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Conformal bi-slant submersions

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Abstract: In this paper, we study conformal bi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds as a generalized of conformal anti-invariant, conformal semi-invariant, conformal semi-slant, conformal slant, and conformal hemi-slant submersions. We investigate the integrability of distributions and obtain necessary and sufficient conditions for the maps to have totally geodesic fibers. Also, we consider some decomposition theorems for the new submersion and study the total geodesicity of such maps. Finally, we find curvature relations between the base space and the total space.

Key words: Bi-slant submersion, conformal bi-slant submersion, almost Hermitian manifold

1. Introduction

In complex geometry, as a generalization of holomorphic and totally real immersions, slant immersions were defined by Chen [12]. Cabrerizo et al. [11] defined bi-slant submanifolds in almost contact metric manifolds. In [34], Uddin et al. studied warped product bi-slant immersions in Kaehler manifolds. They proved that there do not exist any warped product bi-slant submanifolds of Kaehler manifolds other than hemi-slant warped products and CR-warped products.

The theory of Riemannian submersions as an analogue of isometric immersions was initiated by O'Neill [24] and Gray[16]. The Riemannian submersions are important in physics due to applications in the Yang–Mills theory, Kaluza–Klein theory, robotic theory, supergravity, and superstring theories. In Kaluza-Klein theory, the general solution of a recent model is given in point of harmonic maps satisfying Einstein equations (see [9, 10, 13, 19, 20, 23, 36]). Altafini [6] expressed some applications of submersions in the theory of robotics and Şahin [28] also investigated some applications of Riemannian submersions on redundant robotic chains. On the other hand, Riemannian submersions are very useful in studying the geometry of Riemannian manifolds equipped with differentiable structures. In [35], Watson defined the notion of almost Hermitian submersions between almost complex manifolds. He obtained some geometric properties between base manifold and total manifold as well as fibers. Şahin [29] introduced anti-invariant Riemannian submersions from almost Hermitian manifolds. He showed that such maps have some geometric properties. Also, he studied slant submersions from almost Hermitian manifolds onto a Riemannian manifolds [31]. Recently, considering different conditions on Riemannian submersions, many studies have been done (see [4, 7, 25–27, 30, 32, 33]).

As special horizontally conformal maps which were introduced independently by Fuglede [15] and Ishihara [21], horizontally conformal submersions are defined as follows: (M_1, g_1) and (M_2, g_2) are Riemannian manifolds

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of dimension m_1 and m_2, respectively. A smooth submersion $f:(M_1,g_1)\to(M_2,g_2)$ is called a horizontally conformal submersion if there is a positive function λ such that

$$\lambda^2 g_1(X_1,X_2)=g_2(f_*X_1,f_*X_2)$$

for all $X_1,X_2\in\Gamma\left((\ker f_*)^{\perp}\right)$. Here a horizontally conformal submersion f is called horizontally homothetic if the $grad\lambda$ is vertical, i.e.

$$\mathcal{H}(grad\lambda)=0.$$

We denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distributions $(\ker f_*)$ and horizontal distributions $(\ker f_*)^{\perp}$. It can be said that Riemannian submersion is a special horizontally conformal submersion with $\lambda=1$. Recently, Akyol and Şahin have introduced conformal anti-invariant submersions [2], conformal semi-invariant submersion[3], conformal slant submersion [5], and conformal semi-slant submersions[1]. Also, the geometry of conformal submersions have been studied by several authors [18, 22].

In Section 2, we review basic formulas and definitions needed for this paper. In Section 3, we define a new conformal bi-slant submersion from almost Hermitian manifolds onto Riemannian manifolds and we present an example. We investigate the geometry of the horizontal distribution and the vertical distribution. Finally, we obtain necessary and sufficient conditions for a conformal bi-slant submersion to be totally geodesic and give some curvature relations between the base space and the total space.

2. Preliminaries

Let (M_1,g_1,J) be an almost Hermitian manifold. Then this means that M_1 admits a tensor field J of type $(1,1)$ on M_1 which satisfies

$$J^2=-I,\quad g_1(JE_1,JE_2)=g_1(E_1,E_2) \tag{2.1}$$

for $E_1,E_2\in\Gamma(TM_1)$. An almost Hermitian manifold M_1 is called Kaehlerian manifold if

$$(\nabla_{E_1}J)E_2=0,\quad E_1,E_2\in\Gamma(TM_1)$$

where ∇ is the operator of Levi-Civita covariant differentiation.

Now, we will give some definitions and theorems about the concept of (horizontally) conformal submersions.

Definition 2.1 *Let (M_1,g_1) and (M_2,g_2) be two Riemannian manifolds with the dimension m_1 and m_2, respectively. A smooth map $f:(M_1,g_1)\to(M_2,g_2)$ is called horizontally weakly conformal or semi-conformal at $q\in M$ if, either*

i. $df_q=0$, *or*

ii. df_q *is surjective and there exists a number* $\Omega(q)\neq 0$ *satisfying*

$$g_2(df_qX_1,df_qX_2)=\Omega(q)g_1(X_1,X_2)$$

for $X_1,X_2\in\Gamma(\ker(df))^{\perp}$.

Here we say that a point q satisfying type (i) is a critical point and we say that a point q satisfying type (ii) is a regular point. The rank of df_q at a critical point is 0; the rank of df_q at a regular point is m_2 and f is a submersion. Also, the number $\Omega(q)$ is called the square dilation. Its square root $\lambda(q) = \sqrt{\Omega(q)}$ is called the dilation. The map f is called horizontally weakly conformal or semi-conformal on M_1 if it is horizontally weakly conformal at every point of M_1. It is said to be a (horizontally) conformal submersion if f has no critical point.

Let $f : M_1 \to M_2$ be a submersion. A vector field X_1 on M_1 is called a basic vector field if $X_1 \in \Gamma\left((\ker f_*)^{\perp}\right)$ and f-related with a vector field X_2 on M_2 i.e $f_*(X_{1q}) = X_{2f(q)}$ for $q \in M_1$.

The two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ are given by the formulas

$$\mathcal{T}(E_1, E_2) = \mathcal{T}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2 \tag{2.2}$$

$$\mathcal{A}(E_1, E_2) = \mathcal{A}_{E_1}E_2 = \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2 + \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 \tag{2.3}$$

for $E_1, E_2 \in \Gamma(TM_1)$ [14].

Note that a Riemannian submersion $f : M_1 \longrightarrow M_2$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically.

Considering the equations (2.2) and (2.3), one can write

$$\nabla_{U_1}U_2 = \mathcal{T}_{U_1}U_2 + \bar{\nabla}_{U_1}U_2 \tag{2.4}$$

$$\nabla_{U_1}X_1 = \mathcal{H}\nabla_{U_1}X_1 + \mathcal{T}_{U_1}X_1 \tag{2.5}$$

$$\nabla_{X_1}U_1 = \mathcal{A}_{X_1}U_1 + \mathcal{V}\nabla_{X_1}U_1 \tag{2.6}$$

$$\nabla_{X_1}X_2 = \mathcal{H}\nabla_{X_1}X_2 + \mathcal{A}_{X_1}X_2 \tag{2.7}$$

for $X_1, X_2 \in \Gamma\left((\ker f_*)^{\perp}\right)$ and $U_1, U_2 \in \Gamma(\ker f_*)$, where $\bar{\nabla}_{U_1}U_2 = \mathcal{V}\nabla_{U_1}U_2$. Then we can easily see that $\mathcal{T}_{U_1}$ and $\mathcal{A}_{X_1}$ are skew-symmetric, i.e. $g_1(\mathcal{A}_{X_1}E_1, E_2) = -g_1(E_1, \mathcal{A}_{X_1}E_2)$ and $g_1(\mathcal{T}_{U_1}E_1, E_2) = -g_1(E_1, \mathcal{T}_{U_1}E_2)$ for any $E_1, E_2 \in \Gamma(TM_1)$. For the special case where f is horizontally conformal, the following proposition can be given:

Proposition 2.2 *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a horizontally conformal submersion with dilation λ and $X_1, X_2 \in \Gamma\left((\ker f_*)^{\perp}\right)$, then*

$$\mathcal{A}_{X_1}X_2 = \frac{1}{2}\left(\mathcal{V}[X_1, X_2] - \lambda^2 g_1(X_1, X_2)\, grad_{\mathcal{V}}\left(\frac{1}{\lambda^2}\right)\right). \tag{2.8}$$

Let $f : (M_1, g_1) \to (M_2, g_2)$ be a smooth map between (M_1, g_1) and (M_2, g_2) Riemannian manifolds. Then the second fundamental form of f is given by

$$(\nabla f_*)(E_1, E_2) = \nabla^f_{E_1} f_*(E_2) - f_*\left(\nabla^{M_1}_{E_1}E_2\right) \tag{2.9}$$

for any $E_1, E_2 \in \Gamma(TM_1)$. It is known that the second fundamental form of f is symmetric [8].

The smoooth map f is called a totally geodesic map if $(\nabla f_*)(E_1, E_2) = 0$ for $E_1, E_2 \in \Gamma(TM)$ [8].

Lemma 2.3 *Suppose that $f : M_1 \to M_2$ is a horizontally conformal submersion. Then for $X_1, X_2 \in \Gamma\left((\ker f_*)^\perp\right)$ and $U_1, U_2 \in \Gamma(\ker f_*)$, we have*

i. $(\nabla f_*)(X_1, X_2) = X_1(\ln\lambda) f_* X_2 + X_2(\ln\lambda) f_* X_1 - g_1(X_1, X_2) f_*(\nabla \ln\lambda)$,

ii. $(\nabla f_*)(U_1, U_2) = -f_*(\mathcal{T}_{U_1} U_2)$,

iii. $(\nabla f_*)(X_1, U_1) = -f_*\left(\nabla^{M_1}_{X_1} U_1\right) = -f_*(\mathcal{A}_{X_1} U_1)$.

We assume that g is a Riemannian metric tensor on the manifold $M = M_1 \times M_2$ and the canonical foliations D_{M_1} and D_{M_2} intersect vertically everywhere. Then g is the metric tensor of a usual product of Riemannian manifold if and only if D_{M_1} and D_{M_2} are totally geodesic foliations.

Now we recall the following curvature relations for a conformal submersion from [17];

Theorem 2.4 *Suppose that (M_1, g_1), (M_2, g_2) are two Riemannian manifolds with the corresponding curvature tensors R and $\hat{R}$, respectively. Let $f : M_1 \to M_2$ be a horizontally conformal submersion with dilation $\lambda : M_1 \to \mathbb{R}^+$ and R^* the curvature tensor of the fibers of f. Then*

$$\begin{aligned} g_1(R(U_1, U_2)U_3, U_4) = {} & g_1(R^*(U_1, U_2)U_3, U_4) + g_1(\mathcal{T}_{U_1} U_3, \mathcal{T}_{U_2} U_4) \\ & - g_1(\mathcal{T}_{U_2} U_3, \mathcal{T}_{U_1} U_4), \end{aligned} \tag{2.10}$$

$$g_1(R(U_1, U_2)U_3, X_1) = g_1((\nabla_{U_1}\mathcal{T})_{U_2} U_3, X_1) - g_1((\nabla_{U_2}\mathcal{T})_{U_1} U_3, X_1), \tag{2.11}$$

$$\begin{aligned} g_1(R(U_1, X_1)X_2, U_2) = {} & g_1\left((\nabla_{U_1}\mathcal{A})_{X_1} X_2, U_2\right) + g_1(\mathcal{A}_{X_1} U_1, \mathcal{A}_{X_2} U_2) \\ & - g_1\left((\nabla_{X_1}\mathcal{T})_{U_1} X_2, U_2\right) - g_1(\mathcal{T}_{U_2} X_2, \mathcal{T}_{U_1} X_1) \\ & + \lambda^2 g(\mathcal{A}_{X_1} X_2, U_1)\, g_1(U_2, grad_{\mathcal{V}}(\frac{1}{\lambda^2})) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} g_1(R(X_1, X_2)X_3, X_4) = {} & \frac{1}{\lambda^2} g_2\left(\hat{R}(f_* X_1, f_* X_2) f_* X_3, f_* X_4\right) + \frac{1}{4}[g_1(\mathcal{V}[X_1, X_3], \mathcal{V}[X_2, X_4]) \\ & - g_1(\mathcal{V}[X_2, X_3], \mathcal{V}[X_1, X_4]) + 2 g_1(\mathcal{V}[X_1, X_2], \mathcal{V}[X_3, X_4])] \\ & + \frac{\lambda^2}{2}[g_1(X_1, X_3)\, g_1(\nabla_{X_2} grad(\frac{1}{\lambda^2}), X_4) - g_1(X_2, X_3)\, g_1(\nabla_{X_1} grad(\frac{1}{\lambda^2}), X_4) \\ & + g_1(X_2, X_4)\, g_1(\nabla_{X_1} grad(\frac{1}{\lambda^2}), X_3) - g_1(X_1, X_4)\, g_1(\nabla_{X_2} grad(\frac{1}{\lambda^2}), X_3)] \\ & + \frac{\lambda^4}{4}[(g_1(X_1, X_4)\, g_1(X_2, X_3) - g_1(X_2, X_4)\, g_1(X_1, X_3)) \|grad(\frac{1}{\lambda^2})\|^2 \\ & + g_1(X_1(\frac{1}{\lambda^2})X_2 - X_2(\frac{1}{\lambda^2})X_1, X_4(\frac{1}{\lambda^2})X_3 - X_3(\frac{1}{\lambda^2})X_4)] \end{aligned} \tag{2.13}$$

for any vertical vector fields U_1, U_2, U_3, U_4 and horizontal vector fields X_1, X_2, X_3, X_4.

3. Conformal bi-slant submersions

Definition 3.1 *Let (M_1, g_1, J) be an almost Hermitian manifold and (M_2, g_2) a Riemannian manifold. A horizontal conformal submersion $f : M_1 \longrightarrow M_2$ is called a conformal bi-slant submersion if D and $\bar{D}$ are slant distributions with the slant angles θ and $\bar{\theta}$, respectively, such that $\ker f_* = D \oplus \bar{D}$. f is called proper if its slant angles satisfy $\theta, \bar{\theta} \neq 0, \frac{\pi}{2}$.*

If we denote the dimension of D and $\bar{D}$ by m and $\bar{m}$, respectively, then we have the following particular cases.

i. If $m = 0$ and $\bar{\theta} = \frac{\pi}{2}$, then f is a conformal anti-invariant submersion,

ii. If $m, \bar{m} \neq 0$, $\theta = 0$ and $\bar{\theta} = \frac{\pi}{2}$, then f is a conformal semi-invariant submersion,

iii. If $m, \bar{m} \neq 0$, $\theta = 0$ and $0 < \bar{\theta} < \frac{\pi}{2}$, then f is a conformal semi-slant submersion,

iv. If $m, \bar{m} \neq 0$, $\theta = \frac{\pi}{2}$ and $0 < \bar{\theta} < \frac{\pi}{2}$, then f is a conformal hemi-slant submersion.

We now give a example of a proper conformal bi-slant submersion.

Example 3.2 *We consider the compatible almost complex structure J_ω on $\mathbb{R}^8$ such that*

$$J_\omega = (\cos\omega) J_1 + (\sin\omega) J_2, \ \ 0 < \omega \leq \frac{\pi}{2}$$

where

$$J_1 (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7)$$
$$J_2 (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6)$$

Consider a submersion $f : \mathbb{R}^8 \to \mathbb{R}^4$ defined by

$$f (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \pi^5 \left(\frac{x_1 - x_3}{\sqrt{2}}, x_4, \frac{x_5 - x_6}{\sqrt{2}}, x_7 \right)$$

Then it follows that

$$D = span\{U_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \right), U_2 = \frac{\partial}{\partial x_2}\}$$
$$\bar{D} = span\{U_3 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \right), U_4 = \frac{\partial}{\partial x_8}\}$$

Thus, f is a conformal bi-slant submersion with θ and $\bar{\theta}$ such that $\cos\theta = \frac{1}{\sqrt{2}} \cos\omega$ and $\cos\bar{\theta} = \frac{1}{\sqrt{2}} \sin\omega$.

Suppose that f is a conformal bi-slant submersion from a almost Hermitian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). For $U_1 \in \Gamma(\ker f_*)$, we have

$$U_1 = PU_1 + QU_1, \tag{3.1}$$

where $PU_1 \in \Gamma(D_1)$ and $QU_1 \in \Gamma(D_2)$.
Also, for $U_1 \in \Gamma(\ker f_*)$, we write

$$JU_1 = \xi U_1 + \eta U_1, \tag{3.2}$$

where $\xi U_1 \in \Gamma(\ker f_*)$ and $\eta U_1 \in \Gamma\left((\ker f_*)^{\perp}\right)$.

For $X_1 \in \Gamma\left((\ker f_*)^{\perp}\right)$, we have

$$JX_1 = \mathcal{B}X_1 + \mathcal{C}X_1, \tag{3.3}$$

where $\mathcal{B}X_1 \in \Gamma(\ker f_*)$ and $\mathcal{C}X_1 \in \Gamma\left((\ker f_*)^{\perp}\right)$.

The horizontal distribution $(\ker f_*)^{\perp}$ is decompesed as

$$(\ker f_*)^{\perp} = \eta D_1 \oplus \eta D_2 \oplus \mu,$$

where μ is the complementary distribution to $\eta D_1 \oplus \eta D_2$ in $(\ker f_*)^{\perp}$.

Considering Definition 3.1, we can give the following result that we will use throughout the article.

Theorem 3.3 *Suppose that f is a conformal bi-slant submersion from an almost Hermitian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then we have*

i) $\xi^2 U_1 = -\left(\cos^2\theta\right) U_1$ *for* $U_1 \in \Gamma(D)$

ii) $\xi^2 V_1 = -\left(\cos^2\bar{\theta}\right) V_1$ *for* $V_1 \in \Gamma\left(\bar{D}\right)$

Proof The proof of this theorem is similar to slant immersions [11, 12]. □

Theorem 3.4 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then*

i) the distribution D is integrable if and only if

$$\begin{aligned}\lambda^{-2} g_2\left(\nabla f_*\left(U_1, \eta U_2\right), f_* \eta V_1\right) = & g_1\left(\mathcal{T}_{U_2} \eta \xi U_1 - \mathcal{T}_{U_1} \eta \xi U_2, V_1\right) \\ & + g_1\left(\mathcal{T}_{U_1} \eta U_2 - \mathcal{T}_{U_2} \eta U_1, \xi V_1\right) \\ & + \lambda^{-2} g_2\left(\nabla f_*\left(U_2, \eta U_1\right), f_* \eta V_1\right),\end{aligned}$$

ii) the distribution $\bar{D}$ is integrable if and only if

$$\begin{aligned}\lambda^{-2} g_2\left(\nabla f_*\left(V_1, \eta V_2\right), f_* \eta U_1\right) = & g_1\left(\mathcal{T}_{V_2} \eta \xi V_1 - \mathcal{T}_{V_1} \eta \xi V_2, U_1\right) \\ & + g_1\left(\mathcal{T}_{V_1} \eta V_2 - \mathcal{T}_{V_2} \eta V_1, \xi U_1\right) \\ & + \lambda^{-2} g_2\left(\nabla f_*\left(V_2, \eta V_1\right), f_* \eta U_1\right),\end{aligned}$$

where $U_1, U_2 \in \Gamma(D)$, $V_1, V_2 \in \Gamma\left(\bar{D}\right)$.

Proof *i)* From $U_1, U_2 \in \Gamma(D)$ and $V_1 \in \Gamma\left(\bar{D}\right)$, we have

$$g_1([U_1,U_2],V_1) = g_1(\nabla_{U_1}\xi U_2, JV_1) + g_1(\nabla_{U_1}\eta U_2, JV_1) - g_1(\nabla_{U_2}\xi U_1, JV_1) - g_1(\nabla_{U_2}\eta U_1, JV_1).$$

Considering Theorem 3.3, we reach

$$\sin^2\theta g_1([U_1,U_2],V_1) = -g_1(\nabla_{U_1}\eta\xi U_2, V_1) + g_1(\nabla_{U_1}\eta U_2, JV_1) + g_1(\nabla_{U_2}\eta\xi U_1, V_1) - g_1(\nabla_{U_2}\eta U_1, JV_1).$$

By using the equations (2.5) and (2.9), we obtain

$$\begin{aligned}\sin^2\theta g_1([U_1,U_2],V_1) =& g_1(\mathcal{T}_{U_2}\eta\xi U_1 - \mathcal{T}_{U_1}\eta\xi U_2, V_1) + g_1(\mathcal{T}_{U_1}\eta U_2 - \mathcal{T}_{U_2}\eta U_1, \xi V_1)\\ &- \lambda^{-2} g_2(\nabla f_*(U_1,\eta U_2), f_*\eta V_1)\\ &+ \lambda^{-2} g_2(\nabla f_*(U_2,\eta U_1), f_*\eta V_1).\end{aligned}$$

The proof of *(ii)* can be made by applying similar calculations. □

Theorem 3.5 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then the distribution D defines a totally geodesic foliation if and only if*

$$\lambda^{-2} g_2(\nabla f_*(\eta U_2, U_1), f_*\eta V_1) = -g_1(\mathcal{T}_{U_1}\eta\xi U_2, V_1) + g_1(\mathcal{T}_{U_1}\eta U_2, \xi V_1). \tag{3.4}$$

and

$$\begin{aligned}\lambda^{-2} g_2\left(\nabla^f_{X_1} f_*\eta U_1, f_*\eta U_2\right) =& -\sin^2\theta g_1([U_1,X_1], U_2) + g_1(\mathcal{A}_{X_1}\eta\xi U_1, U_2)\\ &+ g_1(grad(\ln\lambda), X_1)\, g_1(\eta U_1, \eta U_2)\\ &+ g_1(grad(\ln\lambda), \eta U_1)\, g_1(X_1, \eta U_2)\\ &- g_1(grad(\ln\lambda), \eta U_2)\, g_1(X_1, \eta U_1)\\ &- g_1(\mathcal{A}_{X_1}\eta U_1, \xi U_2)\end{aligned} \tag{3.5}$$

where $U_1, U_2 \in \Gamma(D)$, $V_1 \in \Gamma\left(\bar{D}\right)$ and $X_1 \in \Gamma\left((\ker f_*)^\perp\right)$.

Proof

For $U_1, U_2 \in \Gamma(D)$ and $V_1 \in \Gamma\left(\bar{D}\right)$, we have

$$g_1(\nabla_{U_1}U_2, V_1) = -g_1\left(\nabla_{U_1}\xi^2 U_2, V_1\right) - g_1(\nabla_{U_1}\eta\xi U_2, V_1) + g_1(\nabla_{U_1}\eta U_2, JV_1).$$

Thus, from (2.5), we can write

$$\sin^2\theta g_1(\nabla_{U_1}U_2, V_1) = -g_1(\mathcal{T}_{U_1}\eta\xi U_2, V_1) + g_1(\mathcal{T}_{U_1}\eta U_2, \xi V_1) + g_1(\mathcal{H}\nabla_{U_1}\eta U_2, \eta V_1).$$

Using (2.9), we obtain

$$\sin^2\theta g_1\left(\nabla_{U_1}U_2,V_1\right)=-\,g_1\left(\mathcal{T}_{U_1}\eta\xi U_2,V_1\right)+g_1\left(\mathcal{T}_{U_1}\eta U_2,\xi V_1\right)$$
$$-\,\lambda^{-2}g_2\left(\nabla f_*\left(\eta U_2,U_1\right),f_*\eta V_1\right).$$

which is the first equation in Theorem 3.5.

On the other hand, any $U_1,U_2\in\Gamma(D)$ and $X_1\in\Gamma\left(\left(\ker f_*\right)^{\perp}\right)$, we can write

$$g_1\left(\nabla_{U_1}U_2,X_1\right)=-\,g_1\left(\left[U_1,X_1\right],U_2\right)-g_1\left(\nabla_{X_1}U_1,U_2\right)$$
$$=-\,g_1\left(\left[U_1,X_1\right],U_2\right)+g_1\left(\nabla_{X_1}J\xi U_1,U_2\right)-g_1\left(\nabla_{X_1}\eta U_1,JU_2\right).$$

Using Theorem 3.3, we reach the following equation:

$$g_1\left(\nabla_{U_1}U_2,X_1\right)=-\,g_1\left(\left[U_1,X_1\right],U_2\right)-\cos^2\theta g_1\left(\nabla_{X_1}U_1,U_2\right)$$
$$+\,g_1\left(\nabla_{X_1}\eta\xi U_1,U_2\right)-g_1\left(\nabla_{X_1}\eta U_1,JU_2\right)$$

Then by (2.7), we get

$$\sin^2\theta g_1\left(\nabla_{U_1}U_2,X_1\right)=-\sin^2\theta g_1\left(\left[U_1,X_1\right],U_2\right)+g_1\left(\mathcal{A}_{X_1}\eta\xi U_1,U_2\right)$$
$$-\,g_1\left(\mathcal{A}_{X_1}\eta U_1,\xi U_2\right)-g_1\left(\mathcal{H}\nabla_{X_1}\eta U_1,\eta U_2\right)$$

Hence, from Lemma 2.3, we have

$$\begin{aligned}\sin^2\theta g_1\left(\nabla_{U_1}U_2,X_1\right)=&-\sin^2\theta g_1\left(\left[U_1,X_1\right],U_2\right)+g_1\left(\mathcal{A}_{X_1}\eta\xi U_1,U_2\right)\\&-g_1\left(\mathcal{A}_{X_1}\eta U_1,\xi U_2\right)-\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta U_1,f_*\eta U_2\right)\\&+g_1\left(grad(\ln\lambda),X_1\right)g_1\left(\eta U_1,\eta U_2\right)\\&+g_1\left(grad(\ln\lambda),\eta U_1\right)g_1\left(X_1,\eta U_2\right)\\&-g_1\left(grad(\ln\lambda),\eta U_2\right)g_1\left(X_1,\eta U_1\right)\end{aligned}$$

This completes the proof. □

Theorem 3.6 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1,g_1,J) onto a Riemannian manifold (M_2,g_2) with slant angles $\theta,\bar{\theta}$. Then the distribution $\bar{D}$ defines a totally geodesic foliation if and only if*

$$\lambda^{-2}g_2\left(\nabla f_*\left(\eta V_2,V_1\right),f_*\eta U_1\right)=-g_1\left(\mathcal{T}_{V_1}\eta\xi V_2,U_1\right)+g_1\left(\mathcal{T}_{V_1}\eta V_2,\xi U_1\right)\tag{3.6}$$

and

$$\begin{aligned}\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta V_1,f_*\eta V_2\right)=&-\sin^2\bar{\theta}g_1\left(\left[V_1,X_1\right],V_2\right)+g_1\left(\mathcal{A}_{X_1}\eta\xi V_1,V_2\right)\\&+g_1\left(grad(\ln\lambda),X_1\right)g_1\left(\eta V_1,\eta V_2\right)\\&+g_1\left(grad(\ln\lambda),\eta V_1\right)g_1\left(X_1,\eta V_2\right)\\&-g_1\left(grad(\ln\lambda),\eta V_2\right)g_1\left(X_1,\eta V_1\right)\\&-g_1\left(\mathcal{A}_{X_1}\eta V_1,\xi V_2\right),\end{aligned}\tag{3.7}$$

where $U_1 \in \Gamma(D)$, $V_1, V_2 \in \Gamma(\bar{D})$ *and* $X_1 \in \Gamma\left((\ker f_*)^\perp\right)$.

Proof The proof of this theorem is similar to the proof of Theorem 3.5. □

Theorem 3.7 *Suppose that* f *is a proper conformal bi-slant submersion from a Kaehlerian manifold* (M_1, g_1, J) *onto a Riemannian manifold* (M_2, g_2) *with slant angles* $\theta, \bar{\theta}$*. Then, the vertical distribution* $(\ker f_*)$ *is a locally product* $M_D \times M_{\bar{D}}$ *if and only if the equations (3.4)–(3.7) are hold where* M_D *and* $M_{\bar{D}}$ *are integral manifolds of the distributions* D *and* $\bar{D}$*, respectively.*

Theorem 3.8 *Suppose that* f *is a proper conformal bi-slant submersion from a Kaehlerian manifold* (M_1, g_1, J) *onto a Riemannian manifold* (M_2, g_2) *with slant angles* $\theta, \bar{\theta}$*. Then the distribution* $(\ker f_*)^\perp$ *defines a totally geodesic foliation if and only if*

$$\begin{aligned}\lambda^{-2} g_2\left(\nabla^f_{X_1} f_*\eta U_1, f_*CX_2\right) =& - g_1\left(\mathcal{A}_{X_1}\eta U_1, BX_2\right) + \lambda^{-2} g_2\left(\nabla^f_{X_1} f_*\eta\xi U_1, f_*X_2\right)\\ &- g_1(grad \ln\lambda, X_1)\, g_1(\eta\xi U_1, X_2)\\ &- g_1(grad \ln\lambda, \eta\xi U_1)\, g_1(X_1, X_2)\\ &+ g_1(X_1, \eta\xi U_1)\, g_1(grad \ln\lambda, X_2)\\ &+ g_1(grad \ln\lambda, \eta U_1)\, g_1(X_1, CX_2)\\ &- g_1(X_1, \eta U_1)\, g_1(grad \ln\lambda, CX_2)\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\lambda^{-2} g_2\left(\nabla^f_{X_1} f_*\eta V_1, f_*CX_2\right) =& - g_1\left(\mathcal{A}_{X_1}\eta V_1, BX_2\right) + \lambda^{-2} g_2\left(\nabla^f_{X_1} f_*\eta\xi V_1, f_*X_2\right)\\ &- g_1(grad \ln\lambda, X_1)\, g_1(\eta\xi V_1, X_2)\\ &- g_1(grad \ln\lambda, \eta\xi V_1)\, g_1(X_1, X_2)\\ &+ g_1(X_1, \eta\xi V_1)\, g_1(grad \ln\lambda, X_2)\\ &+ g_1(grad \ln\lambda, \eta V_1)\, g_1(X_1, CX_2)\\ &- g_1(X_1, \eta V_1)\, g_1(grad \ln\lambda, CX_2),\end{aligned} \tag{3.9}$$

where $X_1, X_2 \in \Gamma(\ker f_*)^\perp$, $U_1 \in \Gamma(D)$, *and* $V_1 \in \Gamma(\bar{D})$.

Proof For $X_1, X_2 \in \Gamma(\ker f_*)^\perp$ and $U_1 \in \Gamma(D)$, we can write

$$g_1(\nabla_{X_1} X_2, U_1) = -g_1(\nabla_{X_1}\xi U_1, JX_2) - g_1(\nabla_{X_1}\eta U_1, JX_2).$$

From Theorem 3.3, we have

$$\begin{aligned}g_1(\nabla_{X_1} X_2, U_1) =& -\cos^2\theta g_1(\nabla_{X_1} U_1, X_2) + g_1(\nabla_{X_1}\eta\xi U_1, X_2)\\ &- g_1(\nabla_{X_1}\eta U_1, JX_2).\end{aligned}$$

By using the equation (2.7), we derive

$$\begin{aligned}\sin^2\theta g_1\left(\nabla_{X_1}X_2,U_1\right)=&g_1\left(\mathcal{H}\nabla_{X_1}\eta\xi U_1,X_2\right)-g_1\left(\mathcal{H}\nabla_{X_1}\eta U_1,CX_2\right)\\&-g_1\left(\nabla_{X_1}\eta U_1,BX_2\right).\end{aligned}$$

Since f is a conformal bi-slant submersion, it follows from Lemma 2.3 that

$$\begin{aligned}\sin^2\theta g_1\left(\nabla_{X_1}X_2,U_1\right)=&-g_1\left(\mathcal{A}_{X_1}\eta U_1,BX_2\right)+\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta\xi U_1,f_*X_2\right)\\&-g_1\left(grad\ln\lambda,X_1\right)g_1\left(\eta\xi U_1,X_2\right)\\&-g_1\left(grad\ln\lambda,\eta\xi U_1\right)g_1\left(X_1,X_2\right)\\&+g_1\left(X_1,\eta\xi U_1\right)g_1\left(grad\ln\lambda,X_2\right)\\&-\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta U_1,f_*CX_2\right)\\&+g_1\left(grad\ln\lambda,\eta U_1\right)g_1\left(X_1,CX_2\right)\\&-g_1\left(X_1,\eta U_1\right)g_1\left(grad\ln\lambda,CX_2\right).\end{aligned}$$

Thus, we have the first desired equation. Similarly, for $X_1,X_2\in\Gamma\left((\ker f_*)^\perp\right)$ and $V_1\in\left(\bar{D}\right)$, we find

$$\begin{aligned}\sin^2\bar{\theta}g_1\left(\nabla_{X_1}X_2,V_1\right)=&-g_1\left(\mathcal{A}_{X_1}\eta V_1,BX_2\right)+\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta\xi V_1,f_*X_2\right)\\&-g_1\left(grad\ln\lambda,X_1\right)g_1\left(\eta\xi V_1,X_2\right)\\&-g_1\left(grad\ln\lambda,\eta\xi V_1\right)g_1\left(X_1,X_2\right)\\&+g_1\left(X_1,\eta\xi V_1\right)g_1\left(grad\ln\lambda,X_2\right)\\&-\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta V_1,f_*CX_2\right)\\&+g_1\left(grad\ln\lambda,\eta V_1\right)g_1\left(X_1,CX_2\right)\\&-g_1\left(X_1,\eta V_1\right)g_1\left(grad\ln\lambda,CX_2\right)\end{aligned}$$

which completes the proof. □

Theorem 3.9 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1,g_1,J) onto a Riemannian manifold (M_2,g_2) with slant angles $\theta,\bar{\theta}$. Then the distribution $(\ker f_*)$ defines a totally geodesic foliation on M_1 if and only if*

$$\begin{aligned}\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta U_1,f_*\eta U_2\right)=&\left(\cos^2\theta-\cos^2\bar{\theta}\right)g_1\left(\nabla_{X_1}QU_1,U_2\right)-g_1\left(\mathcal{A}_{X_1}U_2,\eta\xi U_1\right)\\&+g_1\left(\mathcal{A}_{X_1}\xi U_2,\eta U_1\right)-\sin^2\theta g_1\left(\left[U_1,X_1\right],U_2\right)\\&-g_1\left(X_1,\eta U_1\right)g_1\left(grad\ln\lambda,\eta U_2\right)\\&+g_1\left(grad\ln\lambda,X_1\right)g_1\left(\eta U_1,\eta U_2\right)\\&+g_1\left(grad\ln\lambda,\eta U_1\right)g_1\left(X_1,\eta U_2\right),\end{aligned}\tag{3.10}$$

where $X_1\in\Gamma\left((\ker f_*)^\perp\right)$ and $U_1,U_2\in\Gamma(\ker f_*)$.

Proof Given $X_1 \in \Gamma\left((\ker f_*)^\perp\right)$ and $U_1, U_2 \in (\ker f_*)$. Then we obtain

$$g_1(\nabla_{U_1}U_2, X_1) = -g_1([U_1, X_1], U_2) + g_1(J\nabla_{X_1}\xi U_1, U_2) - g_1(\nabla_{X_1}\eta U_1, JU_2).$$

By using Theorem 3.3, we have

$$\begin{aligned} g_1(\nabla_{U_1}U_2, X_1) = &- g_1([U_1, X_1], U_2) - \cos^2\theta g_1(\nabla_{X_1}PU_1, U_2) \\ &- \cos^2\bar{\theta} g_1(\nabla_{X_1}QU_1, U_2) + g_1(\nabla_{X_1}\eta\xi U_1, U_2) \\ &- g_1(\nabla_{X_1}\eta U_1, \xi U_2) - g_1(\nabla_{X_1}\eta U_1, \eta U_2). \end{aligned}$$

Then we obtain

$$\begin{aligned} \sin^2\theta g_1(\nabla_{U_1}U_2, X_1) = &\left(\cos^2\theta - \cos^2\bar{\theta}\right) g_1(\nabla_{X_1}QU_1, U_2) \\ &+ g_1(\nabla_{X_1}\eta\xi U_1, U_2) - \sin^2\theta g_1([U_1, X_1], U_2) \\ &- g_1(\nabla_{X_1}\eta U_1, \xi U_2) - g_1(\nabla_{X_1}\eta U_1, \eta U_2). \end{aligned}$$

From the equation (2.6) and Lemma 2.3, we obtain

$$\begin{aligned} \sin^2\theta g_1(\nabla_{U_1}U_2, X_1) = &\left(\cos^2\theta - \cos^2\bar{\theta}\right) g_1(\nabla_{X_1}QU_1, U_2) - g_1(\mathcal{A}_{X_1}U_2, \eta\xi U_1) \\ &- \sin^2\theta g_1([U_1, X_1], U_2) + g_1(\mathcal{A}_{X_1}\xi U_2, \eta U_1) \\ &+ g_1(grad \ln\lambda, X_1) g_1(\eta U_1, \eta U_2) \\ &+ g_1(grad \ln\lambda, \eta U_1) g_1(X_1, \eta U_2) \\ &- g_1(X_1, \eta U_1) g_1(grad \ln\lambda, \eta U_2) \\ &- \lambda^{-2} g_2\left(\nabla^f_{X_1} f_*\eta U_1, f_*\eta U_2\right). \end{aligned}$$

Using the above equation, the desired equality is achieved. □

Theorem 3.10 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then, the total space M_1 is a locally product $M_{1D} \times M_{1\bar{D}} \times M_{1(\ker f_*)^\perp}$ if and only if the equations (3.4)–(3.9) hold where M_{1D}, $M_{1\bar{D}}$, and $M_{1(\ker f_*)^\perp}$ are integral manifolds of the distributions D, $\bar{D}$, and $(\ker f_*)^\perp$, respectively.*

Theorem 3.11 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then, the total space M_1 is a locally product $M_{1\ker f_*} \times M_{1(\ker f_*)^\perp}$ if and only if the equations (3.8)–(3.10) hold where $M_{1\ker f_*}$ and $M_{1(\ker f_*)^\perp}$ are integral manifolds of the distributions $\ker f_*$ and $(\ker f_*)^\perp$, respectively.*

Theorem 3.12 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then f is a totally geodesic map if and only if* *i.*

$$\lambda^{-2}g_2\left(\nabla^f_{\eta U_2}f_*\eta U_1, f_*JCX_1\right) = \left(\cos^2\theta - \cos^2\bar{\theta}\right)g_1\left(\mathcal{T}_{U_1}QU_2, X_1\right) + \lambda^{-2}g_2\left(\nabla f_*\left(\xi U_1, \eta U_2\right), f_*JCX_1\right) - g_1\left(\eta U_1, \eta U_2\right)g_1\left(grad\ln\lambda, JCX_1\right) - \lambda^{-2}g_2\left(\nabla f_*\left(U_1, \eta\xi U_2\right), f_*X_1\right) - g_1\left(\mathcal{T}_{U_1}\eta U_2, BX_1\right),$$

ii.

$$\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta U_1, f_*CX_2\right) = \left(\cos^2\theta - \cos^2\bar{\theta}\right)g_1\left(\mathcal{A}_{X_1}QU_1, X_2\right) + \lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta\xi U_1, f_*X_2\right) - g_1\left(grad\ln\lambda, X_1\right)g_1\left(\eta\xi U_1, X_2\right) - g_1\left(grad\ln\lambda, \eta\xi U_1\right)g_1\left(X_1, X_2\right) + g_1\left(X_1, \eta\xi U_1\right)g_1\left(grad\ln\lambda, X_2\right) + g_1\left(grad\ln\lambda, \eta U_1\right)g_1\left(X_1, CX_2\right) - g_1\left(X_1, \eta U_1\right)g_1\left(grad\ln\lambda, CX_2\right) + g_1\left(\mathcal{A}_{X_1}BX_2, \eta U_1\right),$$

iii. f is a horizontally homothetic map,

where $X_1, X_2 \in \Gamma\left((\ker f_*)^\perp\right)$ and $U_1, U_2 \in \Gamma(\ker f_*)$.

Proof Given $U_1, U_2 \in \Gamma(\ker f_*)$ and $X_1 \in \Gamma\left((\ker f_*)^\perp\right)$, we write

$$\lambda^{-2}g_2\left(\nabla f_*(U_1, U_2), f_*X_1\right) = -\lambda^{-2}g_2\left(f_*\left(\nabla_{U_1}U_2\right), f_*X_1\right).$$

From Theorem 3.3, we obtain

$$\left(\sin^2\theta\right)\lambda^{-2}g_2\left(\nabla f_*\left(U_1, U_2\right), f_*X_1\right) = \left(\cos^2\theta - \cos^2\bar{\theta}\right)g_1\left(\nabla_{U_1}QU_2, X_1\right) - g_1\left(\nabla_{U_1}\eta U_2, JX_1\right) + g_1\left(\nabla_{U_1}\eta\xi U_2, X_1\right).$$

Considering (2.4), (2.5), and Lemma 2.3 we find

$$\left(\sin^2\theta\right)\lambda^{-2}g_2\left(\nabla f_*\left(U_1, U_2\right), f_*X_1\right) = \left(\cos^2\theta - \cos^2\bar{\theta}\right)g_1\left(\mathcal{T}_{U_1}QU_2, X_1\right) + \lambda^{-2}g_2\left(\nabla f_*\left(\xi U_1, \eta U_2\right), f_*JCX_1\right) - g_1\left(\eta U_1, \eta U_2\right)g_1\left(grad\ln\lambda, JCX_1\right) - \lambda^{-2}g_2\left(\nabla^f_{\eta U_2}f_*\eta U_1, f_*JCX_1\right) - \lambda^{-2}g_2\left(\nabla f_*\left(U_1, \eta\xi U_2\right), f_*X_1\right) - g_1\left(\mathcal{T}_{U_1}\eta U_2, BX_1\right).$$

Therefore, we obtain the first equation of Theorem 3.12.

On the other hand, for $X_1, X_2 \in \Gamma\left((\ker f_*)^{\perp}\right)$ and $U_1 \in \Gamma(\ker f_*)$, we can write

$$\begin{aligned}\left(\sin^2\theta\right)\lambda^{-2}g_2\left(\nabla f_*\left(U_1,X_1\right),f_*X_2\right) &= \left(\cos^2\theta-\cos^2\bar{\theta}\right)g_1\left(\nabla_{X_1}QU_1,X_2\right)\\ &+g_1\left(\nabla_{X_1}\eta\xi U_1,X_2\right)+g_1\left(\nabla_{X_1}\eta U_1,BX_2\right)\\ &-g_1\left(\nabla_{X_1}\eta U_1,CX_2\right).\end{aligned}$$

By using the equation (2.6) and Lemma 2.3, we obtain

$$\begin{aligned}\left(\sin^2\theta\right)\lambda^{-2}g_2\left(\nabla f_*\left(U_1,X_1\right),f_*X_2\right) &= \left(\cos^2\theta-\cos^2\bar{\theta}\right)g_1\left(\mathcal{A}_{X_1}QU_1,X_2\right)\\ &+\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta\xi U_1,f_*X_2\right)\\ &-g_1\left(grad\ln\lambda,X_1\right)g_1\left(\eta\xi U_1,X_2\right)\\ &-g_1\left(grad\ln\lambda,\eta\xi U_1\right)g_1\left(X_1,X_2\right)\\ &+g_1\left(X_1,\eta\xi U_1\right)g_1\left(grad\ln\lambda,X_2\right)\\ &-\lambda^{-2}g_2\left(\nabla^f_{X_1}f_*\eta U_1,f_*CX_2\right)\\ &+g_1\left(grad\ln\lambda,\eta U_1\right)g_1\left(X_1,CX_2\right)\\ &-g_1\left(X_1,\eta U_1\right)g_1\left(grad\ln\lambda,CX_2\right)\\ &+g_1\left(\mathcal{A}_{X_1}BX_2,\eta U_1\right).\end{aligned}$$

Finally, we show that λ is a constant on $\Gamma\left(\eta\bar{D}\right)$. For $V_1, V_2 \in \Gamma(\bar{D})$ from Lemma 2.3, we obtain

$$(\nabla f_*)(\eta V_1,\eta V_2)=\eta V_1(\ln\lambda)f_*\eta V_2+\eta V_2(\ln\lambda)f_*\eta V_1-g_1(\eta V_1,\eta V_2)f_*(grad\ln\lambda).$$

If V_1 is written instead of V_2, we get

$$(\nabla f_*)(\eta V_1,\eta V_1)=2\eta V_1(\ln\lambda)f_*\eta V_1-g_1(\eta V_1,\eta V_1)f_*(grad\ln\lambda).$$

Applying the inner product with $f_*\eta V_1$ to the above equation,

$$2g_1(grad\ln\lambda,\eta V_1)g_2(f_*\eta V_1,f_*\eta V_1)-g_1(\eta V_1,\eta V_1)g_2(f_*(grad\ln\lambda),f_*\eta V_1)=0.$$

Then we get that λ is a constant on $\Gamma(\eta\bar{D})$. Similarly, it can be shown that λ is a constant on $\Gamma(D), \Gamma(\mu)$. □

4. Curvature relations

In the present section of the paper, we examine the sectional curvatures of the total space, the base space, and the fibers of a conformal bi-slant submersions. Let f be a conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). The sectional curvature K is defined by the following formula

$$K\left(X_1,X_2\right)=\frac{R(X_1,X_2,X_2,X_1)}{\|X_1\|^2\|X_2\|^2}$$

for any pair of nonzero orthogonal vectors X_1 and X_2 on M_1. We denote the sectional curvatures of M_1, M_2, and any fiber $f^{-1}(x)$ by K, $\hat{K}$ and K^*, respectively.

Theorem 4.1 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then we obtain the following equations*

$$\begin{aligned}
K(U_1,U_2) &= K^*(\xi U_1,\xi U_2) + \frac{1}{\lambda^2}\hat{K}(f_*\eta U_1, f_*\eta U_2) - g_1(\mathcal{T}_{\xi U_1}\xi U_1, \mathcal{T}_{\xi U_2}\xi U_2)\\
&+ \cos^4\theta\|\mathcal{T}_{U_1}PU_2\|^2 + 2\cos^2\theta\cos^2\bar{\theta}g_1(\mathcal{T}_{U_1}PU_2, \mathcal{T}_{U_1}QU_2) + \cos^4\bar{\theta}\|\mathcal{T}_{U_1}QU_2\|^2\\
&- 2\cos^2\theta g_1(\mathcal{T}_{U_1}PU_2, \eta[\xi U_2, U_1] - \mathcal{H}\nabla_{U_1}\eta\xi U_2 - \mathcal{H}\nabla_{\xi U_2}\eta U_1)\\
&- 2\cos^2\bar{\theta} g_1(\mathcal{T}_{U_1}QU_2, \eta[\xi U_2, U_1] - \mathcal{H}\nabla_{U_1}\eta\xi U_2 - \mathcal{H}\nabla_{\xi U_2}\eta U_1)\\
&+ \|\eta[\xi U_2, U_1] - \mathcal{H}\nabla_{U_1}\eta\xi U_2 - \mathcal{H}\nabla_{\xi U_2}\eta U_1\|^2 + g_1((\nabla_{\xi U_1}\mathcal{A})_{\eta U_2}\eta U_2, \xi U_1)\\
&+ \|\mathcal{A}_{\eta U_2}\xi U_1\|^2 - g_1((\nabla_{\eta U_2}\mathcal{T})_{\xi U_1}\eta U_2, \xi U_1)\\
&- \|\xi[\xi U_1, U_2] - \cos^2\theta\bar{\nabla}_{U_2}PU_1 - \cos^2\bar{\theta}\bar{\nabla}_{U_2}QU_1 + \mathcal{T}_{U_2}\eta\xi U_1 - \bar{\nabla}_{\xi U_1}\xi U_2\|^2\\
&+ g_1((\nabla_{\xi U_2}\mathcal{A})_{\eta U_1}\eta U_1, \xi U_2) + \|\mathcal{A}_{\eta U_1}\xi U_2\|^2 - g_1((\nabla_{\eta U_1}\mathcal{T})_{\xi U_2}\eta U_1, \xi U_2)\\
&- \|\xi[\xi U_2, U_1] - \cos^2\theta\bar{\nabla}_{U_1}PU_2 - \cos^2\bar{\theta}\bar{\nabla}_{U_1}QU_2 + \mathcal{T}_{U_1}\eta\xi U_2 - \bar{\nabla}_{\xi U_2}\xi U_1\|^2\\
&- \frac{3}{4}\|\mathcal{V}[\eta U_1, \eta U_2]\|^2 - \frac{\lambda^2}{2}[g_1(\eta U_1, \eta U_2)g_1(\nabla_{\eta U_2}grad(\frac{1}{\lambda^2}), \eta U_1)\\
&- g_1(\eta U_2, \eta U_2)g_1(\nabla_{\eta U_1}grad(\frac{1}{\lambda^2}), \eta U_1) + g_1(\eta U_2, \eta U_1)g_1(\nabla_{\eta U_1}grad(\frac{1}{\lambda^2}), \eta U_2)\\
&- g_1(\eta U_1, \eta U_1)g_1(\nabla_{\eta U_2}grad(\frac{1}{\lambda^2}), \eta U_2)] + \frac{\lambda^4}{4}[(g_1(\eta U_1, \eta U_1)g_1(\eta U_2, \eta U_2)\\
&- g_1(\eta U_2, \eta U_1)g_1(\eta U_1, \eta U_2))\|grad(\frac{1}{\lambda^2})\|^2 + \|\eta U_1(\frac{1}{\lambda^2})\eta U_2 - \eta U_2(\frac{1}{\lambda^2})\eta U_1\|^2] \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
K(X_1,U_1) =& K^*(BX_1,\xi U_1) + \frac{1}{\lambda^2}\hat{K}(f_*CX_1, f_*\eta U_1) - g_1(\mathcal{T}_{\xi U_1}\xi U_1, \mathcal{T}_{BX_1}BX_1) - \|\mathcal{T}_{BX_1}\eta U_1\|^2\\
&+ \cos^4\theta\|\mathcal{A}_{X_1}PU_1\|^2 + 2\cos^2\theta\cos^2\bar{\theta}g_1(\mathcal{A}_{X_1}PU_1, \mathcal{A}_{X_1}QU_1) + \cos^4\bar{\theta}\|\mathcal{A}_{X_1}QU_1\|^2\\
&- 2\cos^2\theta g_1(\mathcal{A}_{X_1}PU_1, \eta[\xi U_1, X_1] + \mathcal{H}\nabla_{X_1}\eta\xi U_1 - \mathcal{H}\nabla_{\xi U_1}CX_1)\\
&- 2\cos^2\bar{\theta} g_1(\mathcal{A}_{X_1}QU_1, \eta[\xi U_1, X_1] + \mathcal{H}\nabla_{X_1}\eta\xi U_1 - \mathcal{H}\nabla_{\xi U_1}CX_1)\\
&+ \|\eta[\xi U_1, X_1] + \mathcal{H}\nabla_{X_1}\eta\xi U_1 - \mathcal{H}\nabla_{\xi U_1}CX_1\|^2 + g_1((\nabla_{BX_1}\mathcal{A})_{\eta U_1}\eta U_1, BX_1)\\
&+ \|\mathcal{A}_{\eta U_1}BX_1\|^2 - g_1((\nabla_{\eta U_1}\mathcal{T})_{BX_1}\eta U_1, BX_1) + g_1((\nabla_{\xi U_1}\mathcal{A})_{CX_1}CX_1, \xi U_1)\\
&+ \|\mathcal{A}_{CX_1}\xi U_1\|^2 - g_1((\nabla_{CX_1}\mathcal{T})_{\xi U_1}CX_1, \xi U_1)\\
&- \|\xi[\xi U_1, X_1] - \cos^2\theta\mathcal{V}\nabla_{X_1}PU_1 - \cos^2\bar{\theta}\mathcal{V}\nabla_{X_1}QU_1 + \mathcal{A}_{X_1}\eta\xi U_1 - \bar{\nabla}_{\xi U_1}BX_1\|^2\\
&- \frac{3}{4}\|\mathcal{V}[CX_1, \eta U_1]\|^2 - \frac{\lambda^2}{2}[g_1(\eta U_1, \eta U_1)g_1(\nabla_{CX_1}grad(\frac{1}{\lambda^2}), CX_1)\\
&+ g_1(CX_1, CX_1)g_1(\nabla_{\eta U_1}grad(\frac{1}{\lambda^2}), \eta U_1)] + \frac{\lambda^4}{4}[g_1(CX_1, CX_1)g_1(\eta U_1, \eta U_1)\|grad(\frac{1}{\lambda^2})\|^2\\
&+ \|CX_1(\frac{1}{\lambda^2})\eta U_1 - \eta U_1(\frac{1}{\lambda^2})CX_1\|^2] \qquad (4.2)
\end{aligned}$$

$$
\begin{aligned}
K(X_1,X_2) =& K^*(BX_1,BX_2)+\frac{1}{\lambda^2}\hat{K}(f_*CX_1,f_*CX_2)+\|\mathcal{T}_{BX_1}BX_2\|^2-g_1(\mathcal{T}_{BX_2}BX_2,\mathcal{T}_{BX_1}BX_1)\\
&+g_1((\nabla_{BX_1}\mathcal{A})_{CX_2}CX_2,BX_1)+\|\mathcal{A}_{CX_2}BX_1\|^2-g_1((\nabla_{CX_2}\mathcal{T})_{BX_1}CX_2,BX_1)\\
&-\|\mathcal{T}_{BX_1}CX_2\|^2+g_1((\nabla_{BX_2}\mathcal{A})_{CX_1}CX_1,BX_2)+\|\mathcal{A}_{CX_1}BX_2\|^2\\
&-g_1((\nabla_{CX_1}\mathcal{T})_{BX_2}CX_1,BX_2)-\|\mathcal{T}_{BX_2}CX_1\|^2-\frac{3}{4}\|\mathcal{V}[CX_1,CX_2]\|^2\\
&+\frac{\lambda^2}{2}[g_1(CX_1,CX_2)g_1(\nabla_{CX_2}grad(\frac{1}{\lambda^2}),CX_1)-g_1(CX_2,CX_2)g_1(\nabla_{CX_1}grad(\frac{1}{\lambda^2}),CX_1)\\
&+g_1(CX_2,CX_1)g_1(\nabla_{CX_1}grad(\frac{1}{\lambda^2}),CX_2)-g_1(CX_1,CX_1)g_1(\nabla_{CX_2}grad(\frac{1}{\lambda^2}),CX_2)]\\
&+\frac{\lambda^4}{4}[(g_1(CX_1,CX_1)g_1(CX_2,CX_2)-g_1(CX_2,CX_1)g_1(CX_1,CX_2))\|grad(\frac{1}{\lambda^2})\|^2\\
&+\|CX_1(\frac{1}{\lambda^2})CX_2-CX_2(\frac{1}{\lambda^2})CX_1\|^2] \qquad (4.3)
\end{aligned}
$$

for ortonormal vector fields $U_1,U_2\in\Gamma(\ker f_*)$ *and* $X_1,X_2\in\Gamma((\ker f_*)^{\perp})$.

Proof Since M_1 is a Kaehlerian manifold, we have

$$K(U_1,U_2)=K(\xi U_1,\xi U_2)+K(\xi U_1,\eta U_2)+K(\eta U_1,\xi U_2)+K(\eta U_1,\eta U_2) \qquad (4.4)$$

for ortonormal vector fields $U_1,U_2\in\Gamma(\ker f_*)$. From (2.10), we have

$$
\begin{aligned}
K(\xi U_1,\xi U_2)=&g_1(R(\xi U_1,\xi U_2)\xi U_2,\xi U_1)=g_1(R^*(\xi U_1,\xi U_2)\xi U_2,\xi U_1)-g_1(\mathcal{T}_{\xi U_1}\xi U_1,\mathcal{T}_{\xi U_2}\xi U_2)\\
&+\cos^4\theta\|\mathcal{T}_{U_1}PU_2\|^2+2\cos^2\theta\cos^2\bar\theta g_1(\mathcal{T}_{U_1}PU_2,\mathcal{T}_{U_1}QU_2)+\cos^4\bar\theta\|\mathcal{T}_{U_1}QU_2\|^2\\
&-2\cos^2\theta g_1(\mathcal{T}_{U_1}PU_2,\eta[\xi U_2,U_1]-\mathcal{H}\nabla_{U_1}\eta\xi U_2-\mathcal{H}\nabla_{\xi U_2}\eta U_1)\\
&-2\cos^2\bar\theta g_1(\mathcal{T}_{U_1}QU_2,\eta[\xi U_2,U_1]-\mathcal{H}\nabla_{U_1}\eta\xi U_2-\mathcal{H}\nabla_{\xi U_2}\eta U_1)\\
&+\|\eta[\xi U_2,U_1]-\mathcal{H}\nabla_{U_1}\eta\xi U_2-\mathcal{H}\nabla_{\xi U_2}\eta U_1\|^2.
\end{aligned}
$$

Similarly by using (2.12), we obtain

$$
\begin{aligned}
K(\xi U_1,\eta U_2)=&g_1((\nabla_{\xi U_1}\mathcal{A})_{\eta U_2}\eta U_2,\xi U_1)+\|\mathcal{A}_{\eta U_2}\xi U_1\|^2-g_1((\nabla_{\eta U_2}\mathcal{T})_{\xi U_1}\eta U_2,\xi U_1)\\
&-\left\|\xi[\xi U_1,U_2]-\cos^2\theta\bar\nabla_{U_2}PU_1-\cos\bar\theta\bar\nabla_{U_2}QU_1+\mathcal{T}_{U_2}\eta\xi U_1-\bar\nabla_{\xi U_1}\xi U_2\right\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
K(\eta U_1,\xi U_2)=&g_1((\nabla_{\xi U_2}\mathcal{A})_{\eta U_1}\eta U_1,\xi U_2)+\|\mathcal{A}_{\eta U_1}\xi U_2\|^2-g_1((\nabla_{\eta U_1}\mathcal{T})_{\xi U_2}\eta U_1,\xi U_2)\\
&-\left\|\xi[\xi U_2,U_1]-\cos^2\theta\bar\nabla_{U_1}PU_2-\cos^2\bar\theta\bar\nabla_{U_1}QU_2+\mathcal{T}_{U_1}\eta\xi U_2-\bar\nabla_{\xi U_2}\xi U_1\right\|^2.
\end{aligned}
$$

Finally, using the equation (2.13), we have

$$\begin{aligned}
K(\eta U_1,\eta U_2) &= \frac{1}{\lambda^2}\hat{K}(f_*\eta U_1, f_*\eta U_2) - \frac{3}{4}\left\|\mathcal{V}[\eta U_1,\eta U_2]\right\|^2 \\
&- \frac{\lambda^2}{2}[g_1(\eta U_1,\eta U_2)g_1(\nabla_{\eta U_2} grad(\frac{1}{\lambda^2}),\eta U_1) - g_1(\eta U_2,\eta U_2)g_1(\nabla_{\eta U_1} grad(\frac{1}{\lambda^2}),\eta U_1) \\
&+ g_1(\eta U_2,\eta U_1)g_1(\nabla_{\eta U_1} grad(\frac{1}{\lambda^2}),\eta U_2) - g_1(\eta U_1,\eta U_1)g_1(\nabla_{\eta U_2} grad(\frac{1}{\lambda^2}),\eta U_2)] \\
&+ \frac{\lambda^4}{4}[(g_1(\eta U_1,\eta U_1)g_1(\eta U_2,\eta U_2) - g_1(\eta U_2,\eta U_1)g_1(\eta U_1,\eta U_2))\|grad(\frac{1}{\lambda^2})\|^2 \\
&+ \|\eta U_1(\frac{1}{\lambda^2})\eta U_2 - \eta U_2(\frac{1}{\lambda^2})\eta U_1\|^2].
\end{aligned}$$

If $K(\xi U_1,\xi U_2), K(\xi U_1,\eta U_2), K(\eta U_1,\xi U_2),$ and $K(\eta U_1,\eta U_2)$ are written instead in (4.4), we obtain the equation (4.1). Using a similar way, the equations (4.2) and (4.3) are obtained. □

Now by using Theorem 4.1, the following inequalities can be given.

Corollary 4.2 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then we obtain*

$$\begin{aligned}
K(U_1,U_2) &\geq K^*(\xi U_1,\xi U_2) + \frac{1}{\lambda^2}\hat{K}(f_*\eta U_1, f_*\eta U_2) - g_1(\mathcal{T}_{\xi U_1}\xi U_1, \mathcal{T}_{\xi U_2}\xi U_2) \\
&+ 2\cos^2\theta\cos^2\bar{\theta} g_1(\mathcal{T}_{U_1}PU_2, \mathcal{T}_{U_1}QU_2) \\
&- 2\cos^2\theta g_1(\mathcal{T}_{U_1}PU_2, \eta[\xi U_2,U_1] - \mathcal{H}\nabla_{U_1}\eta\xi U_2 - \mathcal{H}\nabla_{\xi U_2}\eta U_1) \\
&- 2\cos^2\bar{\theta} g_1(\mathcal{T}_{U_1}QU_2, \eta[\xi U_2,U_1] - \mathcal{H}\nabla_{U_1}\eta\xi U_2 - \mathcal{H}\nabla_{\xi U_2}\eta U_1) \\
&+ \|\eta[\xi U_2,U_1] - \mathcal{H}\nabla_{U_1}\eta\xi U_2 - \mathcal{H}\nabla_{\xi U_2}\eta U_1\|^2 + g_1((\nabla_{\xi U_1}\mathcal{A})_{\eta U_2}\eta U_2, \xi U_1) \\
&+ \left\|\mathcal{A}_{\eta U_2}\xi U_1\right\|^2 - g_1((\nabla_{\eta U_2}\mathcal{T})_{\xi U_1}\eta U_2, \xi U_1) \\
&- \|\xi[\xi U_1,U_2] - \cos^2\theta\bar{\nabla}_{U_2}PU_1 - \cos^2\bar{\theta}\bar{\nabla}_{U_2}QU_1 + \mathcal{T}_{U_2}\eta\xi U_1 - \bar{\nabla}_{\xi U_1}\xi U_2\|^2 \\
&+ g_1((\nabla_{\xi U_2}\mathcal{A})_{\eta U_1}\eta U_1, \xi U_2) + \|\mathcal{A}_{\eta U_1}\xi U_2\|^2 - g_1((\nabla_{\eta U_1}\mathcal{T})_{\xi U_2}\eta U_1, \xi U_2) \\
&- \|\xi[\xi U_2,U_1] - \cos^2\theta\bar{\nabla}_{U_1}PU_2 - \cos^2\bar{\theta}\bar{\nabla}_{U_1}QU_2 + \mathcal{T}_{U_1}\eta\xi U_2 - \bar{\nabla}_{\xi U_2}\xi U_1\|^2 \\
&- \frac{3}{4}\left\|\mathcal{V}[\eta U_1,\eta U_2]\right\|^2 - \frac{\lambda^2}{2}[g_1(\eta U_1,\eta U_2)g_1(\nabla_{\eta V} grad(\frac{1}{\lambda^2}),\eta U_1) \\
&- g_1(\eta U_2,\eta U_2)g_1(\nabla_{\eta U_1} grad(\frac{1}{\lambda^2}),\eta U_1) + g_1(\eta U_2,\eta U_1)g_1(\nabla_{\eta U_1} grad(\frac{1}{\lambda^2}),\eta U_2) \\
&- g_1(\eta U_1,\eta U_1)g_1(\nabla_{\eta U_2} grad(\frac{1}{\lambda^2}),\eta U_2)] + \frac{\lambda^4}{4}[(g_1(\eta U_1,\eta U_1)g_1(\eta U_2,\eta U_2) \\
&- g_1(\eta U_2,\eta U_1)g_1(\eta U_1,\eta U_2))\|grad(\frac{1}{\lambda^2})\|^2 + \|\eta U_1(\frac{1}{\lambda^2})\eta U_2 - \eta U_2(\frac{1}{\lambda^2})\eta U_1\|^2]
\end{aligned}$$

for $U_1, U_2 \in \Gamma(\ker f_*)$. The equality case is satisfied if and only if f is a conformal anti-invariant submersion or the fibers are totally geodesic.

Corollary 4.3 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then we have*

$$\begin{aligned}K(X_1, U_1) \geq & K^*(BX_1, \xi U_1) + \frac{1}{\lambda^2}\hat{K}(f_*CX_1, f_*\eta U_1) - g_1(\mathcal{T}_{\xi U_1}\xi U_1, \mathcal{T}_{BX_1}BX_1) \\ & + 2\cos^2\theta\cos^2\bar{\theta} g_1(\mathcal{A}_{X_1}PU_1, \mathcal{A}_{X_1}QU_1) - \|\mathcal{T}_{BX_1}\eta U_1\|^2 \\ & - 2\cos^2\theta g_1(\mathcal{A}_{X_1}PU_1, \eta[\xi U_1, X_1] + \mathcal{H}\nabla_{X_1}\eta\xi U_1 - \mathcal{H}\nabla_{\xi U_1}CX_1) \\ & - 2\cos^2\bar{\theta} g_1(\mathcal{A}_{X_1}QU_1, \eta[\xi U_1, X_1] + \mathcal{H}\nabla_{X_1}\eta\xi U_1 - \mathcal{H}\nabla_{\xi U_1}CX_1) \\ & + \|\eta[\xi U_1, X_1] + \mathcal{H}\nabla_{X_1}\eta\xi U_1 - \mathcal{H}\nabla_{\xi U_1}CX_1\|^2 + g_1((\nabla_{BX_1}\mathcal{A})_{\eta U_1}\eta U_1, BX_1) \\ & + \|\mathcal{A}_{\eta U_1}BX_1\|^2 - g_1((\nabla_{\eta U_1}\mathcal{T})_{BX_1}\eta U_1, BX_1) + g_1((\nabla_{\xi U_1}\mathcal{A})_{CX_1}CX_1, \xi U_1) \\ & + \|\mathcal{A}_{CX_1}\xi U_1\|^2 - g_1((\nabla_{CX_1}\mathcal{T})_{\xi U_1}CX_1, \xi U_1) \\ & - \|\xi[\xi U_1, X_1] - \cos^2\theta\mathcal{V}\nabla_{X_1}PU_1 - \cos^2\bar{\theta}\mathcal{V}\nabla_{X_1}QU_1 + \mathcal{A}_{X_1}\eta\xi U_1\|^2 \\ & - \frac{3}{4}\|\mathcal{V}[CX_1, \eta U_1]\|^2 - \frac{\lambda^2}{2}[g_1(\eta U_1, \eta U_1)g_1(\nabla_{CX_1}grad(\frac{1}{\lambda^2}), CX_1) \\ & + g_1(CX_1, CX_1)g_1(\nabla_{\eta U_1}grad(\frac{1}{\lambda^2}), \eta U_1)] + \frac{\lambda^4}{4}[g_1(CX_1, CX_1)g_1(\eta U_1, \eta U_1)\|grad(\frac{1}{\lambda^2})\|^2 \\ & + \|CX_1(\frac{1}{\lambda^2})\eta U_1 - \eta U_1(\frac{1}{\lambda^2})CX_1\|^2]\end{aligned}$$

for $U_1 \in \Gamma(\ker f_*)$ and $X_1 \in \Gamma(\ker f_*)^{\perp}$. The equality case is satisfied if and only if f is a conformal anti-invariant submersion or $\mathcal{A}_{X_1}PU_1 = 0$ and $\mathcal{A}_{X_1}QU_1 = 0$.

Corollary 4.4 *Suppose that f is a proper conformal bi-slant submersion from a Kaehlerian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with slant angles $\theta, \bar{\theta}$. Then we get*

$$\begin{aligned}K(X_1, X_2) \geq & K^*(BX_1, BX_2) + \frac{1}{\lambda^2}\hat{K}(f_*CX_1, f_*CX_2) - g_1(\mathcal{T}_{BX_2}BX_2, \mathcal{T}_{BX_1}BX_1) \\ & + g_1((\nabla_{BX_1}\mathcal{A})_{CX_2}CX_2, BX_1) - g_1((\nabla_{CX_2}\mathcal{T})_{BX_1}CX_2, BX_1) \\ & - \|\mathcal{T}_{BX_1}CX_2\|^2 + g_1((\nabla_{BX_2}\mathcal{A})_{CX_1}CX_1, BX_2) + \|\mathcal{A}_{CX_1}BX_2\|^2 \\ & - g_1((\nabla_{CX_1}\mathcal{T})_{BX_2}CX_1, BX_2) - \|\mathcal{T}_{BX_2}CX_1\|^2 - \frac{3}{4}\|\mathcal{V}[CX_1, CX_2]\|^2 \\ & + \frac{\lambda^2}{2}[g_1(CX_1, CX_2)g_1(\nabla_{CX_2}grad(\frac{1}{\lambda^2}), CX_1) - g_1(CX_2, CX_2)g_1(\nabla_{CX_1}grad(\frac{1}{\lambda^2}), CX_1) \\ & + g_1(CX_2, CX_1)g_1(\nabla_{CX_1}grad(\frac{1}{\lambda^2}), CX_2) - g_1(CX_1, CX_1)g_1(\nabla_{CX_2}grad(\frac{1}{\lambda^2}), CX_2)] \\ & + \frac{\lambda^4}{4}[(g_1(CX_1, CX_1)g_1(CX_2, CX_2) - g_1(CX_2, CX_1)g_1(CX_1, CX_2))\|grad(\frac{1}{\lambda^2})\|^2 \\ & + \|CX_1(\frac{1}{\lambda^2})CX_2 - CX_2(\frac{1}{\lambda^2})CX_1\|^2]\end{aligned}$$

for $X_1, X_2 \in \Gamma((\ker f_*)^{\perp})$. The equality case is satisfied if and only if $\mathcal{T}_{BX_1}BX_2 = 0$ and $\mathcal{A}_{CX_2}BX_1 = 0$.

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