpha^k(z)) + \mu(\alpha^k(x),\alpha^k(y),D(z)))$$
$$= \mu(\theta D(x),\alpha^{k+k'}(y),\alpha^{k+k'}(z)) + \mu(\theta\alpha^k(x),D\alpha^{k'}(y),\alpha^{k+k'}(z)) + \mu(\theta\alpha^k(x),\alpha^{k+k'}(y),D\alpha^{k'}(z)). \quad (7.12)$$

By making the difference of Eqs. (7.11) and (7.12), we get

$$[D,\theta](\mu(x,y,z)) = \mu([D,\theta](x),\alpha^{k+k'}(y),\alpha^{k+k'}(z)).$$

Similarly we can prove that, for all $x,y,z \in T$,

$$[D,\theta](\mu(x,y,z)) = \mu(\alpha^{k+k'}(x),[D,\theta](y),\alpha^{k+k'}(z)) \text{ and } [D,\theta](\mu(x,y,z)) = \mu(\alpha^{k+k'}(x),\alpha^{k+k'}(y),[D,\theta](z)).$$

Now, let $\Psi \in Der^{\Delta}_{\alpha^k}(L)$, $\Theta \in C^{\Delta}_{\alpha^{k'}}(L)$, we have

$$\Delta\circ\Theta\circ\Psi = (\Theta\otimes\alpha^{k'}\otimes\alpha^{k'})\circ\Delta\circ\Psi$$
$$= (\Theta\circ\Psi\otimes\alpha^{k+k'}\otimes\alpha^{k+k'} + \Theta\circ\alpha^k\otimes\Psi\circ\alpha^{k'}\otimes\alpha^{k+k'} + \Theta\circ\alpha^k\otimes\alpha^{k+k'}\otimes\Psi\circ\alpha^{k'})\circ\Delta, \quad (7.13)$$

and

$$\Delta\circ\Psi\circ\Theta = (\Psi\otimes\alpha^k\otimes\alpha^k + \alpha^k\otimes\Psi\otimes\alpha^k + \alpha^k\otimes\alpha^k\otimes\Psi)\circ\Delta\circ\Theta$$
$$= (\Psi\circ\Theta\otimes\alpha^{k+k'}\otimes\alpha^{k+k'} + \Theta\circ\alpha^k\otimes\Psi\circ\alpha^{k'}\otimes\alpha^{k+k'} + \Theta\circ\alpha^k\otimes\alpha^{k+k'}\otimes\Psi\circ\alpha^{k'})\circ\Delta. \quad (7.14)$$

By making the difference of Eqs. (7.13) and (7.14), we get

$$\Delta\circ[\Psi,\Theta] = ([\Psi,\Theta]\otimes\alpha^{k+k'}\otimes\alpha^{k+k'})\circ\Delta.$$

Similarly we can prove that, for all $x,y,z \in L$,

$$\Delta\circ[\Psi,\Theta] = ([\Psi,\Theta]\otimes\alpha^{k+k'}\otimes\alpha^{k+k'})\circ\Delta \text{ and } \Delta\circ[\Psi,\Theta] = (\alpha^{k+k'}\otimes\alpha^{k+k'}\otimes[\Psi,\Theta])\circ\Delta.$$

□

**Remark 7.27** *The same definitions and results could be stated for generalized multiplicative 3-Hom-Lie bialgebras.*

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