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On the spectra of generalized Fibonomial and Jacobsthal-binomial graphs

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Abstract: In this work, we first give a more general form of the binomial, Fibonomial, and balance-binomial graphs that is called generalized Fibonomial graph. We also argue the spectra of generalized Fibonomial graph. Next, we introduce a new type of graph on Jacobsthal numbers that is called Jacobsthal-binomial graph and denoted by JB_n. We obtain the adjacency, Laplacian and signless Laplacian characteristic polynomials of JB_n, respectively. We lastly give inequalities for the adjacency, Laplacian and signless Laplacian energies of JB_n.

Key words: Adjacency matrix, Fibonomial graph, balance-binomial graph, spectral characterization, graph energy

1. Introduction

A graph G consists of a vertex set $V(G) = \{v_1, \ldots, v_n\}$ and an edge set $E(G)$, and it is denoted by $G = (V(G), E(G))$. The order of G is the number of vertices in G, that is $|V(G)|$, and the size of G is the number of edges in G that is $|E(G)|$. The degree of a vertex $v_i \in V(G)$ is the number of vertices in $V(G)$ which are adjacent to v_i, and it is denoted by $deg(v_i)$. If a vertex in G is adjacent to itself, then there occurs a loop. If a graph has order $n \in \mathbb{Z}^+$ and size 0, then it is called a null graph with n vertices. Also, if all of the vertices of a graph with order n are adjacent to each other (except itself), then it is called a complete graph and denoted by K_n. Adjacency matrix of a graph G is $A(G) = [a_{ij}]_{n\times n}$ such that

$$a_{ij} = \begin{cases} 1, & (v_i, v_j) \in E(G); \\ 0, & (v_i, v_j) \notin E(G). \end{cases}$$

Degree matrix of a graph G is the diagonal matrix $D(G) = diag(deg(v_1), \ldots, deg(v_n))$. Laplacian matrix of a graph G is $L(G) = D(G) - A(G)$ and the signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. For a graph matrix $M(G)$, the M-spectrum of G is the set of the eigenvalues of $M(G)$. For more information about spectral graph theory, we refer to [2]. Energy of a graph was first given by Gutman [6]. This concept has recently had great attention by the researchers such as [5, 7, 8, 12]. For a given graph $G = (V(G), E(G))$, let $M(G)$ be the adjacency matrix, Laplacian matrix, or signless Laplacian matrix of G, and $\lambda_1 \geq \ldots \geq \lambda_n$ consists of the spectrum of $M(G)$. Then we compute the energy as follows:

$$\varepsilon(M(G)) = \begin{cases} \sum_{i=1}^{n} |\lambda_i|, & \text{if } G \text{ is simple;} \\ \sum_{i=1}^{n} |\lambda_i - \frac{trace(M(G))}{|V_n|}|, & \text{otherwise.} \end{cases} \tag{1.1}$$

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Binomial graphs were first introduced in 1997 by Christopher and Kennedy [4]. For $0 \leq n \in \mathbb{Z}$, a binomial graph $B_n = (V_n, E_n)$ is defined with $V_n = \{v_j : j = 0, \ldots, 2^n - 1\}$ and $E_n = \{(v_i, v_j) : \left[\begin{array}{c} i+j \\ j \end{array}\right] \equiv 1(\text{mod } 2)\}$. Adjacency spectrum of B_n is obtained, and closed walks in B_n are also investigated. Similar to this concept, in 2014, Akbulak et al. [1] introduced a new type of graph called Fibonomial graph. They used Fibonomial coefficents instead of binomial coefficents. Spectral properties and energy of Fibonomial graphs were investigated in the same paper. Next, balance-binomial graphs were introduced in [9] whose entries are dependent on the balancing numbers. Balance-binomial coefficents were used instead of Fibonomial coefficents in [9]. However, actually, balance-binomial graphs are exactly the same with binomial graphs. This could be easily seen by comparing [4] and [9]. By the motivation of these results, in this work, we first examine the situation for the generalized Fibonacci numbers and so define the generalized Fibonomial graphs with a more general perspective. Also, we define the Jacobsthal-binomial graphs by using Jacobsthal binomial coefficents. We investigate the spectra and energy of Jacobsthal-binomial graphs according to the adjacency, Laplacian, and signless Laplacian matrices, respectively. Characteristic polynomials and inequalities for the energies of these matrices are obtained here.

2. Generalized Fibonomial graphs

The well-known Fibonacci sequence is defined as:

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \tag{2.1}$$

where F_n denotes the nth Fibonacci number. Similar to the binomial coefficents, Fibonomial coefficents are defined as:

$$\left[\begin{array}{c} n \\ k \end{array}\right]_F = \frac{F_n \cdot \ldots \cdot F_{n-k+1}}{F_k \cdot \ldots \cdot F_1} \tag{2.2}$$

where $n \geq k \geq 1$, $\left[\begin{array}{c} n \\ 0 \end{array}\right]_F = 1$ and $\left[\begin{array}{c} n \\ k \end{array}\right]_F = 0$ for $n < k$.

For $0 \leq n \in \mathbb{Z}$, a Fibonomial graph $G_n = (V_n, E_n)$ is defined by Akbulak et al. [1] with the vertex set $V_n = \{v_t : t = 0, 1, \ldots, 3 \cdot 2^n - 1\}$ and the edge set $E_n = \{(v_i, v_j) : \left[\begin{array}{c} i+j \\ j \end{array}\right]_F \equiv 1(\text{mod } 2)\}$. Chen and Sagan [3] examined the fractal nature of the Fibonomial triangle. As a result of this investigation, adjacency matrix of a Fibonomial graph could be written in terms of Kronecker product, that is given in [1] as follows:

$$A_n = \left[\bigotimes_{i=1}^{n} \mathfrak{B}\right] \bigotimes A_0 \text{ where } \mathfrak{B} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right] \text{ and } A_0 = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right]. \tag{2.3}$$

The balancing numbers are defined by the recursion formulae

$$B_n = 6B_{n-1} - B_{n-2} \text{ with initial values } B_1 = 1 \text{ and } B_2 = 6 \tag{2.4}$$

. By using this sequence, a balance-binomial graph $G'_n = (V'_n, E'_n)$ on 2^n vertices, with vertex set $V_n = \{v_t : t = 0, 1, \ldots, 2^n - 1\}$ and the edge set $E_n = \{(v_i, v_j) : \left[\begin{array}{c} i+j \\ j \end{array}\right]_B \equiv 1(\text{mod } 2)\}$ is given in [9] such that the balance-binomial coefficents are defined as:

$$\left[\begin{array}{c} n \\ k \end{array}\right]_B = \frac{B_n \cdot \ldots \cdot B_{n-k+1}}{B_k \cdot \ldots \cdot B_1} \tag{2.5}$$

for $n \geq k \geq 1$ with $\left[\begin{array}{c} n \\ 0 \end{array}\right]_B = 1$ and $\left[\begin{array}{c} n \\ k \end{array}\right]_B = 0$ for $n < k$. Similar to the Fibonomial graphs, adjacency matrix of a balance-binomial graph is obtained by using Kronecker product as follows:

$$A'_n = \left[\bigotimes_{i=1}^{n} \mathfrak{B}\right] \bigotimes A'_0 \text{ where } A'_0 = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right] = \mathfrak{B}. \tag{2.6}$$

The generalized Fibonacci sequence is defined as:

$$GF_n = pGF_{n-1} + qGF_{n-2} \text{ for } n \geq 2, p, q \in \mathbb{Z}^+ \text{ with } GF_0 = 0 \text{ and } GF_1 = 1. \tag{2.7}$$

Generalized Fibonomial coefficents are similarly defined as:

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{GF} = \frac{GF_n \cdot \ldots \cdot GF_{n-k+1}}{GF_k \cdot \ldots \cdot GF_1} \tag{2.8}$$

for $n \geq k \geq 1$ with $\left[\begin{array}{c} n \\ 0 \end{array}\right]_{GF} = 1$ and $\left[\begin{array}{c} n \\ k \end{array}\right]_{GF} = 0$ for $n < k$.

Here we define the generalized Fibonomial graphs $GF_n = (V_{GF_n}, E_{GF_n})$ with the vertex set V_{GF_n} and the edge set

$$E_{GF_n} = \{(v_i, v_j) : \left[\begin{array}{c} i+j \\ j \end{array}\right]_{GF} \equiv 1 (\text{mod } 2) \text{ such that } v_i, v_j \in V_{GF_n}\}.$$

All of the situations of the generalized Fibonomial coefficents according to the modulo 2 are explained in [3] by Chen and Sagan. Thus, we obtain four kinds of adjacency matrices for the generalized Fibonomial graphs. If p, q are both odd, then we obtain the Fibonomial graph with $3 \cdot 2^n$ vertices. If p is even and q is odd, then we have the balance-binomial graph with 2^n vertices (actually, this graph could also be easily obtained from Pascal's triangle). If p is odd and q is even, then we obtain that the adjacency matrix is the all one matrix $\mathbf{J}_n$. If p, q are both even, then we get that the adjacency matrix is the all zero matrix $\mathbf{0}_n$. Hence, we have four kinds of generalized Fibonomial graphs. These are Fibonomial graphs, balance-binomial graphs (or shortly binomial graphs), the graph obtained from a complete graph by adding loop to all vertices, and the null graph, respectively.

Thus, if we talk about the spectra of the generalized Fibonomial graphs, there are four kinds of the adjacency spectrum according to the p and q. Adjacency spectrum of the Fibonomial graphs and balance-binomial graphs are given in [1] and [4, 9], respectively. Clearly $\mathbf{J}_n$ has eigenvalues $0^{n-1}, n$ and $\mathbf{0}_n$ has eigenvalues 0^n. Based on these results, in the next section, we define a new graph on the Jacobsthal numbers that is different from generalized Fibonomial graphs, and we also obtain the spectral properties of this graph.

3. Jacobsthal-binomial graphs

3.1. Jacobsthal numbers

The Jacobsthal numbers are given by the following recursion formula:

$$J_0 = 0, J_1 = 1, J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2 \tag{3.1}$$

where J_n denotes the nth Jacobsthal number. Here we give some properties of Jacobsthal numbers that are used in the next sections. The following table for the remainders of Jacobsthal numbers is obtained in [17].

Lemma 3.1 *[10] Let n be an integer. The well-known Binet-like formula of the Jacobsthal numbers is:*

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{3.2}$$

Theorem 3.2 *[10] For all $n, k \in \mathbb{Z}^+$, $J_n \mid J_{kn}$.*

3.2. Construction of Jacobsthal-binomial graphs

Similar to the binomial coefficents, here we define the Jacobsthal-binomial coefficents for $n \geq k \geq 1$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_J = \frac{J_n \cdot \ldots \cdot J_{n-k+1}}{J_k \cdot \ldots \cdot J_1} \tag{3.3}$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_J = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_J = 0$ for $n < k$. From Theorem 3.2, one can easily see that $\begin{bmatrix} n \\ k \end{bmatrix}_J$ always takes integer values. Thus, we define the Jacobsthal-binomial graph $JB_n = (V_{JB_n}, E_{JB_n})$ with the vertex set $V_{JB_n} = \{v_j : j = 0, 1, \ldots, 2n\}$ and the edge set

$$E_{JB_n} = \{(v_i, v_j) : \begin{bmatrix} i+j \\ j \end{bmatrix}_J \equiv 1(\bmod 4)\}.$$

In this way, we obtain that the adjacency matrix of the Jacobsthal-binomial graph $A(JB_n) = [a_{ij}]_{m \times m}$ such that $m = 2n + 1$ and

$$a_{ij} = \begin{cases} 0, & \text{if } \begin{bmatrix} i+j \\ j \end{bmatrix}_J \equiv 3(\bmod 4) \text{ ;} \\ 1, & \text{if } \begin{bmatrix} i+j \\ j \end{bmatrix}_J \equiv 1(\bmod 4). \end{cases}$$

From the definition of the Jacobsthal-binomial graph, the first row and the first column of $A(JB_n)$ are all-one vectors. In addition, we can easily obtain the second row and the second column of $A(JB_n)$ from Figure 1. Now, we find other rows and columns of $A(JB_n)$ by the following theorem.

Theorem 3.3 *For $i, j \in \{1, 2, \ldots, 2n\}$, let $\begin{bmatrix} i+j \\ j \end{bmatrix}_J = s$. If i, j are both odd, then $s \equiv 1(mod\ 4)$. Otherwise, $s \equiv 3(mod\ 4)$.*

Proof We give the proof of the statement by induction on j in two cases. Assume that the statement holds for j. Then, we need to show that the equality holds for $j+2$.

Case 1: Let j be an odd integer. For $j = 1$,

$$\begin{bmatrix} i+1 \\ 1 \end{bmatrix}_J = \frac{J_{i+1}}{J_1} = \frac{J_{i+1}}{1} = J_{i+1}$$

From Figure 1, it can easily be seen that if i is odd then $J_{i+1} \equiv 1(\bmod 4)$; otherwise, $J_{i+1} \equiv 3(\bmod 4)$. Now, we check whether the statement holds for $j+2$. Let us consider the situation in two subcases.

Subcase 1: Suppose that i is odd. Then,

$$\left[\begin{array}{c} i+j+2 \\ j+2 \end{array}\right]_J = \frac{J_{i+j+2} \cdot J_{i+j+1} \cdot \ldots \cdot J_{i+1}}{J_{j+2} \cdot J_{j+1} \cdot \ldots \cdot J_1}$$

Since the statement holds for j, if $\left[\begin{array}{c} i+j \\ j \end{array}\right]_J = m$ then we have $m \equiv 1 (\bmod\ 4)$. Also, by using the Binet-like formula (3.2) of the Jacobsthal numbers, we get

$$\frac{J_{i+j+2} \cdot J_{i+j+1}}{J_{j+2} \cdot J_{j+1}} = \frac{(2^{i+j+2}-1)(2^{i+j+1}+1)}{(2^{j+2}+1)(2^{j+1}-1)} = \frac{2^{2i+2j+3}+2^{i+j+1}-1}{2^{2j+3}-2^{j+1}-1}.$$

If $\frac{2^{2i+2j+3}+2^{i+j+1}-1}{2^{2j+3}-2^{j+1}-1} = k$, then we can see easily that $k \equiv 1 (\bmod\ 4)$. Hence, we get $\left[\begin{array}{c} i+j+2 \\ j+2 \end{array}\right]_J \equiv 1 (\bmod\ 4)$ when i and j are both odd.

Subcase 2: Let i is even. Thus, we have $\left[\begin{array}{c} i+j \\ j \end{array}\right]_J = m \equiv 3 (\bmod\ 4)$. Therefore,

$$\left[\begin{array}{c} i+j+2 \\ j+2 \end{array}\right]_J = \frac{J_{i+j+2} \cdot J_{i+j+1}}{J_{j+2} \cdot J_{j+1}} \cdot m = \frac{2^{2i+2j+3}-2^{i+j+1}-1}{2^{2j+3}-2^{j+1}-1} \cdot m$$

Thus, we get $\left[\begin{array}{c} i+j+2 \\ j+2 \end{array}\right]_J \equiv 3 (\bmod\ 4)$.

Case 2: Let j be even. For $j=2$, we get

$$\left[\begin{array}{c} i+2 \\ 2 \end{array}\right]_J = \frac{J_{i+2} \cdot J_{i+1}}{J_2 \cdot J_1} = J_{i+2} \cdot J_{i+1} \equiv 3 (\bmod\ 4).$$

We assume again that the statement holds for j and check for $j+2$. Thus, $\left[\begin{array}{c} i+j \\ j \end{array}\right]_J = m \equiv 3 (\bmod\ 4)$. We will consider the situation in two subcases.

Subcase 1: Suppose that i is even. Then, we have

$$\left[\begin{array}{c} i+j+2 \\ j+2 \end{array}\right]_J = \frac{J_{i+j+2} \cdot J_{i+j+1}}{J_{j+2} \cdot J_{j+1}} \cdot m = \frac{2^{2i+2j+3}+2^{i+j+1}-1}{2^{2j+3}+2^{j+1}-1} \cdot m$$

Thus, we get $\left[\begin{array}{c} i+j+2 \\ j+2 \end{array}\right]_J \equiv 3 (\bmod\ 4)$.

Subcase 2: Suppose that i is odd. Thus, we get

$$\left[\begin{array}{c} i+j+2 \\ j+2 \end{array}\right]_J = \frac{2^{2i+2j+3}-2^{i+j+1}-1}{2^{2j+3}+2^{j+1}-1} \cdot m \equiv 3 (\bmod\ 4)$$

Consequently, we obtain that if i and j are both odd, then $\left[\begin{array}{c} i+j \\ j \end{array}\right]_J \equiv 1 (\bmod\ 4)$; otherwise, $\left[\begin{array}{c} i+j \\ j \end{array}\right]_J \equiv 3 (\bmod\ 4)$.

□

By the help of the Theorem 3.3 , we get $A(JB_n)$ with $2n+1$ vertices in the following form.

$$A(JB_n) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{3.4}$$

Remark 3.4 *For all* $v_i \in V_{JB_n}$,

$$deg(v_i) = \begin{cases} 2n+2, & \textit{if i=0;} \\ n+2, & \textit{if i is odd;} \\ 1, & \textit{if i is even.} \end{cases}$$

and $|E_{JB_n}| = n^2 + 5n + 2$.

J_n / m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	...
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
3	1	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	
5	1	2	1	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	
11	1	2	3	1	5	4	3	2	1	0	11	11	11	11	11	11	11	11	
21	1	0	1	1	3	0	5	3	1	10	9	8	7	6	5	4	3	2	
43	1	1	3	3	1	1	3	7	3	10	7	4	1	13	11	9	7	5	
85	1	1	1	0	1	1	5	4	5	8	1	7	1	10	5	0	13	9	
171	1	0	3	1	3	3	3	0	1	6	3	2	3	6	11	1	9	0	
341	1	2	1	1	5	5	5	8	1	0	5	3	5	11	5	1	17	18	
683	1	2	3	3	5	4	3	8	3	1	11	7	11	8	11	3	17	18	
1365	1	0	1	0	3	0	5	6	5	1	9	0	7	0	5	5	15	16	
2731	1	1	3	1	1	1	3	4	1	3	7	1	1	1	11	11	13	14	
5461	1	1	1	1	1	1	5	7	1	5	1	1	1	1	5	4	7	8	
10923	1	0	3	3	3	3	3	6	3	0	3	3	3	3	11	9	15	17	
21845	1	2	1	0	5	5	5	2	5	10	5	5	5	5	5	0	11	14	
43691	1	2	3	1	5	4	3	5	1	10	11	11	11	11	11	1	5	10	
87381	1	0	1	1	3	0	5	0	1	8	9	8	7	6	5	1	9	0	
174763	1	1	3	3	1	1	3	1	3	6	7	4	1	13	11	3	1	1	
349525	1	1	1	0	1	1	5	1	5	0	1	7	1	10	5	5	1	1	
⋮																			

Figure 1. First four Jacobsthal-binomial graphs.

The pineapple graph K_p^q is obtained by appending q pendant edges to a vertex of a complete graph K_p $(q \geq 1, p \geq 3)$[14, 15]. From (3.4) and Figure 2, we can see easily that JB_n could also be obtained from a pineapple graph K_{n+1}^n by adding loop to the all of vertices of its maximum clique K_{n+1}.

Lemma 3.5 *The characteristic polynomial of the adjacency matrix of Jacobsthal-binomial graph with 2n+1 vertices is as follows:*

$$A(JB_n)(x) = x^{2n-2}(-x^3 + (n+1)x^2 + nx - n^2) \tag{3.5}$$

Proof Let $\mathbf{1}$ denote the all-one vector, $\mathbf{J}_n$ denote the all-one $n \times n$ matrix, and $\mathbf{0}$ denote the all-zero $n \times n$ matrix. It is clear that $A(JB_n)$ could be written in the following form.

$$A(JB_n) = \begin{bmatrix} \mathbf{1} & \mathbf{1}^T & \mathbf{1}^T \\ \mathbf{1} & \mathbf{J}_n & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}_{(2n+1)\times(2n+1)}$$

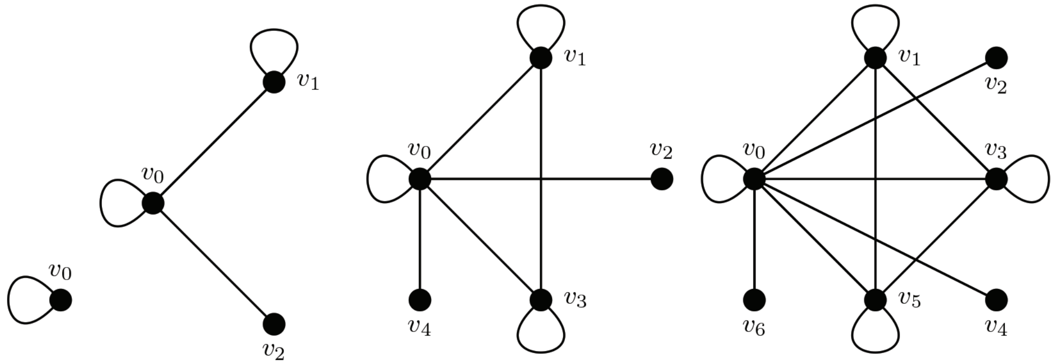

Figure 2. Sets of the remainder of Jacobsthal numbers for modulo m.

One can easily see that the rank of $A(JB_n)$ could be equal to at most 3. Hence, x^{2n-2} is a factor of the $A(JB_n)(x)$. Also, an equitable partition of $A(JB_n)$ is as follows:

$$P_n = \begin{bmatrix} 1 & n & n \\ 1 & n & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3\times 3}$$

The characteristic polynomial of P_n is equal to $P_n(x) = -x^3 + (n+1)x^2 + nx - n^2$. Since $P_n(x)$ is a divisor of $A(JB_n)(x)$, we get that

$$A(JB_n)(x) = x^{2n-2}(-x^3 + (n+1)x^2 + nx - n^2)$$

□

Lemma 3.6 *The Laplacian characteristic polynomial of the Jacobsthal-binomial graph with 2n+1 vertices is given below.*

$$L(JB_n)(x) = (x-(n+2))^{n-1}(x-1)^{n-1}(-x^3 + (2n+4)x^2 - (4n+5)x + n + 2) \qquad (3.6)$$

Proof Let I_n denote the $n \times n$ identity matrix. Since $L(JB_n) - I_n$ and $L(JB_n) - (n+2)I_n$ contains n identical rows, $(x-1)^{n-1}$ and $(x-(n+2))^{n-1}$ are factors of $L(JB_n)(x)$, respectively. Clearly, $L(JB_n)$ could be written in the following form:

$$L(JB_n) = \begin{bmatrix} 2n+1 & \mathbf{-1}^T & \mathbf{-1}^T \\ \mathbf{-1} & (n+2)\mathbf{I}_n - \mathbf{J}_n & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \mathbf{I}_n \end{bmatrix}_{(2n+1)\times(2n+1)}$$

Hence, an equatible partition of $L(JB_n)$ is as follows:

$$P'_n = \begin{bmatrix} 2n+1 & -n & -n \\ -1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}_{3\times 3}$$

The characteristic polynomial of P'_n is equal to $P'_n(x) = -x^3 + (2n+4)x^2 - (4n+5)x + n + 2$. Thus, we obtain that

$$L(JB_n)(x) = (x-(n+2))^{n-1}(x-1)^{n-1}(-x^3 + (2n+4)x^2 - (4n+5)x + n + 2)$$

□

Lemma 3.7 *The signless Laplacian characteristic polynomial of the Jacobsthal-binomial graph with 2n+1 vertices is in the following form:*

$$\begin{aligned} Q(JB_n)(x) &= (x-(n+2))^{n-1}(x-1)^{n-1} \\ &\quad (-x^3+(4n+6)x^2-(4n^2+12n+11)x+2n^2+7n+6) \end{aligned} \tag{3.7}$$

Proof $Q(JB_n)-I_n$ and $Q(JB_n)-(n+2)I_n$ have n identical rows, respectively. Hence, $(x-1)^{n-1}$ and $(x-(n+2))^{n-1}$ are factors of $Q(JB_n)(x)$. Also, $Q(JB_n)$ could be written as below:

$$Q(JB_n) = \begin{bmatrix} 2n+3 & \mathbf{1}^T & \mathbf{1}^T \\ \mathbf{1} & (n+2)\mathbf{I}_n + \mathbf{J}_n & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{I}_n \end{bmatrix}_{(2n+1)\times(2n+1)}$$

Thus, we can obtain an equatible partition of $Q(JB_n)$ as follows:

$$P''_n = \begin{bmatrix} 2n+3 & n & n \\ 1 & 2n+2 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Hereby the characteristic polynomial of P''_n is equal to $P''_n(x) = -x^3+(4n+6)x^2-(4n^2+12n+11)x+2n^2+7n+6$. Thus, we conclude that

$$\begin{aligned} Q(JB_n)(x) &= (x-(n+2))^{n-1}(x-1)^{n-1} \\ &\quad (-x^3+(4n+6)x^2-(4n^2+12n+11)x+2n^2+7n+6) \end{aligned}$$

□

3.3. Energies of Jacobsthal-binomial graph

Theorem 3.8 *Let $\varepsilon(JB_n)$, $\varepsilon_L(JB_n)$, and $\varepsilon_Q(JB_n)$ denote the energy, Laplacian energy, and signless Laplacian energy of the Jacobsthal-binomial graph with 2n+1 vertices, respectively. Then, the following inequalities hold:*

(i) $2n+2 < \varepsilon(JB_n) < 2n+3$

(ii) $n^2+\frac{5n}{2}-\frac{5}{4} < \varepsilon_L(JB_n) < n^2+\frac{5n}{2}+\frac{3}{4}$

(iii) $n^2+\frac{7n}{2}+\frac{23}{4} < \varepsilon_Q(JB_n) < n^2+\frac{7n}{2}+\frac{31}{4}$

Proof **(i)** Let us denote the eigenvalues of $A(JB_n)$ with $\lambda_1 \geq \ldots \geq \lambda_{2n+1}$. Then, from (3.5) we have $\lambda_1 = n+1+\alpha$, $\lambda_2 = 1+\beta$, $\lambda_3 = \ldots = \lambda_{2n} = 0$, $\lambda_{2n+1} = -1-\alpha-\beta$ such that $0 < \alpha, \beta < 1$. We know that

$\varepsilon(JB_n) = \sum_1^{2n+1} |\lambda_i - \frac{s}{2n+1}|$ where $s = tr(A(JB_n)) = n+1$. Hence,

$$\begin{aligned}
\varepsilon(JB_n) &= \sum_{i=1}^{2}(\lambda_i - \frac{n+1}{2n+1}) + (\frac{n+1}{2n+1} - \lambda_{2n+1}) + (2n-2)(\frac{n+1}{2n+1}) \\
&= n+3+2(\alpha+\beta)+(2n-3)\frac{n+1}{2n+1} \\
&= n+3+2(\alpha+\beta)+n-1-\frac{2}{2n+1} \\
&= 2n+2+2(\alpha+\beta)-\frac{2}{2n+1} \\
&= 2n+2+\gamma
\end{aligned}$$

where $0 < \gamma = 2(\alpha+\beta) - \frac{2}{2n+1} < 1$. Thus, we get $2n+2 < \varepsilon(JB_n) < 2n+3$.

(ii) If we denote the eigenvalues of the $L(JB_n)$ with $\mu_1 \geq \ldots \geq \mu_{2n+1}$, then from (3.6) we get $\mu_1 = 2n+1+\alpha_1$, $\mu_2 = \ldots = \mu_n = n+2$, $\mu_{n+1} = 1+\beta_1$, $\mu_{n+2} = \ldots = \mu_{2n} = 1$, $\mu_{2n+1} = 2-\alpha_1-\beta_1$ such that $0 < \alpha_1, \beta_1 < 1$. Also, $s_L = tr(L(JB_n)) = 2n+1+n(n+1)+n = n^2+4n+1$. Thus, we calculate the Laplacian energy of JB_n as follows:

$$\begin{aligned}
\varepsilon(L(JB_n)) &= \sum_{i=1}^{2n+1} |\mu_i - \frac{s_L}{2n+1}| \\
&= \sum_{i=1}^{2n+1} |\mu_i - \frac{n^2+4n+1}{2n+1}| \\
&= |2n+1+\alpha_1 - \frac{n^2+4n+1}{2n+1}| + (n-1)|n+2-\frac{n^2+4n+1}{2n+1}| \\
&+ |1+\beta_1 - \frac{n^2+4n+1}{2n+1}| + (n-1)|1-\frac{n^2+4n+1}{2n+1}| \\
&+ |2-\alpha_1-\beta_1-\frac{n^2+4n+1}{2n+1}| \\
&= 2n-2+2\alpha_1+\frac{n^2+4n+1}{2n+1}+(n-1)(n+1) \\
&= n^2+\frac{5n}{2}-\frac{5}{4}+2\alpha_1+\frac{1}{8n+4}
\end{aligned}$$

Hence, we get $n^2 + \frac{5n}{2} - \frac{5}{4} < \varepsilon_L(JB_n) < n^2 + \frac{5n}{2} + \frac{3}{4}$.

(iii) We denote the eigenvalues of $Q(JB_n)$ with $q_1 \geq \ldots \geq q_{2n+1}$. From (3.7), we obtain that $q_1 = 2n+a$, $q_2 = 2n+b$, $q_3 = \ldots = q_{n+1} = n+2$, $q_{n+2} = \ldots = q_{2n} = 1$, $q_{2n+1} = \alpha_2 = 6-a-b$, and $0 < \alpha_2 < 1$. Also,

$s_Q = tr(Q(JB_n)) = n^2 + 6n + 3$. Hence, we may calculate $\varepsilon_Q(JB_n)$ as below:

$$
\begin{aligned}
\varepsilon(Q(JB_n)) &= \sum_{i=1}^{2n+1} |q_i - \frac{s_Q}{2n+1}| \\
&= \sum_{i=1}^{2n+1} |q_i - \frac{n^2+6n+3}{2n+1}| \\
&= |q_1 - \frac{n^2+6n+3}{2n+1}| + |q_2 - \frac{n^2+6n+3}{2n+1}| \\
&+ (n-1)|n+2-\frac{n^2+6n+3}{2n+1}| \\
&+ (n-1)|1-\frac{n^2+6n+3}{2n+1}| + |q_{2n+1} - \frac{n^2+6n+3}{2n+1}| \\
&= q_1 + q_2 - q_{2n+1} - \frac{n^2+6n+3}{2n+1} + (n-1)(n+1) \\
&= 4n + 2(a+b) - 6 + n^2 - \frac{n}{2} + \frac{7}{4} - \frac{1}{8n+4} \\
&= n^2 + \frac{7n}{2} + 2(a+b) - \frac{17}{4} - \frac{1}{8n+4}
\end{aligned}
$$

Since $0 < \alpha_2 < 1$, we get $n^2 + \frac{7n}{2} + \frac{23}{4} < \varepsilon_Q(JB_n) < n^2 + \frac{7n}{2} + \frac{31}{4}$.

□

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