Ideal triangulation and disc unfolding of a singular flat surface

İsmail SAĞLAM*⬤

Adana Alparslan Türkeş Science and Technology University, Adana, Turkey

Abstract: An ideal triangulation of a singular flat surface is a geodesic triangulation such that its vertex set is equal to the set of singular points of the surface. Using the fact that each pair of points in a surface has a finite number of geodesics having length $\leq L$ connecting them, where L is any positive number, we prove that each singular flat surface has an ideal triangulation provided that the surface has singular points when it has no boundary components, or each of its boundary components has a singular point. Also, we prove that such a surface contains a finite number of geodesics which connect its singular points so that when we cut the surface through these arcs we get a flat disc with a nonsingular interior.

Key words: Flat surface, raw length spectrum, separatrix, ideal triangulation, unfolding, saddle connection, translation surface

1. Introduction

Let $S_{\mathfrak{g},\mathfrak{n}}$ be a closed orientable surface of genus $\mathfrak{g} \geq 0$ with $\mathfrak{n} \geq 0$ punctured points. It is well known that $S_{\mathfrak{g},\mathfrak{n}}$ admits a complete finite-area Riemannian metric of constant curvature equal to -1 when the Euler characteristic $\chi(S_{\mathfrak{g},\mathfrak{n}})$ of $S_{\mathfrak{g},\mathfrak{n}}$ is negative. Indeed, in this case there are many Riemannian metrics of constant curvature equal -1 on $S_{\mathfrak{g},\mathfrak{n}}$. The space of equivalence classes of this metrics is called Teichmüller space of $S_{\mathfrak{g},\mathfrak{n}}$ and denoted by $\mathbb{T}(S_{\mathfrak{g},\mathfrak{n}})$. Also, the surface $S_{1,0}$ admits a Riemannian metric of the constant curvature equal to 0.

It is known that $\mathbb{T}(S_{\mathfrak{g},\mathfrak{n}})$ is homeomorphic to $\mathbb{R}^{6g-6+2n}$ when $\chi(S_{\mathfrak{g},\mathfrak{n}}) < 0$. There are several ways to see this. For example, one can form a pants decomposition of $S_{\mathfrak{g},\mathfrak{n}}$ and consider the length parameters and the twist parameters. The coordinates obtained this way are called the Fenchel-Nielsen coordinates. See [12] or [3] for a proof of this fact.

If the surface is not compact, that is $\mathfrak{n} > 0$, and $\chi(S_{\mathfrak{g},\mathfrak{n}}) < 0$, then there is another way to parametrize the Teichmüller space. The main ingredient of this parametrization is an ideal triangulation of a given hyperbolic structure. An ideal triangulation of a hyperbolic surface is a triangulation such that each edge is a geodesic and the vertex set of the triangulation is the set of punctures of the surface. Any hyperbolic structure on such a surface is formed by an ideal triangulation and one may parametrize the Teichmüller space by using ideal triangulations. See [8] for details.

Let $\mathrm{Diff}^+(S_{\mathfrak{g},\mathfrak{n}})$ be the group of all orientation preserving diffeomorphisms of $S_{\mathfrak{g},\mathfrak{n}}$ and $\mathrm{Diff}_0(S_{\mathfrak{g},\mathfrak{n}})$ be the subgroup of diffeomorphisms which are isotopic to the identity. $\mathrm{Diff}^+(S_{\mathfrak{g},\mathfrak{n}})/\mathrm{Diff}_0(S_{\mathfrak{g},\mathfrak{n}})$ is called the mapping class group and denoted by $\mathfrak{Mod}(S_{\mathfrak{g},\mathfrak{n}})$.

---

*Correspondence: isaglamtrfr@gmail.com

If one endowes Teichmüller space with the Teichmüller's metric, then the mapping class group acts on it by isometries. Also, this action is properly discontinuous. The proof of this fact is nontrivial, and the following lemma is one of the main ingredients of the proof. Before stating the lemma, let us recall what the raw length spectrum of a hyperbolic metric is.

Let S be a complete hyperbolic surface which has a finite area. Let c be the isotopy class of a simple closed curve which is not homotopic to a point or a puncture. We call such a simple closed curve essential. Then it is a well-known fact that c contains a unique geodesic. Let $l_S(c)$ be the length of this simple closed geodesic. Let us define the raw length spectrum of S as the set

$$\mathfrak{rls}(S) = \{l_S(c)\} \subset \mathbb{R}_+,$$

where c ranges over all isotopy classes of simple closed curves which are essential.

Lemma 1.1 ([3]) *Let S be a hyperbolic surface which has a finite area. Then the set $\mathfrak{rls}$ is a closed, discrete subset of $\mathbb{R}$. Furthermore, for each $L \in \mathbb{R}$ the set*

$$\{c : c \text{ is an isotopy class of an essential simple curve in } S \text{ with } l_S(c) \leq L\}$$

is finite.

Before stating the main results of this paper, let us recall what a singular flat surface is. Roughly speaking, a singular flat surface is a surface obtained by gluing Euclidean triangles along their edges appropriately. It is known that any compact surface with genus greater than or equal to one admits a singular flat metric. A geodesic arc on a singular flat surface is a continuous map from a closed interval to the surface which is locally length minimizing.

It is well-known that the space of singular flat surfaces having the same topology and prescribed angle data can be identified with the Teichmüller space of a punctured surface (see [14]). Therefore, singular flat surfaces form a nice family to study the Teichmüller theory.

In [5], the authors proved a result about singular flat surfaces which is similar to Lemma 1.1. The proof is analytic and based on the Arzelo-Ascoli Theorem. The authors use this result to obtain a generalized flip algorithm to construct a Delaunay triangulation and a Voronoi diagram on a given singular flat surface. Although this result was proven for only surfaces without boundary, the reasoning of the authors is valid for singular flat surfaces with a boundary component. Here is the result we mentioned above:

Proposition 1.2 ([5]) *Let S be a compact singular flat surface. For any pair of points $p, q \in S$ and for any $L \geq 0$, the number of geodesics arcs of length $\leq L$ joining p and q is finite.*

We now state the main result of the present paper. It is well known that a compact singular flat surface can be triangulated with Euclidean triangles. A proof of this fact may be found in [13]. Indeed, one can cut any singular flat surface through piecewise geodesic arcs to obtain a disc with no singular points in its interior, where an interior point is called singular if it has an angle not equal to 2π (see Section 2). Then one can show that such a disc can be triangulated and this triangulation gives us a triangulation of the surface that we started with. However, there are several problems with such a triangulation. First of all, there may be a lot of triangles required and we may not keep track of the number of triangles involved. Also, it does not give us any clue to construct a singular flat surface from a disc or a family of triangles.

If the surface has genus 0 and the angle at each singular point is less than 2π, then it can be triangulated in a nice way. Indeed, Thurston [11] proved that such a surface can be triangulated in a way that the vertex set of the triangulation coincides with the set of singular points of the surface. He proved this result by considering Voronoi diagrams in flat surfaces. He used this result to parametrize the space of equivalence classes of singular flat spheres with prescribed angle data. We give an example of such a triangulation in Figure 1. Glue the edges of the triangle with the same label to get a flat sphere with four singular points having angle π. Then the given triangulation of the triangle induces a triangulation of the tetrahedron.
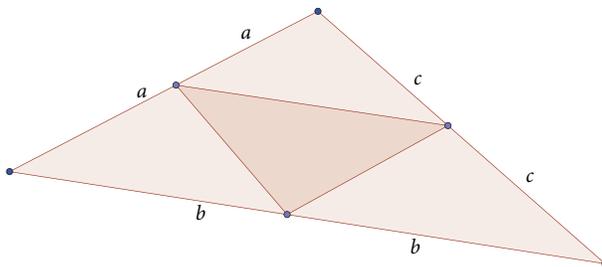

Figure 1. A triangulation of the flat sphere with four singular points.

In this paper, we generalize Thurston's result to arbitrary flat surfaces with or without boundary. More precisely, we prove that any singular flat surface without a boundary component has a triangulation such that the set of singular points coincides with the vertex set of the triangulation provided that the surface has a singular point. If the surface has a boundary component, we show that the surface has such a triangulation if each of its boundary components has a singular point. Note that we do not use the result of Thurston in the proof. Instead, we use Proposition 1.2 to reduce the problem to the case of a disc by an inductive reasoning. Also, we consider nonorientable surfaces with or without boundary. This result was proven in [6] and [10], but the proof given here is different.

The second main result of the present paper may be considered a weak generalization of the Alexandrov's unfolding process (see [1] and [4]). Alexandrov's unfolding theorem states that every flat sphere with an angle less than 2π at each singular point can be cut through some geodesic connecting some of its vertices so that the resulted shape is a disc and this disc can be embedded into the Euclidean plane. It implies that any flat sphere with appropriate angle data can be obtained from a planar polygon of a suitable type. We give an example of such a polygon in Figure 2. Glue the edges of the polygon having the same label to obtain a flat sphere with 5 singular points.

We show that any singular flat surface can be obtained from a flat disc with a nonsingular interior with a relatively small number of edges. More precisely, we prove that any singular flat surface can be cut through a finite number of geodesics which may intersect only at their end points so that the resulting flat surface is a disc, provided that the initial singular flat surface has enough singular points. Our result is a weak generalization of Alexandrov's unfolding theorem since it says nothing about embedding the resulting disc into Euclidean plane.

2. Singular flat surface

Let S be a compact surface perhaps with boundary. A singular flat metric on S is a singular Riemannian metric so that each interior point of the surface has a neighborhood that is isometric to that of a neighborhood of the apex of a cone, and each boundary point has a neighborhood isometric to that of a neighborhood of the
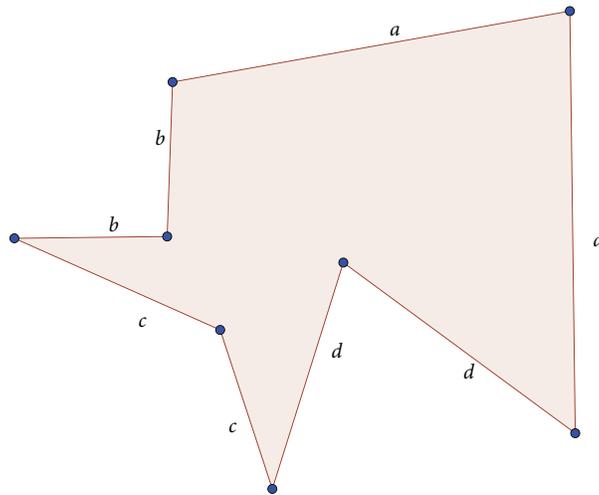
Figure 2. A flat sphere from a planer polygon.

apex of a cut of a cone. See Figure 3. A surface with a singular flat metric is called singular flat surface.

In particular, there is a well-defined notion of angle for each point in a singular flat surface. Let us denote the angle at $p \in S$ by θ_p. An interior point p of S is called singular if $\theta_p \neq 2\pi$. Otherwise, it is called nonsingular. A boundary point p is called singular if $\theta_p \neq \pi$. Otherwise, it is called nonsingular. The values $2\pi - \theta_p$ and $\pi - \theta_p$ are called the curvature of S at p if p is an interior point or a boundary point, respectively. Thus, a point is nonsingular if and only if its curvature is 0.

It is known that a compact singular flat surface can be triangulated. That is, it can be obtained by gluing Euclidean triangles along their edges appropriately. See [13] for a proof. Note that the triangulation given in [13] is not necessarily ideal. That is, it may not be true that the set of vertices of the triangulation coincides with the set of singular points of the surface. In [9], it was proven that any complete singular flat metric in a noncompact surface of finite type can be triangulated by finitely many isometry types of Euclidean triangles. Note that a surface is called finite type if it is obtained from a compact surface by removing finitely many points.
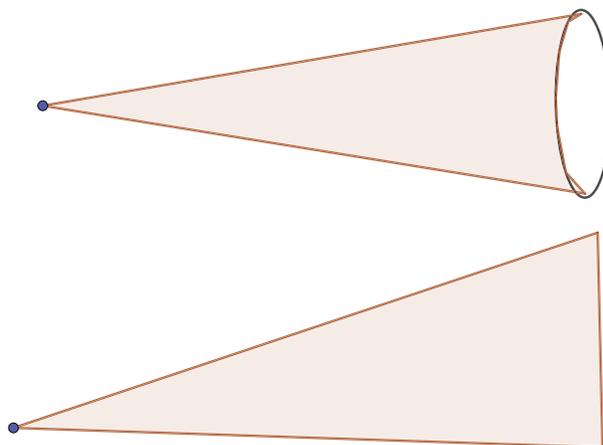

Figure 3. A cone and a cut of a cone.

The following formula is called the Gauss–Bonnet formula. Its proof may be found in [13].

Proposition 2.1 *Let S be a compact orientable singular flat surface. Then*

$$\sum_{p \in S} \kappa_p = 2\pi\chi(S).$$

A singular flat surface comes with additional structures. First of all, each path in S has a length. This makes a singular Euclidean surface a length space. That is, we can define a metric on S by considering the infimum of the lengths of the curves that join two points on the surface. When S is compact, it follows from the general properties of length spaces that for any two points there is a length minimizing path between them and each homotopy class of paths joining two points has a length minimizing representative. Also, there is a length minimizing geodesic in any free homotopy class of loops (see [2]). Note that by a geodesic, we mean a locally length minimizing path.

3. Labeled singular flat surfaces and their raw length spectrum

In this section, we define labeled singular flat surfaces and their raw length spectra. First, we introduce the notion of the labeled singular flat surface which is essentially a singular flat surface together with a finite set of points which we call the label set. This notion is necessary to prove the existence of a triangulation with desired properties. Proof of this fact is given by induction on the number of singular points and genus of a given surface, and it is obtained by cutting the surface through arcs. When such a cutting operation is performed on a surface, one can get nonsingular points from the singular ones. For example, if we cut a singular interior point having angle π through a geodesic which starts at that point, then we get a nonsingular boundary point. Thus, to keep track of these points, we introduce labeled singular flat surfaces.

Definition 3.1 *A labeled singular flat surface is a compact singular flat surface S together with a finite set of points $P \subset S$ so that*

- *P is nonempty,*

- *each singular point of S is in P,*

- *each boundary component of S has a nonempty intersection with P.*

Definition 3.2 *An arc on a labeled flat surface (S, P) is a geodesic which joins two points of P and whose interior is a subset of $S - P$.*

Note that we assume that constant paths are not arcs. Each arc has a constant speed parametrization, and we regard two different constant speed parametrization of the same arc equivalent.

Definition 3.3 *Let (S, P) be a labeled singular flat surface. The raw length spectrum of (S, P) is the set of lengths of arcs on S, that is,*

$$\mathfrak{rls}(S, P) = \{l_S(c) : c \text{ is an arc on } S\} \subset \mathbb{R},$$

where $l_S(c)$ length of the arc c.

Corollary 3.4 *The raw length spectrum* $\mathfrak{rls}(S, P)$ *of* (S, P) *is a discrete and closed subset of* $\mathbb{R}$. *Indeed, for any* $L \geq 0$ *the set*

$$\{c : c \text{ is an arc such that } l_S(c) \leq L\}$$

is finite.

Proof These statements follow immediately from Proposition 1.2. □

Note that when L is sufficiently large, $\mathfrak{rls}(S, P)$ contains a positive number.

Definition 3.5 *Let* (S, p) *be a labeled flat singular surface. A multiple arc is a path which is a finite connected union of arcs. We define the multiple raw length spectrum as*

$$\mathfrak{mrls} = \{l_S(c) : c \text{ is a multiple arc on } S\},$$

where $l_S(c)$ *is the length of* c *with respect to singular flat metric on* S.

The following corollary is immediate.

Corollary 3.6 $\mathfrak{mrls}(S, P)$ *is a discrete and closed subset of* $\mathbb{R}$. *Indeed, for any* $L \geq 0$ *the set*

$$\{c : c \text{ is an multiple arc such that } l_S(c) \leq L\}$$

Consider the set of geodesics which start and end at the same point of P. Call this set $Loop(S, P)$. Note that we do not include constant loops to this set. Each element in $Loop(S, P)$ passes through a point in P finitely many times. This means that each element in $Loop(S, P)$ is a finite union of arcs. Therefore, we have that

$$Loop(S, P) \subset \{l_S(c) : c \text{ is a multiple arc on } S\}.$$

Hence, the following corollary is immediate.

Corollary 3.7 *For any* $L \geq 0$, *the set*

$$\{c \in Loop(S, P) : \text{ length of } c \text{ is less than or equal to } L\}$$

is finite.

A singular flat surface which has a trivial holonomy group is called a translation surface. A geodesic segment which has nonsingular interior and connects two singular points of the surface is called a saddle connection. It is known that the set of holonomy vectors of saddle connections of translation surface is a discrete subset of $\mathbb{R}^2$. See [7] for a proof.

4. Ideal triangulation

In this section, we show that each flat surface with enough singular points have a triangulation whose set of vertices coincides with the set of singular points of the surface. We will make what it means that a surface has enough singular points clear. This amounts to discarding some surfaces which obviously do not have such a triangulation. For example, flat tori cannot have such triangulations because they do not have singular points.

Also, a singular flat surface with a nonsingular boundary component cannot have such a triangulation. Indeed, we show that any singular flat surface with boundary having a singular point in each of its boundary component have such a nice triangulation. Furthermore, if the surface does not have boundary components, then it can be triangulated in this manner when it contains singular points.

The proof of existence of such a triangulation is based on induction on the number of boundary components, genus, and the number of singular points of the surface. We will cut a given singular flat surface several times through geodesics that join its singular points to apply the induction hypothesis. It may happen that, if one cuts a singular flat surface through a geodesic starting at a singular point, then one gets nonsingular points. This will be the point where the notion of labeled singular flat surface will help us.

Now we define explicitly what we mean by a nice triangulation.

Definition 4.1 *Let S be a labeled singular flat surface with the label set P. An ideal triangulation of S is a triangulation of S such that*

1. *each edge of the triangulation is a simple arc,*

2. *the vertex set of the triangulation is P,*

Note that we say that a path $c : [a, b] \to S$ is simple if $t < t' \in [a, b]$ with $c(t) = c(t')$ implies that $t = a$ and $t = b$. We will prove that any labeled singular flat surface has an ideal triangulation. Note that we do not discard nonorientable surfaces and surfaces with boundary.

Some triangles of an ideal triangulation may have two edges and two vertices. For example, one can glue the edges of an isosceles triangle which have equal lengths to get an ideal triangulation on a singular flat surface.

4.1. Orientable surfaces

Now we start to prove existence of an ideal triangulation. We start with the simplest case; the case of a flat disc. Recall that a flat disc is a labeled singular flat surface which is homeomorphic to a closed disc and has a nonsingular interior. Note that polygons are examples of flat discs, but it is not true that any flat disc can be embedded into the Euclidean plane.

Definition 4.2 *A labeled flat disc is a singular flat disc which is homeomorphic to a closed disc and does not have any labeled points on its interior.*

Lemma 4.3 *Each labeled flat disc has an ideal triangulation.*

Proof Any flat disc has at least three singular boundary points. Therefore, if the label set contains three points, then the disc is indeed a triangle; hence, it has an obvious ideal triangulation.

We do induction on the number of labeled points. Assume that a flat disc has a label set which contains more than three points. Take a point v which is on the boundary and singular. For each other labeled point, consider a length minimizing path joining v and this point. It follows that there exists a labeled point w such that the path joining v and w is not completely contained on the boundary. Note that this is not true if the label set has only three points. Since the path is length minimizing, it follows that it has a subpath such that the end points of this subpath are (not necessarily the same) labeled points and its interior is contained in the interior of the disc. If we cut the disc through this path, then we get two flat discs so that each of them has less labeled points than the original disc. By the induction hypothesis, these discs have ideal triangulations, and it is clear that we can glue these discs back to get an ideal triangulation of the original disc. □

Now we consider singular flat surfaces which are homeomorphic to a closed disc possibly with singular interiors.

Lemma 4.4 *A labeled flat surface which is homeomorphic to a closed disc has an ideal triangulation.*

Proof If the disc has no labeled interior points, in Lemma 4.3 we proved that it has an ideal triangulation. Now we do induction on the number of labeled interior points. Since the base case is done, we assume that the disc has labeled interior points. Take a labeled boundary point of the disc and call it v. Also, take a labeled interior point w. There is a (not necessarily unique) length minimizing path between these two points. This path has a subpath which connects a labeled boundary point and a labeled interior point so that interior of it has an empty intersection with P. Note that these new points may be different from v and w. Cut the surface through this subpath and form a new labeled singular flat surface by adding the two new boundary points to the label set. This new labeled singular flat surface has less labeled interior points than the one that we started with. So it has an ideal triangulation by the induction hypothesis. This triangulation induces an ideal triangulation of the surface that we started with.

$\square$

Now we consider the case of the sphere. Note that Thurston [11] also considered this case and obtained an ideal triangulation for each singular flat sphere having nonnegative curvature data. His method depends on existence of Voronoi regions on a given flat surface. Our approach is quite different from his approach.

Lemma 4.5 *Any labeled singular flat surface which is homeomorphic to a sphere has an ideal triangulation.*

Proof Note that the Gauss–Bonnet formula implies that any singular flat metric on the sphere has at least three singular points. Take two labeled points in such a surface and consider a length minimizing path joining them. Then this path has a subpath which joins two labeled points and has an interior having nonempty intersection with the label set. Cutting the sphere through this path, we obtain a labeled singular flat surface which is homeomorphic to a closed c. Lemma 4.4 implies that this disc has an ideal triangulation and it is clear that this triangulation induces an ideal triangulation of the sphere. $\square$

If S is a surface, we denote the surface obtained by removing interiors of b disjoint closed discs from the interior of S by $S[b]$. Now we consider a singular flat metric on a surface of genus zero with several boundary components. An example of a such flat surface may be found in Figure 4.
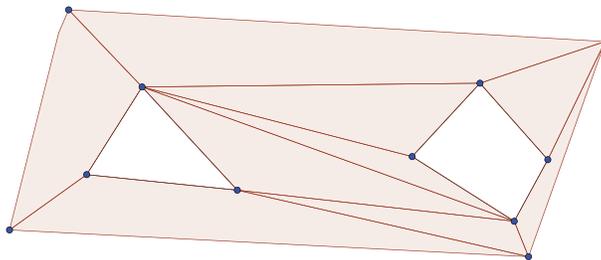

Figure 4. A singular flat surface which is homeomorphic to $S^2[3]$.

Lemma 4.6 *If a labeled singular flat surface is homeomorphic to $S^2[b]$, $b \geq 0$, then it admits an ideal triangulation, where S^2 is the 2-sphere.*

Proof The case $b \leq 1$ was done in Lemmas 4.4 and 4.5. We do induction on b. Assume that $b > 1$. Take two distinct boundary components. Let w and w' be labeled points in these boundary components so that they do not lie in the same component. Consider a length minimizing geodesic which connects w and w'. Note that if this geodesic makes a nonzero angle between any boundary component at any point distinct from w and w', then this point is a singular point. Furthermore, since the geodesic is length minimizing, it does not intersect itself. Now consider the intersection of P with this geodesic. If we cut the surface through this geodesic, then we get a singular flat surface which is homeomorphic to $S^2[b']$ for some $b' < b$. Each point on the intersection set of the geodesic and P gives us two new points on the boundary of the resulting surface, we add these points to P to form a new label set. This new surface together with this new label set has an ideal triangulation, and this triangulation induces an ideal triangulation of the original labeled flat surface. □

Lemma 4.7 *Let (S, P) be a labeled flat surface whose genus is greater than or equal to 1 and which has at most one boundary component. Let $\mathfrak{S}_P$ be the subset of $Loop(S, P)$ which consists of the curves that are not homotopic to a point and loops winding about the boundary component several times. Then the shortest element of $\mathfrak{S}_P$ is a simple geodesic.*

Proof Since the genus of the surface is positive, $\mathfrak{S}_P$ is not empty. Note that such an element is not unique and its existence follows by Corollary 3.7. Firstly, assume that the surface has no boundary components. Assume that this geodesic is not simple. Consider the parametric representation $\alpha : [0, 1] \to S$ of this geodesic such that $\alpha(0) = \alpha(1) = x \in P$. Let $b \in (0, 1)$ be the smallest value so that there exists $b' > b$ with $\alpha(b) = \alpha(b')$. Call this point $\alpha(b) = \alpha(b') = y$. Then the loop which is obtained by restricting α to the interval $[b, b']$ is not homotopic to a point since the path α is length minimizing. Also, the point y cannot be in P. Indeed, if y is in P, then the curve obtaining by restricting α to $[b, b']$ is in $\mathfrak{S}_P$ and has a length less than the length of α. In particular, y is a nonsingular point. Now consider the path obtaining by the juxtaposition of the curves $\alpha|_{[0,b]}$ and $\alpha|_{[b',1]}$. If this loop is homotopic to a point, then there is a piecewise geodesic path based at x which has length less than the length of α and which is not homotopically trivial. See Figure 5. But this means that there is a loop based at x whose length is less then the length of α, which is impossible by the choice of α.

Now assume that the surface has one boundary component. Assume that the geodesic is not simple. Consider the parametric representation $\alpha : [0, 1] \to S$ of this geodesic such that $\alpha(0) = \alpha(1) = x \in P$. As before, let $b \in (0, 1)$ be the smallest value so that there exists $b' > b$ with $\alpha(b) = \alpha(b')$. Call this point $\alpha(b) = \alpha(b') = y$. The curve obtaining by restricting α to the interval $[b, b']$ cannot be homotopic to a point since α is length minimizing. Assume that it is homotopic to a curve winding about the boundary component several times. Then the curve obtaining by juxtaposing the curves $\alpha|_{[0,b]}$ and $\alpha|_{[b',1]}$ cannot be homotopic to a point or the boundary component. However, the length of this curve is less than the length of the curve α, which is a contradiction. Therefore, the curve obtained by restricting α to $[b, b']$ is not homotopic to a point or the boundary. As above, it follows that y is not in P. In particular, y is not a singular point. Now let α' be the curve joining y and x and which traces the same set with $\alpha|_{[0,b]}$, but in opposite direction. Assume that α' has a length less than or equal to $\alpha|_{[b',1]}$. Then the curve obtained by juxtaposing the curves $\alpha|_{[0,b']}$ and α' is not homotopic to the boundary or a single point. However, it has length less than or equal to the length of α. Also, since y is nonsingular, we can modify this curve near y to get a curve in the same homotopy class but with a strictly smaller length. This contradicts with the choice of α. If the length of $\alpha|_{[b',1]}$ is less than the length of α', we can proceed in a similar way to get a contradiction. □
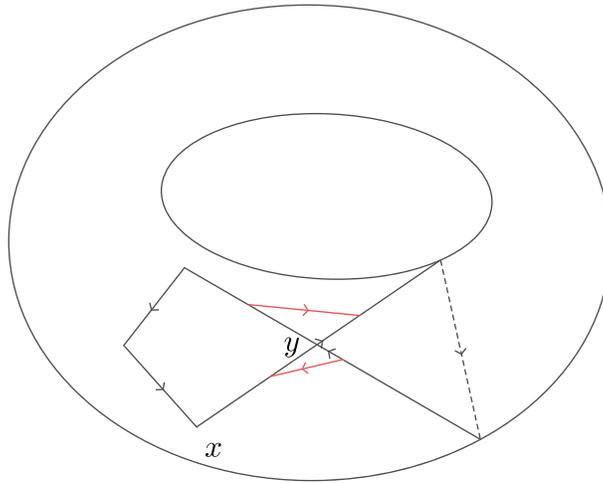
Figure 5. The polygonal loop α is described in a torus. Consider the quadrilateral formed by red line segments. If we modify α by replacing its diagonals by red segments, then we get a path whose length is strictly less than the length of α.

Note that there is a notion of genus for nonorientable surfaces. Lemma 4.7 does not discard the case of nonorientable surfaces since in the proof we made no assumption about orientability of S.

Proposition 4.8 *Let (S, P) be an orientable labeled singular surface. Then S has an ideal triangulation.*

Proof   Assume that S has more than one boundary component. In that case, as in the proof Lemma 4.6, we may take two points in two different boundary components and cut the surface through a length minimizing geodesic between these points. Each point in the intersection of P and this geodesic give us two points in the new surface. By including these points to the label set, we can get a new label set. Note that after this operation, the number of the boundary components decreases, so we may repeat the process until we get one boundary component. Therefore, we assume that S has at most one boundary component. Now we do induction on the genus of S. If S has genus zero, then it has an ideal triangulation by Lemma 4.6. Assume that genus of S is positive. Lemma 4.7 implies that there exists a point $x \in P$ and a simple geodesic loop based at x which is not homotopic to a point or a boundary component. Cut the surface through this geodesic loop. Then there are two possible cases.

1. This geodesic is separating. In this case, we obtain two surfaces so that each of them has a smaller genus than the original surface. We can form a labeled set for each of these surfaces accordingly. By induction hypothesis, these surfaces admit ideal triangulations. It is clear that these triangulations induce an ideal triangulation of the original surface.

2. The geodesic is nonseparating. In this case we obtain a surface which has a genus less than the genus of the original surface. Note that each point on the geodesic induces two points on the boundary of new surface. By adding these points to the labeled set, we may form a new labeled set. By induction hypothesis, this surface has an ideal triangulation and this ideal triangulation induces an ideal triangulation of the original surface.

□

The following corollaries are immediate.

Corollary 4.9 *Let S be a compact orientable surface with boundary together with a singular flat metric. If each boundary component of S has a singular point, then S has an ideal triangulation with respect to its set of singular points.*

Corollary 4.10 *Let S be a closed orientable surface without boundary and with a singular flat metric. If the set of singular points of S is not empty, then S admits an ideal triangulation with respect to its set of singular points.*

4.2. Nonorientable surfaces

Our next objective is to show that any nonorientable labeled flat surface has an ideal triangulation.

Remark 4.11 *A nonsingular Möbius strip with a label set P having cardinality $n \geq 1$ has an ideal triangulation. In Figure 6, you can find an example of such a triangulation when the cardinality of P is 1.*
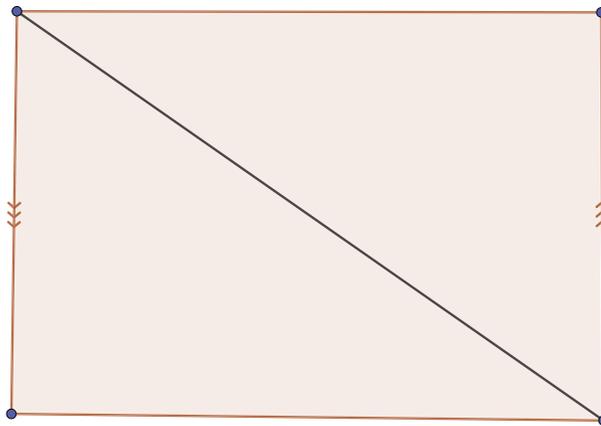

Figure 6. An ideal triangulation of the Möbius band with one labeled point.

Lemma 4.12 *A labeled singular flat surface which is homeomorphic to the Möbius band admits an ideal triangulation.*

Proof Take a labeled boundary point of the surface and consider a length minimizing geodesic which is in the homotopy class of the loop shown in Figure 6. Note that since the geodesic is length minimizing, it does not have self-intersection, that is, it is simple. After this operation we get a disc with a singular flat metric on it. We can form a new label set and get an ideal triangulation of this disc. This triangulation induces an ideal triangulation of the Möbius band. □

Theorem 4.13 *Any compact labeled singular flat surface admits an ideal triangulation.*

Proof If the surface is orientable, then the statement was proven in Proposition 4.8. Assume that the surface is nonorientable and it is not homeomorphic to the Möbius band. We do the following operation on the surface:

1. Reduce the number of boundary components: if the surface has more than one boundary component, we reduce the number of boundary components to one by cutting the surface through arcs joining its two boundary components repeatedly.

2. If the surface has less than or equal to one boundary component, then we cut the surface through the loop whose existence is given by Lemma 4.7.

After this process, we get a singular flat surface with one or two connected components. If any of these singular flat surfaces is orientable or homeomorphic to a Möbius band, then we apply no more cutting operation to it. Otherwise, we apply these operations again and arrange a new set of labeled points for it. Since the surface is of finite type, after finitely many runs of the process, we will get a finite number of singular flat surfaces each of which is either homeomorphic to the Möbius band or orientable. By Lemma 4.12 and Proposition 4.8, these surfaces admit ideal triangulations. These triangulations give an ideal triangulation of the initial surface.

□

5. Disc unfolding

Consider a singular flat surface which is homeomorphic to the sphere S^2 such that curvature at each singular point is positive. Assume that it has n singular points and the curvature at each of these points is positive. Pick a singular point and call it the base point. Consider a length minimizing geodesic for each other singular point which connects the base point and this singular point. Then these geodesics do not have self-intersection, and they do not intersect with each other except at the base point. The surface we obtain after cutting the sphere through these geodesics is a flat disc with a nonsingular interior. Alexandrov's unfolding theorem states [1] that this flat disc can be embedded isometrically into the Euclidean plane. Shortly, it states that any singular flat surface can be obtained from a planar polygon.

In Section 3, we obtained any singular flat surface from a flat disc. However, in the construction, we used an arbitrary number of geodesic segments. Actually, we cut the surface through geodesics which are not necessarily arcs. In this section, we show that we can do this unfolding by using only arcs.

Definition 5.1 *Let (S, P) be a labeled singular flat surface. We say that (S, P) admits a disc unfolding if there exists a finite set of simple arcs of S which can intersect only at their end points so that when we cut the surface through these arcs, we get a flat disc with a nonsingular interior.*

Lemma 5.2 *Any singular flat surface homeomorphic to $S^2[b]$ admits a disc unfolding.*

Proof The proof of this statement is similar to the proof of Lemma 4.6. Therefore, we omit the proof. □

Proposition 5.3 *Any orientable labeled flat surface admits a disc unfolding.*

Proof We do induction on the genus. The case $g = 0$ is explained in Lemma 5.2. Assume that the surface has a genus greater than zero. If the surface has more than one boundary component, then we can cut the surface through length minimizing geodesics, which are formed indeed by finite unions of arcs, to reduce the number of the boundary components. Therefore, we may assume that the surface has at most one boundary component. Lemma 4.7 implies that there exists a simple multiple arc which is not homotopic to the boundary component and a point. Now there are two cases to consider.

1. The arc is nonseparating. If we cut the surface through this arc. We get a surface with a smaller genus. By induction hypothesis, this surface admits a disc unfolding, and we are done.

2. The arc is separating. In this case, we obtain two surfaces such that each of them has a smaller genus. Each of these surfaces admits disc unfolding. Glue these discs through the corresponding edges, the edges which are induced by the simple arc, to get a disc together with a flat metric. Now this disc admits a disc unfolding, and we are done.

□

Lemma 5.4 *Any labeled singular flat surface which is homeomorphic to the Möbius band admits a disc unfolding.*

Proof The proof Lemma 6 shows that we can cut this surface through a loop based at its boundary so that the resulting surface is a disc. Then the result follows from Proposition 5.3. □

Theorem 5.5 *Any labeled singular flat surface admits a disc unfolding.*

Proof If the surface is orientable, we proved the statement in Proposition 5.3. Assume that the surface is not orientable. As in the proof of Theorem 4.13, we can cut the surface through the simple arcs to get a finite number of orientable surfaces and Möbius band. By Proposition 5.3 and Lemma 5.4, each of these surfaces admits a disc unfolding. By gluing these unfoldings accordingly, we may get a disc unfolding. □

Acknowledgements

References

[1] Alexandrov AD. Convex Polyhedra. Berlin, Germany: Springer Monographs in Mathematics Springer-Verlag, 2005.

[2] Burago D, Burago Y, Ivanov S. A course in metric geometry. Providence, RI, USA: Graduate Studies in Mathematics, 33. American Mathematical Society, 2001.

[3] Farb B, Margalit D. A Primer on Mapping Class Groups. Princeton, NJ, USA: Princeton University Press, 2012.

[4] Fillastre F. From spaces of polygons to spaces of polyhedra following Bavard, Ghys and Thurston. L'Enseigenement Mathématique, 2011; 57 (1): 23-56. doi: 10.4171/LEM/57-1-2

[5] Indermitte C, Liebling TM, Clémençon H, Troyanov M. Voronoi Diagrams on piecewise flat surfaces and an application to biological growth. Theoretical Computer Science, 2001; 263 (1-2): 263-274. doi: 10.1016/S0304-3975(00)00248-6

[6] Izmestiev I. A simple proof of an isoperimetric inequality for Euclidean and hyperbolic cone-surfaces. Differential Geometry and its Applications, 2015; 43: 95-101. doi: 10.1016/j.difgeo.2015.09.007

[7] Hubert P, Schmidt T. An introduction to Veech surfaces. In: Hasselblatt B, Katok A (editors). Handbook of dynamical systems 1B. North-Holland, Netherlands: Elsevier, 2006, pp. 501-526.

[8] Papadopoulos A, Théret G. On Teichmüller's metric and Thurston's asymmetric metric. In: Papadopoulos A (editor). Handbook of Teichmüller Theory 1. Zurich, Switzerland: European Mathematical Society, 2007, pp. 111-204.

[9] Sağlam İ. Complete flat cone metrics on punctured surfaces. Turkish Journal of Mathematics, 2019; 43: 813-832. doi: 10.3906/mat-1806-23

[10] Tahar G. Geometric triangulations and flips, Comptes Rendus Mathematique, 2019; 357 (7): 620-623. doi: 10.1016/j.crma.2019.07.001

[11] Thurston WP. Shapes of polyhedra and triangulations of the sphere. Geometry and Topology Monographs, 1998, 1: 511-549. doi: 10.2140/gtm.1998.1.511

[12] Thurston WP. Three-dimensional geometry and topology. Princeton, NJ, USA: Princeton University Press, 1997.

[13] Troyanov M. Les surfaces euclidienne a singularites coniques, L'Enseignement Mathématique, 1986, 32 (1-2): 79-94 (in French). doi: 10.5169/seals-55079

[14] Troyanov M. On the moduli space of singular Euclidean surfaces. In: Papadopoulos A (editor). Handbook of Teichmuller Theory 1. Zurich, Switzerland: European Mathematical Society, 2007, pp. 507-540.