Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1626 – 1634
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doi:10.3906/mat-2008-19

Ranks of nilpotent subsemigroups of order-preserving and decreasing transformation semigroups

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Received: 07.08.2020 • **Accepted/Published Online:** 07.05.2021 • **Final Version:** 27.07.2021

Abstract: Let $\mathcal{C}_n$ be the semigroup of all order-preserving and decreasing transformations on $X = \{1, \ldots, n\}$ under its natural order, and let $N(\mathcal{C}_n)$ be the subsemigroup of all nilpotent elements of $\mathcal{C}_n$. For $1 \leq r \leq n-1$, let

$$\begin{aligned} N(\mathcal{C}_{n,r}) &= \{\alpha \in N(\mathcal{C}_n) : |\operatorname{im}(\alpha)| \leq r\}, \\ N_r(\mathcal{C}_n) &= \{\alpha \in N(C_n) : \alpha \text{ is an } m\text{-potent for any } 1 \leq m \leq r\}. \end{aligned}$$

In this paper we find the cardinality and the rank of the subsemigroup $N(\mathcal{C}_{n,r})$ of $\mathcal{C}_n$. Moreover, we show that the set $N_r(\mathcal{C}_n)$ is a subsemigroup of $N(\mathcal{C}_n)$ and then, we find a lower bound for the rank of $N_r(\mathcal{C}_n)$.

Key words: Order-preserving and decreasing transformation, nilpotent subsemigroups, m-potent element, generating set, rank

1. Introduction

Let $\mathcal{T}_n$ be the (full) transformation semigroup (under composition) on $X_n = \{1, \ldots, n\}$ under its natural order. A transformation $\alpha \in \mathcal{T}_n$ is called *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$ and *decreasing* (*increasing*) if $x\alpha \leq x$ ($x\alpha \geq x$) for all $x \in X_n$. The subsemigroup of all order-preserving transformations in $\mathcal{T}_n$ is denoted by $\mathcal{O}_n$, and the subsemigroup of all order-preserving and decreasing (increasing) transformations in $\mathcal{T}_n$ is denoted by $\mathcal{C}_n$ ($\mathcal{C}_n^+$). It is a well known fact that $\mathcal{C}_n$ and $\mathcal{C}_n^+$ are isomorphic semigroups (for example, see Remarks on [10, page 290]). There have been many applications of $\mathcal{C}_n$, especially in computer science. For example, to find the cardinality, it is given a specific connection between C_n and the set of all binary trees on n source nodes in [10].

The *index* and the *period* of an element a of a finite semigroup are defined as the smallest values of $m \geq 1$ and $r \geq 1$ such that $a^{m+r} = a^m$. In particular, an element with index m and period 1 is called an m-*potent* element. An element e of a semigroup S is called *idempotent* if $e^2 = e$, and the set of all idempotents in S is denoted by $E(S)$. An element a of a finite semigroup S with zero is called *nilpotent* if $a^m = 0$ for some positive integer m, and moreover, if $a^{m-1} \neq 0$, then a is called an m-*nilpotent* element of S. The set of all nilpotent elements of S is denoted by $N(S)$. A semigroup S with 0 is called nilpotent if there exists $m \in \mathbb{Z}^+$ such that $S^m = \{0\}$. It is clear that the m-potent and m-nilpotent elements are equivalent in any nilpotent

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2010 *AMS Mathematics Subject Classification:* 20M20

semigroup. For any non-empty subset A of a semigroup S, let $\langle A\rangle$ denote the subsemigroup generated by A, i.e. the smallest subsemigroup of S containing A. If there exists a finite subset A of S such that $S=\langle A\rangle$, then S is called a *finitely generated semigroup*. The *rank* of a finitely generated semigroup S is defined by

$$\operatorname{rank}(S)=\min\{\,|A|:\langle A\rangle=S\}.$$

An element a in $S\setminus S^2$ (if there exists) is called *indecomposable*. It is clear that every generating set must contain all indecomposable elements. In other words, if $S=\langle A\rangle$, then $S\setminus S^2\subseteq A$. Therefore, if $S=\langle S\setminus S^2\rangle$, then $S\setminus S^2$ is the minimum generating set of S, and so $\operatorname{rank}(S)=|S\setminus S^2|$.

The *image*, the *fix* and the *kernel* of any transformation α in $\mathcal{T}_n$ are defined by

$$\begin{aligned}\operatorname{im}(\alpha) &= \{x\alpha : x\in X_n\},\\ \operatorname{fix}(\alpha) &= \{x\in X_n : x\alpha=x\}\ \text{ and}\\ \ker(\alpha) &= \{(x,y): x\alpha=y\alpha \text{ for } x,y\in X_n\},\end{aligned}$$

respectively. Let $\varepsilon\in\mathcal{C}_n$ be the constant map to 1. Then ε is the zero element of $\mathcal{C}_n$, and moreover, an element α in $\mathcal{C}_n$ is nilpotent if and only if $\operatorname{fix}(\alpha)=\{1\}$ (see [11, Lemma 2.2]). It is also a well-known fact that $\alpha\in\mathcal{T}_n$ is an idempotent if and only if $\operatorname{fix}(\alpha)=\operatorname{im}(\alpha)$. A partition $P=\{A_1,\dots,A_r\}$ of X_n for $1\le r\le n$ is called an *ordered partition*, and written $P=(A_1<\cdots<A_r)$, if $x<y$ for all $x\in A_i$ and $y\in A_{i+1}$ $(1\le i\le r-1)$, (the idea of ordering a family of sets appeared on page 335 of [14]). For any $\alpha\in N(\mathcal{C}_n)$, let $\operatorname{im}(\alpha)=\{a_1,\dots,a_r\}$ with $1=a_1<a_2<\cdots<a_r$, and let $A_i=a_i\alpha^{-1}$ for every $1\le i\le r$. Then the set of kernel classes of α, $K(\alpha)=X_n/\ker(\alpha)=\{A_1,\dots,A_r\}$, is an ordered partition of X_n. Moreover, since α is order-preserving, each class A_i is convex, and since α is decreasing, $a_i\le x\alpha$ for all $x\in A_i$ for every $1\le i\le r$. In particular, for all $x\in A_1$, $x\alpha=1$. For $\alpha\in N(\mathcal{C}_n)$, since $\operatorname{fix}(\alpha)=\{1\}$, it follows that $a_i\gneqq x\alpha$ for all $x\in A_i$ and all $2\le i\le r$. If we use the following tabular form:

$$\alpha=\begin{pmatrix} A_1 & A_2 & \cdots & A_r\\ 1 & a_2 & \cdots & a_r\end{pmatrix}, \tag{1.1}$$

then it is clear that $\alpha\in N(\mathcal{C}_n)$ is r-potent (and equivalently r-nilpotent) for any $1\le r\le n-1$ if and only if $a_i\in A_{i-1}$ for each $2\le i\le r$.

Recall that nth Catalan number C_n and Narayana number $N(n,r)$ are defined by

$$\begin{aligned} C_n &= \frac{1}{n+1}\binom{2n}{n}=\frac{1}{n}\binom{2n}{n-1}\ \text{ for } n\ge 1,\ \text{ and}\\ N(n,r) &= \frac{1}{n}\binom{n}{r}\binom{n}{r-1}=\frac{1}{n-r+1}\binom{n-1}{r-1}\binom{n}{r}\ \text{ for } 1\le r\le n,\end{aligned}$$

respectively. It is well known that $\sum\limits_{r=1}^{n}N(n,r)=C_n$ (see, [8]). Also, from [11, Theorem 2.1 and Proposition 2.3], we have that

$$|\mathcal{C}_n|=|\mathcal{C}_n^+|=C_n,\ \text{ and }\ |N(\mathcal{C}_n)|=|N(\mathcal{C}_n^+)|=C_{n-1}.$$

For other terms not explained here we refer to [7, 9].

Some important problems in combinatorial algebra are to find the cardinalities, minimum generating sets (if there exists) and ranks of some semigroups, which play useful roles in semigroup theory. It is also an important problem to investigate the nilpotent subsemigroups of nilpotent semigroups. Combinatorial and rank properties of finite semigroups have been widely studied. The calculation of $\operatorname{rank}(\mathcal{C}_n) = n-1$ and $\operatorname{rank}(N(\mathcal{C}_n)) = C_{n-1} - C_{n-2}$ were given in [12] and [16], respectively. For any $\xi \in E(\mathcal{C}_n)$, the cardinality and rank of the subsemigroup $\mathcal{C}_n(\xi) = \{\, \alpha \in \mathcal{C}_n : \alpha^m = \xi \text{ for some } m \in \mathbb{Z}^+ \,\}$ were determined by Yağcı and Korkmaz in [16, Theorem 3 and 5]. For any non-empty subset Y of X_n, the cardinality of the set $\mathcal{C}_{n,Y} = \{\, \alpha \in \mathcal{C}_n : \operatorname{fix}(\alpha) = Y \,\}$ has been computed by Ayık, Ayık and Koç in [2]. The number of nilpotent elements in $\mathcal{C}_n$ have been calculated by Laradji and Umar in [11]. The number of m-potent elements and (m,r)-potent elements in ST_n, the subsemigroup of all singular transformations of T_n, were computed by Ayık, Ayık, Ünlü and Howie in [3]. The combinatorial results relating the cardinalities of the subsets $\{\alpha \in \mathcal{C}_n : |\operatorname{im}(\alpha)| = r \text{ and } n\alpha = k\}$ and $\{\alpha \in \mathcal{C}_n : \operatorname{im}(\alpha) = r\}$ were given by Umar in [12, Propositions 3.4 and 3.6]. The rank of $\mathcal{C}_{n,r} = \{\alpha \in \mathcal{C}_n : |\operatorname{im}(\alpha)| \le r\}$ were given in [12, Proposition 4.1]. Among more recent contributions are [1],[4],[5] and [6].

Since $N(\mathcal{C}_1) = N(\mathcal{C}_2)$ consists of only the zero element ε, we suppose that n is an integer at least 3 throughout this paper. For $1 \le r \le n-1$, let

$$\begin{aligned} N(\mathcal{C}_{n,r}) &= \{\alpha \in N(\mathcal{C}_n) : |\operatorname{im}(\alpha)| \le r\}, \\ N_r(\mathcal{C}_n) &= \{\alpha \in N(C_n) : \alpha \text{ is an } m\text{-potent for any } 1 \le m \le r\}. \end{aligned}$$

In the second section of this paper, we find the cardinality and the rank of the nilpotent subsemigroup $N(\mathcal{C}_{n,r})$ of $N(\mathcal{C}_n)$, and so of $\mathcal{C}_n$. In the third section, we show that the set $N_r(\mathcal{C}_n)$ is a nilpotent subsemigroup of $N(\mathcal{C}_n)$ and then, we find a lower bound for the rank of $N_r(\mathcal{C}_n)$. Since $N(\mathcal{C}_{n,1}) = N_1(\mathcal{C}_n)$ consists of only the zero element, and since $N(\mathcal{C}_{n,n-1}) = N_{n-1}(\mathcal{C}_n) = N(\mathcal{C}_n)$ and the rank of $N(\mathcal{C}_n)$ were given in [16, Theorem 2], we consider the case $2 \le r \le n-2$ while we calculate the rank of $N(\mathcal{C}_{n,r})$ and find a lower bound for the rank of $N_r(\mathcal{C}_n)$.

2. Cardinality and rank of $N(\mathcal{C}_{n,r})$

For $1 \le r \le n-1$, let α and β be two elements in

$$N(\mathcal{C}_{n,r}) = \{\alpha \in N(\mathcal{C}_n) : |\operatorname{im}(\alpha)| \le r\}.$$

Since $\operatorname{im}(\alpha\beta) \subseteq \operatorname{im}(\beta)$, and so $|\operatorname{im}(\alpha\beta)| \le r$, it follows that $N(\mathcal{C}_{n,r})$ is a nilpotent subsemigroup of both $\mathcal{C}_n$ and $N(\mathcal{C}_n)$ with the zero element ε. For $1 \le r \le n-1$, our main goal in this section is to find a formula for $|N(\mathcal{C}_{n,r})|$. Moreover, we determine the minimum generating set, and so we find the rank of $N(\mathcal{C}_{n,r})$ by using this formula.

Lemma 2.1 *For $1 \le r \le n-1$,*

$$|N(\mathcal{C}_{n,r})| = \sum_{k=1}^{r} N(n-1,k) = \frac{1}{n-1} \sum_{k=1}^{r} \binom{n-1}{k}\binom{n-1}{k-1}.$$

Proof For $\alpha \in \mathcal{C}_{n-1}$, define $\widehat{\alpha} : X_n \to X_n$ by $1\widehat{\alpha} = 1\alpha = 1$ and $i\widehat{\alpha} = (i-1)\alpha$ for all $2 \le i \le n$. Then it is clear that $\widehat{\alpha} \in N(\mathcal{C}_n)$. Moreover, consider the function $\varphi : \mathcal{C}_{n-1} \to N(\mathcal{C}_n)$ defined by $(\alpha)\varphi = \widehat{\alpha}$ for all $\alpha \in \mathcal{C}_{n-1}$. It is shown in [11, Propsition 2.3] that φ is a bijection, and so $|\mathcal{C}_{n-1}| = |N(\mathcal{C}_n)|$. For $1 \leq k \leq r \leq n-1$, consider the sets

$$\begin{aligned} \mathcal{C}_n(k) &= \{\alpha \in \mathcal{C}_n : |\mathrm{im}\,(\alpha)| = k\} \text{ and} \\ N(\mathcal{C}_n(k)) &= \{\alpha \in N(\mathcal{C}_n) : |\mathrm{im}\,(\alpha)| = k\}. \end{aligned}$$

It is shown in [12, Theorem 3.1] that $|\mathcal{C}_n(k)| = \frac{1}{n-k+1}\binom{n-1}{k-1}\binom{n}{k} = N(n,k)$. From the bijection defined above, it is clear that $\alpha \in \mathcal{C}_{n-1}(k)$ if and only if $\widehat{\alpha} \in N(\mathcal{C}_n(k))$, and so

$$|N(\mathcal{C}_n(k))| = N(n-1,k) = \frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}.$$

Since $N(\mathcal{C}_{n,r})$ is the union of disjoint sets $N(\mathcal{C}_n(k))$ for $1 \leq k \leq r$, the proof is now completed. □

For any finite semigroup S with the zero element 0, it is a well known fact that the following conditions are equivalent:

(i) S is nilpotent,

(ii) every element $a \in S$ is nilpotent, and

(iii) the unique idempotent of S is the zero element

(see, for example [7]). Moreover, it is proved in [13, Lemma 2.0.2] that $S \setminus S^2$ is the minimum generating set of a nilpotent semigroup S, and so

$$\mathrm{rank}\,(S) = |S| - |S^2|.$$

Now our aim is to find the rank of $N(\mathcal{C}_{n,r})$. As mentioned above, since $N(\mathcal{C}_{n,1}) = \{\varepsilon\}$ and since $\mathrm{rank}\,(N(\mathcal{C}_{n,n-1})) = \mathrm{rank}\,(N(\mathcal{C}_n)) = C_{n-1} - C_{n-2}$ were found in [16, Theorem 2], we take account of the case where $2 \leq r \leq n-2$. Moreover, since $N(\mathcal{C}_{n,r})$ is a nilpotent subsemigroup, it is enough to find the cardinality of $N(\mathcal{C}_{n,r}) \setminus N(\mathcal{C}_{n,r})^2$. For any $\alpha \in N(\mathcal{C}_n)$, recall that $1\alpha = 1$, and that $i\alpha \leq i-1$ for each $i \geq 2$.

Theorem 2.2 *For* $2 \leq r \leq n-2$,

$$\mathrm{rank}\,(N(\mathcal{C}_{n,r})) = \sum_{m=1}^{r} N(n-1,m) - \sum_{m=1}^{r} N(n-2,m).$$

Proof Let $2 \leq r \leq n-2$. For any $\alpha, \beta \in N(\mathcal{C}_n)$, since $(i\alpha)\beta \leq (i-1)\beta \leq (i-2)$ for each $3 \leq i \leq n$, it follows that if $\gamma \in N(\mathcal{C}_{n,r})^2$, then $1\gamma = 2\gamma = 3\gamma = 1$ and $i\gamma \leq i-2$ for each $3 \leq i \leq n$. In particular, $n\gamma \leq n-2$. Then, for any $\gamma \in N(\mathcal{C}_{n,r})^2$, define $\widetilde{\gamma} : X_{n-1} \to X_{n-1}$ by

$$i\widetilde{\gamma} = (i+1)\gamma \quad \text{for} \quad 1 \leq i \leq n-1.$$

Then it is clear that $\widetilde{\gamma} \in N(\mathcal{C}_{n-1,r})$. Moreover, for any $\lambda \in N(\mathcal{C}_{n-1,r})$, define $\widehat{\lambda} : X_n \to X_n$ by

$$1\widehat{\lambda} = 1 \quad \text{and} \quad i\widehat{\lambda} = (i-1)\lambda \quad \text{for} \quad 2 \leq i \leq n,$$

as in Lemma 2.1. Now we use the tabular form given in (1.1) for the rest of the proof. If $\lambda = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ 1 & a_2 & \cdots & a_s \end{pmatrix} \in N(\mathcal{C}_{n-1,r})$ where $1 \leq s \leq r$, then it follows that

$$\widehat{\lambda} = \begin{pmatrix} A_1 \cup \{b_2\} & (A_2 \setminus \{b_2\}) \cup \{b_3\} & \cdots & (A_s \setminus \{b_s\}) \cup \{n\} \\ 1 & a_2 & \cdots & a_s \end{pmatrix}.$$

where $b_i = \min A_i$ for $2 \leq i \leq s$. Since $\lambda \in N(\mathcal{C}_{n-1,r})$, $a_s \leq n-2$, and $a_i \lneqq x\lambda$ for all $x \in A_i$ where $2 \leq i \leq s$. Then it is clear that $\widehat{\lambda} \in N(\mathcal{C}_{n,r})$. Next we consider the transformations

$$\mu = \begin{pmatrix} A_1 \cup \{b_2\} & (A_2 \setminus \{b_2\}) \cup \{b_3\} & \cdots & (A_s \setminus \{b_s\}) \cup \{n\} \\ 1 & b_2 & \cdots & b_s \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} A_1 & A_2 & \cdots & A_{s-1} & A_s \cup \{n\} \\ 1 & a_2 & \cdots & a_{s-1} & a_s \end{pmatrix}.$$

Then, we see that μ, $\tau \in N(\mathcal{C}_{n,r})$ and $\mu\tau = \widehat{\lambda}$, and so $\widehat{\lambda} \in N(\mathcal{C}_{n,r})^2$. Therefore, the mapping

$$\psi : N(\mathcal{C}_{n,r})^2 \to N(\mathcal{C}_{n-1,r}),$$

defined by $\gamma\psi = \widetilde{\gamma}$ for all $\gamma \in N(\mathcal{C}_{n,r})^2$, is a well-defined bijection. Hence, it follows from Lemma 2.1 that $|N(\mathcal{C}_{n,r})^2| = |N(\mathcal{C}_{n-1,r})| = \sum\limits_{m=1}^{r} N(n-2,m)$, and so we obtain

$$\operatorname{rank}(N(\mathcal{C}_{n,r})) = \sum_{m=1}^{r} N(n-1,m) - \sum_{m=1}^{r} N(n-2,m),$$

as required. $\square$

Notice that since $\binom{m}{m+1} = 0$, and so $N(m,m+1) = 0$ for any positive integer m, it follows that $\sum\limits_{m=1}^{n-1} N(n-2,m) = \sum\limits_{m=1}^{n-2} N(n-2,m) = C_{n-2}$. Moreover, since $\sum\limits_{m=1}^{n-1} N(n-1,m) = C_{n-1}$ and $\operatorname{rank}(N(\mathcal{C}_{n,n-1})) = \operatorname{rank}(N(\mathcal{C}_n)) = C_{n-1} - C_{n-2}$, the above theorem is valid for also $r = n-1$.

3. Rank of $N_r(\mathcal{C}_n)$

In this section, we first show that for any $1 \leq r \leq n-1$, the set

$$N_r(\mathcal{C}_n) = \{\alpha \in N(\mathcal{C}_n) : \alpha \text{ is an } m\text{-potent for any } 1 \leq m \leq r\}$$

is a subsemigroup of $\mathcal{C}_n$ with the zero element ε. Then, we find the minimum generating set and a lower bound for the rank of $N_r(\mathcal{C}_n)$.

Lemma 3.1 *For $1 \leq r \leq n-1$, $N_r(\mathcal{C}_n)$ is an ideal (and so a subsemigroup) of $\mathcal{C}_n$ with the zero element ε.*

Proof It is clear that $\varepsilon \in N_r(\mathcal{C}_n)$. For $1 \leq m \leq r$, let $\alpha \in N_r(\mathcal{C}_n)$ be an m-potent, and let β be any element in $\mathcal{C}_n$ (or in $N(\mathcal{C}_n)$). For $k \in \mathbb{Z}^+$, we first show by induction on k that $x(\alpha\beta)^k \leq x\alpha^k$ and $x(\beta\alpha)^k \leq x\alpha^k$ for any $x \in X_n$. For $k = 1$, since $x\alpha \leq x$ and $x\beta \leq x$ for any $x \in X_n$, it follows that $x(\alpha\beta) = (x\alpha)\beta \leq x\alpha$ and that $x(\beta\alpha) = (x\beta)\alpha \leq x\alpha$. Now suppose that $x(\alpha\beta)^k \leq x\alpha^k$ and $x(\beta\alpha)^k \leq x\alpha^k$ for $k \in \mathbb{Z}^+$. Then

$$\begin{aligned} x(\alpha\beta)^{k+1} &= (x(\alpha\beta)^k)(\alpha\beta) \leq (x\alpha^k)(\alpha\beta) = (x\alpha^{k+1})\beta \leq x\alpha^{k+1} \text{ and} \\ x(\beta\alpha)^{k+1} &= (x(\beta\alpha)^k)(\beta\alpha) \leq (x\alpha^k)(\beta\alpha) = ((x\alpha^k)\beta)\alpha \leq x\alpha^{k+1}, \end{aligned}$$

as required. Since $\alpha^m = \varepsilon$, it follows that $x(\alpha\beta)^m, x(\beta\alpha)^m \leq x\alpha^m = x\varepsilon = 1$ for all $x \in X_n$, and so $(\alpha\beta)^m = (\beta\alpha)^m = \varepsilon$. Therefore, both $\alpha\beta$ and $\beta\alpha$ are elements of $N_r(\mathcal{C}_n)$, as required. □

Similarly, one can prove that, for any $\alpha, \beta \in \mathcal{C}_n$ and for any $x \in X_n \setminus \operatorname{fix}(\alpha)$, we have $x\alpha^k \lneqq x$, and so $x(\alpha\beta)^k \lneqq x$ for any $k \in \mathbb{Z}^+$. Moreover, we have the following corollary:

Corollary 3.2 *For $1 \leq r \leq n-1$, $N(\mathcal{C}_{n,r})$ is a subsemigroup of $N_r(\mathcal{C}_n)$.*

Proof If $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ 1 & a_2 & \cdots & a_s \end{pmatrix} \in N(\mathcal{C}_{n-1,r})$ where $1 \leq s \leq r$, then since $a_k \in \bigcup_{i=1}^{k-1} A_i$ for each $2 \leq k \leq s$, it follows that $\alpha^s = \varepsilon$, and so $\alpha^r = \varepsilon$, as required. □

As noticed after the proof of [15, Theorem 2.2], there exists only one $(n-1)$-potent element in $N(\mathcal{C}_n)$, namely $\mu_0 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 2 & \cdots & n-1 \end{pmatrix}$. Since μ_0 is the unique element with image of size $n-1$, and since $N(\mathcal{C}_{n,n-1}) = N_{n-1}(\mathcal{C}_n) = N(\mathcal{C}_n)$, it follows that

$$N(\mathcal{C}_{n,n-2}) = N_{n-2}(\mathcal{C}_n) = N(\mathcal{C}_n) \setminus \{\mu_0\}.$$

However, we have the fact $N(\mathcal{C}_{n,r}) \neq N_r(\mathcal{C}_n)$ for any $2 \leq r \leq n-3$. Indeed if we consider

$$\lambda_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & r+2 & r+3 & \cdots & n \\ 1 & 1 & 1 & 2 & \cdots & r & r+1 & \cdots & r+1 \end{pmatrix} \in N(\mathcal{C}_n),$$

then we see that $\lambda_0 \in N_r(\mathcal{C}_n)$ and that $\lambda_0 \notin N(\mathcal{C}_{n,r})$.

We need some background and notations in order to find the rank of $N_r(\mathcal{C}_n)$. For $m, n \in \mathbb{Z}^+$ with $m \leq n$, let $S(n,m)$ be the set of all positive integer solutions of the equation $x_1 + \cdots + x_m = n$, that is

$$S(n,m) = \{\, (s_1, \ldots, s_m) : s_1, \ldots, s_m \in \mathbb{Z}^+ \text{ and } s_1 + \cdots + s_m = n \,\}.$$

It is clear that the cardinality of $S(n,m)$ is $\binom{n-1}{m-1}$, and that the number of non-negative integer solutions of the equation $x_1 + \cdots + x_m = n$ is $\binom{n+m-1}{m-1}$. It is clear that $\alpha \in N(\mathcal{C}_n)$ is a 1-nilpotent element if and only if $\alpha = \varepsilon$, and so the number of 1-nilpotent elements is 1. The number of m-potent (and equivalently, m-nilpotent) elements in $N(\mathcal{C}_n)$ for $2 \leq m \leq n-1$ were computed in [15, Theorem 2.2] as follows.

Theorem 3.3 *For $2 \leq m \leq n-1$, the number of m-potent (and equivalently m-nilpotent) elements in $N(\mathcal{C}_n)$ is*

$$\sum_{(s_1,\ldots,s_m) \in S_{(n-1,m)}} \prod_{i=2}^{m} \binom{s_i + s_{i-1} - 1}{s_i}.$$

□

Now our aim is to find a lower bound for the rank of $N_r(\mathcal{C}_n)$. As mentioned above, we take account of the case where $2 \leq r \leq n-2$. Moreover, since $N_r(\mathcal{C}_n)$ is also a nilpotent subsemigroup, it is enough to find a lower bound for the cardinality of $N_r(\mathcal{C}_n) \setminus N_r(\mathcal{C}_n)^2$.

Theorem 3.4 *For* $2 \leq r \leq n-2$,

$$\begin{aligned} \operatorname{rank}(N_r(\mathcal{C}_n)) \quad \geq \quad & \sum_{m=2}^{r} \left(\sum_{(s_1,\ldots,s_m)\in S_{(n-1,m)}} \prod_{i=2}^{m} \binom{s_i + s_{i-1} - 1}{s_i} \right) \\ & - \sum_{m=2}^{r} \left(\sum_{(t_1,\ldots,t_m)\in S_{(n-2,m)}} \prod_{i=2}^{m} \binom{t_i + t_{i-1} - 1}{t_i} \right). \end{aligned}$$

Proof Since the numbers of m-potent elements in $N_r(\mathcal{C}_n)$ and $N(\mathcal{C}_n)$ are the same, it follows from Theorem 3.3 that the number of m-potent elements in $N_r(\mathcal{C}_n)$ is

$$\sum_{(s_1,\ldots,s_m)\in S_{(n-1,m)}} \prod_{i=2}^{m} \binom{s_i + s_{i-1} - 1}{s_i}$$

for $2 \leq m \leq r \leq n-2$. Then, this yields

$$|N_r(\mathcal{C}_p)| \quad = \quad 1 + \sum_{m=2}^{r} \left(\sum_{(s_1,\ldots,s_m)\in S_{(p-1,m)}} \prod_{i=2}^{m} \binom{s_i + s_{i-1} - 1}{s_i} \right), \tag{3.1}$$

where $p = n-1,\, n$. As it is shown in the proof of Theorem 2.2, if α is an element in $N_r(\mathcal{C}_n)^2$, then α has the following tabular form:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 1 & 1 & 4\alpha & 5\alpha & \cdots & n\alpha \end{pmatrix}$$

where $i\alpha \leq i-2$ for each $i \geq 3$ and $1 \leq 4\alpha \leq \cdots \leq n\alpha \leq n-2$. Next, for each α in $N_r(\mathcal{C}_n)^2$, consider

$$\widetilde{\alpha} = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 \\ 1 & 1 & 4\alpha & 5\alpha & \cdots & n\alpha \end{pmatrix},$$

as defined in the proof of Theorem 2.2. Similarly, it is also clear that $\widetilde{\alpha} \in N_r(\mathcal{C}_{n-1})$. Now we consider the function

$$\Psi : N_r(\mathcal{C}_n)^2 \to N_r(\mathcal{C}_{n-1})$$

defined by the rule $(\alpha)\Psi = \widetilde{\alpha}$ for all $\alpha \in N_r(\mathcal{C}_n)^2$. It is easy to check that Ψ is a well-defined one-to-one function. However, Ψ is not onto in general. Therefore, the result follows from the fact that $N_r(\mathcal{C}_n) \setminus N_r(\mathcal{C}_n)^2$ is the minimum generating set of $N_r(\mathcal{C}_n)$ and that $|N_r(\mathcal{C}_n) \setminus N_r(\mathcal{C}_n)^2| = |N_r(\mathcal{C}_n)| - |N_r(\mathcal{C}_n)^2| \geq |N_r(\mathcal{C}_n)| - |N_r(\mathcal{C}_{n-1})|$, as required. $\square$

Finally, we give a counter example, which shows that $\Psi : N_r(\mathcal{C}_n)^2 \to N_r(\mathcal{C}_{n-1})$ is not onto in general.

Example 3.5 Assume that

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 2 & 3 \end{pmatrix} \in N_2(\mathcal{C}_6)^2.$$

Then, there are $\mu, \tau \in N_2(\mathcal{C}_6)$ such that $\mu\tau = \beta$. Since $1\tau = 2\tau = 3\tau = 1$, and $5\mu \leq 4$, it follows that $5\mu = 4$. Since $5\beta \neq 6\beta$, we must have $6\mu \neq 4$, and so similarly, $6\mu = 5$. Thus, μ is a 3-potent which is a contradiction. Therefore, $\beta \notin N_2(\mathcal{C}_6)^2$. Since $\widetilde{\beta} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix} \in N_2(\mathcal{C}_5)$, it follows that $\Psi : N_2(\mathcal{C}_6)^2 \to N_2(\mathcal{C}_5)$ is not onto.

We also have the following example:

Example 3.6 The function $\Psi : N_2(\mathcal{C}_5)^2 \to N_2(\mathcal{C}_4)$ defined above is a bijection. Indeed,

$$\begin{aligned} N_2(\mathcal{C}_5) \quad = \quad & \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 3 \end{pmatrix}, \right. \\ & \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix}, \\ & \left. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 2 \end{pmatrix} \right\}, \end{aligned}$$

$$\begin{aligned} N_2(\mathcal{C}_5)^2 \quad = \quad & \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 3 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 2 \end{pmatrix} \right\} \text{ and} \end{aligned}$$

$$\begin{aligned} N_2(\mathcal{C}_4) \quad = \quad & \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix} \right\}, \end{aligned}$$

as required. Moreover, we have $\operatorname{rank}(N_2(\mathcal{C}_5)) = 4$.

Finally, one may ask if the sets

$$\begin{aligned} P_r(\mathcal{C}_n) &= \{\alpha \in \mathcal{C}_n : \alpha \text{ is an } m\text{-potent for some } 1 \leq m \leq r\} \text{ and} \\ P_r(\mathcal{O}_n) &= \{\alpha \in \mathcal{O}_n : \alpha \text{ is an } m\text{-potent for some } 1 \leq m \leq r\}, \end{aligned}$$

where $1 \leq r \leq n$, are subsemigroups of $\mathcal{C}_n$ and $\mathcal{O}_n$, respectively. If we consider the transformations $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{pmatrix}$, then we see that both α and β in $P_2(\mathcal{C}_4)$ and $P_2(\mathcal{O}_4)$, but $\alpha\beta$ neither in $P_2(\mathcal{C}_4)$ nor in $P_2(\mathcal{O}_4)$. Thus, neither $P_r(\mathcal{C}_n)$ nor $P_r(\mathcal{O}_n)$ is a subsemigroup.

Acknowledgment

We would like to thank referees for the comments and suggestions, which significantly contributed to improving the quality of the publication.

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