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On congruences related to trinomial coefficients and harmonic numbers

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Abstract: In this paper, we establish some congruences involving the trinomial coefficients and harmonic numbers. For example, for any prime $p > 3$,

$$\sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k}_2 H_k \equiv 0 \pmod{p}.$$

Key words: Congruences, binomial coefficients, trinomial coefficients, harmonic numbers

1. Introduction

The harmonic numbers H_n are defined by

$$H_0 = 0 \text{ and } H_n = \sum_{i=1}^{n} \frac{1}{i} \text{ for } n \geq 1.$$

In [4], for arbitrary integer $m \geq 1$ and complex number n,

$$\left(1 + x + x^2 + \cdots + x^m\right)^n := \sum_{k \geq 0} \binom{n}{k}_m x^k.$$

For $m = 2$, $\binom{n}{k}_2$ is the trinomial coefficient. It is seen ([4, 7, 8]) that

$$\binom{n}{k}_m = \sum_{i=\lceil k/m \rceil}^{k} \binom{i}{k-i}_{m-1} \binom{n}{i},$$

where $\lceil . \rceil$ denote ceiling functions. The congruence properties for the trinomial coefficients have been investigated by several authors (see [1, 3, 13]). Recently Elkhiri and Mihoubi gave the following identity (see [6])

$$\binom{n}{k}_2 = \sum_{i=0}^{k} \binom{n}{i} \binom{n}{k-i} \cos \frac{(k-2i)\pi}{3}.$$

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Apagodu and Liu [2] gave that for any prime $p \geq 5$ and integer j with $0 < j < p$,

$$\sum_{k=0}^{p-1} \binom{k}{j}_2 \equiv (-1)^{\frac{p-j-1}{2}} \frac{(-1)^j + 1}{2} \pmod{p}.$$

If p is a prime and a is an integer not divisible by p, Fermat little theorem is given by $a^{p-1} \equiv 1 \pmod{p}$. This is the origin of the definition of the Fermat quotient of p to base a,

$$q_p(a) := \frac{a^{p-1} - 1}{p},$$

which is an integer according to Fermat little theorem.
For an odd prime p and an integer a, the Legendre symbol is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p|\,a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

Note that

$$\left(\frac{p}{3}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{3}, \\ -1 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Some sums of harmonic numbers are given as follows [12]: For any positive integer m,

$$\sum_{k=1}^{n} k^{\underline{m}} H_k = n^{\underline{m}} \frac{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right), \tag{1.1}$$

$$\sum_{k=1}^{n} (-1)^k H_k = \left(\frac{(-1)^n - 1}{2}\right) H_n + \frac{1}{2} H_{\lfloor n/2 \rfloor}, \tag{1.2}$$

where $k^{\underline{m}} = k(k-1)\cdots(k-m+1)$ and L.J denote floor functions.

Let p be any prime and n be integer not divided by p. For $0 \leq k \leq p-1$,

$$\binom{np-1}{k} = (-1)^k \prod_{j=1}^{k}\left(1 - \frac{np}{j}\right) \equiv (-1)^k (1 - npH_k) \pmod{p^2}. \tag{1.3}$$

Let p be an odd prime. The following results are well-known:

$$q_p(2) \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \pmod{p}, \tag{1.4}$$

and for $0 \leq k \leq p-1$,

$$H_{p-1-k} \equiv H_k \pmod{p}. \tag{1.5}$$

Lehmer [10] gave that for any prime $p > 3$,

$$H_{\lfloor (p-1)/3 \rfloor} \equiv -\frac{3}{2} q_p(3) \pmod{p}. \tag{1.6}$$

Elkhiri et al. [5] proved that for any prime $p>3$,

$$\sum_{k=1}^{\lfloor (p-1)/3\rfloor} H_{3k} \equiv \begin{cases} \frac{1}{3}-\frac{1}{3}q_p(3) \pmod p & \text{if } p\equiv 1 \pmod 3, \\ \frac{1}{3}-\frac{1}{6}q_p(3) \pmod p & \text{if } p\equiv 2 \pmod 3. \end{cases} \tag{1.7}$$

The Catalan numbers are given by

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n+1},\quad n\in\mathbb{N}=\{0,1,2,...\}.$$

Koparal and Ömür [9] established that for any odd prime p,

$$\sum_{k=1}^{(p-1)/2} C_k H_k x^{k+1} \equiv \frac{2^p\left((1-4x)^{(p+1)/2}+1\right)-\left(\sqrt{1-4x}+1\right)^{p+1}-\left(\sqrt{1-4x}-1\right)^{p+1}}{2p} \pmod p, \tag{1.8}$$

where x is an integer not divisible by p.
Elkhiri and Mihoubi [6] showed following congruences that for any prime $p>3$,

$$\binom{np-1}{3k}_2 \equiv 1-np\left(\frac{2}{3}H_k+\sum_{i=0}^{k-1}\frac{1}{3i+2}\right) \pmod{p^2}, \tag{1.9}$$

$$\binom{np-1}{3k+1}_2 \equiv -1+np\left(\frac{2}{3}H_k+\sum_{i=0}^{k}\frac{1}{3i+1}\right) \pmod{p^2}, \tag{1.10}$$

$$\binom{np-1}{3k+2}_2 \equiv np\left(\sum_{i=0}^{k}\frac{1}{3i+2}-\sum_{i=0}^{k}\frac{1}{3i+1}\right) \pmod{p^2}, \tag{1.11}$$

where n and k are positive integers. They obtained that for any prime $p>3$,

$$\sum_{k=0}^{p-1}\binom{np-1}{k}_2 \equiv \begin{cases} 1+npq_p(3) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3, \\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3, \end{cases} \tag{1.12}$$

and

$$\sum_{k=1}^{\lfloor p/3\rfloor -1}\frac{1}{3k+1} \equiv \begin{cases} \frac{1}{2}q_p(3)-1 \pmod p & \text{if } p\equiv 1 \pmod 3, \\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3, \end{cases} \tag{1.13}$$

$$\sum_{k=1}^{\lfloor p/3\rfloor -1}\frac{1}{3k+2} \equiv \begin{cases} -\frac{1}{2} \pmod p & \text{if } p\equiv 1 \pmod 3, \\ \frac{1}{2}\left(q_p(3)-1\right) \pmod p & \text{if } p\equiv 2 \pmod 3. \end{cases} \tag{1.14}$$

2. On congruences

In this section, firstly we will start with some lemmas for further use:

Lemma 2.1 *For any prime $p > 3$, we have*

$$\sum_{k=1}^{\lfloor (p-1)/3 \rfloor} \frac{(-1)^k}{3k-1} \equiv \frac{2}{3} q_p(2) \pmod{p}, \tag{2.1}$$

$$\sum_{k=0}^{\lfloor (p-2)/3 \rfloor} \frac{(-1)^k}{3k+1} \equiv \frac{2}{3} q_p(2) \pmod{p}. \tag{2.2}$$

Proof We will give proof of (2.1) for $p \equiv 1 \pmod 3$. Consider that

$$\sum_{k=1}^{\lfloor (p-1)/3 \rfloor} \frac{(-1)^k}{3k-1} = -\sum_{k=0}^{(p-4)/3} \frac{(-1)^k}{3k+2}. \tag{2.3}$$

With the help of the congruence $\sum\limits_{k=0}^{(p-4)/3} \frac{(-1)^k}{3k+1} \equiv \sum\limits_{k=0}^{(p-4)/3} \frac{(-1)^k}{3k+3} \pmod{p}$, we have

$$\begin{aligned} \sum_{k=0}^{(p-4)/3} \frac{(-1)^k}{3k+2} &= \sum_{k=1}^{p-1} \frac{(-1)^k}{k} + \sum_{k=0}^{(p-4)/3} \frac{(-1)^k}{3k+1} + \sum_{k=0}^{(p-4)/3} \frac{(-1)^k}{3k+3} \\ &\equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} - \frac{2}{3} \sum_{k=1}^{(p-1)/3} \frac{(-1)^k}{k} \pmod{p}. \end{aligned}$$

By (1.4), (2.3) and $\sum\limits_{k=1}^{(p-1)/3} \frac{(-1)^k}{k} \equiv -2q_p(2) \pmod{p}$, we have the result. Similarly, for $p \equiv 2 \pmod 3$, using the equality

$$\frac{1}{3} \sum_{k=1}^{(p-2)/3} \frac{(-1)^k}{k} - \sum_{k=0}^{(p-2)/3} \frac{(-1)^k}{3k+1} - \sum_{k=1}^{(p-2)/3} \frac{(-1)^k}{3k-1} = \sum_{k=1}^{p-1} \frac{(-1)^k}{k},$$

the desired result is obtained. Proof of (2.2) is similar to proof of (2.1). Thus, the proof of Lemma 2.1 is complete. □

Lemma 2.2 *For integer numbers $n \geq 0$ and $m > 1$, we have*

$$\sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{k(m+1)} \binom{n}{k} \binom{n}{mk} = \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{k} \binom{n}{n-k}_{m-1}.$$

Proof Consider that

$$\begin{aligned} \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{k(m+1)} \binom{n}{k} \binom{n}{mk} &= [x^n] \left\{ (1+x)^n \left(1 + (-1)^{m+1} x^m \right)^n \right\} \\ &= [x^n] \left\{ (1+x)^{2n} \left(1 - x + x^2 - \ldots + (-x)^{m-1} \right)^n \right\} \\ &= [x^n] \left(\sum_{k=0}^{\infty} \binom{2n}{k} x^k \right) \left(\sum_{k=0}^{\infty} \binom{n}{k}_{m-1} (-x)^k \right). \end{aligned}$$

By product of generating functions, we get

$$\begin{aligned}\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{k(m+1)}\binom{n}{k}\binom{n}{mk} &= [x^n]\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{2n}{i}\binom{n}{k-i}_{m-1}\right)x^k \\ &= \sum_{i=0}^{n}(-1)^{n-i}\binom{2n}{i}\binom{n}{n-i}_{m-1},\end{aligned}$$

as claimed. □

Corollary 2.3 *For any prime* $p>3$, *we have*

$$\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}_2 \equiv (-1)^{(p-1)/2}\sum_{k=0}^{\lfloor (p-1)/6\rfloor}\frac{1}{2^{8k}}\binom{2k}{k}\binom{6k}{3k} \pmod p,$$

$$\sum_{k=1}^{\lfloor (p+3)/6\rfloor}\frac{k(3k-2)}{2^{8(k-1)}}\binom{2k}{k}\binom{6k-4}{3k-2} \equiv (-1)^{(p+1)/2}\sum_{k=0}^{(p-1)/2}\binom{k+1}{2}\binom{(p-3)/2}{k-1}_2 \pmod p. \tag{2.4}$$

Proof We will give proof of (2.4). Setting $n=(p-3)/2$, $m=3$ in Lemma 2.2, we write

$$\sum_{k=0}^{\lfloor (p-3)/6\rfloor}\binom{(p-3)/2}{k}\binom{(p-3)/2}{3k} = \frac{(-1)^{(p-1)/2}}{p-2}\sum_{k=0}^{(p-1)/2}k(-1)^k\binom{p-2}{k}\binom{(p-3)/2}{(p-1)/2-k}_2.$$

In view of equality $\binom{(p-1)/2}{k+1}=\frac{p-1}{2(k+1)}\binom{(p-3)/2}{k}$, we have

$$\begin{aligned}&4\sum_{k=0}^{\lfloor (p-3)/6\rfloor}\frac{(k+1)(3k+1)}{(p-1)^2}\binom{(p-1)/2}{k+1}\binom{(p-1)/2}{3k+1} \\ &= \frac{(-1)^{(p-1)/2}}{(p-2)(p-1)}\sum_{k=0}^{(p-1)/2}k(k+1)(-1)^k\binom{p-1}{k+1}\binom{(p-3)/2}{(p-1)/2-k}_2.\end{aligned}$$

By the congruences $(-1)^k\binom{p-2}{k}\equiv -\frac{k+1}{p-1} \pmod p$ and for $1\le k\le (p-1)/2$, $\binom{(p-1)/2}{k}\equiv \frac{1}{(-4)^k}\binom{2k}{k} \pmod p$, we have the proof. Similarly, the other congruence is given. This concludes the proof. □

Lemma 2.4 *Let* $p>3$ *be a prime number and* n *be a positive integer. Then*

$$3\sum_{k=0}^{\lfloor (p-1)/3\rfloor}k\binom{np-1}{3k}_2 \equiv \begin{cases} -\frac{1}{3}+p\left(\frac{5}{18}n-\frac{1}{3}nq_p(3)+\frac{1}{6}\right) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3, \\ -\frac{1}{3}+p\left(\frac{13}{18}n-\frac{1}{3}nq_p(3)-\frac{1}{6}\right) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3, \end{cases} \tag{2.5}$$

$$\sum_{k=1}^{\lfloor (p-2)/3\rfloor}(3k+1)\binom{np-1}{3k+1}_2 \equiv \begin{cases} \frac{1}{6}(3p+4)-\frac{1}{3}np\left(q_p(3)+\frac{11}{3}\right) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3, \\ 1-\frac{p}{18}(17n+3) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3, \end{cases} \tag{2.6}$$

and

$$\sum_{k=1}^{\lfloor (p-3)/3\rfloor}(3k+2)\binom{np-1}{3k+2}_2 \equiv \begin{cases} \frac{17}{18}np \pmod{p^2} & \text{if } p\equiv 1 \pmod 3, \\ \frac{5}{9}np \pmod{p^2} & \text{if } p\equiv 2 \pmod 3. \end{cases} \tag{2.7}$$

Proof Firstly, we will give the proof of (2.5). From (1.9), we have

$$\begin{aligned}\sum_{k=0}^{\lfloor (p-1)/3\rfloor} k\binom{np-1}{3k}_2 &\equiv \sum_{k=0}^{\lfloor (p-1)/3\rfloor} k\left(1-np\left(\frac{2}{3}H_k+\sum_{i=0}^{k-1}\frac{1}{3i+2}\right)\right)\\ &= \sum_{k=0}^{\lfloor (p-1)/3\rfloor} k-\frac{2}{3}np\sum_{k=0}^{\lfloor (p-1)/3\rfloor} kH_k-np\sum_{k=0}^{\lfloor (p-1)/3\rfloor} k\sum_{i=1}^{k}\frac{1}{3i-1}\\ &= \sum_{k=0}^{\lfloor (p-1)/3\rfloor} k-\frac{2}{3}np\sum_{k=0}^{\lfloor (p-1)/3\rfloor} kH_k-np\sum_{i=1}^{\lfloor (p-1)/3\rfloor}\frac{1}{3i-1}\sum_{k=i}^{\lfloor (p-1)/3\rfloor} k \pmod{p^2},\end{aligned}$$

and using some elementary operations, we get

$$\begin{aligned}&\sum_{k=0}^{\lfloor (p-1)/3\rfloor} k\binom{np-1}{3k}_2\\ \equiv\ & \sum_{k=0}^{\lfloor (p-1)/3\rfloor} k-\frac{2}{3}np\sum_{k=0}^{\lfloor (p-1)/3\rfloor} kH_k-np\sum_{k=1}^{\lfloor (p-1)/3\rfloor}\frac{1}{3k-1}\left(\frac{1}{2}\left\lfloor\frac{p-1}{3}\right\rfloor\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)-\frac{k(k-1)}{2}\right)\\ =\ & \left(1+\frac{np}{6}\right)\sum_{k=1}^{\lfloor (p-1)/3\rfloor} k-\frac{2}{3}np\sum_{k=0}^{\lfloor (p-1)/3\rfloor} kH_k-\frac{np}{9}\sum_{k=1}^{\lfloor (p-1)/3\rfloor}1-\frac{np}{9}\sum_{k=1}^{\lfloor (p-1)/3\rfloor}\frac{1}{3k-1}\\ &\qquad -\frac{np}{2}\left\lfloor\frac{p-1}{3}\right\rfloor\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)\sum_{k=1}^{\lfloor (p-1)/3\rfloor}\frac{1}{3k-1}\\ =\ & \frac{1}{2}\left(1+\frac{np}{6}\right)\left\lfloor\frac{p-1}{3}\right\rfloor\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)-\frac{2}{3}np\sum_{k=0}^{\lfloor (p-1)/3\rfloor} kH_k-\frac{np}{9}\left\lfloor\frac{p-1}{3}\right\rfloor\\ &\qquad -\frac{np}{2}\left(\frac{2}{9}+\left\lfloor\frac{p-1}{3}\right\rfloor\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)\right)\sum_{k=1}^{\lfloor (p-1)/3\rfloor}\frac{1}{3k-1} \pmod{p^2}.\end{aligned}$$

By $\frac{2}{9}+\left\lfloor\frac{p-1}{3}\right\rfloor\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)\equiv 0 \pmod p$ and (1.1), we write

$$\begin{aligned}\sum_{k=0}^{\lfloor (p-1)/3\rfloor} k\binom{np-1}{3k}_2 &\equiv \frac{1}{2}\left(1+\frac{np}{6}\right)\left\lfloor\frac{p-1}{3}\right\rfloor\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)-\frac{np}{9}\left\lfloor\frac{p-1}{3}\right\rfloor\\ &\quad +\frac{np}{6}\left\lfloor\frac{p-1}{3}\right\rfloor\left(\left\lfloor\frac{p-1}{3}\right\rfloor-1-2\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)H_{\lfloor (p-1)/3\rfloor}\right) \pmod{p^2}.\end{aligned}$$

(1.6) yields that

$$\sum_{k=0}^{\lfloor (p-1)/3\rfloor} 3k\binom{np-1}{3k}_2 \equiv \frac{3}{2}\left(1+\frac{np}{6}+npq_p(3)\right)\left\lfloor\frac{p-1}{3}\right\rfloor\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)-\frac{np}{3}\left\lfloor\frac{p-1}{3}\right\rfloor + \frac{np}{2}\left\lfloor\frac{p-1}{3}\right\rfloor\left(\left\lfloor\frac{p-1}{3}\right\rfloor-1\right) \pmod{p^2}.$$

According to the cases of p, the proof of (2.5) is clearly obtained. With the help of (1.6), (1.10) and (1.13), the proof of (2.6) is similar to the proof of (2.5). Also from (1.11), (1.13) and (1.14), the proof of (2.7) is obtained. □

Lemma 2.5 *Let $p>3$ be a prime number and n be a positive integer. Then*

$$\sum_{k=0}^{\lfloor (p-1)/3\rfloor} (-1)^k\binom{np-1}{3k}_2 \equiv \begin{cases} 1+np\left(\frac{1}{2}q_p(3)+\frac{1}{3}q_p(2)\right) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ np\left(\frac{1}{3}q_p(2)-\frac{1}{4}q_p(3)\right) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{\lfloor (p-2)/3\rfloor} (-1)^k\binom{np-1}{3k+1}_2 \equiv np\begin{cases} \frac{1}{4}q_p(3)-\frac{1}{3}q_p(2) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ \frac{1}{2}q_p(3)-\frac{1}{3}q_p(2) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

and

$$\sum_{k=0}^{\lfloor (p-3)/3\rfloor} (-1)^k\binom{np-1}{3k+2}_2 \equiv np\left(\frac{1}{4}q_p(3)-\frac{2}{3}q_p(2)\right) \pmod{p^2}.$$

Proof Using (1.9), (1.10), (1.11), together with (1.2), (1.14), Lemma 2.1, the proof is similar to proof of Lemma 2.4. □

Lemma 2.6 *Let $p>3$ be a prime number and n be a positive integer. Then*

$$\sum_{k=0}^{\lfloor (p-1)/3\rfloor} (3k)^2\binom{np-1}{3k}_2 \equiv \frac{1}{3^2}\begin{cases} -1+p\left(\frac{n-3}{2}-nq_p(3)\right) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ 1-\frac{3}{2}p+np\left(q_p(3)-\frac{13}{6}\right) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=1}^{\lfloor (p-2)/3\rfloor} (3k+1)^2\binom{np-1}{3k+1}_2 \equiv \begin{cases} \frac{1}{18}(20-9p)-np\left(\frac{23}{27}-\frac{1}{9}q_p(3)\right) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ \frac{1}{18}(3p+22)-np\left(\frac{3}{2}-\frac{2}{9}q_p(3)\right) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

and

$$\sum_{k=1}^{\lfloor (p-3)/3\rfloor} (3k+2)^2\binom{np-1}{3k+2}_2 \equiv \frac{1}{54}\begin{cases} 115np \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ 130np \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Proof With the help of (1.1), (1.6), (1.9), (1.10), (1.11) and (1.13), the proof of Lemma 2.6 is similar to proof of Lemma 2.4. □

Now, we will give main theorems.

Theorem 2.7 *Let $p>3$ be a prime number and n be a positive integer. Then*

$$\sum_{k=0}^{p-1}(-1)^k\binom{np-1}{k}_2 \equiv \begin{cases} 1+\frac{np}{2}q_p(3) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3, \\ -\frac{np}{2}q_p(3) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3, \end{cases} \tag{2.8}$$

$$\sum_{k=0}^{p-1}k\binom{np-1}{k}_2 \equiv -\frac{1}{3}\begin{cases} 2\left(1+p\left(nq_p\left(3\right)-1\right)\right) \pmod{p^2} & \text{if } p\equiv 1 \pmod 3, \\ 1+p\left(nq_p\left(3\right)-n+1\right) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3, \end{cases} \tag{2.9}$$

and

$$\sum_{k=0}^{p-1}k^2\binom{np-1}{k}_2 \equiv \frac{1}{3}\begin{cases} \left(n-2\right)p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3, \\ 1+np\left(q_p\left(3\right)-1\right) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3. \end{cases}$$

Proof We will give the proof of (2.9). Consider that

$$\begin{aligned} &\sum_{k=0}^{p-1}k\binom{np-1}{k}_2 \\ = &\sum_{k=0}^{\lfloor (p-1)/3\rfloor}3k\binom{np-1}{3k}_2+\sum_{k=0}^{\lfloor (p-2)/3\rfloor}(3k+1)\binom{np-1}{3k+1}_2+\sum_{k=0}^{\lfloor (p-3)/3\rfloor}(3k+2)\binom{np-1}{3k+2}_2 \\ = &\sum_{k=0}^{\lfloor (p-1)/3\rfloor}3k\binom{np-1}{3k}_2+\sum_{k=1}^{\lfloor (p-2)/3\rfloor}(3k+1)\binom{np-1}{3k+1}_2+\binom{np-1}{1}_2 \\ &+\sum_{k=1}^{\lfloor (p-3)/3\rfloor}(3k+2)\binom{np-1}{3k+2}_2+2\binom{np-1}{2}_2. \end{aligned}$$

The equalities $\binom{np-1}{1}_2=\binom{np-1}{1}$ and $\binom{np-1}{2}_2=\binom{np-1}{1}^2-\binom{np-1}{2}$ yield that

$$\begin{aligned} &\sum_{k=0}^{p-1}k\binom{np-1}{k}_2 \\ \equiv\ &-1+\sum_{k=0}^{\lfloor (p-1)/3\rfloor}3k\binom{np-1}{3k}_2+\sum_{k=1}^{\lfloor (p-2)/3\rfloor}(3k+1)\binom{np-1}{3k+1}_2+\sum_{k=1}^{\lfloor (p-3)/3\rfloor}(3k+2)\binom{np-1}{3k+2}_2 \pmod{p^2}. \end{aligned}$$

With the help of Lemma 2.4, for $p\equiv 1 \pmod 3$,

$$\begin{aligned} &\sum_{k=0}^{p-1}k\binom{np-1}{k}_2 \\ \equiv\ &-1-\frac{1}{3}+p\left(\frac{5}{18}n-\frac{1}{3}nq_p\left(3\right)+\frac{1}{6}\right)+\frac{1}{6}\left(3p+4\right)-\frac{1}{3}np\left(q_p\left(3\right)+\frac{11}{3}\right)+\frac{17}{18}np \\ =\ &\frac{2}{3}p\left(1-nq_p\left(3\right)\right)-\frac{2}{3} \pmod{p^2}, \end{aligned}$$

and for $p \equiv 2 \pmod 3$,

$$\sum_{k=0}^{p-1} k \binom{np-1}{k}_2 \equiv -1 - \frac{1}{3} + \frac{13}{18}pn - \frac{1}{3}pnq_p(3) - \frac{1}{6}p + 1 - \frac{17}{18}pn - \frac{p}{6} + \frac{5}{9}np$$
$$= -\frac{1}{3}\left(1 + p\left(nq_p(3) - n + 1\right)\right) \pmod{p^2}.$$

So, the proof of the congruence is complete. Similarly, with the help of Lemmas 2.5 and 2.6, proofs of the other congruences are clearly obtained. □

Theorem 2.8 *For any prime $p > 3$, we have*

$$\sum_{k=0}^{p-1} \binom{p-1}{k}_2 H_k \equiv \begin{cases} -\frac{1}{2}q_p(3) \pmod p & \text{if } p \equiv 1 \pmod 3, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Proof By product of generating functions, we obtain

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n}{k}_2 &= [x^n] \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} (-1)^i \binom{n}{i}_2 \binom{n}{k-i} \right) x^k \\ &= [x^n] \left(\sum_{k=0}^{\infty} (-1)^k \binom{n}{k}_2 x^k \right) \left(\sum_{k=0}^{\infty} \binom{n}{k} x^k \right) \\ &= [x^n] \left(1 - x + x^2\right)^n (1+x)^n \\ &= [x^n] \left(1 + x^3\right)^n = [x^n] \left(\sum_{k=0}^{\infty} \binom{n}{k} x^{3k} \right) \\ &= \begin{cases} \binom{n}{n/3} & \text{if } n = 3k, \\ 0 & \text{if } n \neq 3k. \end{cases} \end{aligned}$$

This identity with $n = p-1$ and (1.3) yield that

$$\sum_{k=0}^{p-1} \binom{p-1}{k}_2 - p \sum_{k=0}^{p-1} H_k \binom{p-1}{k}_2 \equiv \begin{cases} (-1)^{(p-1)/3} \left(1 - pH_{(p-1)/3}\right) \pmod{p^2} & p \equiv 1 \pmod 3, \\ 0 \pmod{p^2} & p \equiv 2 \pmod 3. \end{cases}$$

From here, using (1.6) and (1.12), the desired result is clearly obtained. □

Theorem 2.9 *For any prime $p > 3$, we have*

$$\sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k}_2 H_k \equiv 0 \pmod p.$$

Proof It is known (see the sequence A082759 in the OEIS) that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{k}_2 = \sum_{k=0}^{n} \binom{2n-k}{k}\binom{n}{k}. \tag{2.10}$$

Substuting $n = p-1$ in this equality, by (1.3) and for $0 \le k \le p-1$, $\binom{2p-2-k}{k} \equiv (-1)^k \binom{2k+1}{k+1} \pmod{p}$, we have

$$p\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 H_k \equiv \sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 - \sum_{k=0}^{p-1}\binom{2k+1}{k+1} + p\sum_{k=0}^{p-1}\binom{2k+1}{k+1}H_k \pmod{p^2}.$$

From $\sum\limits_{k=0}^{p-1}\binom{2k+1}{k+1} = 2\sum\limits_{k=0}^{p-1}\binom{2k}{k} - \sum\limits_{k=0}^{p-1}\frac{1}{k+1}\binom{2k}{k} \equiv \frac{1}{2}\left(1+\left(\frac{p}{3}\right)\right) \pmod{p^2}$ [14], we write

$$\begin{aligned} & p\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 H_k \\ \equiv\ & \sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 - \frac{1}{2}\left(1+\left(\frac{p}{3}\right)\right) + p\sum_{k=0}^{p-1}\left(2-\frac{1}{k+1}\right)\binom{2k}{k}H_k \\ =\ & \sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 - \frac{1}{2}\left(1+\left(\frac{p}{3}\right)\right) + p\sum_{k=0}^{p-1}\left(2-\frac{1}{k+1}\right)\binom{2k}{k}H_k \\ =\ & \sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 - \frac{1}{2}\left(1+\left(\frac{p}{3}\right)\right) \\ & \qquad + p\left(2\sum_{k=0}^{p-1}\binom{2k}{k}H_k - \left(\sum_{k=0}^{(p-1)/2} C_k H_k + \sum_{k=1}^{(p-1)/2} C_{p-k}H_{p-k}\right)\right) \pmod{p^2}, \end{aligned}$$

and by for $1 \le k \le (p-1)/2$, $\binom{2(p-k)}{p-k} \equiv 0 \pmod{p}$,

$$\begin{aligned} & p\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 H_k \\ \equiv\ & \sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 - \frac{1}{2}\left(1+\left(\frac{p}{3}\right)\right) + p\left(2\sum_{k=0}^{p-1}\binom{2k}{k}H_k - \sum_{k=0}^{(p-1)/2} C_k H_k\right) \pmod{p^2}. \end{aligned}$$

With the help of (1.8) and $\sum\limits_{k=0}^{p-1}\binom{2k}{k}H_k \equiv \left(\frac{p}{3}\right)\frac{1-3^{p-1}}{p} \pmod{p}$ [11], we write

$$\begin{aligned} & p\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 H_k \\ \equiv\ & \sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}_2 - \frac{1}{2}\left(1+\left(\frac{p}{3}\right)\right) \\ & +p\left(\frac{\left(\sqrt{-3}+1\right)^{p+1} + \left(\sqrt{-3}-1\right)^{p+1} - 2^p\left((-3)^{(p+1)/2}+1\right)}{2p} + 2\left(\frac{p}{3}\right)\frac{1-3^{p-1}}{p}\right) \pmod{p^2}. \end{aligned}$$

(2.8) and

$$\left(\left(\sqrt{-3}+1\right)^{p+1}+\left(\sqrt{-3}-1\right)^{p+1}\right)/2=\begin{cases} -2^{p} & p\equiv 1 \pmod 3,\\ 2^{p+1} & p\equiv 2 \pmod 3,\end{cases}$$

yield that

$$\sum_{k=0}^{p-1}(-1)^{k}\binom{p-1}{k}_{2}H_{k}$$
$$\equiv 3\begin{cases} \frac{(-12)^{(p-1)/2}-1}{p}-\frac{1}{2}q_{p}(3)-q_{p}(2) \pmod p & \text{if } p\equiv 1 \pmod 3,\\ \frac{1}{2}q_{p}(3)+q_{p}(2)+\frac{1+(-12)^{(p-1)/2}}{p} \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

From here, for $p\equiv 1 \pmod 3$, we have

$$\begin{aligned}\sum_{k=0}^{p-1}(-1)^{k}\binom{p-1}{k}_{2}H_{k} &\equiv \frac{3\left(2^{p}(-3)^{(p-1)/2}-3^{p-1}+1-2^{p}\right)}{2p} \pmod p\\ &= \frac{3\left(1-\left((-3)^{(p-1)/2}\right)^{2}-2^{p}\left(1-(-3)^{(p-1)/2}\right)\right)}{2p}\\ &= \frac{3\left(1-(-3)^{(p-1)/2}\right)\left(1-2^{p}+(-3)^{(p-1)/2}\right)}{2p}\\ &\equiv -\frac{3\left(1-(-3)^{(p-1)/2}\right)^{2}}{2p} \pmod p.\end{aligned}$$

By $\left(\frac{-3}{p}\right)\equiv 1 \pmod p$, the desired result is complete. Similarly, by $\left(\frac{-3}{p}\right)\equiv -1 \pmod p$, the other congruence is obtained. Thus we have the proof. □

Theorem 2.10 *For any prime $p>3$, we have*

$$3\sum_{k=1}^{p-1}k\binom{p-1}{k}_{2}H_{k}\equiv\begin{cases} q_{p}(3) \pmod p & \text{if } p\equiv 1 \pmod 3,\\ \frac{1}{2}q_{p}(3)-2 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Proof Setting $n=p-1$, $m=3$ in Lemma 2.2, we have

$$\sum_{k=0}^{\lfloor (p-1)/3\rfloor}\binom{p-1}{k}\binom{p-1}{3k}=\sum_{k=0}^{p-1}(-1)^{k}\binom{2(p-1)}{p-1-k}\binom{p-1}{k}_{2},$$

and from (1.3),

$$\begin{aligned}
&\sum_{k=0}^{\lfloor (p-1)/3\rfloor} \left(1-pH_k-pH_{3k}\right)\\
\equiv\ & \sum_{k=1}^{p-1}(-1)^k\frac{p-k}{2p-1}\binom{2p-1}{p-k}\binom{p-1}{k}_2+\frac{p}{2p-1}\binom{2p-1}{p}\\
\equiv\ & (-1)^p\sum_{k=1}^{p-1}\frac{p-k}{2p-1}\left(1-2pH_{p-k}\right)\binom{p-1}{k}_2+(-1)^p\frac{p}{2p-1}\left(1-2pH_p\right)\\
\equiv\ & (-1)^p\sum_{k=1}^{p-1}\left(k-\left(1-2k\right)p\right)\left(1-2pH_{p-k}\right)\binom{p-1}{k}_2-(-1)^p\frac{p}{2p-1}\\
\equiv\ & \sum_{k=1}^{p-1}\left(2pkH_{p-k}+\left(1-2k\right)p-k\right)\binom{p-1}{k}_2-p \pmod{p^2}.
\end{aligned}$$

Hence we write

$$2p\sum_{k=1}^{p-1}kH_{p-k}\binom{p-1}{k}_2\equiv\sum_{k=0}^{\lfloor (p-1)/3\rfloor}\left(1-pH_k-pH_{3k}\right)-\sum_{k=1}^{p-1}\left(p-2pk-k\right)\binom{p-1}{k}_2+p \pmod{p^2}.$$

By (1.1) and some elementary operations, we have

$$\begin{aligned}
2p\sum_{k=1}^{p-1}kH_{p-k}\binom{p-1}{k}_2\equiv&\left\lfloor\frac{p-1}{3}\right\rfloor+1-p\left(\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)H_{\lfloor (p-1)/3\rfloor}-\left\lfloor\frac{p-1}{3}\right\rfloor\right)+p\\
&-p\sum_{k=0}^{\lfloor (p-1)/3\rfloor}H_{3k}-\sum_{k=1}^{p-1}\left(p-2pk-k\right)\binom{p-1}{k}_2 \pmod{p^2}.
\end{aligned}$$

For $p\equiv 1 \pmod 3$, by (1.7), (1.12) and (2.9),

$$\begin{aligned}
2p\sum_{k=1}^{p-1}kH_{p-k}\binom{p-1}{k}_2\equiv&\left\lfloor\frac{p-1}{3}\right\rfloor+1-p\left(\left(\left\lfloor\frac{p-1}{3}\right\rfloor+1\right)H_{\lfloor (p-1)/3\rfloor}-\left\lfloor\frac{p-1}{3}\right\rfloor\right)+p\\
&-p\left(\frac{1}{3}-\frac{1}{3}q_p\left(3\right)\right)-\frac{2}{3}-\frac{2}{3}p\left(q_p\left(3\right)-1\right)-\frac{4}{3}p \pmod{p^2}.
\end{aligned}$$

With the help of (1.6), we have

$$2p\sum_{k=1}^{p-1}kH_{p-k}\binom{p-1}{k}_2\equiv\frac{2}{3}pq_p\left(3\right) \pmod{p^2},$$

and by (1.5), we have the proof for $p\equiv 1 \pmod 3$. Similarly, for $p\equiv 2 \pmod 3$, the proof complete. □

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