Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1619 – 1625
© TÜBİTAK
doi:10.3906/mat-2001-54

A refinement of the Bergström inequality

Gültekin TINAZTEPE[1,*], İlknur YEŞİLCE[2], Gabil ADİLOV[3]
[1]Vocational School of Technical Sciences, Akdeniz University, Antalya, Turkey
[2]Department of Mathematics, Faculty of Science and Letters, Aksaray University, Aksaray, Turkey
[3]Faculty of Education, Akdeniz University, Antalya, Turkey

Received: 16.01.2020 • **Accepted/Published Online:** 07.05.2021 • **Final Version:** 27.07.2021

Abstract: In this paper, the Bergström inequality is studied, and a refinement of this inequality is obtained by performing the optimality conditions based on abstract concavity. Some numerical experiments are given to illustrate the efficacy of the refinement.

Key words: Abstract concavity, refinement, Bergström inequality, global optimization

1. Introduction

The refinements or extensions of inequalities take place widely in inequality theory. There are many approaches or techniques to obtain new versions of inequalities and studies on the equivalences of inequalities (See [3, 8, 10, 12]). Some of these approaches are based on the convexity of functions as in Jensen type, Hermite-Hadamard type inequalities. In recent years, new convexity types have been developed (See [4, 15, 19]). Many new inequalities and their versions, such as integral, fractional integral, and Hermite-Hadamard type inequalities, have been obtained for the function classes of various convexity types by different authors in [2, 5, 6, 11, 13, 15, 18, 20, 23–27]. Also, sharper versions of the well-known discrete inequalities have been derived by means of the results of abstract convexity in [1, 16, 20–22]. In [1] and [20], the sharper versions for weighted arithmetic, geometric, and harmonic mean inequalities and Hölder inequality are derived with the help of the results in [16]. In this study, we give a refinement of the Bergström inequality.

In [15], A.M. Rubinov gives the definition of abstract concave function as follows:

Let H be the class of functions on $\Omega \subset \mathbb{R}^n$. If there exists a function $h \in H$ such that $h \geq f$, it is said that $f : \Omega \to R \cup \{-\infty\}$ is majorized by H.

Definition 1.1 *A function $f : \Omega \to R \cup \{-\infty\}$ majorized by H is called abstract concave with respect to H (or $H-$concave) if there exists a set $U \subset H$ such that*

$$f(x) = \inf_{h \in U} h(x)$$

for all $x \in \Omega$.

*Correspondence: gtinaztepe@akdeniz.edu.tr
2010 *AMS Mathematics Subject Classification:* 26D07

In [16], the optimality conditions for the function f that is abstract concave function with respect to H where

$$H = \left\{ h : X \to \mathbb{R} \middle| a > 0, l \in X, c \in \mathbb{R}, h(x) = a \left\| x \right\|^2 + \langle l, x \rangle + c \right\}$$

are considered and the following result is obtained:

Theorem 1.2 *[16] Let $\|.\|$ and $\|.\|_\circ$ be two norms on $\mathbb{R}^n$. Let Ω be the subset of $\mathbb{R}^n$ with nonempty interior and let $f \in C^1(\Omega)$. Suppose that $x \longmapsto \nabla f(x)$ is a Lipschitz mapping on Ω and*

$$K := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|} < \infty.$$

Let f have global minimum at $x^* \in$ int Ω. Consider the ball

$$B_\circ(x^*, r) = \{x : \|x - x^*\|_\circ \leq r\} \subset int\Omega$$

and

$$M := \max \left\{ \|\nabla f(x)\|_\circ : x \in B_\circ(x^*, r) \right\}.$$

Let q be a positive number such that $B_\circ(x^*, r + q) \subset \Omega$ and let $a \geq \max\left(K, \dfrac{M}{2q}\right)$. Then

$$\frac{1}{4a} \left\| \nabla f(x) \right\|^2 \leq f(x) - f(x^*), \;\; x \in B_\circ(x^*, r).$$

Theorem 1.2 can be used to obtain sharper versions of well-known inequalities in such a way that each inequality can be written in the form of $f(x) \geq 0$, also if the points satisfying the equation $f(x) = 0$ are known, which are also global minimum points of f, then the theorem can be applied to suitable inequalities. In this paper, we will apply the theorem to obtain a refinement of the Bergström inequality. This inequality is given as follows:

For $x_k \in \mathbb{R}$ and $a_k > 0, \;\; k \in \{1, 2, ..., n\}$,

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \cdots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{a_1 + a_2 + \cdots + a_n}$$

holds. Equality occurs if and only if $x_k = \lambda a_k$ for $k \in \{1, 2, ..., n\}$ and $\lambda \in \mathbb{R}$.

Bergström inequality has stimulated several mathematicians' interest, and various extensions, refinements, and proofs of the inequality have been provided. We refer to [7, 9, 14, 17, 28, 29] and the references given therein.

In this paper, a refinement of this inequality for real numbers x_k is obtained.

2. Main results

Theorem 1.2 has been used to obtain the refinements of some inequalities in [1, 20]. Likewise, by means of the same theorem, the refined version of the Bergström inequality under certain conditions is given below.

Theorem 2.1 *If* $x_k \in \mathbb{R}$ *and* $a_k > 0,\ k \in \{1, 2, ..., n\}$*, then*

$$\sum_{i=1}^{n} \frac{x_i^2}{a_i} \geq \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{\sum\limits_{i=1}^{n} a_i} + \frac{1}{2}\left[\sum_{k=1}^{n} a_k^{-2} - 2\left(\sum_{i=1}^{n} a_i\right)^{-1} \sum_{k=1}^{n} a_k^{-1} + n^2 \left(\sum_{i=1}^{n} a_i\right)^{-2}\right]^{-\frac{1}{2}} \sum_{k=1}^{n}\left(\frac{x_k}{a_k} - \frac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} a_i}\right)^2.$$

Proof Let $a_1, ..., a_n$ and λ be positive numbers. Consider the function f on $\mathbb{R}^n$ such that

$$f(x) = \sum_{i=1}^{n} \frac{x_i^2}{a_i} - \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{\sum\limits_{i=1}^{n} a_i}.$$

The function f has the global minimum at the points x such that $x_i = \lambda a_i$. Taking into account Theorem 1.2 for $f(x)$, we can sharpen the Bergström inequality. Some elementary calculations imply that

$$\nabla f(x) = 2\left[\frac{x_1}{a_1} - \frac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} a_i}, \frac{x_2}{a_2} - \frac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} a_i}, \cdots, \frac{x_n}{a_n} - \frac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} a_i}\right]$$

whence

$$\|\nabla f(x)\|^2 = 4\sum_{k=1}^{n}\left[\frac{x_k}{a_k} - \frac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} a_i}\right]^2.$$

Assume $\|.\|_\circ$ as maximum norm $\|.\|_\infty$, consider the set

$$\begin{aligned} V_{\lambda,d} &= B_\infty(\lambda\mathbf{a}, d) = \{x \in \mathbb{R}^n : \|x - \lambda\mathbf{a}\|_\infty \leq d\} \\ &= \{x \in \mathbb{R}^n : \lambda a_i - d \leq x_i \leq a y_i + d, i = 1, ..., n\} \end{aligned}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. Let us define $\rho_k(x) = \frac{x_k}{a_k} - \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} a_i}$ and estimate $\|\nabla\rho_k(x)\|$ for $x \in V_{\lambda,d}$ to show that $\nabla f(x)$ is Lipschitz mapping on $V_{\lambda,d}$.

$$\frac{\partial\rho_k}{\partial x_k}(x) = \frac{1}{a_k} - \frac{1}{\sum\limits_{i=1}^{n} a_i},$$

$$\frac{\partial\rho_k}{\partial x_j}(x) = -\frac{1}{\sum\limits_{i=1}^{n} a_i} \qquad (k \neq j)$$

so

$$\|\nabla\rho_k(x)\| = \left[\frac{1}{a_k^2} - \frac{2}{a_k \sum\limits_{i=1}^{n} a_i} + n\left(\sum_{i=1}^{n} a_i\right)^{-2}\right]^{\frac{1}{2}}. \tag{2.1}$$

Assume that $x, z \in V_{\lambda,d}$. By means of the mean value theorem and Cauchy-Schwarz inequality, it can be concluded that there exist numbers $\theta_i \in (0,1)$ for $i = 1, 2, ..., n$ such that

$$\begin{aligned}
\|\nabla f(x) - \nabla f(z)\| &= 2\left\|\left[\rho_1(x)-\rho_1(z)\right], \left[\rho_2(x)-\rho_2(z)\right], ..., \left[\rho_n(z)-\rho_n(z)\right]\right\| \\
&= 2\left(\sum_{k=1}^{n}\left[\rho_k(x)-\rho_k(z)\right]^2\right)^{\frac{1}{2}} \\
&= 2\left(\sum_{k=1}^{n}\left[\nabla\rho_k(x+\theta_k(z-x))(x-z)\right]^2\right)^{\frac{1}{2}} \\
&\leq 2\left(\sum_{k=1}^{n}\left\|\nabla\rho_k(x+\theta_k(z-x))\right\|^2\right)^{\frac{1}{2}}\|x-z\| \\
&= 2\left(\sum_{k=1}^{n}\left[\frac{1}{a_k^2}-\frac{2}{a_k\sum\limits_{i=1}^{n}a_i}+n\left(\sum_{i=1}^{n}a_i\right)^{-2}\right]\right)^{\frac{1}{2}}\|x-z\| \\
&= 2\left(\sum_{k=1}^{n}\frac{1}{a_k^2}-\frac{2}{\sum\limits_{i=1}^{n}a_i}\sum_{k=1}^{n}\frac{1}{a_k}+n^2\left(\sum_{i=1}^{n}a_i\right)^{-2}\right)^{\frac{1}{2}}\|x-z\|.
\end{aligned}$$

Now we have

$$\|\nabla f(x) - \nabla f(z)\| \leq A\|x-z\|, \quad x, z \in V_{\lambda,d}$$

where

$$A = 2\left(\sum_{k=1}^{n}a_k^{-2} - 2\left(\sum_{i=1}^{n}a_i\right)^{-1}\sum_{k=1}^{n}a_k^{-1} + n^2\left(\sum_{i=1}^{n}a_i\right)^{-2}\right)^{\frac{1}{2}}.$$

In conclusion, the mapping $x \to \nabla f(x)$ is Lipschitz with the constant $K \leq A$.

Let $r \in (0,d)$ and $q = d - r$. We can estimate $M = \max\left\{\|\nabla f(x)\|_\infty : x \in V_{\lambda,r}\right\}$ as follows:

$$\begin{aligned}
M = \max_{x\in V_{\lambda,r}}\left\{\|\nabla f(x)\|_\infty\right\} &= 2\max_{x\in V_{\lambda,r}}\left\{\max_{1\leq k\leq n}\left|\frac{x_k}{a_k}-\frac{\sum\limits_{i=1}^{n}x_i}{\sum\limits_{i=1}^{n}a_i}\right|\right\} \\
&\leq 2r\max_{1\leq k\leq n}\left\{\frac{1}{a_k}+\frac{n}{\sum\limits_{i=1}^{n}a_i}\right\} \\
&\equiv 2r\left\{\frac{1}{\min\limits_{1\leq k\leq n}a_k}+\frac{n}{\sum\limits_{i=1}^{n}a_i}\right\} = M_0.
\end{aligned}$$

Now we can determine the number a such that $a \geq \max\left(K, \frac{M}{2q}\right)$. Since $\lim_{d \to r^+} \frac{M_0}{2(d-r)} = +\infty$ and the function $d \longmapsto \frac{M_0}{2(d-r)}$ is continuous on (r, λ), we can set the number a in Theorem 1.2 as

$$a = \min_{r<d<\lambda} \max\left\{A, \frac{M_0}{2(d-r)}\right\} = A.$$

Applying Theorem 1.2, we conclude that

$$\sum_{i=1}^{n} \frac{x_i^2}{a_i} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{\sum_{i=1}^{n} a_i} + \frac{1}{A} \sum_{k=1}^{n} \left[\frac{x_k}{a_k} - \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} a_i}\right]^2 \quad \text{for } x \in V_{\lambda,r}. \tag{2.2}$$

It is seen that (2.1) is independent of the values of x, whence A is independent of x. On the other hand, M_0 changes with respect to r, but it does not affect the determination of a since $\frac{M_0}{2(d-r)} \to \infty$ as $d \to r^+$. Thus, we can deduce that (2.2) is valid for all $x \in \mathbb{R}$. □

Remark 2.2 *Assuming $\lambda = 1$ and $r = \max_k |x_k - a_k| + c$ for positive numbers a_k, c and real numbers x_k, one can determine the appropriate region Ω to which theorem is applied.*

3. Numerical examples

In this section, we present some numerical examples to see the numerical efficacy of sharper version of the Bergström inequality. Let f be a function as denoted in the proof and let

$$u(x) = \frac{1}{2}\left[\sum_{k=1}^{n} a_k^{-2} - 2\left(\sum_{i=1}^{n} a_i\right)^{-1} \sum_{k=1}^{n} a_k^{-1} + n^2 \left(\sum_{i=1}^{n} a_i\right)^{-2}\right]^{-\frac{1}{2}} \sum_{k=1}^{n} \left(\frac{x_k}{a_k} - \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} a_i}\right)^2.$$

Since the amount of $u(x)$ can increase with respect to a_k values and n, we present $\frac{u(x)}{f(x)}$ as relative sharpening ratio to see the efficacy of the proposed inequality.

In Table 1, we choose a_i and x_k as positive numbers in increasing order for different n values.

Table 1. $a_i = i,\ x_i = i^3$

n	$\approx f(x)$	$\approx u(x)$	$\approx \frac{u(x)}{f(x)}$ (%)
5	1.050×10^3	2.0141×10^2	19.14
10	5.445×10^4	5.445×10^3	10
20	3.072×10^6	1.593×10^5	5.18
40	1.836×10^8	4.870×10^6	2.65
50	6.903×10^8	1.473×10^7	2.12

In Table 2, we choose a_i in increasing order as in Table 1 and x_k with alternating sign.

Table 2. $a_i = i,\ x_i = (-1)^i i^3$

n	$\approx f(x)$	$\approx u(x)$	$\approx \frac{u(x)}{f(x)}$ (%)
5	3.988×10^3	4.271×10^2	10.71
10	2.148×10^5	1.040×10^4	4.84
20	1.224×10^7	2.882×10^5	2.35
40	7.336×10^8	8.577×10^6	1.16
50	2.760×10^9	2.580×10^7	0.93

In Table 3, we choose a_i and x_i nonmonotonic.

Table 3. $a_i = 100\,|\sin i|\,,\ x_i = i^3 \sin i$

n	$\approx f(x)$	$\approx u(x)$	$\approx \frac{u(x)}{f(x)}$ (%)
5	1.147×10^2	1.165×10^1	10.01
10	1.120×10^4	1.130×10^3	10
20	1.402×10^6	8.762×10^4	6.25
40	5.208×10^7	3.493×10^5	0.67
50	7.675×10^7	4.679×10^5	0.61

According to the numerical examples, one can empirically deduce that when n increases, the sharpening ratio decreases. One reason for this may be the place of n in $u(x)$. The more n increases, the more $u(x)$ decreases, so does sharpening ratio.

References

[1] Adilov GR, Tınaztepe G. The sharpening some inequalities via abstract convexity. Mathematical Inequalities and Applications 2012; 12: 33-51.

[2] Budak H, Sarıkaya MZ. On generalized Ostrowski-type inequalities for functions whose first derivatives absolute values are convex. Turkish Journal of Mathematics 2016; 40 (6): 1193-1210.

[3] Bullen PS, Mitrinovic DS, Vasic PM. Means and Their Inequalities (Mathematics and Its Applications). Dordrecht, the Netherlands: Springer, 2013.

[4] Crouzeix JP, Legaz JEM, Volle M. Generalized Convexity, Generalized Monotonicity: Recent Results. Dordrecht, the Netherlands: Kluwer, 1998.

[5] Dragomir SS, Dutta J, Rubinov AM. Hermite-Hadamard Type inequalities for increasing convex along rays functions. Analysis 2001; 2: 171-181.

[6] Jain S, Mehrez K, Baleanu D, Agarwal P. Certain Hermite–Hadamard Inequalities for logarithmically convex functions with applications. Mathematics 2019; 7: 163.

[7] Li Y, Gu XM, Xiao J. A note on the proofs of generalized Radon inequality. Mathematica Moravica 2018; 22 (2): 59-67.

[8] Li Y, Gu XM, Zhao J. The weighted arithmetic mean-geometric mean inequality is equivalent to the Hölder inequality, Symmetry 2018; 10 (9): 380.

[9] Marghidanu, D. Generalizations and refinements for Bergström and Radon's inequalities. Journal of Science and Arts 2008; 8 (1): 57-62.

[10] Marshall A, Olkin I. Inequalities: theory of majorization and its applications. New York, NY, USA: Springer, 1979.

[11] Özdemir ME, Kavurmacı H, Set E. Ostrowski's type inequalities for $(\alpha,\ \mathrm{m})$ – convex functions. Kyungpook Mathematical Journal 2010; 50 (3): 371-378.

[12] Pachpatte BG. Analytic Inequalities Recent Advances. Paris, France: Atlantis Press, 2012.

[13] Pavic Z, Ardıç MA. The most important inequalities of m-convex functions. Turkish Journal of Mathematics 2017; 41: 625-635.

[14] Pop OT. About Bergström Inequality. Journal of Mathematical Inequalities 2009; 3 (2): 237-242.

[15] Rubinov AM. Abstract Convexity and Global optimization. Dordrecht, the Netherlands: Kluwer, 2000.

[16] Rubinov AM, Wu ZY. Optimality conditions in global optimization and their applications. Mathematical Programming 2009; 120 (1): 101-123.

[17] Sahir, MJS. Formation of versions of some dynamic inequalities unified on time scale calculus. Ural Mathematical Journal 2018; 4 (2): 88-98.

[18] Sarikaya MZ, Set E. Özdemir ME. On new inequalities of Simpson's type for s -convex functions. Computers and Mathematics with Applications 2010; 60 (8): 2191-2199

[19] Singer I. Abstract Convex Analysis. New York, NY, USA: Wiley, 1997.

[20] Tınaztepe G. The sharpening Hölder inequality via abstract convexity. Turkish Journal of Mathematics 2016; 40: 438-444.

[21] Tınaztepe G, Kemali S, Sezer S, Eken Z. The sharper form of Brunn-Minkowski type inequality for boxes. Hacettepe Journal of Mathematics and Statistics 2021; 50 (2): 377-386.

[22] Tınaztepe G, Tınaztepe R. A sharpened version of Aczel inequality and some remarks, Mathematical Inequalities and Applications 2021; 24 (3): 635-643.

[23] Tunç M. On some new inequalities for convex functions. Turkish Journal of Mathematics 2012; 36: 245-251.

[24] Yesilce I. Inequalities for B-convex functions via generalized fractional integral. Journal of Inequalities and Applications 2019; 1: 1-15.

[25] Yesilce I, Adilov G. Hermite-Hadamard inequalities for B-convex and B^{-1}-convex functions. The International Journal of Nonlinear Analysis and Applications 2017; 8: 225-233.

[26] Yesilce I, Adilov G. Fractional Integral inequalities for B-convex functions. Creative Mathematics and Informatics 2017; 26: 345-351.

[27] Yesilce I, Adilov G. Hermite-Hadamard inequalities for $L(j)$-convex functions and $S(j)$-convex functions. Malaya Journal of Matematik 2015; 3: 346-359.

[28] Yau, SF, Bresler Y. A generalization of Bergstrom's inequality and some applications. Linear algebra and its applications 1992; 161: 135-151.

[29] Zhao CJ, Li XY. On mixed discriminants of positively definite matrix. Ars Mathematica Contemporanea 2015; 9 (2): 261-266.