Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1724 – 1737

doi:10.3906/mat-2103-98

Relationship between lattice ordered semigroups and ordered hypersemigroups

Niovi KEHAYOPULU*
Nikomidias 18, 16122 Kesariani, Greece

Received: 23.03.2021 • **Accepted/Published Online:** 23.05.2021 • **Final Version:** 27.07.2021

Abstract: It has been shown in Turk J Math 2019; 43 (5): 2592–2601 that many results on hypersemigroups can be obtained directly as corollaries of more general results from the theory of lattice ordered semigroups, $\vee e$-semigroups or poe-semigroups. The present note shows that although this is not exactly the case for ordered hypersemigroups, even in this case various results may be suggested from analogous results for le, $\vee e$ or poe-semigroups and direct proofs derive along the lines of those le, $\vee e$ or poe-semigroups setting as well; the sets in the investigation provides a further indication that the results on this structure come from the lattice ordered semigroups or ordered semigroups in general. In many cases, whenever we have a look at any result on lattice ordered semigroups, we immediately know if can be transferred to ordered hypersemigroups. We never work on ordered hypersemigroups directly.

Key words: le-semigroup, hypersemigroup, ideal element, ideal, meet-irreducible, semisimple

1. Introduction

It has been shown in [16], that many results on hypersemigroups do not need any proof as they can be obtained from more results in the lattice ordered semigroup or poe-semigroup setting. It may be instructive to prove them directly, just to show how an independent proof works, but this direct, independent proof will follow along the lines of poe-semigroups. The aim of the present paper is to show that although this is not exactly the case for ordered hypersemigroups, even in that case the idea for various results come from the le, poe-semigroup setting, and direct proofs derive along the lines of those in the le, poe-semigroups setting. We illustrate by showing how the results in [6] (see also [14], [17]) can be obtained from the le-semigroups; the sets in the investigation is a further indication to justify what we say. Care should be taken with respect to the hyperoperation and the operation as they are not identical, not be denoted by the same symbol. Symbols of the form "$S \circ b \circ S \circ S \circ a \circ S$" are without meaning. What we present about the results in [6] holds for any published paper on hypersemigroups or ordered hypersemigroups.

When we say "S is a poe-semigroup" we mean that we work on the elements, that is ideal elements, etc. and not on the subsets (that is, ideals, etc.) of the ordered semigroup S.

*Correspondence: nkehayop@math.uoa.gr
2010 *AMS Mathematics Subject Classification:* 06F99, 06F05, 20N99

2. Prerequisites

An *ordered groupoid* (or *po-groupoid*) is a groupoid $(S, \cdot)$ with an order relation $\leq$ on S in which the multiplication is compatible with the ordering (that is, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for every $c \in S$). A $\vee$-*groupoid* is a groupoid at the same time a $\vee$-semilattice such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in S$. If S is not only a $\vee$-semilattice but a lattice, then it is called an l-*groupoid*. If the multiplication on the groupoid is associative, then we have the l-semigroup, po-semigroup etc. One can find these definitions first by Dubreil-Jacotin, Lesieur, Croisot [3] and later by Birkhoff and Fuchs [1, 2, 4]. A po-groupoid, po-semigroup, l-semigroup etc. possessing a greatest element "e" (that is, $e \geq a$ for all a), is called *poe-groupoid poe-semigroup*, *le-semigroup*, respectively. An element a of a poe-groupoid S is called a *right* (resp. *left*) *ideal element* of S if $ae \leq a$ (resp. $ea \leq a$) [7]. An element of S that is both a right and a left ideal element of S is called an *ideal element* of S. For an element a of S, we denote by $r\Big(l(a)\Big)$ the ideal element of S generated by a, that is, $r\Big(l(a)\Big)$ is an ideal element of S, $r\Big(l(a)\Big) \geq a$ and if t is an ideal element of S such that $t \geq a$, then $r\Big(l(a)\Big) \leq t$. We denote by $r(a)$, $l(a)$ the right and the left ideal element of S, respectively generated by a. For a $\vee e$-semigroup S and an element a of S, we have $r(a) = a \vee ae$, $l(a) = a \vee ea$, and $r\Big(l(a)\Big) = a \vee ea \vee ae \vee eae$. We also have $r\Big(l(a)\Big) = l\Big(r(a)\Big)$.

An *hypergroupoid* is a nonempty set S with an hyperoperation "$\circ$" on S (that is, a mapping assigning to each couple (a, b) of elements of S a nonempty subset $a \circ b$ of S) and an operation "$*$" on the set $\mathcal{P}^*(S)$ of nonempty subsets of S (induced by the hyperoperation "$\circ$") defined by $A * B = \bigcup\limits_{a \in A, b \in B} a \circ b$ and it is denoted by $(S, \circ)$ [9]. We have the following: If A, B, C and D are nonempty subsets of S such that $A \subseteq B$ and $C \subseteq D$, then $A * C \subseteq B * D$. For any nonempty subset A of S, we have $S * A \subseteq S$ and $A * S \subseteq S$. For any $a, b \in S$, we have $\{a\} * \{b\} = a \circ b$. For any nonempty subsets A, B of S we have:

(1) If $x \in A * B$, then there exists $a \in A$ and $b \in B$ such that $x \in a \circ b$.

(2) If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.

An hypergroupoid $(S, \circ)$ is said to be *hypersemigroup* if, for every $a, b, c \in S$, we have

$$(a \circ b) * \{c\} = \{a\} * (b \circ c).$$

Moreover the following hold:

(1) If S is an hypersemigroup, then the operation "$*$" on the set $\mathcal{P}^*(S)$ is associative; that is

$$(A * B) * C = A * (B * C) \text{ for any } A, B, C \in \mathcal{P}^*(S) \tag{2.1}$$

An easy proof different than that one given in [13] is the following: If $x \in (A * B) * C$, then $x \in u \circ c$ for some $u \in A * B$, $c \in C$ and $u \in a \circ b$ for some $a \in A$, $b \in B$; thus we have

$$x \in u \circ c = \{u\} * \{c\} \subseteq (a \circ b) * \{c\} = \{a\} * (b \circ c) \subseteq A * (B * C),$$

so $(A * B) * C \subseteq A * (B * C)$. Similarly, $A * (B * C) \subseteq (A * B) * C$.

(2) If S is an hypergroupoid then, for any $A, B, C \in \mathcal{P}^*(S)$, we have

$$(A \cup B) * C = (A * C) \cup (B * C) \text{ and } A * (B \cup C) = (A * B) \cup (A * C) \tag{2.2}$$

and

$$(A \cap B) * C \subseteq (A * C) \cap (B * C) \text{ and } A * (B \cap C) \subseteq (A * B) \cap (A * C) \tag{2.3}$$

Properties (2.1) and (2.2) have been given in [9, 13] for a finite number of nonempty subsets $A_1, A_2, ..., A_n$ of S. Let us prove property (2.3); at a similar way this property for any nonempty subsets $A_1, A_2, ..., A_n$ of S also holds.

Let $x \in (A \cap B) * C$. Then $x \in u \circ c$ for some $u \in A \cap B$, $c \in C$. If $u \in A$, then $u \circ c \subseteq A * C$ and so $x \in A * C$. If $u \in B$, then $u \circ c \subseteq B * C$ and so $x \in B * C$. Thus we have $(A \cap B) * C \subseteq (A * C) \cap (B * C)$. Similarly, $A * (B \cap C) \subseteq (A * B) \cap (A * C)$ and property (2.3) is satisfied.

Remark 2.1 [16] *By properties* (2.1) *and* (2.2), $(\mathcal{P}^*(S), *, \subseteq)$ *is an le-semigroup,* S *being the greatest element of* $\mathcal{P}^*(S)$ *and* $A \cap B$, $A \cup B$ *the infimum and the supremum of* A *and* B *respectively. Since* $(\mathcal{P}^*(S), *, \subseteq)$ *is an le-semigroup it is a* $\vee e$*-semigroup and a poe-semigroup as well. This is because every le-semigroup is a* $\vee e$*-semigroup and every* $\vee e$*-semigroup is a poe-semigroup (indeed, if* $a \leq b$, *then* $a \vee b = b$, *then* $bc = (a \vee b)c = ac \vee bc \geq ac$; *similarly,* $a \leq b$ *implies* $ca \leq cb$ *for any* $c \in S$ *[2]).*

An *ordered hypergroupoid* (*po-hypergroupoid*) is an hypergroupoid $(S, \circ)$ with an order relation $\leq$ on S such that $a \leq b$ implies $a \circ c \preceq b \circ c$ and $c \circ a \preceq c \circ b$ for any $c \in S$ in the sense that for any $u \in a \circ c$ there exists $v \in b \circ c$ such that $u \leq v$ and for every $u \in c \circ a$ there exists $v \in c \circ b$ such that $u \leq v$ and it is denoted by $(S, \circ, \leq)$. If $(S, \circ, \leq)$ is an ordered hypergroupoid such that $(S, \circ)$ is an hypersemigroup, then $(S, \circ, \leq)$ is called an *ordered hypersemigroup* (*po-hypersemigroup*).

The following properties of an ordered groupoid, being not related with the hyperoperation, for an ordered hypergroupoid also hold.

(1) $A \subseteq (A]$ for any $A \subseteq S$

(2) $A \subseteq B$ implies $(A] \subseteq (B]$

(3) $(S] = S$

(4) $\big((A]\big] = (A]$ for any $A \subseteq S$

where, for a nonempty subset A of S, the set $(A]$ is the subset of S defined by:

$$(A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}$$

(see, for example [8, Lemma 1]) .

Moreover, in an hypergroupoid $(S, \circ)$, we have

$$(A] * (B] \subseteq (A * B] \tag{2.4}$$

(this is the Lemma 2.8 in [12]).

A nonempty subset A of an hypergroupoid $(S, \circ)$ is called a *right* (resp. *left*) *ideal* of S, if $A * S \subseteq A$ (resp. $S * A \subseteq A$); equivalently, if $a \in A$ and $s \in S$, then $a \circ s \subseteq A$ (resp. $s \circ a \subseteq A$).

A nonempty subset A of an ordered hypergroupoid $(S, \circ, \leq)$ is called a *right* (resp. *left*) *ideal* of S, if A is a right (resp. left) ideal of the hypergroupoid $(S, \circ)$ and, in addition,

$$a \in A \text{ and } S \ni b \leq a \text{ imply } b \in A$$

that means that $(A] = A$.

In both cases, A is called an *ideal* of S if it is both a right and a left ideal of S. For a nonempty subset A of S, we denote by $I(A)$ the ideal of S generated by A. If S is an hypersemigroup, then $I(A) = A \cup (S * A) \cup (A * S) \cup (S * A * S)$ [11, 13]. If S is an ordered hypersemigroup then, for a nonempty subset A of S,

$$I(A) = \Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big] \text{ [14, Proposition 13].}$$

If S is an hypergroupoid or an ordered hypergroupoid, a nonempty subset T of S is called *prime* if for any nonempty subsets A, B of S such that $A * B \subseteq T$, we have $A \subseteq T$ of $B \subseteq T$, equivalently if for any $a, b \in S$ such that $a \circ b \subseteq T$, we have $a \in T$ or $b \in T$; it is called *weakly prime* if for any ideals A, B of S such that $A * B \subseteq T$, we have $A \subseteq T$ of $B \subseteq T$ [14]. A nonempty subset T of an hypergroupoid or an ordered hypergroupoid S is called *semiprime* if for any nonempty subset A of S such that $A * A \subseteq T$, we have $A \subseteq T$, equivalently if for any $a \in T$ such that $a \circ a \subseteq T$, we have $a \in T$ [14]; it is called *weakly semiprime* if for any ideal A of S such that $A * A \subseteq T$, we have $A \subseteq T$.

3. Main results

Definition 3.1 *An element t of a po-groupoid S is called prime if for every $a, b \in S$ such that $ab \leq t$, we have $a \leq t$ or $b \leq t$ [2, p. 329]. The element t is called weakly prime if for every ideal elements $a, b \in S$ such that $ab \leq t$, we have $a \leq t$ or $b \leq t$. An element t of a po-groupoid S is called semiprime [7] if for every $a \in S$ such that $a^2 \leq t$, we have $a \leq t$; it is called weakly semiprime if for every ideal element a of S such that $a^2 \leq t$, we have $a \leq t$.*

Proposition 3.2 *We have the following:*

(1) If S is a poe-semigroup, a is a left ideal element and b is a right ideal element of S, then ab is an ideal element of S.

(2) If S is a poe-groupoid at the same time a $\wedge$-semilattice and a, b are ideal elements of S, then the element $a \wedge b$ is an ideal element of S.

(3) If S is a $\vee e$-groupoid and a, b are ideal elements of S, then $a \vee b$ is an ideal element of S.

(4) If S is a poe-semigroup, then eae is an ideal element of S for every $a \in S$.

Proof (1) Let a be a left ideal element and b a right ideal element of S. Then $(ab)e = a(be) \leq ab$ and $e(ab) = (ea)b \leq ab$, thus ab is an ideal element of S.

(2) Let a, b be ideal elements of S. Then

$$(a \wedge b)e \leq ae \wedge be \leq a \wedge b \text{ and } e(a \wedge b) \leq ea \wedge eb \leq a \wedge b,$$

thus $a \wedge b$ is an ideal element of S.

(3) Let a, b be ideal elements of S. Then

$$(a \vee b)e = ae \vee be \le a \vee b \text{ and } e(a \vee b) = ea \vee eb \le a \vee b,$$

thus $a \vee b$ is an ideal element of S.

(4) We have $(eae)e = eae^2 \le eae$ and $e(eae) = e^2ae \le eae$, thus eae is an ideal element of S. □

Corollary 3.3 *If S is an hypersemigroup, A is a left ideal and B is a right ideal of S, then the set $A * B$ is an ideal of S. If S is an hypergroupoid and A, B are ideals of S, then the sets $A \cap B$ and $A \cup B$ are also ideals of S. If S is an hypersemigroup then, for every nonempty subset A of S, the set $S * A * S$ is an ideal of S.*

Proof If S is an hypersemigroup, A is a left ideal and B is a right ideal of S, then $(\mathcal{P}^*(S), *, \subseteq)$ is a *poe*-semigroup, A is a left ideal element and B is a right ideal element of $\mathcal{P}^*(S)$. By Proposition 3.2(1), the set $A * B$ is an ideal element of $\mathcal{P}^*(S)$ and so it is an ideal of S.

If S is an hypergroupoid and A and B are ideals of S, then the intersection $A \cap B$ is nonempty (see [13, Proposition 17]), $(\mathcal{P}^*(S), *, \subseteq)$ is an *le*-groupoid and A, B are ideal elements of $\mathcal{P}^*(S)$. By Proposition 3.2(2) and (3), the sets $A \cap B$ and $A \cup B$ are ideal elements of $(\mathcal{P}^*(S), *, \subseteq)$ and so they are ideals of S.

Finally, let S be an hypersemigroup and A a nonempty subset of S. Then A is an element of the *poe*-semigroup $(\mathcal{P}^*(S), *, \subseteq)$. By Proposition 3.2(4), the set $S * A * S$ is an ideal element of $\mathcal{P}^*(S)$ and so $S * A * S$ is an ideal of S. □

Proposition 3.4 *(see [6, Lemma 2.3]) If S is an ordered hypersemigroup, A is a left ideal and B is a right ideal of S, then the set $(A * B]$ is an ideal of S. If S is an ordered hypergroupoid and A, B are ideals of S, then the sets $A \cap B$ and $A \cup B$ are also ideals of S. If S is an ordered hypersemigroup then, for every nonempty subset A of S, the set $(S * A * S]$ is an ideal of S.*

The proof of this proposition, on the line of Proposition 3.2, is as follows, and if we delete the ( ] from its proof, we get the direct proof of Corollary 3.3 again on the line of Proposition 3.2.

Proof Let $(S, \circ, \le)$ be an ordered hypersemigroup, A be a left ideal and B a right ideal of S. Then

$$(A * B] * S = (A * B] * (S] \subseteq \Big((A * B) * S\Big] = \Big(A * (B * S)\Big] \subseteq (A * B],$$

$$S * (A * B] = (S] * (A * B] \subseteq \Big(S * (A * B)\Big] = \Big((S * A) * B\Big] \subseteq (A * B],$$

also $\Big((A * B]\Big] = (A * B]$ (as $\big((X]\big] = (X]$ for any nonempty subset X of S), thus $(A * B]$ is an ideal of S.

Let $(S, \circ, \le)$ be an ordered hypergroupoid and A, B ideals of $(S, \circ, \le)$. Since $(S, \circ)$ is an hypergroupoid and A, B ideals of $(S, \circ)$, as we have already seen, the sets $A \cap B$ and $A \cup B$ are ideals of $(S, \circ)$. If $x \in A \cap B$ and $S \ni y \le x$ then, since $x \in A$ and A is an ideal of $(S, \circ, \le)$, we have $y \in A$; since $x \in B$, we have $y \in B$ and so $y \in A \cap B$. Let $x \in A \cup B$ and $S \ni y \le x$. If $x \in A$, then $y \in A$; if $x \in B$, then $y \in B$ and so $y \in A \cup B$. Thus the sets $A \cap B$ and $A \cup B$ are ideals of $(S, \circ, \le)$.

If $(S, \circ, \le)$ is an ordered hypersemigroup and A a nonempty subset of S, then

$$(S * A * S] * S = (S * A * S] * (S] \subseteq (S * A * S * S] \subseteq (S * A * S],$$

similarly $S*(S*A*S] \subseteq (S*A*S]$. Also $\Big((S*A*S]\Big] = (S*A*S]$, so $(S*A*S]$ is an ideal of S. $\square$

Theorem 3.5 *[10, Proposition 7] Let S be a $\vee e$-semigroup and t an ideal element of S. Then t is prime if and only if it is semiprime and weakly prime. If S is a commutative $\vee e$-semigroup, then an ideal element t of S is prime if and only if it is weakly prime.*

An hypergroupoid or ordered hypergroupoid S is said to be *commutative* if $A*B = B*A$ for any nonempty subsets A, B of S; that is, we have $u \in a \circ b$ for some $a \in A$ and $b \in B$ if and only if $u \in c \circ d$ for some $b \in B$, $d \in A$.

Corollary 3.6 *If $(S, \circ)$ is an hypersemigroup, then an ideal of S is prime if and only if it is semiprime and weakly prime. In particular, if S is commutative, then the prime and weakly prime ideals of S coincide.*

Proof Let T be a semiprime and weakly prime ideal of $(S, \circ)$. Then $(\mathcal{P}^*(S), *, \subseteq)$ is a $\vee e$-semigroup and T is a semiprime and weakly prime ideal element of $(\mathcal{P}^*(S), *, \subseteq)$. By Theorem 3.5, T is prime ideal element of $\mathcal{P}^*(S)$ and so it is a prime ideal of S. The rest of the proof, at the similar way, is easy. $\square$

Theorem 3.7 *[6, Theorem 2.5] Let $(S, \circ, \leq)$ be an ordered hypersemigroup and T be an ideal of S. Then T is prime if and only if it is semiprime and weakly prime. In particular, if S is commutative, then the prime and weakly prime ideals of S coincide.*

The proof of this theorem, on the line of Theorem 3.5, is as follows, and if we delete the ( ] from its proof, we get the direct proof of Corollary 3.6 again on the line of Theorem 3.5.

Proof The $\Rightarrow$-part is obvious.
$\Longleftarrow$. Let T be semiprime and weakly prime and A, B nonempty subsets of S such that $A*B \subseteq T$. Then

$$\begin{aligned}(B*S*A]*(B*S*A] &\subseteq \Big((B*S*A)*(B*S*A)\Big] \\ &= \Big((B*S)*(A*B)*(S*A)\Big] \subseteq \Big(S*(A*B)*S\Big] \\ &\subseteq \Big((S*T)*S\Big] \subseteq (T*S] \subseteq (T] = T.\end{aligned}$$

Since T is semiprime, we have $(B*S*A] \subseteq T$. Then we have

$$(S*B*S]*(S*A*S] \subseteq \Big(S*(B*S*A)*S\Big] \subseteq (S*T*S] \subseteq T.$$

Since $(S*B*S]$ and $(S*A*S]$ are ideals of S, and T is weakly prime, we have $(S*B*S] \subseteq T$ or $(S*A*S] \subseteq T$.

Let $(S * A * S] \subseteq T$. Then we have

$$\begin{aligned}
I(A) * I(A) &= \Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big] * \Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big] \\
&\subseteq \Big(\Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big) * \Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big)\Big] \\
&= \Big((A * A) \cup (S * A * A) \cup (A * S * A) \cup (S * A * S * A) \cup (A * A * S) \\
&\cup \quad (S * A * A * S) \cup (A * S * A * S) \cup (S * A * S * A * S)\Big] \\
&\subseteq \Big((S * A) \cup (S * A * S)\Big].
\end{aligned}$$

Then

$$\begin{aligned}
\Big(I(A) * I(A)\Big) * I(A) &\subseteq \Big((S * A) \cup (S * A * S)\Big] * \Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big] \\
&\subseteq \Big((S * A * A) \cup (S * A * S * A) \cup (S * A * A * S) \cup (S * A * S * A * S)\Big] \\
&\subseteq (S * A * S] \subseteq T.
\end{aligned}$$

Since $I(A) * I(A)$ and $I(A)$ are ideals of S and T is weakly prime, we have $I(A) * I(A) \subseteq T$ or $I(A) \subseteq T$. If $I(A) \subseteq T$, then we have $A \subseteq T$. Let $I(A) * I(A) \subseteq T$. Since $I(A)$ is an ideal of S and T is semiprime, we have $I(A) \subseteq T$ and so $A \subseteq T$. If $S * B * S \subseteq T$, in a similar way we get $B \subseteq T$. Hence T is prime.

Finally, let S be a commutative hypersemigroup, T be a weakly prime ideal of S and A, B nonempty subsets of S such that $A * B \subseteq T$. Then

$$\begin{aligned}
I(A) * I(B) &= \Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big] * \Big(B \cup (S * B) \cup (B * S) \cup (S * B * S)\Big] \\
&\subseteq \Big((A * B) \cup (S * A * B)\Big] \subseteq I(A * B) \subseteq I(T) = T.
\end{aligned}$$

Since $I(A)$ and $I(B)$ are ideals of S and T is weakly prime, we have $I(A) \subseteq T$ or $I(B) \subseteq T$. Thus we have $A \subseteq T$ or $B \subseteq T$, hence T is prime. □

Definition 3.8 [5] *Let S be a po-groupoid. An element t of S is called meet-irreducible if for any ideal elements a, b of S such that $a \wedge b$ exists and $a \wedge b = t$, we have $a = t$ or $b = t$.*

Proposition 3.9 *Let S be a poe-groupoid at the same time a $\wedge$-semilattice and $t \in S$. If t is weakly prime, then t is meet-irreducible.*

Proof Let a, b be ideal elements of S such that $a \wedge b = t$. Since a is a right ideal element and b is a left ideal element of S, we have $ab \le ae \le a$ and $ab \le eb \le b$, then $ab \le a \wedge b = t$. Since t is weakly prime, a, b are ideal elements of S and $ab \le t$, we have $a \le t$ or $b \le t$. Thus we have $a = t$ or $b = t$ and so t is meet-irreducible. □

Proposition 3.10 *Let S be an le-groupoid at the same time a distributive lattice and t an ideal element of S. If t is weakly semiprime and meet-irreducible, then t is weakly prime.*

Proof Let a, b be ideal elements of S such that $ab \leq t$. By Proposition 3.2(2), $a \wedge b$ is an ideal element of S. On the other hand, $(a \wedge b)(a \wedge b) \leq ab \leq t$. Since t is weakly semiprime and $a \wedge b$ is an ideal element of S, we have $a \wedge b \leq t$. Then $t = t \vee (a \wedge b) = (t \vee a) \wedge (t \vee b)$. By Proposition 3.2(3), $t \vee a$ and $t \vee b$ are ideal elements of S. Since t is meet-irreducible, we have $t \vee a = t$ or $t \vee b = t$. Then $a \leq t$ or $b \leq t$ and so t is weakly prime. □

By Proposition 3.9 and Proposition 3.10, we have the following theorem.

Theorem 3.11 *Let S be an le-groupoid at the same time a distributive lattice. An ideal element t of S is weakly prime if and only if it is weakly semiprime and meet-irreducible.*

If S is an hypergroupoid or an ordered hypergroupoid, a nonempty subset T of S is called *meet-irreducible* (irreducible in the sense of [6]) if for any ideals A, B of S such that $A \cap B = T$, we have $A = T$ or $B = T$.

From Propositions 3.9 and 3.10, we have the following corollary.

Corollary 3.12 *Let $(S, \circ)$ be an hypersemigroup. If a subset of S is weakly prime, then it is meet-irreducible. If an ideal of S is weakly semiprime and meet-irreducible, then it is weakly prime.*

Theorem 3.13 *(see [6, Theorem 2.6]) Let $(S, \circ, \leq)$ be an ordered hypersemigroup. If a subset of S is weakly prime, then it is meet-irreducible. If an ideal of S is weakly semiprime and meet-irreducible, then it is weakly prime.*

The proof of this theorem on line of Propositions 3.9 and 3.10 is as follows:

Proof $\Longrightarrow$. Let T be a weakly prime subset of S and A, B be ideals of S such that $A \cap B = T$. We have $A * B \subseteq A * S \subseteq A$ and $A * B \subseteq S * B \subseteq B$ and so $A * B \subseteq A \cap B = T$. Since T is a weakly prime subset of S and A, B are ideals of S such that $A * B \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$. Then we have $A = T$ or $B = T$ and T is meet-irreducible.

$\Longleftarrow$. Let T be a weakly semiprime and meet-irreducible ideal of S and A, B be ideals of S such that $A * B \subseteq T$. By Proposition 3.4, $A \cap B$ is an ideal of S. On the other hand, $(A \cap B) * (A \cap B) \subseteq A * B \subseteq T$. Since T is weakly semiprime subset of S and $A \cap B$ is an ideal of S, we have $A \cap B \subseteq T$. Then we have $T = T \cup (A \cap B) = (T \cup A) \cap (T \cup B$. Since T, A, B are ideals of S, by Proposition 3.4, $T \cup A$ and $T \cup B$ are ideals of S. Since T is meet-irreducible, we have $T \cup A = T$ or $T \cup B = T$. Then $A \subseteq T$ or $B \subseteq T$ and so T is weakly prime. □

Definition 3.14 [10, 15] *A poe-semigroup S is called semisimple if, for any $a \in S$, we have*

$$a \leq eaeae.$$

Theorem 3.15 *(see also [10, Proposition 9]) Let S be an le-semigroup. The following are equivalent:*

(1) S is semisimple.

(2) $a \wedge b = ab$ for any ideal elements a, b of S.

(3) $a^2 = a$ for any ideal element a of S.

(4) Every ideal element of S is weakly semiprime.

Proof $(1) \Longrightarrow (2)$. Let a,b be ideal elements of S. By hypothesis, we have

$$a \wedge b \quad \leq \quad e(a \wedge b)e(a \wedge b)e \leq (ea)(ebe) \leq ab \leq (ae) \wedge (eb) \leq a \wedge b$$

and so $a \wedge b = ab$.

$(2) \Longrightarrow (3)$. Let a be an ideal element of S. By (2), we have $a = a \wedge a = a^2$ and so $a^2 = a$.

$(3) \Longrightarrow (4)$. Let t be an ideal element of S and a an ideal element of S such that $a^2 \leq t$. By (3), we have $a^2 = a$. Thus we get $a \leq t$ and so t is weakly semiprime.

$(4) \Longrightarrow (1)$. Let $a \in S$. We have

$$r\Big(l(a)\Big)^3 \leq eae,\ r\Big(l(a)\Big)^4 \leq eaea \vee eaeae,\ r\Big(l(a)\Big)^5 \leq eaeae.$$

Since $eaeae$ is an ideal element of S, by (4), it is weakly semiprime. Since $r\Big(l(a)\Big)^4$ and $r\Big(l(a)\Big)$ are ideal elements of S such that $r\Big(l(a)\Big)^4 r\Big(l(a)\Big) \leq eaeae$, we have $r\Big(l(a)\Big)^4 \leq eaeae$ or $r\Big(l(a)\Big) \leq eaeae$. If $r\Big(l(a)\Big) \leq eaeae$, then $a \leq eaeae$. If $r\Big(l(a)\Big)^2 r\Big(l(a)\Big)^2 \leq eaeae$ then, since $r\Big(l(a)\Big)^2$ is an ideal element of S and $eaeae$ is weakly semiprime, we have $r\Big(l(a)\Big)^2 \leq eaeae$. Since $r\Big(l(a)\Big)$ is an ideal element of S and $eaeae$ is weakly semiprime, we have $r\Big(l(a)\Big) \leq eaeae$. Then we have $a \leq eaeae$ and so property (1) is holds.
$\square$

Remark 3.16 *The implication $(1) \Rightarrow (2)$ in Theorem 3.15 holds for any poe-semigroup that is also a $\wedge$-semilattice; in a poe-groupoid having condition (2), condition (3) also holds; the implication $(4) \Rightarrow (1)$ holds in $\vee e$-semigroups in general.*

An hypersemigroup S is called *semisimple* if for every $a \in S$ there exist $x,y,z \in S$ such that $a \in (x \circ a) * (y \circ a) * \{z\}$. This is equivalent to saying that $A \subseteq S * A * S * A * S$ for every nonempty subset A of S [11].

According to Remark 2.1, from Theorem 3.15, we have the following corollary.

Corollary 3.17 *(see [11, Theorem 2.25]) Let S be an hypersemigroup. The following are equivalent:*

(1) S is semisimple.

(2) $A \cap B = A * B$ for all ideals A,B of S.

(3) $A * A = A$ for every ideal A of S.

(4) Every ideal of S is weakly semiprime.

An ordered hypersemigroup $(S, \circ, \leq)$ is called *semisimple* if for every $a \in S$ there exist $x,y,z,t \in S$ such that $t \in (x \circ a) * (y \circ a) * \{z\}$ and $a \leq t$ [14, 15]. This is equivalent to $A \subseteq (S * A * S * A * S]$ for any nonempty subset A of S or to $a \in (S * \{a\} * S * \{a\} * S]$ for any $a \in S$ [14, Proposition 17].

Theorem 3.18 *(see also [14, Theorems 9 and 18], [6, Theorem 3.1]) Let S be an ordered hypersemigroup. The following are equivalent:*

(1) S is semisimple.

(2) $A \cap B = (A * B]$ for all ideals A, B of S.

(3) $(A * A] = A$ for every ideal A of S.

(4) Every ideal of S is weakly semiprime.

The implication $(2) \Rightarrow (3)$ being obvious, the proof of the implications $(1) \Rightarrow (2)$ and $(3) \Rightarrow (4) \Rightarrow (1)$ on the line of the proof of Theorem 3.15 is as follows.

Proof $(1) \Longrightarrow (2)$. Let A, B be ideals of S. Since $A \cap B \neq \emptyset$, we have

$$\begin{aligned} A \cap B &\subseteq \Big(S * (A \cap B) * S * (A \cap B) * S\Big] \subseteq \Big((S * A) * (S * B * S)\Big] \subseteq (A * B] \\ &\subseteq (A * S] \cap (S * B] \subseteq (A] \cap (B] = A \cap B, \end{aligned}$$

and so $A \cap B = (A * B]$.

$(3) \Longrightarrow (4)$. Let A, T be ideals of S such that $A * A \subseteq T$. Then, by (3), $A = (A * A] \subseteq (T] = T$ and T is weakly semiprime.

$(4) \Longrightarrow (1)$. Let A be a nonempty subset of S. We have

$$\begin{aligned} I(A) * I(A) &= \Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big] * \Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big] \\ &\subseteq \bigg(\Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big) * \Big(A \cup (S * A) \cup (A * S) \cup (S * A * S)\Big)\bigg] \\ &\subseteq \Big((S * A) \cup (S * A * S)\Big]. \end{aligned}$$

Then

$I(A)^3 \subseteq (S * A * S]$,

$I(A)^4 \subseteq \Big((S * A * S * A) \cup (S * A * S * A * S)\Big]$,

$I(A)^5 \subseteq (S * A * S * A * S]$.

Since $(S * A * S * A * S]$ is an ideal of S, by hypothesis, it is weakly semiprime and so we have $A \subseteq I(A) \subseteq (S * A * S * A * S]$ and S is semisimple (see the proof of Theorem 3.15). □

If we delete from the proof of Theorem 3.18 the ( ], we get the independent proof of Corollary 3.17 again on the line of Theorem 3.15. It should be mentioned here that the equivalence of conditions (1) and (3) in Theorem 3.1 in [6] is the Theorem 18 in [14] and the equivalence of conditions (2) and (3) in the same theorem in [6] is the Theorem 9 in [14].

Definition 3.19 *[7] A poe-semigroup S is called intra-regular if, for every $a \in S$, we have*

$$a \leq ea^2e.$$

Theorem 3.20 *[10, Proposition 13] A poe-semigroup S is intra-regular if and only if the ideal elements of S are semiprime.*

An hypersemigroup S is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \in (x \circ a) * (a \circ y)$. This is equivalent to saying that $A \subseteq S * A * A * S$ for every nonempty subset A of S [11, 13].

Applying Remark 2.1, we get the following corollary.

Corollary 3.21 *An hypersemigroup $(S, \circ)$ is intra-regular if and only if the ideals of S are semiprime.*

Theorem 3.20 in case of ordered hypersemigroups is given by the next theorem with its proof on the line of the proof of Theorem 3.20. If we delete from its proof the ( ], then this is the independent proof of Corollary 3.21, again on the line of the proof of Theorem 3.20.

An ordered hypersemigroup $(S, \circ, \leq)$ is called *intra-regular* if for every $a \in S$ there exist $x, y, t \in S$ such that $t \in (x \circ a) * (a \circ y)$ and $a \leq t$ [12]. This is equivalent to saying that $A \subseteq (S * A * A * S]$ for any nonempty subset A of S or $a \in (S * \{a\} * \{a\} * S]$ for every $a \in S$ [14].

Theorem 3.22 *(see [6, Theorem 3.2]) An ordered hypersemigroup $(S, \circ, \leq)$ is intra-regular if and only if the ideals of S are semiprime.*

Proof $\Longrightarrow$. Let T be an ideal of S and A a nonempty subset of S such that $A * A \subseteq T$. Since S is intra-regular, we have $A \subseteq (S * A * A * S] \subseteq (S * T * S] \subseteq (T] = T$, so $A \subseteq T$ and T is semiprime.
$\Longleftarrow$. Let A be a nonempty subset of S. Since $A * A * A * A \subseteq (S * A * A * S]$ and $(S * A * A * S]$ is an ideal of S, by hypothesis, we have $A * A \subseteq (S * A * A * S]$ and $A \subseteq (S * A * A * S]$, thus S is intra-regular. $\square$

Theorem 3.23 *(see also [10, Theorem 12]) Let S be an le-semigroup. Then the ideal elements of S are weakly prime if and only if S is semisimple and the ideal elements of S form a chain.*

Proof $\Longrightarrow$. Let a be an ideal element of S. Then a^2 is an ideal element of S and, by hypothesis, a^2 is weakly prime. As $a^2 \leq a^2$, we have $a \leq a^2 \leq ae \leq a$ and so $a^2 = a$. Thus, the ideal elements of S are idempotent. Then, by Theorem 3.15 $(3) \Rightarrow (1)$, S is semisimple. Let now a, b be ideal elements of S. Then ab is an ideal element of S. Since $ab \leq ab$ and ab is weakly prime, we have $a \leq ab \leq ae \leq a$ or $b \leq ab \leq ae \leq a$. Thus the ideal elements of S form a chain.
$\Longleftarrow$. Let t, a, b be ideal elements of S such that $ab \leq t$. Since a, b are ideal elements of S and S is semisimple, by Theorem 3.5 $(1) \Rightarrow (2)$, we have $a \wedge b = ab$. Since the ideal elements of S form a chain, we have $a \leq b$ or $b \leq a$. Then we have $a = a \wedge b = ab \leq t$ or $b = a \wedge b = ab \leq t$. Then $a \leq t$ or $b \leq t$ and so t is weakly prime.
$\square$

Corollary 3.24 *Let S be an hypersemigroup. Then the ideals of S are weakly prime if and only if S is semisimple and the ideals of S form a chain.*

Theorem 3.25 *([14, Theorem 18], [14, Theorem 19], [6, Theorem 3.3]) Let S be an ordered hypersemigroup. Then the ideals of S are weakly prime if and only if S is semisimple and the ideals of S form a chain.*

The proof of this theorem, on line of Theorem 3.23 is as follows:

Proof $\Longrightarrow$. Let A be an ideal of S. By Proposition 3.4, the set $(A * A]$ is an ideal of S and, by hypothesis, $(A * A]$ is weakly prime. As $A * A \subseteq (A * A]$, we have $A \subseteq (A * A] \subseteq (A * S] \subseteq (A] = A$ and so $(A * A] = A$; thus the ideals of S are idempotent. Then, by Theorem 3.18(3) $\Rightarrow$ (1), S is semisimple. Let now A, B be ideals of S. By Proposition 3.4, $(A * B]$ is an ideal of S. Since $A * B \subseteq (A * B]$ and $(A * B]$ is weakly prime, we have $A \subseteq (A * B] \subseteq (A * S] \subseteq (A] = A$ or $B \subseteq (A * B] \subseteq (A * S] \subseteq (A] = A$; and the ideals of S form a chain.
$\Longleftarrow$. Let T, A, B be ideals of S such that $A * B \subseteq T$. Since A, B are ideals of S and S is semisimple, by Theorem 3.18(1) $\Rightarrow$ (2), we have $A \cap B = (A * B]$. Since the ideals of S form a chain, we have $A \subseteq B$ or $B \subseteq A$. Then we have $A = A \cap B = (A * B] \subseteq (T] = T$ or $B = A \cap B = (A * B] \subseteq (T] = T$. Then $A \subseteq T$ or $B \subseteq T$ and T is weakly prime. $\square$

Theorem 3.26 *[10, Theorem 16] Let S be an le-semigroup. The ideal elements of S are prime if and only if they form a chain and S is intra-regular.*

From Theorem 3.26, we have the following corollary.

Corollary 3.27 *Let $(S, \circ)$ be an hypersemigroup. Then the ideals of S are prime if and only if S is intra-regular and the ideals of S form a chain.*

As far as the ordered hypersemigroup is concerned, again on the line of the proof of Theorem 3.26, one can prove the next theorem.

Theorem 3.28 *([14, Corollary 24],[6, Theorem3.4]) Let S be an ordered hypersemigroup. Then the ideals of S are prime if and only if S is intra-regular and the ideals of S form a chain.*

4. Examples

Example 4.1 We consider the le-semigroup given by Table 1 and the order given by Figure 1.

Table 1. Multiplication table of Example 4.1.

$\cdot$	a	b	c	d	e
a	e	b	c	e	e
b	b	b	b	b	b
c	c	b	c	c	c
d	a	b	c	d	e
e	e	b	c	e	e

This is a semisimple and intra-regular le-semigroup and the ideal elements of S are the elements b, c and e. The results of the paper concerning the le-semigroups can be applied.

Under the methodology described in [15], the set $S = \{a, b, c, d, e\}$ with the hyperoperation given by Table 2 and the same order given by Figure 1 is a semisimple and intra-regular ordered hypersemigroup and the ideals of $(S, \circ, \leq)$ are the sets $\{b\}$, $\{b, c\}$ and S. The results of the paper related to ordered hypersemigroups can be also applied.

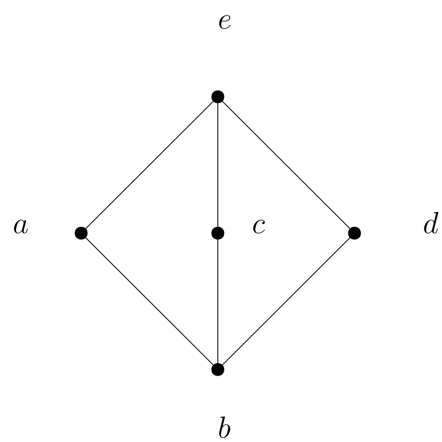

Figure 1. Figure of Example 4.1.

Table 2. The hyperoperation of Example 4.1.

$\circ$	a	b	c	d	e
a	S	$\{b\}$	$\{b,c\}$	S	S
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
c	$\{b,c\}$	$\{b\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
d	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,d\}$	S
e	S	$\{b\}$	$\{b,c\}$	S	S

Note. The fact that the results on ordered hypersemigroups (or hypersemigroups) given above have been proved using sets and not elements, is a further indication that they are based on lattice ordered semigroups, $\vee e$-semigroups or poe-semigroups.

With my thanks to the anonymous referees for their time to read the paper.

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