Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1738 – 1756
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doi:10.3906/mat-2012-66

Global attractivity of delay difference equations in Banach spaces via fixed-point theory

Abdullah KALKAN*
Department of Mathematics, University Klagenfurt, Klagenfurt am Wörthersee, Austria

Received: 17.12.2020 • **Accepted/Published Online:** 25.05.2021 • **Final Version:** 27.07.2021

Abstract: We formulate initial value problems for delay difference equations in Banach spaces as fixed-point problems in sequence spaces. By choosing appropriate sequence spaces various types of attractivity can be described. This allows us to establish global attractivity by means of fixed-point results. Finally, we provide an application to delay integrodifference equations in the space of continuous functions over a compact domain.

Key words: Delay difference equations, contractive difference equations, fixed-point theory, population dynamics, integrodifference equations, global attractivity

1. Introduction

Delay difference equations have become a popular topic in dynamical systems. There are many applications of delay difference equations as models in mathematical life science, e.g., population models. For example, [5, 7] consider difference equations where solutions are frequently real-valued. The work of [15] contains several other applications to social science models. However, recent applications also involve spatial effects such as dispersal. This requires attractivity conditions valid for problems in general Banach spaces.

In [11, 12] linear and nonlinear delay difference equations are considered as operator equations in a separable Hilbert space, in particular, the space and subspace of square summable sequences. This functional analytic approach has been applied to investigate the asymptotic behaviour of delay difference equations by applying fixed-point theorem for holomorphic mappings. While the topology of the considered sequence spaces is given by an inner product, [13] consider spaces of convergent or exponentially bounded sequences. Attractivity properties are obtained by showing the well-definedness of a nonlinear operator on an ambient space and using the contraction mapping principle, which provide assumptions guaranteeing the existence of fixed-points.

In this paper, we extend results from [13] and formulate initial value problems for delay difference equations as fixed point problems. We establish attractivity properties of difference equations on Banach spaces of infinite sequences. These difference equations involve finite delays. Three different norms are considered on the base Banach space. Both explicit and implicit difference equations are investigated, the latter requiring a fixed point argument to establish the existence and uniqueness of solutions. These results are then applied to general integrodifference equations and illustrated with two examples.

In order to obtain global attractivity, we discuss several spaces of sequences converging to zero.

*Correspondence: abukalkan@gmail.com

2010 *AMS Mathematics Subject Classification:* 34N05, 39A60, 45G10, 45G15, 45P99, 47B01, 92D25

Let X be a Banach space and $\ell_d(X)$, $d \in \mathbb{N}_0$, the space of all sequences $(\Phi_t)_{t \geq -d}$ with values $\Phi_t \in X$ for all $t \geq -d$. In detail we consider the space of

- ω-bounded sequences $\ell^\infty_\omega(X) := \{\Phi \in \ell_d(X) : \sup_{t \geq -d} \omega_t^{-1} \|\Phi_t\| < \infty\}$ with $\omega_t > 0$, which allows us to capture various form of decay to 0 and even boundedness for the limiting case $\omega_t \equiv 1$,
- zero convergent sequences $c_0(X) := \{\Phi \in \ell_d(X) : \lim_{t \to \infty} \|\Phi_t\| = 0\}$,
- p-summable sequences $\ell^p(X) := \{\Phi \in \ell_d(X) : \sum_{t \geq -d} \|\Phi_t\|^p < \infty\}$, where elements allow us to describe a subexponential decay to 0.

A particularly relevant special case is given in terms of integrodifference equations, see [8]. Our abstract theory can be applied to delay integrodifference equations in Banach spaces of continuous functions over a compact domain Ω. Integrodifference equations are used in theoretical ecology to model growth and spatial dispersal of species living in a habitat Ω. In the following, we consider integrodifference equations given in implicit form by

$$u_{t+1}(x) = \int_\Omega f_t(x, y, u_t(x), u_{t+1}(y), u_t(y), \ldots, u_{t-d}(y))dy + h_t(x),$$

$t \in \mathbb{N}_0$, where u_t is a real or vector valued continuous function, the kernel f_t and the inhomogeneity h_t are continuous and $\Omega \subseteq \mathbb{R}^\kappa$ is a compact domain. The inhomogeneity function h_t is an external increase in the population. This class of problems contains three relevant special cases:

i) the continuous functions f_t depend on $x, u_t(x)$ only. This leads to the explicit spatial difference equations

$$u_{t+1}(x) = \int_\Omega f_t(x, u_t(x))dy + h_t(x) = \lambda_\kappa(\Omega) f_t(x, u_t(x)) + h_t(x),$$

which model sedentary populations [9], where $\lambda_\kappa(\Omega)$ denotes the Lebesgue measure of Ω,

ii) the continuous functions f_t depend on $x, y, u_t(y)$ only. This leads us to the explicit integrodifference equations

$$u_{t+1}(x) = \int_\Omega f_t(x, y, u_t(y))dy + h_t(x),$$

which model dispersive populations. For background, see [9],

iii) the continuous functions f_t depend on $x, y, u_{t+1}(y)$ only. This leads us to the nonlinear Urysohn integral equation

$$u_{t+1}(x) = \int_\Omega f_t(x, y, u_{t+1}(y))dy + h_t(x);$$

for background see [10, 16].

These special cases illustrate how classical iterative methods for the solution of nonlinear Urysohn integral equations extend to more contemporary problems related to the long-term behavior of nonautonomous integrodifference equations.

In order to illustrate our results we use appropriate discretisation to replace the integral equations by the Nyström method with trapezoidal quadrature rule, for instance, see [2, 14]. This method is also used in [6], where simulations are given for pullback and forward attractors of contractive integrodifference equations.

The contents of this paper are as follows: In Section 2, we first establish the necessary terminology concerning attractivity notions and the required operator settings. Associate initial value problems for delay difference equations are formulated in this section as fixed-point problems using an operator $G_{\mathcal{F}}$. We show the well-definedness of this operator in the appropriate sequence spaces. This allows us to apply a variant of the Banach contraction principle to obtain global attractivity. Followed by that, Section 3 addresses the related notions for integrodifference equations of Urysohn type and presents an application of our abstract theory. Finally, we provide two examples, namely the delay Ricker and the delay Beverton–Holt integrodifference equation, including numerical simulations.

Notation. Now we provide some standard notions. Let $\mathbb{N}_0 := \{0, 1, 2, ...\}$ be the set of nonnegative integers. Let Ω be a compact and nonempty set in $\mathbb{R}^\kappa$, $\kappa \in \mathbb{N}$. Consider a Banach space X with $\|\cdot\|$ and a closed subset $D \subseteq X$ containing 0 throughout. Writing Y for another Banach space, the space of linear bounded mappings between X and Y is $L(X,Y)$ with norm $\|\cdot\|_L$, we abbreviate $L(X) := L(X,X)$. We denote the open ball with center $x \in X$ and radius $\rho > 0$ as $B_\rho(x)$.

Sequence spaces. Let $\ell_d(X) := \{(\Phi_t)_{t\geq -d} : \Phi_t \in X \text{ for all } t \geq -d\}$, $d \in \mathbb{N}_0$, denote the linear space of all sequences $\Phi = (\Phi_t)_{t\geq -d}$ in X. Moreover, we define the subset $\ell_d(D) := \{(\Phi_t)_{t\geq -d} : \Phi_t \in D \text{ for all } t \geq -d\}$. A real sequence $\omega = (\omega_t)_{t\geq -d}$ with positive values is called a weight sequence, if

$$\Upsilon(\omega) := \sup_{t\geq -d} \frac{\omega_t}{\omega_{t+1}} < \infty.$$

With the positive sequence ω, we define the Banach space of ω-bounded sequences

$$\ell_\omega^\infty(X) = \{\Phi \in \ell_d(X) : \sup_{t\geq -d} \omega_t^{-1}\|\Phi_t\| < \infty\},$$

with norm

$$\|\Phi\|_{\ell_\omega^\infty(X)} := \sup_{t\geq -d} \omega_t^{-1}\|\Phi_t\|$$

and the closed subset

$$\ell_\omega^\infty(D) := \{\Phi \in \ell_\omega^\infty(X) : \Phi_t \in D \text{ for all } t \geq -d\}.$$

For simplicity we write $\|\cdot\|_\omega$ instead of $\|\cdot\|_{\ell_\omega^\infty(X)}$. Note that $\ell_\omega^\infty(X)$ coincides with the bounded sequences for $\omega_t \equiv 1$. Furthermore consider the Banach space of sequences converging to zero

$$c_0(X) = \{\Phi \in \ell_d(X) : \lim_{t\to\infty} \|\Phi_t\| = 0\}$$

as a normed subspace of $\ell_1^\infty(X)$, i.e. with the norm $\|\Phi\|_0 := \sup_{t\geq -d} \|\Phi_t\|$ and the closed subset

$$c_0(D) := \{\Phi \in c_0(X) : \Phi_t \in D \text{ for all } t \geq -d\}.$$

With a real $p \geq 1$ we define the Banach spaces of p-summable sequences

$$\ell^p(X) = \{\Phi \in \ell_d(X) : \sum_{t \geq -d} \|\Phi_t\|^p < \infty\},$$

and the norm

$$\|\Phi\|_{\ell^p(X)} := \left(\sum_{t \geq -d} \|\Phi_t\|^p \right)^{1/p};$$

where we usually write $\|\cdot\|_p$ rather than $\|\cdot\|_{\ell^p(X)}$. Again we abbreviate the closed subset

$$\ell^p(D) := \{\Phi \in \ell^p(X) : \Phi_t \in D \text{ for all } t \geq -d\}.$$

It is clear that $\ell^p(D) \subseteq c_0(D)$.

Example 1.1 (weight sequences) *Consider the sequence ω defined by $\omega_t := \gamma^t$ with $\gamma > 0$. Then $\Upsilon(\omega) = \sup_{t \in \mathbb{N}_0} \frac{1}{\gamma} < \infty$ and thus the sequence ω is a weight sequence. If we consider the sequence ω defined by $\omega_t := \gamma^{t^2}$, then we have $\Upsilon(\omega) = \sup_{t \in \mathbb{N}_0} \frac{1}{\gamma^{2t+1}}$. This exists if and only if $\gamma \geq 1$. Hence, we obtain a weight sequence for $\gamma \geq 1$ only. Also polynomial sequences (in t) are weight sequences, if the coefficients are positive.*

The following proposition enables us to prove our results. It is a variant of the Banach contraction principle [4, p.10, (1.1) Theorem].

Proposition 1.2 (uniform contraction principle) *Suppose (Y, d_Y) is a complete metric space and (Z, d_Z) is a metric space. If there exist $L \in [0, 1)$ and $\lambda \geq 0$ such that $G : Y \times Z \to Y$ satisfies*

i) $d_Y(G(y, z), G(\bar{y}, z)) \leq L d_Y(y, \bar{y})$

ii) $d_Y(G(y, z), G(y, \bar{z})) \leq \lambda d_Z(z, \bar{z})$

for all $y, \bar{y} \in Y, z, \bar{z} \in Z$, then the following holds

a) there exists a unique function $\varphi : Z \to Y$ satisfying $G(\varphi(z), z) \equiv \varphi(z)$ on Z.

b) $d_Y(\varphi(z), \varphi(\bar{z})) \leq \frac{\lambda}{1-L} d_Z(z, \bar{z})$ *for all $z, \bar{z} \in Z$.*

Proof a) The existence of a unique fixed-point function $\varphi : Z \to Y$ is an immediate consequence of the Banach contraction principle [4, p.10, (1.1) Theorem].

b) For all $z, \bar{z} \in Z$ the triangle inequality yields

$$\begin{aligned} d_Y(\varphi(z), \varphi(\bar{z})) &= d_Y(G(\varphi(z), z), G(\varphi(\bar{z}), \bar{z})) \\ &\leq d_Y(G(\varphi(z), z), G(\varphi(z), \bar{z})) + d_Y(G(\varphi(z), \bar{z}), G(\varphi(\bar{z}), \bar{z})) \\ &\overset{i)}{\leq} d_Y(G(\varphi(z), z), G(\varphi(z), \bar{z})) + L d_Y(\varphi(z), \varphi(\bar{z})) \end{aligned}$$

and consequently $L \in [0,1)$ implies

$$d_Y(\varphi(z), \varphi(\bar{z})) \leq \frac{1}{1-L} d_Y(G(\varphi(z), z), G(\varphi(z), \bar{z}))$$
$$\overset{ii)}{\leq} \frac{\lambda}{1-L} d_Z(z, \bar{z}).$$

This is the claim. □

2. Delay difference equations

Consider an abstract implicit delay difference equation of the following form

$$u_{t+1} = \mathcal{F}_t(u_{t+1}, u_t, u_{t-1}, \dots, u_{t-d}), \tag{2.1}$$

with right-hand side $\mathcal{F}_t : D^{d+2} \to D$, $t \geq 0$, $d \in \mathbb{N}_0$ and the initial conditions

$$u_0 = \xi_0, u_{-1} = \xi_1, \dots, u_{-d} = \xi_d \tag{2.2}$$

with initial values $\xi_0, \cdots, \xi_d \in D$. A *forward solution* to (2.1) is a sequence $\Phi = (\Phi_t)_{t \geq -d}$ in D satisfying

$$\Phi_{t+1} \equiv \mathcal{F}_t(\Phi_{t+1}, \Phi_t, \Phi_{t-1}, \dots, \Phi_{t-d})$$

for all $t \geq 0$. In case $\Phi_0 = \xi_0, \Phi_{-1} = \xi_1, \dots, \Phi_{-d} = \xi_d$ holds, we say that Φ solves the initial value problem (2.1), (2.2). In addition, (2.1) is called *well-posed* on D, if for all $\xi_0, \xi_1, \dots, \xi_d \in D$, there exists a unique forward solution of (2.1), (2.2). Note that the continuous dependence on the initial condition will be a part of our result below.
Let us write $\varphi(\xi_0, \xi_1, \dots, \xi_d)$ for the general solution of (2.1), i.e. the unique forward solution of the above initial value problem.

Attractivity notions. Let $\mathcal{Y}$ be a subspace of $\ell_d(D)$. We say a delay difference equation (2.1) is $\mathcal{Y}$-attractive, if $\varphi(\xi_0, \xi_1, \dots, \xi_d) \in \mathcal{Y}$ holds for all initial values $\xi_0, \xi_1, \dots, \xi_d \in D$. Note that, in case $\mathcal{Y} \subseteq c_0(X)$, the solutions actually converge to 0 as $t \to \infty$.

Operator setting We introduce the operators:

- the linear embedding operator $E : X^{d+1} \to \ell_d(X)$ defined by

$$E(\xi_0, \xi_1, \dots, \xi_d) := (\xi_d, \dots, \xi_0, 0, 0, \dots),$$

- the linear right shift operator $S : c_0(X) \to \ell_d(X)$ defined by

$$S\Phi := (0, \Phi_0, \Phi_1, \Phi_2, \dots),$$

- the nonlinear substitution operator $F_{\mathcal{F}} : \ell_d(D) \to c_0(D)$ defined by

$$F_{\mathcal{F}}(\Phi) := (\mathcal{F}_t(\Phi_{t+1}, \Phi_t, \dots, \Phi_{t-d}))_{t \in \mathbb{N}_0},$$

and $G_{\mathcal{F}} : \ell_d(D) \times D^{d+1} \to \ell_d(D)$ given by

$$G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) := E(\xi_0, \dots, \xi_d) + S^{d+1} F_{\mathcal{F}}(\Phi). \tag{2.3}$$

This brings us to the following almost trivial but central observation:

Theorem 2.1 *Let $\xi_0, \dots, \xi_d \in D$. A sequence $\Phi \in \ell_d(D)$ solves the initial value problem (2.1), (2.2) if and only if $G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) = \Phi$.*

Proof Let $\xi_0, \dots, \xi_d \in D$. If $\Phi \in \ell_d(D)$ solves (2.1), (2.2), then

$$\begin{pmatrix} \Phi_{-d} \\ \vdots \\ \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \xi_d \\ \vdots \\ \xi_0 \\ \mathcal{F}_0(\Phi_1, \Phi_0, \dots, \Phi_{-d}) \\ \mathcal{F}_1(\Phi_2, \Phi_1, \dots, \Phi_{-d+1}) \\ \vdots \end{pmatrix} = E(\xi_0, \dots, \xi_d) + S^{d+1} F_{\mathcal{F}}(\Phi)$$

and conversely. □

Hypothesis. We suppose the right-hand side $\mathcal{F}_t$ of (2.1) satisfies the assumption

($\mathcal{A}$) for all $t \geq 0$ there exists a real $L_t \geq 0$ such that

$$\begin{aligned} &\|\mathcal{F}_t(u_1, u_0, \dots, u_{-d}) - \mathcal{F}_t(\bar{u}_1, \bar{u}_0, \dots, \bar{u}_{-d})\| \\ &\qquad \leq L_t \|(u_1, u_0, \dots, u_{-d}) - (\bar{u}_1, \bar{u}_0, \dots, \bar{u}_{-d})\| \end{aligned}$$

for all $u_1, u_0, \dots, u_{-d}, \bar{u}_1, \bar{u}_0, \dots, \bar{u}_{-d} \in D$.

In the following subsections we compute the operator norms which we use in order to establish Lipschitz conditions on the operators $G_{\mathcal{F}}$ in respective sequence spaces. This enables us to solve $G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) = \Phi$, for instance using Proposition 1.2.

2.1. Attractivity on ℓ_ω^∞

In this subsection we equip X^{d+1} with the maximum norm $\|x\| = \max_{0 \leq i \leq d} \|x_i\|$, where we use the component notation $x = (x_0, \dots, x_d)$.

Lemma 2.2 *One has $E \in L(X^{d+1}, \ell_\omega^\infty(X))$ and $S \in L(\ell_\omega^\infty(X))$ with*

i) $\|E\|_L \leq \max\{\omega_0^{-1}, \dots, \omega_{-d}^{-1}\}$,

ii) $\|S^{d+1}\|_L \leq \Upsilon(\omega)$.

Proof We compute the operator norm of E and S.

i) Let $\xi_0, \dots, \xi_d \in X$. One has

$$\|E(\xi_0, \dots, \xi_d)_t\|\omega_t^{-1} \le \max\{\omega_0^{-1}, \dots, \omega_{-d}^{-1}\}\|(\xi_0, \dots, \xi_d)\|$$

for all $t \ge -d$ and this implies

$$\|E(\xi_0, \dots, \xi_d)\|_\omega \le \max\{\omega_0^{-1}, \dots, \omega_{-d}^{-1}\} \max_{0\le i\le d} \|\xi_i\|.$$

Thus, we get $E \in L(X^{d+1}, \ell^\infty_\omega(X))$ with

$$\|E\|_L \le \max\{\omega_0^{-1}, \dots, \omega_{-d}^{-1}\}.$$

ii) Let $\Phi \in \ell^\infty_\omega(X)$. One has

$$\begin{aligned}\|(S^{d+1}\Phi)_t\|\omega_t^{-1} &= \begin{cases} 0, & -d \le t \le 0,\\ \omega_{t+1}^{-1}\|\Phi_t\|, & t > 0\end{cases}\\ &\le \Upsilon(\omega)\begin{cases} 0, & -d \le t \le 0,\\ \omega_t^{-1}\|\Phi_t\|, & t > 0\end{cases}\\ &\le \Upsilon(\omega)\|\Phi\|_\omega\end{aligned}$$

for all $t \ge -d$. This implies $ii)$.

This shows the lemma. $\square$

Hypothesis. We suppose that the assumption $(\mathcal{A})$ holds with

$(\mathcal{B})$ $\tilde{L} := \sup_{t\ge 0} L_t \max\{\frac{\omega_{t+1}}{\omega_t}, 1, \frac{\omega_{t-1}}{\omega_t}, \dots, \frac{\omega_{t-d}}{\omega_t}\} < \infty.$

Lemma 2.3 (Lipschitz condition of $F_{\mathcal{F}}$) *If the assumption $(\mathcal{B})$ holds, then one has*

$$F_{\mathcal{F}}(\Phi) - F_{\mathcal{F}}(\bar{\Phi}) \in \ell^\infty_\omega(X) \quad \textit{with} \quad \|F_{\mathcal{F}}(\Phi) - F_{\mathcal{F}}(\bar{\Phi})\|_\omega \le \tilde{L}\, \|\Phi - \bar{\Phi}\|_\omega$$

for all $\Phi, \bar{\Phi} \in \ell^\infty_\omega(D)$.

Proof Using the assumption $(\mathcal{B})$ we have

$$\begin{aligned}&\|F_{\mathcal{F}}(\Phi)_t - F_{\mathcal{F}}(\bar{\Phi})_t\|\omega_t^{-1} = \|\mathcal{F}_t(\Phi_{t+1}, \Phi_t, \dots, \Phi_{t-d}) - \mathcal{F}_t(\bar{\Phi}_{t+1}, \bar{\Phi}_t, \dots, \bar{\Phi}_{t-d})\|\omega_t^{-1}\\ &\quad\le L_t||(\Phi_{t+1}, \Phi_t, \dots, \Phi_{t-d}) - (\bar{\Phi}_{t+1}, \bar{\Phi}_t, \dots, \bar{\Phi}_{t-d})\|\omega_t^{-1}\\ &\quad= L_t \max\{\|\Phi_{t+1} - \bar{\Phi}_{t+1}\|\omega_t^{-1}, \|\Phi_t - \bar{\Phi}_t\|\omega_t^{-1}, \dots, \|\Phi_{t-d} - \bar{\Phi}_{t-d}\|\omega_t^{-1}\}\\ &\quad= L_t \max\{||\Phi_{t+1} - \bar{\Phi}_{t+1}\|\omega_{t+1}^{-1}\frac{\omega_{t+1}}{\omega_t}, \|\Phi_t - \bar{\Phi}_t\|\omega_t^{-1}, \dots, \|\Phi_{t-d} - \bar{\Phi}_{t-d}\|\omega_{t-d}^{-1}\frac{\omega_{t-d}}{\omega_t}\}\\ &\quad\le L_t \max\{\frac{\omega_{t+1}}{\omega_t}, 1, \frac{\omega_{t-1}}{\omega_t}, \dots, \frac{\omega_{t-d}}{\omega_t}\}\|\Phi - \bar{\Phi}\|_\omega\\ &\quad\le \tilde{L}\, \|\Phi - \bar{\Phi}\|_\omega\end{aligned}$$

for all $t \ge 0$ and $\Phi, \bar{\Phi} \in \ell^\infty_\omega(D)$. Passing over the supremum over $t \ge 0$ yields the claim. $\square$

Lemma 2.4 (Lipschitz condition of $G_{\mathcal{F}}$) *Let $\Phi, \bar{\Phi} \in \ell^\infty_\omega(D)$ and $\xi_0, \dots, \xi_d, \bar{\xi}_0, \dots, \bar{\xi}_d \in D$. If the assumptions of Lemma 2.3 hold with a weight sequence ω, then*

$$G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \dots, \xi_d) \in \ell^\infty_\omega(X),$$
$$G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\Phi, \bar{\xi}_0, \dots, \bar{\xi}_d) \in \ell^\infty_\omega(X)$$

with

$$\|G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \dots, \xi_d)\|_\omega \le \Upsilon(\omega)\tilde{L}\ \|\Phi - \bar{\Phi}\|_\omega,$$
$$\|G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\Phi, \bar{\xi}_0, \dots, \bar{\xi}_d)\|_\omega \le \max\{\omega_0^{-1}, \dots, \omega_{-d}^{-1}\} \max_{0 \le i \le d} \|\xi_i - \bar{\xi}_i\|.$$

Proof Let $\Phi, \bar{\Phi} \in \ell^\infty_\omega(D)$ and $\xi_0, \dots, \xi_d \in D$. One has with (2.3) that

$$G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \dots, \xi_d) = S^{d+1}F_{\mathcal{F}}(\Phi) - S^{d+1}F_{\mathcal{F}}(\bar{\Phi})$$

Using Lemma 2.3 and Lemma 2.2 *ii)* yields

$$\|G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \dots, \xi_d)\|_\omega \le \Upsilon(\omega)\tilde{L}\ \|\Phi - \bar{\Phi}\|_\omega.$$

Moreover, for $\Phi \in \ell^\infty_\omega(D)$ and $\xi_0, \dots, \xi_d$, $\bar{\xi}_0, \dots, \bar{\xi}_d \in D$ one has

$$\begin{aligned}\|G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\Phi, \bar{\xi}_0, \dots, \bar{\xi}_d)\|_\omega &= \|E(\xi_0, \dots, \xi_d) - E(\bar{\xi}_0, \dots, \bar{\xi}_d)\| \\ &\le \max\{\omega_0^{-1}, \dots, \omega_{-d}^{-1}\} \max_{0 \le i \le d} \|\xi_i - \bar{\xi}_i\|.\end{aligned}$$

This completes the proof of the lemma. □

Theorem 2.5 (ℓ^∞_ω-attraction) *Let ω be a weight sequence and suppose that the right-hand side $\mathcal{F}_t : D^{d+2} \to D$ of the delay difference equation (2.1) satisfies $(\mathcal{B})$. If*

$$\Upsilon(\omega)\tilde{L} < 1$$

and $F_{\mathcal{F}}(0) \in \ell^\infty_\omega(D)$, then the following holds for all $\xi_0, \dots, \xi_d \in D$:

(a) The delay difference equation (2.1) is well-posed.

(b) $\varphi(\xi_0, \dots, \xi_d) \in \ell^\infty_\omega(D)$

(c) The inequality

$$\|\varphi(\xi_0, \dots, \xi_d)_t - \varphi(\bar{\xi}_0, \dots, \bar{\xi}_d)_t\| \le \frac{\max\{\omega_0^{-1}, \dots, \omega_{-d}^{-1}\}}{1 - \Upsilon(\omega)\tilde{L}} \max_{0 \le i \le d} \|\xi_i - \bar{\xi}_i\|\omega_t$$

holds for all $t \ge -d$, $\bar{\xi}_0, \dots, \bar{\xi}_d \in D$.

Proof Let $\Phi, \bar{\Phi} \in \ell^\infty_\omega(D)$ and $\xi_0, \dots, \xi_d$, $\bar{\xi}_0, \dots, \bar{\xi}_d \in D$.

(a) Claim: We first show the inclusion $G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) \in \ell^\infty_\omega(D)$. By definition (2.3) and the assumption $F_{\mathcal{F}}(0) \in \ell^\infty_\omega(D)$,

$$G_{\mathcal{F}}(0, \xi_0, \ldots, \xi_d) = E(\xi_0, \ldots, \xi_d) + S^{d+1} F_{\mathcal{F}}(0) \in \ell^\infty_\omega(D)$$

holds, thanks to Lemma 2.3 we have

$$G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) - G_{\mathcal{F}}(0, \xi_0, \ldots, \xi_d) = S^{d+1}(F_{\mathcal{F}}(\Phi) - F_{\mathcal{F}}(0)) \in \ell^\infty_\omega(D),$$

and consequently

$$G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) = G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) - G_{\mathcal{F}}(0, \xi_0, \ldots, \xi_d) + G_{\mathcal{F}}(0, \xi_0, \ldots, \xi_d) \in \ell^\infty_\omega(D).$$

(b) Next we apply Proposition 1.2 with the complete metric spaces

$$Y := \ell^\infty_\omega(D), \qquad d_Y(\Phi, \bar{\Phi}) := \|\Phi - \bar{\Phi}\|_\omega,$$
$$Z := D^{d+1}, \qquad d_Z(\xi, \bar{\xi}) := \max_{0 \le i \le d} \|\xi_i - \bar{\xi}_i\|$$

and the mapping $G = G_{\mathcal{F}} \colon Y \times Z \to Y$. Due to Lemma 2.4, the operator $G_{\mathcal{F}}(\cdot, \xi_0, \ldots, \xi_d)$ has contraction constant $L := \Upsilon(\omega)\tilde{L} < 1$, while $G_{\mathcal{F}}(\Phi, \cdot)$ has the Lipschitz constant $\lambda := \max\{\omega_0^{-1}, \ldots, \omega_{-d}^{-1}\}$. Now, the assertion follows from Proposition 1.2 yielding a unique fixed point $\varphi(\xi_0, \ldots, \xi_d) \in Y$ of $G_{\mathcal{F}}(\Phi, \cdot)$, which by Theorem 2.1 is the desired solution in $\ell^\infty_\omega(D)$. Thus, (a) and (b) hold.

(c) With Lemma 2.4 again one computes

$$\begin{aligned}\|\varphi(\xi_0, \ldots, \xi_d)_t - \varphi(\bar{\xi}_0, \ldots, \bar{\xi}_d)_t\| \|\omega_t^{-1} &\le d_Y(\varphi(\xi_0, \ldots, \xi_d), \varphi(\bar{\xi}_0, \ldots, \bar{\xi}_d)) \\ &\le \frac{\lambda}{1 - L} \max_{0 \le i \le d} \|\xi_i - \bar{\xi}_i\|\end{aligned}$$

for all $t \ge -d$. This implies the assertion (c).

This shows the theorem. □

Corollary 2.6 *If the assumption $\mathcal{B}$ holds with $\sup_{t \ge 0} \tilde{L} < 1$ and $(\mathcal{F}_t(0, \ldots, 0))_{t \ge 0}$ is a bounded sequence, then the unique forward solution of (2.1), (2.2) is bounded for all $\xi_0, \ldots, \xi_d \in D$.*

Proof Since $\ell^\infty_1(D)$ is the set of the bounded sequences in D, the assertion follows from Theorem 2.5 with $\omega_t \equiv 1$. □

2.2. Attractivity on c_0

On X^{d+1} we use the same maximum norm as in the previous subsection.

Lemma 2.7 *One has $E \in L(X^{d+1}, c_0(X))$ and $S \in L(c_0(X))$ with*

i) $\|E\|_L = 1$,

ii) $\|S\|_L = 1$.

Proof We compute the operator norm of E and S.

i) For all $\xi_0, \dots, \xi_d \in X$ one has

$$\|E(\xi_0, \dots, \xi_d)\|_0 = \sup_{t \geq -d} \|E(\xi_0, \dots, \xi_d)_t\| = \max\{\|\xi_0\|, \dots, \|\xi_d\|\} = \|(\xi_0, \dots, \xi_d)\|$$

and this implies the claim.

ii) Given $\Phi \in c_0(X)$ it holds

$$\|S\Phi\|_0 = \sup_{t \geq 0} \|\Phi_t\| = \|\Phi\|_0,$$

and, thus, we have $S \in L(c_0(X))$ with $\|S\|_L = 1$.

$\square$

Lemma 2.8 (Lipschitz condition on $F_{\mathcal{F}}$) *If the assumption $(\mathcal{B})$ holds with $\tilde{L} := \sup_{t \geq 0} L_t < \infty$, then one has*

$$F_{\mathcal{F}}(\Phi) - F_{\mathcal{F}}(\bar{\Phi}) \in c_0(X) \quad \textit{with} \quad \|F_{\mathcal{F}}(\Phi) - F_{\mathcal{F}}(\bar{\Phi})\|_0 \leq \tilde{L} \, \|\Phi - \bar{\Phi}\|_0$$

for all $\Phi, \bar{\Phi} \in c_0(D)$.

Proof Using the assumption $(\mathcal{B})$ with $\tilde{L} = \sup_{t \geq 0} L_t < \infty$, we have

$$\begin{aligned}
&\|F_{\mathcal{F}}(\Phi)_t - F_{\mathcal{F}}(\bar{\Phi})_t\| = \|\mathcal{F}_t(\Phi_{t+1}, \Phi_t, \dots, \Phi_{t-d}) - \mathcal{F}_t(\bar{\Phi}_{t+1}, \bar{\Phi}_t, \dots, \bar{\Phi}_{t-d})\| \\
&\quad \leq L_t \, \|(\Phi_{d+1}, \Phi_d, \dots, \Phi_0) - (\bar{\Phi}_{d+1}, \bar{\Phi}_d, \dots, \bar{\Phi}_0)|| \\
&\quad \leq L_t \max\{\|\Phi_{d+1} - \bar{\Phi}_{d+1}\|, \|\Phi_d - \bar{\Phi}_d\|, \dots, ||\Phi_0 - \bar{\Phi}_0\|\} \\
&\quad \leq L_t \, \|\Phi - \bar{\Phi}\|_0 \\
&\quad = \tilde{L} \, \|\Phi - \bar{\Phi}\|_0
\end{aligned}$$

for all $t \geq 0$ and $\Phi, \bar{\Phi} \in c_0(D)$. Taking the supremum over $t \geq 0$ of both sides yields the result immediately. $\square$

Lemma 2.9 (Lipschitz condition of $G_{\mathcal{F}}$) *Let $\Phi, \bar{\Phi} \in c_0(D)$ and $\xi_0, \dots, \xi_d, \bar{\xi}_0, \dots, \bar{\xi}_d \in D$. If the assumptions of Lemma 2.8 hold, then*

$$G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \dots, \xi_d) \in c_0(X),$$
$$G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\Phi, \bar{\xi}_0, \dots, \bar{\xi}_d) \in c_0(X)$$

with

$$\|(G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \dots, \xi_d)\|_0 \leq \tilde{L} \, \|\Phi - \bar{\Phi}\|_0,$$
$$\|G_{\mathcal{F}}(\Phi, \xi_0, \dots, \xi_d) - G_{\mathcal{F}}(\Phi, \bar{\xi}_0, \dots, \bar{\xi}_d)||_0 \leq \max_{0 \leq i \leq d} \|\xi_i - \bar{\xi}_i\|.$$

Proof Let $\Phi, \bar{\Phi} \in c_0(D)$ and $\xi_0, \ldots, \xi_d \in D$. One has with (2.3) that

$$G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \ldots, \xi_d) = S^{d+1}(F_{\mathcal{F}}(\Phi) - F_{\mathcal{F}}(\bar{\Phi})).$$

Using Lemma 2.8 and Lemma 2.7 *ii)* yields

$$\|G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \ldots, \xi_d)\|_0 \leq \tilde{L}\, \|\Phi - \bar{\Phi}\|_0.$$

Moreover, for $\Phi, \bar{\Phi} \in c_0(D)$ and $\xi_0, \ldots, \xi_d, \bar{\xi}_0, \ldots, \bar{\xi}_d \in D$ one has

$$\begin{aligned} \|G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) - G_{\mathcal{F}}(\Phi, \bar{\xi}_0, \ldots, \bar{\xi}_d)\|_0 &= \|E(\xi_0, \ldots, \xi_d) - E(\bar{\xi}_0, \ldots, \bar{\xi}_d)\|_0 \\ &= \max_{0 \leq i \leq d} \|\xi_i - \bar{\xi}_i\|. \end{aligned}$$

This completes the proof of the lemma. □

Theorem 2.10 (c_0-attraction) *Suppose that the right-hand side $\mathcal{F}_t : D^{d+2} \to D$ of the delay difference equation (2.1) satisfies $(\mathcal{B})$ with $\tilde{L} := \sup_{t \geq 0} L_t < \infty$. If*

$$\sup_{t \geq 0} L_t < 1$$

and $F_{\mathcal{F}}(0) \in c_0(D)$, then the following holds for all $\xi_0, \ldots, \xi_d \in D$:

(a) The delay difference equation (2.1) is well-posed.

(b) $\varphi(\xi_0, \ldots, \xi_d) \in c_0(D)$

(c) The inequality

$$\|\varphi(\xi_0, \ldots, \xi_d)_t - \varphi(\bar{\xi}_0, \ldots, \bar{\xi}_d)_t\| \leq \frac{1}{1 - \sup_{t \geq 0} L_t} \max_{0 \leq i \leq d} \|\xi_i - \bar{\xi}_i\|$$

holds for all $t \geq -d$, $\bar{\xi}_0, \ldots, \bar{\xi}_d \in D$.

Remark 2.11 *The assumption $\sup_{t \geq 0} L_t < 1$ in Theorem 2.10 implies that $\mathcal{F}_t$ is a contraction mapping.*

Proof For $\Phi, \bar{\Phi} \in c_0(D)$ and $\xi_0, \ldots, \xi_d$, $\bar{\xi}_0, \ldots, \bar{\xi}_d \in D$ we proceed as in the proof of Theorem 2.5.

(a) The inclusion $G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) \in c_0(D)$ follows from Lemma 2.8.

(b) Now we apply Proposition 1.2 with the complete metric space

$$Y := c_0(D), \qquad d_Y(\Phi, \bar{\Phi}) := \|\Phi - \bar{\Phi}\|_0.$$

As a result of Lemma 2.9, the operator $G_{\mathcal{F}}(\cdot, \xi_0, \ldots, \xi_d)$ has Lipschitz constant $L := \tilde{L} < 1$ and $G_{\mathcal{F}}(\Phi, \cdot)$ possesses the Lipschitz constant $\lambda = 1$. The unique fixed point results from Proposition 1.2. Hence, (a) and (b) hold.

(c) Due to Lemma 2.9 one obtains

$$\|\varphi(\xi_0, \ldots, \xi_d)_t - \varphi(\bar{\xi}_0, \ldots, \bar{\xi}_d)_t\| \leq \frac{\lambda}{1 - L} \max_{0 \leq i \leq d} \|\xi_i - \bar{\xi}_i\|$$

for all $t \geq -d$. This implies (c).

The proof of the theorem is completed. □

2.3. Attractivity on ℓ^p

We handle this case differently to the previous cases, and we now equip the product X^{d+1} with the p-norm $\|x\|_p = \left(\sum_{i=0}^{d} \|x_i\|^p\right)^{1/p}$, where we assume $p \geq 1$ throughout.

Lemma 2.12 *One has $E \in L(X^{d+1}, \ell^p(X))$ and $S \in L(\ell^p(X))$ with*

i) $\|E\|_L = 1$,

ii) $\|S\|_L = 1$.

Proof We compute the operator norm of E and S.

i) For $\xi_0, \ldots, \xi_d \in D$ one has

$$\|E(\xi_0, \ldots, \xi_d)\|_p^p = \sum_{i=0}^{d} \|\xi_i\|^p = \|(\xi_0, \ldots, \xi_d)\|^p$$

and this implies $E \in L(X^{d+1}, \ell^p(X))$ with $\|E\|_L = 1$.

ii) Given $\Phi \in \ell^p(D)$ one obtains

$$\|(S\Phi)_t\|_p^p = \sum_{t \geq -d} \|\Phi_t\|^p$$

and hence $S \in L(\ell^p(X))$ with $\|S\|_L = 1$.

$\square$

Lemma 2.13 (Lipschitz condition on $F_{\mathcal{F}}$) *If the assumption $(\mathcal{B})$ holds with $\tilde{L} := \sup_{t \geq 0} L_t$, then one has*

$$F_{\mathcal{F}}(\Phi) - F_{\mathcal{F}}(\bar{\Phi}) \in \ell^p(X) \quad \textit{with} \quad \|F_{\mathcal{F}}(\Phi) - F_{\mathcal{F}}(\bar{\Phi})\|_p \leq \tilde{L}(d+2)^{1/p} \, \|\Phi - \bar{\Phi}\|_p$$

for all $\Phi, \bar{\Phi} \in \ell^p(D)$.

Proof Using the assumption $(\mathcal{B})$ with $\tilde{L} := \sup_{t \geq 0} L_t$, we have

$$\begin{aligned}
\sum_{t \geq 0} \|F_{\mathcal{F}}(\Phi)_t - F_{\mathcal{F}}(\bar{\Phi})_t\|^p &= \sum_{t \geq 0} \|\mathcal{F}_t(\Phi_{t+1}, \Phi_t, \ldots, \Phi_{t-d}) - \mathcal{F}_t(\bar{\Phi}_{t+1}, \bar{\Phi}_t, \ldots, \bar{\Phi}_{t-d})\|^p \\
&\leq \sum_{t \geq 0} L_t^p \, \|(\Phi_{t+1}, \Phi_t, \ldots, \Phi_{t-d}) - (\bar{\Phi}_{t+1}, \bar{\Phi}_t, \ldots, \bar{\Phi}_{t-d})\|^p \\
&= \sum_{t \geq 0} L_t^p \sum_{i=t-d}^{t+1} \|\Phi_i - \bar{\Phi}_i\|^p \\
&\leq \tilde{L}^p \sum_{t \geq 0} \left(\|\Phi_{t-d} - \bar{\Phi}_{t-d}\|^p + \ldots + \|\Phi_{t+1} - \bar{\Phi}_{t+1}\|^p\right) \\
&\leq \tilde{L}^p (d+2) \|\Phi - \bar{\Phi}\|_p^p
\end{aligned}$$

and consequently

$$\|F_{\mathcal{F}}(\Phi) - F_{\mathcal{F}}(\bar{\Phi})\|_p \leq \tilde{L}(d+2)^{1/p}\|\Phi - \bar{\Phi}\|_p$$

for all $\Phi, \bar{\Phi} \in \ell^p(D)$. □

Lemma 2.14 (Lipschitz condition of $G_{\mathcal{F}}$) *Let $\Phi, \bar{\Phi} \in \ell^p(D)$ and $\xi_0, \ldots, \xi_d, \bar{\xi}_0, \ldots, \bar{\xi}_d \in D$. If the assumptions of Lemma 2.13 hold, then*

$$G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \ldots, \xi_d) \in \ell^p(X),$$
$$G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) - G_{\mathcal{F}}(\Phi, \bar{\xi}_0, \ldots, \bar{\xi}_d) \in \ell^p(X)$$

with

$$\|G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) - G_{\mathcal{F}}(\bar{\Phi}, \xi_0, \ldots, \xi_d)\|_p \leq \tilde{L}(d+2)^{1/p}\ \|\Phi - \bar{\Phi}\|_p,$$
$$\|G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) - G_{\mathcal{F}}(\Phi, \bar{\xi}_0, \ldots, \bar{\xi}_d)\|_p \leq \|\xi - \bar{\xi}\|_p.$$

Proof The proof of this result is similar to the proof of Lemma 2.9 or Lemma 2.4 using our suitable Lemma 2.13 and Lemma 2.12 *ii*). □

Theorem 2.15 (ℓ^p-attraction) *Suppose that the right-hand side $\mathcal{F}_t : D^{d+2} \to D$ of the delay difference equation (2.1) satisfies $(\mathcal{B})$ with $\tilde{L} := \sup_{t \geq 0} L_t$. If*

$$(d+2)^{1/p} \sup_{t \geq 0} L_t < 1$$

and $F_{\mathcal{F}}(0) \in \ell^p(D)$, then the following holds for all $\xi_0, \ldots, \xi_d \in D$:

(a) The delay difference equation (2.1) is well-posed.

(b) $\varphi(\xi_0, \ldots, \xi_d) \in \ell^p(D)$

(c) The inequality

$$\|\varphi(\xi_0, \ldots, \xi_d)_t - \varphi(\bar{\xi}_0, \ldots, \bar{\xi}_d)_t\| \leq \frac{1}{1 - (d+2)^{1/p} \sup_{t \geq 0} L_t}\|\xi - \bar{\xi}\|_p$$

holds for all $t \geq -d$, $\bar{\xi}_0, \ldots, \bar{\xi}_d \in D$.

Remark 2.16 *Our assumption $(2+d)^{1/p} \sup_{t \geq 0} L_t < 1$ covers the worst possible case. Indeed, as the alert reader concludes from the proof of Lemma 2.13, the constant $(2+d)^{1/p}$ can be replaced by $n^{1/p}$, where n is the number of variables in $\{u_{t+1}, u_t, \ldots, u_{t-d}\}$ on which $\mathcal{F}_t$ actually depends.*

Proof Given $\Phi, \bar{\Phi} \in \ell^p(D)$ and $\xi_0, \ldots, \xi_d$, $\bar{\xi}_0, \ldots, \bar{\xi}_d \in D$ we again proceed as in the proof of Theorem 2.5.

(a) The inclusion $G_{\mathcal{F}}(\Phi, \xi_0, \ldots, \xi_d) \in \ell^p(D)$ results from Lemma 2.13

(b) Now apply Proposition 1.2 with the complete metric space

$$Y := \ell^p(D), \qquad\qquad d_Y(\Phi, \bar{\Phi}) := \|\Phi - \bar{\Phi}\|_p.$$

Due to 2.14 the operator $G_{\mathcal{F}}(\cdot, \xi_0, \dots, \xi_d)$ has Lipschitz constant $L := (d+2)^{1/p} \sup_{t \geq 0} L_t < 1$ and $G_{\mathcal{F}}(\Phi, \cdot)$ possesses the Lipschitz constant $\lambda = 1$. Again, the assertion follows from Proposition 1.2.

(c) The inequality

$$\|\varphi(\xi_0, \dots, \xi_d)_t - \varphi(\bar{\xi}_0, \dots, \bar{\xi}_d)_t\| \leq \frac{\lambda}{1-L} \left(\sum_{i=0}^{d} \|\xi_i - \bar{\xi}_i\|^p \right)^{1/p}$$

holds thanks to Lemma 2.14.

This establishes the proof. □

3. Integrodifference equations

This section applies the abstract theory above to concrete integrodifference equations (shortly IDEs). We study a class of IDEs involving Hammerstein integral operators with Ricker and Beverton–Holt growth functions satisfying a global Lipschitz condition and being defined on the space of continuous functions over a compact domain.

First, we deal with scalar delay integrodifference equations of Urysohn type defined on the space of all continuous functions $C(\Omega)$ over a compact domain $\Omega \subseteq \mathbb{R}^\kappa$ equipped with the sup-norm $\|u\| := \sup_{x \in \Omega} |u(x)|$. Moreover, given a closed interval $I \in \{\mathbb{R}_+, \mathbb{R}\}$ we introduce the closed set $D := \{u \in C(\Omega) \mid u(x) \in I \text{ for all } x \in \Omega\}$.

We consider nonautonomous delay integrodifference equations of form (2.1) given by

$$u_{t+1}(x) = \int_\Omega f_t(x, y, u_{t+1}(y), u_t(y), \dots, u_{t-d}(y))dy + h_t(x) \tag{3.1}$$

with right-hand side defined as

$$\mathcal{F}_t(u_1, u_0, \dots, u_{-d}) := \int_\Omega f_t(\cdot, y, u_{t+1}(y), u_t(y), \dots, u_{t-d}(y))dy + h_t$$

under the following standing assumptions for all $t \in \mathbb{N}_0$:

- the kernel function $f_t : \Omega^2 \times I^{d+2} \to I$ is continuous,
- the inhomogeneity $h_t : \Omega \to I$ is continuous,
- there exists a continuous function $l_t : \Omega^2 \to \mathbb{R}_+$ such that for all $x, y \in \Omega$ and $z_1, z_0, \dots, z_{-d}, \bar{z}_1, \bar{z}_0, \dots, \bar{z}_{-d} \in I$, one has

$$|f_t(x, y, z_1, z_0, \dots, z_{-d}) - f_t(x, y, \bar{z}_1, \bar{z}_0, \dots, \bar{z}_{-d})| \leq l_t(x, y) \max_{-d \leq i \leq 1} |z_i - \bar{z}_i|$$

- and additionally $f_t(x, y, 0, \dots, 0) = 0$ for all $x, y \in \Omega$.

Lemma 3.1 *Under the above assumptions, the Urysohn integral operator $\mathcal{F}_t : D^{d+2} \to D$ is well-defined and satisfies*

$$\|\mathcal{F}_t(u_1, u_0, \dots, u_{-d}) - \mathcal{F}_t(\bar{u}_1, \bar{u}_0, \dots, \bar{u}_{-d})\| \le \sup_{x\in\Omega} \int_\Omega l_t(x,y)dy \max_{-d\le i\le 1} \|u_i - \bar{u}_i\|$$

for all $t \in \mathbb{N}_0$ and $u_1, u_0, \dots, u_{-d}, \bar{u}_1, \bar{u}_0, \dots, \bar{u}_{-d} \in D$.

Proof Let $t \in \mathbb{N}_0$. Using the above assumptions, we compute the Lipschitz constant of the integral operator $\mathcal{F}_t$ as follows:

$$\begin{aligned}
&|\mathcal{F}_t(u_1, u_0, \cdots, u_{-d})(x) - \mathcal{F}_t(\bar{u}_1, \bar{u}_0, \cdots, \bar{u}_{-d})(x)| \\
&\le \int_\Omega |f_t(x, y, u_1(y), u_0(y), \dots, u_{-d}(y)) - f_t(x, y, \bar{u}_1(y), \bar{u}_0(y), \dots, \bar{u}_{-d}(y))|dy \\
&\le \int_\Omega l_t(x,y) \max_{-d\le i\le 1} |u_i(y) - \bar{u}_i(y)|dy \\
&\le \int_\Omega l_t(x,y)dy \max_{-d\le i\le 1} \|u_i - \bar{u}_i\| \\
&\le \sup_{x\in\Omega} \int_\Omega l_t(x,y)dy \max_{-d\le i\le 1} \|u_i - \bar{u}_i\|,
\end{aligned}$$

and passing to the supremum over all $x \in \Omega$ yields the claim. □

Proposition 3.2 (ℓ^∞_ω-attraction for IDEs) *Under the above assumptions with*

i) $\Upsilon(\omega) \sup_{t\ge 0} \max\{\frac{\omega_{t+1}}{\omega_t}, 1, \frac{\omega_{t-1}}{\omega_t}, \dots, \frac{\omega_{t-d}}{\omega_t}\} \sup_{x\in\Omega} \int_\Omega l_t(x,y)dy < 1$ *and*

ii) $\sup_{t\ge 0} \sup_{x\in\Omega} |h_t(x)|\omega_t^{-1} < \infty$,

the assertions of Theorem 2.5 hold for the solutions of IDEs (3.1).

Proof With the above assumptions one has $L_t = \sup_{x\in\Omega} \int_\Omega l_t(x,y)dy$ and

$$F_{\mathcal{F}}(0)_t = \mathcal{F}_t(0, \cdots, 0) = h_t \in C(\Omega)$$

and Theorem 2.5 applies. □

Proposition 3.3 (c_0-attraction for IDEs) *Under the above assumptions with*

i) $\sup_{t\ge 0} \sup_{x\in\Omega} \int_\Omega l_t(x,y)dy < 1$ *and*

ii) $\lim_{t\to\infty} \sup_{x\in\Omega} |h_t(x)| = 0$,

the assertions of Theorem 2.10 hold for the solutions of IDEs (3.1).

Proof The proof follows as in Proposition 3.2, and Theorem 2.10 applies. □

Proposition 3.4 (ℓ^p-attraction for IDEs) *Under the above assumptions with* *i)* $(2+d)^{1/p}\sup_{t\geq 0} L_t < 1$ *and*

ii) $\sum_{t=0}^{\infty}\sup_{x\in\Omega}|h_t(x)|^p < \infty$,

the assertions of Theorem 2.15 hold for the solutions of IDEs (3.1).

Proof Thanks to Lemma 3.1 we have

$$\begin{aligned}\|\mathcal{F}_t(u_1,u_0,\ldots,u_{-d})-\mathcal{F}_t(\bar{u}_1,\bar{u}_0,\ldots,\bar{u}_{-d})\| &\leq \sup_{x\in\Omega}\int_\Omega l_t(x,y)dy \max_{-d\leq i\leq 1}\|u_i-\bar{u}_i\| \\ &\leq \sup_{x\in\Omega}\int_\Omega l_t(x,y)dy\Big(\sum_{i=-d}^{1}\|u_i-\bar{u}_i\|^p\Big)^{1/p}\end{aligned}$$

for all $t\in\mathbb{N}_0$ and $u_1,u_0,\ldots,u_{-d},\bar{u}_1,\bar{u}_0,\ldots,\bar{u}_{-d}\in D$. Given this the proof is akin to Proposition 3.2 and Theorem 2.15 applies. □

In the remaining paper we retreat to delay IDEs (3.1) of Hammerstein type, i.e.

$$\mathcal{F}_t(u_0,\ldots,u_{-d}) := \int_\Omega k_t(\cdot,y)g_t(u_0(y),\ldots,u_{-d}(y))dy + h_t$$

under the following standing assumptions for all $t\in\mathbb{N}_0$:

- the kernel $k_t:\Omega^2\to\mathbb{R}$ is continuous and satisfies $\sup_{x\in\Omega}\int_\Omega|k_t(x,y)|dy<\infty$,
- $h_t:\Omega^2\to\mathbb{R}$ is continuous,
- there exists a $\tilde{l}_t\geq 0$ such that the growth function $g_t:I^{d+1}\to I$ satisfies

  $$|g_t(z_0,\ldots,z_{-d})-g_t(\bar{z}_0,\ldots,\bar{z}_{-d})|\leq \tilde{l}_t\max_{-d\leq i\leq 0}|z_i-\bar{z}_i|$$

  for all $z_0,\ldots,z_{-d},\bar{z}_0,\ldots,\bar{z}_{-d}\in I$ and $g_t(0,\ldots,0)=0$.

We provide two concrete examples of such IDEs with Ricker and Beverton–Holt growth functions. Using the Nyström method with the trapezoidal quadrature rule (having 100 nodes), the solutions to those equations are illustrated in Figure 1-2.

Example 3.5 (Delay Ricker) *The delay IDE with Ricker growth function*

$$u_{t+1}(x) = a_t\int_{-2}^{2}\frac{1}{2}e^{-|x-y|}u_{t-d}(y)e^{-u_{t-d}(y)}dy+\cos(\frac{\pi x}{4}) \tag{3.2}$$

fits into our setting with $\Omega=[-2,2]$, $I:=\mathbb{R}_+$, *the cone*

$$D=\{u\in C[-2,2]: u(x)\geq 0 \text{ for all } x\in[-2,2]\}$$

and

$$\mathcal{F}_t(u_0,\ldots,u_{-d})(x) := a_t\int_\Omega k_t(x,y)u_{-d}(y)e^{-u_{-d}(y)}dy + h_t(x),$$

with

- *the Laplace kernel* $k_t(x,y) := \frac{1}{2}e^{-|x-y|}$,
- *the Ricker growth function* $g_t(z_0,\dots,z_{-d}) := a_t z_{-d} e^{-z_{-d}}$ *with positive coefficients* $a_t \in \mathbb{R}$,
- *the inhomogeneity* $h_t(x) := \cos(\frac{\pi x}{4})$ *being constant in time.*

For the Lipschitz constant of the growth function one finds

$$|g_t(z_0,\dots,z_{-d}) - g_t(\bar{z}_0,\dots,\bar{z}_{-d})| \le C a_t |z_{-d} - \bar{z}_{-d}|$$

with $C := \max_{z\ge 0} z e^{-z} = e^{-1}$ *and thus* $\tilde{l}_t = e^{-1}a_t$. *Hence, for the Lipschitz constant of* $\mathcal{F}_t$ *one has*

$$\begin{aligned}\|\mathcal{F}_t(u_0,\dots,u_{-d}) - \mathcal{F}_t(\bar{u}_0,\dots,\bar{u}_{-d})\| &\le \tilde{l}_t \sup_{x\in\Omega}\int_\Omega |k_t(x,y)|dy \|u_{-d} - \bar{u}_{-d}\| \\ &\le L_t \|u_{-d} - \bar{u}_{-d}\|\end{aligned}$$

with $L_t := \tilde{l}_t \sup_{x\in\Omega}\int_\Omega |k_t(x,y)|dy = \frac{e^{-1}a_t}{1-e^{-2}}$ *for all* $t \in \mathbb{N}_0$. *We choose* $a_t = \frac{1}{2}\frac{e^1}{1-e^{-2}}$, *then for instance Proposition 3.2 applies with the weight sequence* $\omega_t \equiv 1$ *and guarantees that all forward solutions are bounded. In Figure 1 we illustrate the behaviour of a solution to the delay Ricker IDE (3.2) with delay* $d = 10$ *or* $d = 20$ *and the initial conditions* $\xi_t(x) = -\frac{t}{6}x^2 + 13$ *for* $t = 0,\dots,20$, *which apparently converges.*

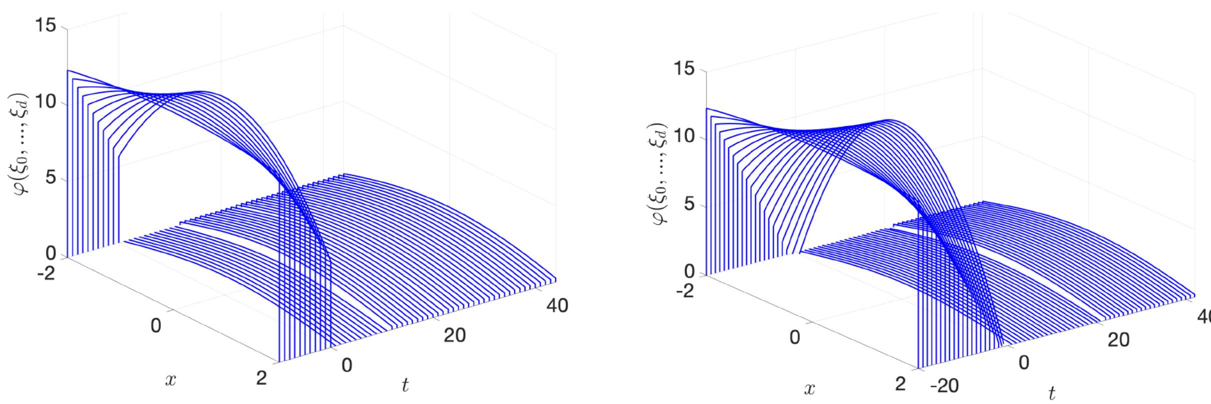

Figure 1. The solution $\varphi(\xi_0,\xi_1,\dots,\xi_d)$ to the inhomogeneous delay Ricker IDE (3.2) with $d = 10$ (left), $d = 20$ (right) and constant inhomogeneity $h_t(x) = \cos(\frac{\pi x}{4})$ exhibiting convergent behaviour.

In the next example we show the delay Beverton–Holt IDE. The delay Beverton–Holt equation without spatial effects is considered in [3]. Apparently, this model coincides with the classical Beverton–Holt model for $u_{t-d}(y) \ge 0$. We however wanted to provide a C^1 nonlinearity being bounded on $\mathbb{R}$.

Example 3.6 (Delay Beverton–Holt) *The delay IDE with Beverton–Holt growth function*

$$u_{t+1}(x) = a_t \int_{-2}^{2} \frac{1}{\sqrt{\pi}} e^{-(x-y)^2} \frac{u_{t-d}(y)}{1+|u_{t-d}(y)|} dy + \frac{1}{\sqrt{1+t}} \sin(\frac{t\pi x}{4}) \tag{3.3}$$

fits into our setting with $\Omega = [-2,2]$, $I := \mathbb{R}$, $D = C[-2,2]$ *and*

$$\mathcal{F}_t(u_0, \ldots, u_{-d})(x) := a_t \int_\Omega k_t(x,y) \frac{u_{t-d}(y)}{1 + |u_{t-d}(y)|} dy + h_t(x),$$

where

- *the Gauss kernel* $k_t(x,y) := \frac{1}{\sqrt{\pi}} e^{-(x-y)^2}$,
- *the Beverton–Holt growth function* $g_t(z_0, \ldots, z_{-d}) := a_t \frac{z_{-d}}{1+|z_{-d}|}$ *with reals* a_t,
- *the inhomogeneity* $h_t(x) := \frac{1}{\sqrt{1+t}} \sin(\frac{t\pi x}{4})$,

The Lipschitz constant of the growth function is $C := \max_z \frac{z}{1+|z|} = 1$. *Therefore, one has* $l_t = |a_t|$ *and* $L_t = l_t \sup_{x\in\Omega} \int_{-2}^{2} k_t(x,y)dy = |a_t| \operatorname{erf}(2)$. *We choose* $a_t = \frac{1}{2\operatorname{erf}(2)}$ *and depending on the decay behaviour of* $(h_t)_{t\in\mathbb{N}_0}$ *the respective Propositions 3.2–3.4 apply. In Figure 2 we illustrate a solution to the delay Beverton–Holt IDE* (3.3) *with delay* $d = 10$ *and the initial conditions* $\xi_t(x) = t$ *for* $t = 0, \ldots, 10$. *The inhomogeneity* $h_t(x) = \frac{1}{\sqrt{1+t}} \sin(\frac{t\pi x}{4})$ *fits both into the setting of*

- *Proposition 3.3 (all solutions tend to* 0 *as* $t \to \infty$*), as well as*
- *Proposition 3.4 (all solutions are* p*-summable with* $p > 2$*); note here that the contractivity condition* $(2+d)^{1/p} \sup_{t\geq 0} L_t < 1$ *simplifies to* $\sup_{t\geq 0} L_t < 1$ *because the growth function depends only on one variable (see Remark 2.16).*

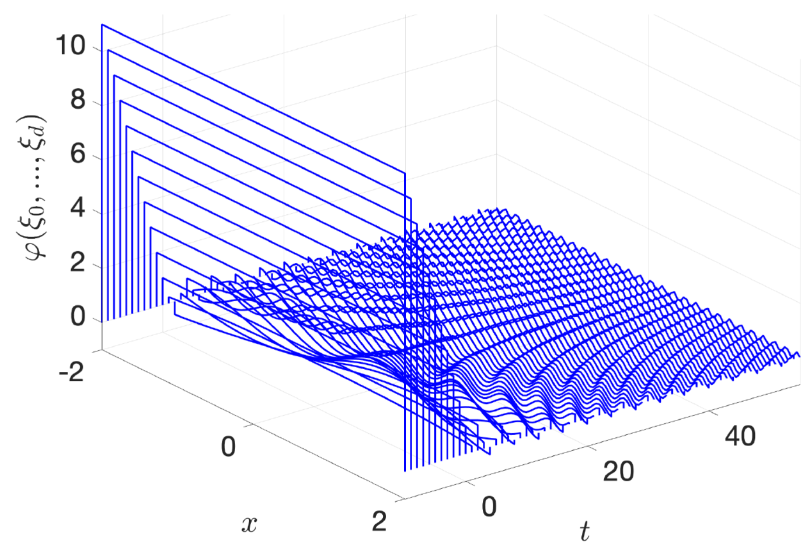

Figure 2. The solution $\varphi(\xi_0, \xi_1, \ldots, \xi_d)$ to the delay Beverton–Holt IDE (3.3) with $d = 10$ and inhomogeneity $h_t(x) = \frac{1}{\sqrt{1+t}} \sin(\frac{t\pi x}{4})$.

Acknowledgment

I am deeply grateful to the editors and the reviewers for their support as well as helpful comments and suggestions on the submitted manuscript. This research project has been supported by the Austrian Science Fund (FWF) under Grant Number P 30874-N35.

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