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General characteristics of a fractal Sturm–Liouville problem

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Abstract: In this paper, we consider a regular fractal Sturm–Liouville boundary value problem. We prove the self-adjointness of the differential operator which is generated by the F^α-derivative introduced in [32]. We obtained the F^α-analogue of Liouville's theorem, and we show some properties of eigenvalues and eigenfunctions. We present examples to demonstrate the efficiency and applicability of the obtained results. The findings of this paper can be regarded as a contribution to an emerging field.

Key words: Fractal calculus, fractal derivative, Sturm–Liouville problem, eigenvalues, eigenfunctions

1. Introduction

Sturm–Liouville problems which have been successfully applied to many fields of science, engineering, and mathematics have shown a considerable development since they were first introduced over 180 years ago [41].

A standard form of the Sturm–Liouville differential equation is given as

$$-\frac{d}{dx}(p(x)\frac{df}{dx}) + q(x)f = \lambda w(x)f,$$

where $p(x)$, $q(x)$, and $w(x)$ are precisely defined functions based on the studies considered, and they are required to satisfy additional conditions.

One of the most important boundary value conditions for this equation are

$$f(a)\cos\alpha + f'(a)\sin\alpha = 0,$$

$$f(b)\cos\beta + f'(b)\sin\beta = 0$$

where α and β are two arbitrary real numbers.

A Sturm–Liouville problem is said to be regular if the interval $[a, b]$ is finite, and the function $q(x)$ in the differential equation is summable on it. Otherwise, if $[a, b]$ is infinite, or if $q(x)$ is not summable on the interval, or both, the Sturm–Liouville problem is said to be singular [26]. Much is already known about Sturm–Liouville problems. Early developments can be found in [28].

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It is a very well-recognized fact that fractals can model many structures found in nature (see [4, 27]) and they are often too irregular to have any smooth differentiable structure defined on them, which results in delivering the methods and techniques of ordinary calculus powerless and inapplicable. Some approaches have been developed to deal with this inapplicability by means of fractional derivatives [6, 10, 21, 23, 24, 34, 38, 42], fractional spaces [1, 7, 20, 35], harmonic analysis [2, 5, 8, 22, 25, 37, 39], and measure theory, non-standard methods, and stochastic process [3, 29–31, 36, 40]. Yet, there has still been a gap in the literature on how to develop an appropriate calculus mainly related to the disconnected fractal subsets of $\mathbb{R}$.

In [32], Parvate and Gangal introduced a new calculus based on fractal subsets of the real line. In this calculus, an integral of order α, $0 < \alpha \leq 1$, called F^α-integral is defined which makes it possible to integrate functions with fractal support F of dimension α. Moreover, a derivative of order α, $0 < \alpha \leq 1$, called F^α-derivative is introduced, which allows us to differentiate functions like the Cantor staircase by changing only on a fractal set. Unlike the classical fractional derivative, the F^α-derivative is local and the F^α-calculus preserves much of the simplicity of ordinary calculus.

Studies concerning the applications of fractal calculus have been an important area of research in recent years. For instance, Golmankhaneh has presented a review and summary of applications in classical mechanics, quantum mechanics and optics in [14], Golmankhaneh and Cattani have introduced the fractal Euler method in order to solve fractal differential equations in [15], Golmankhaneh and Fernandez have defined the integral and derivative of functions on Cantor tartan spaces of different dimensions in [13], Golmankhaneh and Tunç have introduced the analogues of Laplace and Sumudu transforms on fractal calculus in [12], and also we may refer to [11, 16, 17] for other relevant studies.

In this paper, we consider a regular fractal Sturm–Liouville boundary value problem consisting of the equation

$$l^\alpha(f) := -(D_F^\alpha)^2 f(x) + q(x)f(x) = \lambda f(x), \quad x \in [0,\pi], \tag{1.1}$$

together with the boundary conditions

$$U(f) := D_F^\alpha f(0) - hf(0) = 0, \tag{1.2}$$

$$V(f) := D_F^\alpha f(\pi) + Hf(\pi) = 0. \tag{1.3}$$

Here D_F^α indicates the F^α-derivative which was introduced in [32], λ is a spectral parameter, $q(x)$, h and H are real, and $q(x) \in \mathcal{L}_2^\alpha(0,\pi)$ where $\mathcal{L}_2^\alpha(0,\pi)$ is the space of square F^α-integrable functions on $(0,\pi)$, i.e.

$$\int_0^\pi \left|f(x)\right|^2 d_f^\alpha x < \infty$$

holds for $f : [0,\pi] \to \mathbb{R}$ as $Sch(f)$ is an α-perfect set. The space $\mathcal{L}_2^\alpha(0,\pi)$ is a Hilbert space associated with the inner product

$$\langle f|g\rangle = \int_0^\pi f(x)g(x)d_F^\alpha x.$$

We deal with some of the spectral properties of the boundary value problem (1.1)–(1.3). To the best of our knowledge, no work has considered this problem in the manner of F^α-derivative.

We intend that the paper be self-contained; thus, in the next section, we initiate the paper by stating some sufficient terminology from the fractal calculus so the reader does not need previous familiarity. Section

3 is devoted to establish the main results of the paper. In Section 4, we provide some examples to demonstrate the effectiveness of the obtained results, and in Section 5, we close the paper by some concluding remarks. The findings in the paper can be regarded as a contribution to an emerging field.

2. Prelimininaries

In this section, we introduce some of F^{α}-calculus notations which will be used throughout the paper. We borrow the standard notations found in [32].

Definition 2.1 (Definition 1, [32]) *Let F be a fractal subset of $I=[a,b]\subset\mathbb{R}$, then the flag function $\theta(F,I)$ for a set and a closed interval I is given by*

$$\theta(F,I)=\begin{cases}1, & \text{if } F\cap I\neq\emptyset,\\ 0, & \text{otherwise}.\end{cases}$$

Definition 2.2 (Definition 2, [32]) *A subdivision $P_{[a,b]}$ of the interval $[a,b]$, $a<b$ is a finite set of points $\{a=x_0,x_1,x_2,\cdots,x_n=b\}$, $x_i<x_{i+1}$. Any interval of the form $[x_i,x_{i+1}]$ is called a component interval or just a component of the subdivision P. If Q is any subdivision of $[a,b]$ and $P\subset Q$, then we say Q is a refinement of P. If $a=b$, then the set $\{a\}$ is the only subdivision of $[a,b]$.*

Definition 2.3 *[16] For a set F and subdivision $P_{[a,b]}$, $a<b$, $\sigma^{\alpha}[F,P]$ is defined as follows:*

$$\sigma^{\alpha}[F,P]=\sum_{i=0}^{n-1}\Gamma(\alpha+1)\,(x_{i+1}-x_i)^{\alpha}\,\theta\big(F,[x_i,x_{i+1}]\big).$$

If $a=b$, $\sigma^{\alpha}[F,P]$ is defined to be zero.

Now, we introduce the coarse-grained mass, which paves our way to define the mass function in the sequel.

Definition 2.4 (Definition 4, [32]) *Given $\delta>0$ and $a\leq b$, the coarse grained mass $\gamma_{\delta}^{\alpha}(F,a,b)$ of $F\cap[a,b]$ is given by*

$$\gamma_{\delta}^{\alpha}(F,a,b)=\inf_{\{P_{[a,b]}:|P|\leq\delta\}}\sigma^{\alpha}[F,P] \tag{2.1}$$

where $|P|=\max_{0\leq i\leq n-1}(x_{i+1}-x_i)$ for a subdivision P, and the infimum in (2.1) is taken over all subdivisions P of $[a,b]$ satisfying $|P|\leq\delta$.

The mass function is the limit of the coarse-grained mass as $\delta\to\infty$:

Definition 2.5 (Definition 8, [32]) *The mass function $\gamma^{\alpha}(F,a,b)$ is given by*

$$\gamma^{\alpha}(F,a,b)=\lim_{\delta\to 0}\gamma_{\delta}^{\alpha}(F,a,b).$$

Here we note that since $\gamma_{\delta}^{\alpha}(F,a,b)$ increases as δ decreases, $\gamma^{\alpha}(F,a,b)$ always exists and is a nonnegative number which may possibly be $+\infty$.

Now, we introduce one of the central notions of fractal calculus, the integral staircase function for a set F of the order α. This function, which is a generalization of functions like the Lebesgue-Cantor staircase function, describes how the mass of $F \cap [a, b]$ increases as b increases.

Definition 2.6 (Definition 15, [32]) *Let a_0 be an arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$ of order α for a set F is given by*

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F, a_0, x), & \textit{if } x \geq a_0, \\ -\gamma^\alpha(F, x, a_0), & \textit{otherwise}. \end{cases}$$

Now, we are ready to consider the sets for which the mass function $\gamma^\alpha(F, a, b)$ gives the most useful information.

Definition 2.7 (Definition 17, [32]) *The γ-dimension of $F \cap [a, b]$ denoted by $dim_\gamma\big(F \cap [a, b]\big)$ is defined as*

$$dim_\gamma\big(F \cap [a, b]\big) = inf\,\{\alpha : \gamma^\alpha(F, a, b) = 0\} = sup\,\{\alpha : \gamma^\alpha(F, a, b) = \infty\}\,.$$

We say that a point x is a point of change of a function f, if f is not constant over any open interval (c, d) containing x. The set of all points of change of f is called the set of change of f and is denoted by $Schf$.

Let $Sch(S_F^\alpha)$ be a closed set and every point of it be a limit point, then the set $Sch(S_F^\alpha)$ is said to be α-perfect.

In the following, we introduce the notation for limit and continuity using the topology of $F \subset \mathbb{R}$ with the metric inherited from $\mathbb{R}$.

Definition 2.8 (Definition 27, [32]) *Let $F \subset \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $x \in F$. A number l is said to be the limit of f through the points of F, or simply F-limit of f, as $y \to x$, if given any $\epsilon > 0$, there exists $\delta > 0$ such that $y \in F$ and $|y - x| < \delta \Rightarrow |f(y) - l| < \epsilon$. If such a number exists, then it is denoted by $l = \lim_{y \to x} f(y)$.*

This definition does not involve the values $f(y)$ if $y \notin F$. Also, F-limit is not defined at points $x \notin F$.

We now introduce the notion of F-continuity which is continuity as far as the values of the function only on the set F are concerned.

Definition 2.9 (Definition 28, [32]) *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be F-continuous at $x \in F$ if $f(x) = F - \lim\limits_{y \to x} f(y)$ holds.*

We note that the notion of F-continuity is not defined at $x \notin F$. It is clear that continuity of $f : \mathbb{R} \to \mathbb{R}$ at $x \in F$ implies F-continuity at x. However, the converse is not true.

If F is an α-perfect set, then the F^α-derivative of f at the point x is defined as

$$D_F^\alpha f(x) = \begin{cases} F - \lim\limits_{y \to x} \frac{f(y) - f(x)}{S_F^\alpha(y) - S_F^\alpha(x)}, & \texttt{if } x \in F, \\ 0, & \texttt{otherwise}, \end{cases}$$

if the limit exists. The linearity of the F^α-derivative can be easily shown, i.e.

$$D_F^\alpha(af + bg)(x) = aD_F^\alpha f(x) + bD_F^\alpha g(x)$$

holds for F^α-differentiable functions f, g and any arbitrary real numbers a, b.

The F^α-analogue of Leibniz rule will be needed later.

Theorem 2.10 (Theorem 55, [32]) *If the functions $f, g : \mathbb{R} \to \mathbb{R}$ are two F^α-differentiable functions, then the multiplication of these two functions is also F^α-differentiable and*

$$D_F^\alpha(fg)(x) = D_F^\alpha f(x) \cdot g(x) + f(x) \cdot D_F^\alpha g(x)$$

holds.

For a bounded function f on $F \cap [a,b]$, the fractal integral is defined as $g(x) = \int_a^x f(y) d_F^\alpha y$ for all $x \in [a,b]$ and $D_F^\alpha g(x) = f(x)\chi_F(x)$, where $\chi_F(x)$ is the characteristic function of $F \subset \mathbb{R}$.

Theorem 2.11 (Theorem 57, [32]) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, F^α-differentiable function such that $Sch(f)$ is continuous in an α-perfect set F and $h : \mathbb{R} \to \mathbb{R}$ be a F-continuous such that $h(x)\chi_F(x) = D_F^\alpha f(x)$. Then*

$$\int_a^b h(x) d_F^\alpha x = f(b) - f(a).$$

The following theorem states that the F^α-integration can be performed by parts.

Theorem 2.12 (Theorem 58, [32]) *Let the functions $f, g : \mathbb{R} \to \mathbb{R}$ functions such that u is continuous on $[a,b]$ and $Sch(f) \subset F$, $D_F^\alpha f$ exists and is F^α-continuous on $[a,b]$, g is continuous on $[a,b]$. Then,*

$$\int_a^b f(x)g(x) d_F^\alpha x = \left[f(x) \int_a^x f(x') d_F^\alpha x' \right]_a^b - \int_a^b D_F^\alpha f(x) \int_a^x g(x') d_F^\alpha x' d_F^\alpha x$$

holds.

3. Main results

In this section, we establish the main results of the paper. The first theorem is concerned with the self-adjointness of the operator l^α which was defined in (1.1). For a sufficient condition of an operator to be self-adjoint, one may also refer to [18].

Theorem 3.1 *The operator l^α is self-adjoint in $\mathcal{L}_2^\alpha(0,\pi)$.*

Proof Let $f = f(x,\lambda)$ and $g = g(x,\lambda)$ be the solutions of the boundary value problem (1.1)–(1.3). Using the definition of the inner product in $\mathcal{L}_2^\alpha(0,\pi)$ and integrating by part twice yield

$$\begin{aligned}
\langle l^\alpha f | g \rangle &= \int l^\alpha f(x,\lambda) g(x,\lambda) d_f^\alpha x = -\int_0^\pi \left(D_F^\alpha\right)^2 f(x,\lambda) g(x,\lambda) d_F^\alpha x + \int_0^\pi q(x) f(x,\lambda) g(x,\lambda) d_F^\alpha x \\
&= -\left[g(x,\lambda) D_F^\alpha f(x,\lambda)|_0^\pi - \int_0^\pi D_F^\alpha f(x,\lambda) D_F^\alpha g(x,\lambda) d_F^\alpha x \right] + \int_0^\pi q(x) f(x,\lambda) g(x,\lambda) d_F^\alpha x \\
&= -\left[g(x,\lambda) D_F^\alpha f(x,\lambda)|_0^\pi - \left(D_F^\alpha g(x,\lambda) f(x,\lambda)|_0^\pi - \int_0^\pi \left(D_F^\alpha\right)^2 g(x,\lambda) f(x,\lambda) d_F^\alpha x \right) \right] \\
&\quad + \int_0^\pi q(x) f(x,\lambda) g(x,\lambda) d_F^\alpha x.
\end{aligned}$$

Since the functions f and g are the solutions of the boundary value problem (1.1)–(1.3), they satisfy the boundary conditions (1.2), (1.3) and therefore

$$-\, g(x,\lambda) D_F^\alpha f(x,\lambda)|_0^\pi + D_F^\alpha g(x,\lambda) f(x,\lambda)|_0^\pi = 0$$

holds. Thus, we have

$$\begin{aligned}\left\langle l^{\alpha} f|g\right\rangle &= -\int_0^{\pi}\left(D_F^{\alpha}\right)^2 g(x,\lambda)f(x,\lambda)d_F^{\alpha}x + \int_0^{\pi} q(x)f(x,\lambda)g(x,\lambda)d_F^{\alpha}x.\\ &= \int_0^{\pi}\left[-\left(D_F^{\alpha}\right)^2 g(x,\lambda)+q(x)g(x,\lambda)\right]f(x,\lambda)d_F^{\alpha}x = \int_0^{\pi} f(x,\lambda)l^{\alpha}g(x,\lambda)d_F^{\alpha}x = \left\langle f|l^{\alpha}g\right\rangle\end{aligned}$$

and this completes the proof. □

Definition 3.2 *The values of the parameter λ for which boundary value problem (1.1)-(1.3) has nonzero solutions are called the eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions.*

The following results are straightforward from Theorem 3.1.

Corollary 3.3 *The eigenfunctions related to different eigenvalues are orthogonal in $\mathcal{L}_2^{\alpha}(0,\pi)$.*

Corollary 3.4 *The eigenvalues $\{\lambda_n\}$ and the eigenfunctions $f(x,\lambda_n)$, $g(x,\lambda_n)$ are real.*

Let $f(x,\lambda)$ and $g(x,\lambda)$ be the solutions of (1.1) under the initial conditions

$$f(0,\lambda)=1,\ D_F^{\alpha}f(0,\lambda)=h,\ g(\pi,\lambda)=1,\ D_F^{\alpha}g(\pi,\lambda)=-H. \tag{3.1}$$

Clearly,

$$U(f):=D_F^{\alpha}f(0)-hf(0)=0, \quad V(g):=D_F^{\alpha}g(\pi)+Hg(\pi)=0. \tag{3.2}$$

Denote

$$\Delta(\lambda)=\langle g(x,\lambda),f(x,\lambda)\rangle, \tag{3.3}$$

where

$$\langle f,g\rangle = W^{\alpha}(f,g)=fD_F^{\alpha}g-gD_F^{\alpha}f$$

is the α-Wronskian of f and g.

Now, let us consider the fractal differential equation of second order

$$(D_F^{\alpha})^2 f(x)+p_1(x)D_F^{\alpha}f(x)+p_2(x)f(x)=0. \tag{3.4}$$

The following theorem will be useful in the sequel.

Theorem 3.5 (Analogue of Liouville's Theorem) *Let $z_1(x)$ and $z_2(x)$ be any solutions of (3.4). Then the α-Wronskian of these two functions is given by*

$$W^{\alpha}(x)=W^{\alpha}(x_0)exp\left\{-\int_{x_0}^{x}p_1(x')d_F^{\alpha}x'\right\}, \tag{3.5}$$

here x_0 is an arbitrary number.

Proof Since $z_1(x)$ and $z_2(x)$ are the solutions of (3.4), we have

$$(D_F^{\alpha})^2 z_1(x)+p_1(x)D_F^{\alpha}z_1(x)+p_2(x)z_1(x)=0, \tag{3.6}$$

$$(D_F^\alpha)^2 z_2(x) + p_1(x) D_F^\alpha z_2(x) + p_2(x) z_2(x) = 0. \tag{3.7}$$

We know that the α-Wronskian of the functions z_1 and z_2 is

$$W^\alpha(x) = z_1 D_F^\alpha z_2 - z_2 D_F^\alpha z_1.$$

If we apply the F^α-derivative on $W^\alpha(x)$, we get

$$D_F^\alpha W^\alpha(x) = D_F^\alpha\big(z_1 D_F^\alpha z_2 - z_2 D_F^\alpha z_1\big) = z_1 (D_F^\alpha)^2 z_2 - z_2 (D_F^\alpha)^2 z_1. \tag{3.8}$$

From equations (3.6) and (3.7), we have

$$(D_F^\alpha)^2 z_1(x) = -p_1(x) D_F^\alpha z_1(x) - p_2(x) z_1(x),$$

and

$$(D_F^\alpha)^2 z_2(x) = -p_1(x) D_F^\alpha z_2(x) - p_2(x) z_2(x)$$

, respectively. Substituting the LHS of the last two equations in the RHS of (3.8), we have

$$D_F^\alpha W^\alpha(x) = -p_1(x)\big(z_1 D_F^\alpha z_2 - z_2 D_F^\alpha z_1\big) = -p_1(x) W^\alpha(x).$$

Thus, we have

$$\frac{D_F^\alpha W^\alpha(z_1, z_2)}{W^\alpha(z_1, z_2)} = -p_1(x).$$

Using conjugacy between ordinary and fractal calculus [19, 33] and integrating yields

$$W^\alpha(x) = W^\alpha(x_0) exp\left\{-\int_{x_0}^{x} p_1(x') d_F^\alpha x'\right\}.$$

This completes the proof. □

By virtue of Theorem 3.5, $\langle g(x,\lambda), f(x,\lambda)\rangle$ does not depend on x. The function $\Delta(\lambda)$ which is defined by Eq. (3.3) is called the characteristic function of the boundary value problem L^α. Substituting $x = 0$ and $x = \pi$ into (3.3), we get

$$\Delta(\lambda) = V(f) = -U(g). \tag{3.9}$$

The function $\Delta(\lambda)$ is entire in λ and it has an at most countable set of zeros $\{\lambda_n\}$.

Theorem 3.6 *The zeros $\{\lambda_n\}$ of the characteristic function coincide with the eigenvalues of the boundary value problem (1.1)–(1.3). The functions $f(x,\lambda_n)$ and $g(x,\lambda_n)$ are eigenfunctions, and there exists a sequence $\{\beta_n\}$ such that*

$$g(x,\lambda_n) = \beta_n f(x,\lambda_n), \quad \beta_n \neq 0. \tag{3.10}$$

holds.

Proof The proof can be done similar to [9, Theorem 1.1.1, pg. 6]. Indeed, let λ_0 be a zero of the characteristic function $\Delta(\lambda)$. Then, by virtue of (3.2), (3.3), and (3.9), we have $g(x,\lambda_0) = \beta_0 f(x,\lambda_0)$, and the functions

$g(x,\lambda_0)$ and $f(x,\lambda_0)$ satisfy the boundary conditions (1.2), (1.3). Hence, λ_0 is an eigenvalue, and $g(x,\lambda_0)$ and $f(x,\lambda_0)$ are eigenfunctions related to λ_0.

Now, let λ_0 be an eigenvalue of (1.1)–(1.3), and let y_0 be the corresponding eigenfunction. Then $U(y_0) = V(y_0) = 0$ holds. Clearly, $y_0(0) \neq 0$. Without loss of generality, we take $y_0(0) = 1$. Then $y_0'(0) = h$, and consequently $y_0(x) \equiv f(x,\lambda_0)$. Therefore, (3.9) yields $\Delta(\lambda_0) = V\big(f(x,\lambda_0)\big) = V(y_0(x)) = 0$. We have also proved that for each eigenvalue, there exists only one (up to a multiplicative constant) eigenfunction. □

The weight numbers of the boundary value problem (1.1)–(1.3) are denoted by

$$\alpha_n := \int_0^\pi f^2(x,\lambda_n) d_F^\alpha x. \tag{3.11}$$

The numbers $\{\lambda_n, \alpha_n\}$ are called the spectral data of (1.1)–(1.3).

Lemma 3.7 *The following relation holds:*

$$\beta_n \alpha_n = \dot{\Delta}(\lambda_n) \tag{3.12}$$

where the numbers β_n are defined by (3.10) and $\dot{\Delta}(\lambda)$ represents the F^α-derivative of the characteristic function with respect to λ.

Proof Since,

$$-(D_F^\alpha)^2 g(x,\lambda) + q(x) g(x,\lambda) = \lambda g(x,\lambda),$$

$$-(D_F^\alpha)^2 f(x,\lambda_n) + q(x) f(x,\lambda_n) = \lambda_n f(x,\lambda_n)$$

we have

$$\begin{aligned} D_F^\alpha \langle g(x,\lambda), f(x,\lambda_n)\rangle \quad &= g(x,\lambda)(D_F^\alpha)^2 f(x,\lambda_n) - f(x,\lambda_n)(D_F^\alpha)^2 g(x,\lambda) \\ &= g(x,\lambda)[q(x) f(x,\lambda_n) - \lambda_n f(x,\lambda_n)] - f(x,\lambda_n)[q(x) g(x,\lambda) - \lambda g(x,\lambda)] \\ &= (\lambda - \lambda_n) f(x,\lambda_n) g(x,\lambda) \chi_F(x). \end{aligned}$$

Here χ_F is the characteristic function of an α-perfect set $F \subset \mathbb{R}$. Integrating this last equation from 0 to π, using Theorem 2.11 and taking conditions (3.1) with equation (3.9) into consideration yields

$$\begin{aligned} (\lambda - \lambda_n) \textstyle\int_0^\pi g(x,\lambda) f(x,\lambda_n) d_f^\alpha x \quad &= \langle g(x,\lambda), f(x,\lambda_n)\rangle|_0^\pi \\ &= D_F^\alpha f(\pi,\lambda_n) + H f(\pi,\lambda_n) + D_F^\alpha g(0,\lambda) - h g(0,\lambda) \\ &= -\Delta(\lambda). \end{aligned}$$

For $\lambda \to \lambda_n$, this last equation yields

$$\int_0^\pi g(x,\lambda_n) f(x,\lambda_n) d_F^\alpha x = -\dot{\Delta}(\lambda_n).$$

By using (3.10) and (3.11), we arrive at (3.12), which completes the proof. □

Taking the definitions of β_n and α_n into consideration, we have the immediate conclusion of Lemma 3.7 as the following corollary.

Corollary 3.8 *All zeros of $\Delta(\lambda)$ are simple, i.e. $\dot{\Delta}(\lambda_n) \neq 0$ holds.*

4. Examples

In this section, we construct examples to illustrate the effectiveness of our theoretical results.

Example 4.1 *Consider the following fractal Sturm–Liouville boundary value problem*

$$-D_F^{2\alpha}y(x)=\lambda y(x), \tag{4.1}$$

$$y(0)+D_F^{\alpha}y(0)=0,\quad y(1)-D_F^{\alpha}y(1)=0. \tag{4.2}$$

If $\lambda=0$, then the general solution of the boundary value problem (4.1), (4.2) is

$$y(x)=AS_F^{\alpha}(x)+B,$$

where $S_F^{\alpha}(x)$ is the integral staircase function which was given in Definition 6, and A and B are arbitrary real numbers. If $\lambda<0$, then the general solution of the boundary value problem (4.1), (4.2) is

$$y(x)=A\exp\big(\sqrt{-\lambda}S_F^{\alpha}(x)\big)+B\exp\big(-\sqrt{-\lambda}S_F^{\alpha}(x)\big),$$

and if $\lambda>0$, then the general solution of the boundary value problem (4.1), (4.2) is

$$y(x)=A\cos\big(\sqrt{\lambda}S_F^{\alpha}(x)\big)+B\sin\big(\sqrt{\lambda}S_F^{\alpha}(x)\big).$$

Example 4.2 *Consider the following fractal Sturm–Liouville boundary value problem*

$$D_F^{2\alpha}y(x)+3D_F^{\alpha}y(x)+2y(x)+\lambda y(x)=0, \tag{4.3}$$

$$y(0)=0,\quad y(1)=0. \tag{4.4}$$

The characteristic equation of equation (4.3) is

$$r^2+3r+2+\lambda=0,$$

with zeros

$$r_1=\frac{-3+\sqrt{1-4\lambda}}{2},\quad r_2=\frac{-3-\sqrt{1-4\lambda}}{2}.$$

If $\lambda<\frac{1}{4}$, then r_1 and r_2 are real and distinct, so the general solution of equation (4.3) is

$$y(x)=A\exp\big(r_1S_F^{\alpha}(x)\big)+B\exp\big(r_2S_F^{\alpha}(x)\big),$$

where $S_F^{\alpha}(x)$ is the integral staircase function which was given in Definition 2.6, and A and B are arbitrary real numbers. If $\lambda=\frac{1}{4}$, then $r_1=r_2=-\frac{3}{2}$, so the general solution of equation (4.3) is

$$y(x)=\exp\big(\frac{-3S_F^{\alpha}(x)}{2}\big)+B\exp\big(A+BS_F^{\alpha}(x)\big),$$

and if $\lambda > \dfrac{1}{4}$, *then* $r_1 = -\dfrac{3}{2} + iw$ *and* $r_2 = -\dfrac{3}{2} - iw$, *where*

$$w = \frac{\sqrt{4\lambda - 1}}{2}, \tag{4.5}$$

and the general solution of equation (4.3) is

$$y(x) = \exp\big(\frac{-3S_F^{\alpha}(x)}{2}\big)\big(\sin\big(nwS_F^{\alpha}(x)\big)\big), \quad n = 1, 2, 3, \cdots$$

With the help of the boundary conditions and (4.5), the eigenvalues of this case can be calculated as

$$\lambda_n = \frac{\big(1 + 4n^2\pi^2\big)}{4}.$$

Remark 4.3 *We note that if we set* $\alpha = 1$, *all the results lead to the standard ones.*

5. Conclusion

In this paper we deal with a fractal Sturm–Liouville problem on a finite interval. We establish some of the spectral properties of this problem, such as self-adjointness of the differential operator, orthogonality of the eigenfunctions corresponding to different eigenvalues and realness of the eigenvalues. We present examples to demonstrate the effectiveness of the obtained results. We believe that this paper will play an important role to initiate studies related to fractal Sturm–Liouville problems.

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