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Relative conics and their Brianchon points

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Abstract: The purpose of this paper is to study some additional relations between lines and points in the configuration of six lines tangent to the common conic. One of the most famous results concerning with this configuration is Brianchon theorem. It says that three diagonals of a hexagon circumscribing around conic are concurrent. They meet in the so called Brianchon point. In fact, by relabeling the vertices of hexagon, we obtain 60 distinct Brianchon points. We prove, among others, that, in the set of all intersection points of six tangents to the same conic, there exist exactly 10 sextuples of points lying on the common conic, which form the $(10_6, 15_4)$ conic-point configuration. We establish a relation between all Brianchon points of these conics. We use both, algebraic and geometric tools.

Key words: Arrangements of lines, Brianchon theorem, Pascal theorem, conic section

1. Introduction

The Brianchon theorem, proved at the beginning of XIX century, is generally known as a dual version of Pascal theorem. Conditions of Pascal and Brianchon are simple criterions of collinearity of points and concurrency of lines, respectively.

Theorem 1.1 (Pascal theorem, [4], Theorem 6.2.1 **and** 6.2.2**)** *A hexagon is inscribed in a conic if and only if the three intersections of pairs of lines joining the opposite vertices of hexagon are collinear.*

Theorem 1.2 (Brianchon theorem, [4], Theorem 6.2.3**)** *A hexagon is circumscribed around a conic if and only if the three lines passing through the opposite vertices of this hexagon are concurrent.*

In the statements of Theorems 1.1 and 1.2 hexagon is just a set of six points, ordered in some fixed sequence. They are the vertices. Consistently, each line joining two successive vertices (with respect to this fixed order) is a side of hexagon. By inscribing a hexagon in a conic, we understand that the vertices of this hexagon lies on a conic. Analogously, by circumscribing a hexagon around a conic we mean that each side of this hexagon is the tangent to this conic.

In the Figure 1, we present hexagon $P_1P_2P_3P_4P_5P_6$ inscribed in a hyperbola, with Pascal line distinguished by bold solid line. In the Figure 2, one may see hexagon $R_1R_2R_3R_4R_5R_6$ circumscribed around a hyperbola, where points $P_1, \ldots, P_6$ are the points of tangency of the proper sides of hexagon to this hyperbola. The distinguished three bold solid lines meet in a Brianchon point.

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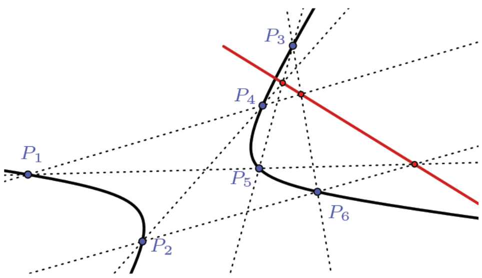

Figure 1. Pascal line

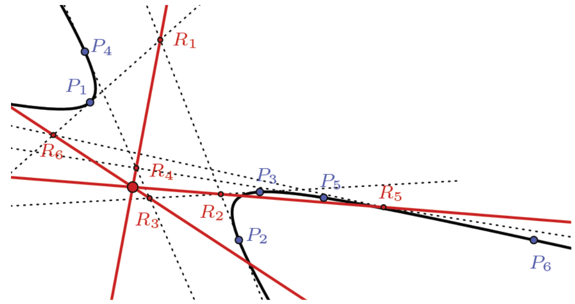

Figure 2. Brianchon point

Fixed order of the vertices means that for example hexagons $F_1 : P_1P_2P_3P_4P_5P_6$ and $F_2 : P_2P_3P_5P_4P_6P_1$ are in fact two distinct hexagons with the same vertices. Indeed, the sides of the hexagon F_1 are P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_6, P_6P_1, while the sides of F_2 are the following lines $P_2P_3, P_3P_5, P_5P_4, P_4P_6, P_6P_1,\ P_1P_2$. These sets of sides are not disjoint, but they are definitely various.

We may relabel six points in 720 distinct ways. Since cyclic permutations and reversal of the sequence describe the same hexagon (in the sense of its sides); thus, there are

$$\frac{5!}{2} = 60$$

distinct hexagons for collection of six vertices. It means that six points on a conic give rise to sixty Pascal lines, or properly sixty Brianchon points (if the initial points are sufficiently general).

With reference to the Pascal theorem nowadays it is a well known characteristic according to the order in which the vertices of hexagon are traversed (see details in [5]). These sixty lines with all incidences form an arrangement called Hexagrammum Mysticum. Although this configuration was studied for many centuries, it is still very popular object of research (see for example [2, 3]).

Many incidences between the points and the lines in the Brianchon configuration are the consequence of the duality relation between Theorems 1.1 and 1.2. Because of that, such characteristic with respect to the order of points on a conic for Brianchon arrangement was not studied separately. We could not trace it down in literature; however, we recognize all these results as generally known.

In this paper, we focus on relations in the Brianchon arrangement, which we do not find as consequence of duality between this configuration and the Pascal arrangement. Our considerations concentrate on the incidences between some conics containing six points, appearing in a set of intersections of six tangents to the common conic. We work on complex numbers.

The paper is organized as follows. In Section 2, we recall the hereditary property for convex hexagons introduced in [1], and we extend it for general hexagon. In Section 3 we present main results. Firstly, we introduce the concept of relative conics and in Theorem 3.1 we prove their existence. We find the collection of relative conics as a configuration of conics with type $(10_6, 15_4)$. Finally, we establish relations between all 60 Brianchon points for collection of relative conics (Theorem 3.3).

2. Hereditary property for hexagons

In [1], the concept of *parent-child* relation between two convex hexagons was established. Namely taking the diagonals of hexagon joining every other vertices, we obtain new hexagon, which vertices are just intersections of mentioned diagonals. The initial hexagon is called a parent of the second one, and conversely, the new hexagon is a child of the initial one. In the Figure 3 the hexagon $P_1P_2P_3P_4P_5P_6$ is a parent of hexagon $R_1R_2R_3R_4R_5R_6$. The sides of an initial hexagon are distinguished with the pointed lines. Diagonals joining every other vertices form two triangles with disjoint sets of vertices, namely $P_1P_3P_5$ and $P_2P_4P_6$.

In fact this concept can be extended for hexagon understood in general sense, i.e. for fixed six points. Let us consider the hexagon $P_2P_1P_3P_5P_4P_6$ with the same vertices as a hexagon in the Figure 3. Its child is hexagon $S_1S_2S_3S_4S_5S_6$ (see Figure 4). This time diagonals form the triangles $P_1P_5P_6$ and $P_2P_3P_4$.

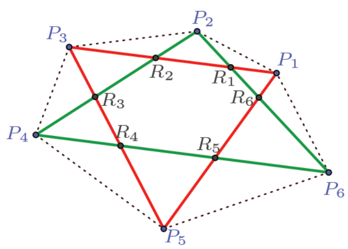

Figure 3. Convex hexagon.

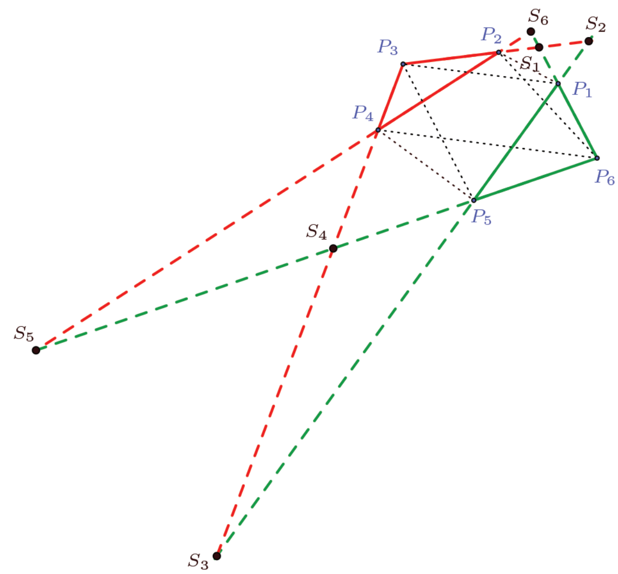

Figure 4. Arbitrary hexagon.

Agarwal and Natarajan proved in [1] an *inheritance theorem* for two hexagons being in parent-child relation.

Theorem 2.1 ([1], Theorem 3) *A parent hexagon is inscribed in an ellipse if and only if its child hexagon is circumscribed around an ellipse, and conversely, a child hexagon is inscribed in an ellipse if and only if its parent hexagon is circumscribed around an ellipse.*

The proof of Agarwal and Natarajan is based on collinearity and concurrency criterions given by Theorems 1.1 and 1.2, which can be alternatively formulated in the language of cross product of vectors. Namely, the line

$$l : Ax + By + Cz = 0$$

on $\mathbb{P}^2(\mathbb{C})$ can be clearly connected with vector $[A, B, C]$. Similarly any point $P = (a : b : c) \in \mathbb{P}^2(\mathbb{C})$ can be identified with vector $[a, b, c]$. From now on, to the end of this section, we automatically identify any point or line with coordinates of associated vector.

If $P = l \cap k$ is the point of intersection of two lines, we have the following relation between their connected vectors

$$P = l \times k.$$

On the other hand, if l is the line connecting two points P_1 and P_2, then their connected vectors fulfill the equality

$$l = P_1 \times P_2.$$

Using this language, the Pascal and Brianchon conditions can be formulated as follows.

Theorem 2.2 ([1], Theorem 1 and 2) *Let $H : P_1P_2P_3P_4P_5P_6$ be a hexagon.*

1) *H is circumscribed around a conic if and only if*

$$[(P_1 \times P_2) \times (P_4 \times P_5), (P_2 \times P_3) \times (P_5 \times P_6), (P_3 \times P_4) \times (P_6 \times P_1)] = 0.$$

2) *H is inscribed in a conic if and only if*

$$[(P_1 \times P_4), (P_2 \times P_5), (P_3 \times P_6)] = 0.$$

The proof of inheritance property in [1] is based on conditions 1) and 2) of Theorem 2.2, with no references to the specific type of the conic curve. Thus an ellipse in the statement of Theorem 2.1 can be replaced by any conic and theorem can be formulated more general. We place it here as a corollary.

Corollary 2.3 *A parent hexagon is inscribed in a conic if and only if its child hexagon is circumscribed around a conic, and conversely, a child hexagon is inscribed in a conic if and only if its parent hexagon is circumscribed around a conic.*

Proof Let $P_1P_2P_3P_4P_5P_6$ be some hexagon. It is enough to notice, that the consecutive sides of its child hexagon are:

$$\begin{array}{lllllllll} u_1 & = & P_3 \times P_5, & u_2 & = & P_4 \times P_6, & u_3 & = & P_5 \times P_1, \\ u_4 & = & P_6 \times P_2, & u_5 & = & P_1 \times P_3, & u_6 & = & P_2 \times P_4. \end{array}$$

Conversely, the vertices $P_1, \ldots, P_6$ of parent hexagon can be written as follows using the sides of child hexagon.

$$\begin{array}{lllllllll} P_1 & = & u_3 \times u_5, & P_2 & = & u_4 \times u_6, & P_3 & = & u_5 \times u_1, \\ P_4 & = & u_6 \times u_2, & P_5 & = & u_1 \times u_3, & P_6 & = & u_2 \times u_4. \end{array}$$

By Theorem 2.2 and some elementary properties of cross product, we obtain thesis. Detailed calculations reader may find in [1]. □

We are ready now to pass to the main part of the paper.

3. Incidences in the Brianchon arrangement

Let us firstly recall how does the Brianchon arrangement look like. The core of the configuration is a conic containing six distinct points (no three of which are collinear) and lines tangent to the conic at these six points. If initial points on a conic are sufficiently general, then all tangents give rise to 15 distinct intersection points.

Let us denote by $P_1, \ldots, P_6$ points lying on a conic and by $l_1, \ldots, l_6$ the tangents to this conic at the proper points P_i (see Figure 5). We introduce the following notation for the intersection points of the tangents

$$R_{i,j} = l_i \cap l_j,$$

where $i, j \in \{1, \ldots, 6\}$. We focus on the incidences occuring in the set of points $\{R_{i,j}\}$.

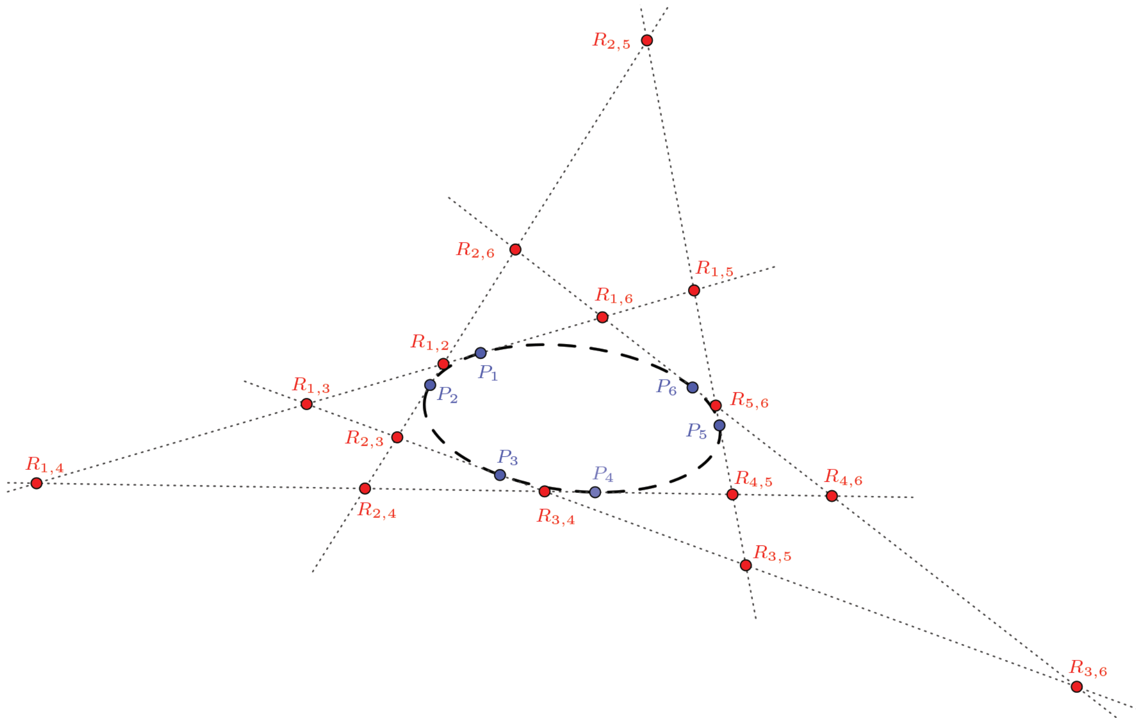

Figure 5. Brianchon arrangement.

3.1. Relative conics

It turns out that in the set $\{R_{i,j}\}$ there exist some sextuples of points lying on an irreducible conics. We call such conics the *relative conics* to the hexagon $P_1P_2P_3P_4P_5P_6$ (or equivalently to the conic $P_1P_2P_3P_4P_5P_6$). We present details in the following theorem.

Theorem 3.1 *For any hexagon inscribed in a conic, there exist exactly* 10 *relative conics.*

Proof Let us fix six distinct points on a conic, no three of which are collinear. We take all tangents to this conic at chosen points. We choose any three of the tangents and we form triangle, which edges are these picked tangents. It gives us automatically the second triangle formed by the remaining three tangents. We label the vertices of triangles by A_1, A_2, A_3 and B_1, B_2, B_3, properly (see Figure 6).

Let us consider the hexagon $A_1B_1A_2B_2A_3B_3$. It can be seen as a parent hexagon of hexagon $T_1T_2T_3T_4T_5T_6$. Indeed, the vertices of each triangle are the every other vertices of hexagon $A_1B_1A_2B_2A_3B_3$ and points $T_1, \ldots, T_6$ appears as the intersections of its proper diagonals, namely:

$$T_1 = A_1A_2 \cap B_1B_2, \qquad T_4 = A_1A_3 \cap B_2B_3,$$

$$T_2 = A_2A_3 \cap B_1B_2, \qquad T_5 = A_1A_3 \cap B_1B_3,$$

$$T_3 = A_2A_3 \cap B_2B_3, \qquad T_6 = A_1A_2 \cap B_1B_3.$$

Since hexagon $T_1T_2T_3T_4T_5T_6$ is circumscribed around a conic (its sides are the tangents to the initial conic), by Corollary 2.3 we conclude that the parent hexagon $A_1B_1A_2B_2A_3B_3$ is inscribed in a conic. In other words, points $A_1, A_2, A_3, B_1, B_2, B_3$ lie on a common conic. In the Figure 7, one may see a hyperbola passing through these concrete points.

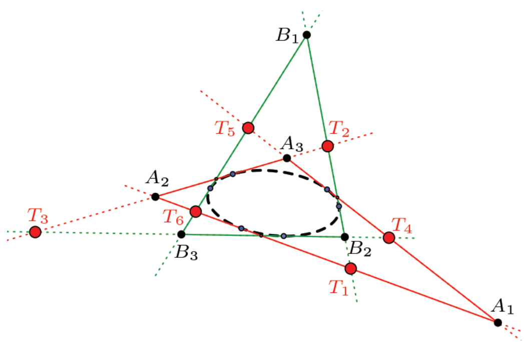

Figure 6. Two triangles.

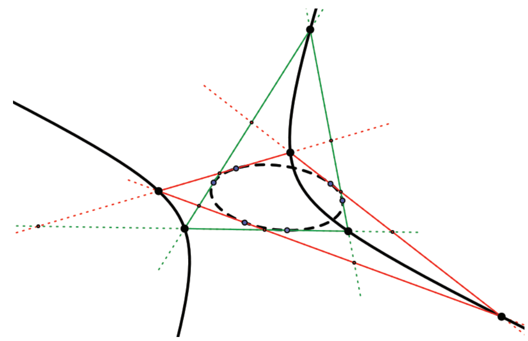

Figure 7. Relative conic.

The collection of six lines can be divided into two triangles with disjoint sets of sides in

$$\frac{\binom{6}{3}}{2} = 10$$

ways, thus finally there exist exactly ten such conics, passing through the vertices of each pair of triangles. Since the proof for all these conics runs analogously, then we omit details here.

□

Keeping the notation introduced at the beginning of Section 3, in the table below we establish the points generating all 10 relative conics together with the vertices of their child hexagons circumscribed on an initial conic $P_1P_2P_3P_4P_5P_6$. We give the points lying on the common curve in the form

$$[l_il_jl_k, l_ml_nl_t],$$

where $l_il_jl_k$ and $l_ml_nl_t$ are triangles with tangents l_i, l_j, l_k and l_m, l_n, l_t as the sides, properly. In the future considerations, to simplify notation, we will refer to the designations $C_1, \ldots, C_{10}$ introduced in the Table 1.

Table 1. Table 3.1 Relative conics and their child hexagons.

Relative conic	Child hexagon
$C_1 : [l_1l_2l_3, l_4l_5l_6]$	$R_{1,5}R_{1,4}R_{3,4}R_{3,6}R_{2,6}R_{2,5}$
$C_2 : [l_1l_2l_4, l_3l_5l_6]$	$R_{1,5}R_{1,3}R_{3,4}R_{4,6}R_{2,6}R_{2,5}$
$C_3 : [l_1l_2l_5, l_3l_4l_6]$	$R_{1,4}R_{1,3}R_{3,5}R_{5,6}R_{2,6}R_{2,4}$
$C_4 : [l_1l_2l_6, l_3l_4l_5]$	$R_{1,3}R_{1,4}R_{4,6}R_{5,6}R_{2,5}R_{2,3}$
$C_5 : [l_1l_3l_4, l_2l_5l_6]$	$R_{1,5}R_{1,2}R_{2,4}R_{4,6}R_{3,6}R_{3,5}$
$C_6 : [l_1l_3l_5, l_2l_4l_6]$	$R_{3,6}R_{3,2}R_{2,5}R_{4,5}R_{1,4}R_{1,6}$
$C_7 : [l_1l_3l_6, l_2l_4l_5]$	$R_{3,5}R_{1,5}R_{1,4}R_{4,6}R_{2,6}R_{2,3}$
$C_8 : [l_1l_4l_5, l_2l_3l_6]$	$R_{1,3}R_{1,2}R_{2,5}R_{5,6}R_{4,6}R_{3,4}$
$C_9 : [l_1l_4l_6, l_2l_3l_5]$	$R_{1,3}R_{1,2}R_{2,6}R_{5,6}R_{4,5}R_{3,4}$
$C_{10} : [l_1l_5l_6, l_2l_3l_4]$	$R_{1,3}R_{1,2}R_{2,6}R_{4,6}R_{4,5}R_{3,5}$

There are some interesting relations between the relative conics of hexagon. First of all, one may see that each of points $R_{i,j}$ is a point of meeting for exactly 4 relative conics. Moreover, among intersections of every two relative conics, there are exactly two points $R_{i,j}$. These incidences can be noticed by an analysis of the Table 1. Making some additional calculations, one may conclude that all remaining intersections of relatives conic are just double points. Indeed, we have $t_2 = 90$ and $t_4 = 15$ where t_i denotes the number of $i-$tuple intersection points. If $i \neq 2$ and $i \neq 4$, then $t_i = 0$.

It can be checked by establishing directly all intersection points. Algebraic tolls necessary to find the equations of relative conics are given in next subsection.

The values of t_i given above satisfy basic combinatorial equality for arrangements of conics with ordinary singularities, namely

$$4 \cdot \binom{k}{2} = \sum_{r \geq 2} \binom{r}{2} t_r,$$

where k is the number of conics (see in [6] also for another combinatorial estimations in this area).

In the Figure 8, we present an example of a hexagon inscribed in a conic with all 10 relative conics distinguished by different patterns and colours. An initial ellipse is denoted with dashed pattern.

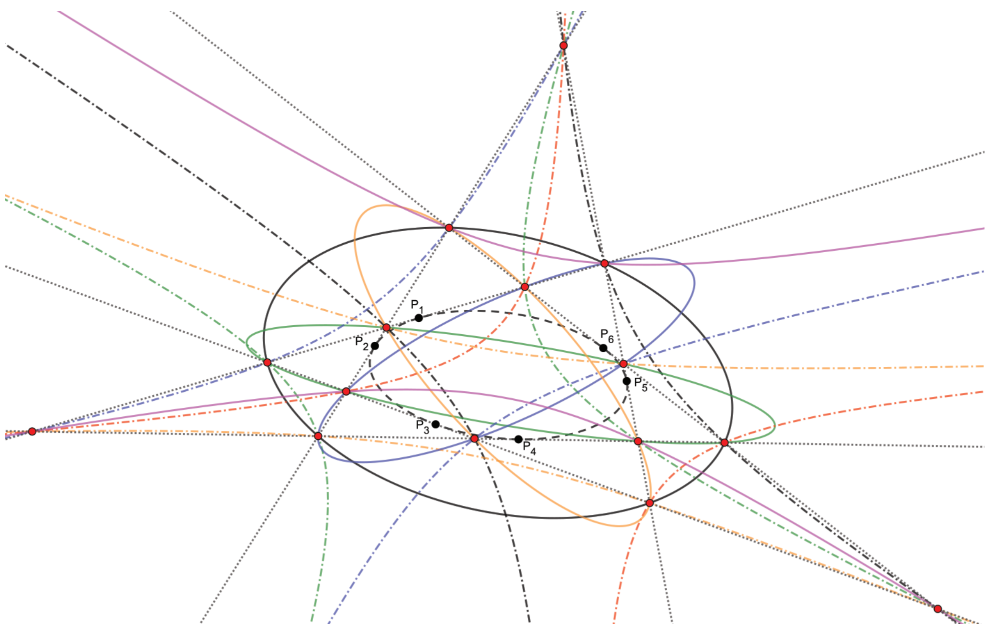

Figure 8. Collection of relative conics for hexagon $P_1P_2P_3P_4P_5P_6$.

Corollary 3.2 *Relative conics of hexagon give point-conic configuration of type* $(10_6, 15_4)$*, i.e. there are* 10 *conics each of which passes through* 6 *points and they meet in fours in* 15 *intersections points (these points are exactly* $R_{i,j}$ *).*

Configurations of conics are rife object of research in geometry (not necessary classical) and algebra. One of the aspects arousing a lot of interest here are incidences between conics in the arrangements, like, for example, the numbers and the multiplicities of intersection points.

In [7] reader can find many arrangements of circles and points with some interesting properties. Since the results of Rigby concern the inverse plane, they can not be literally compared with the properties of relative conics. However, one of his configurations is in some sense symmetric to the configuration of relative conics to the hexagon. This symmetry is between their type, namely Rigby introduced in [8] the configuration of circles with type $(15_4, 10_6)$, and [7] used it to describe another arrangements of circles with interesting combinatorial relations.

One of the consequences of this symmetry is fact that 10 points contained in 15 intersecting circles (each circle passes through 4 points) can be labeled exactly in the same way with numbers $1, \ldots, 6$ as the 10 conics C_i are labeled with the tangents $l_1, \ldots, l_6$ in Table 1. Namely the 6 element set is divided into two disjoint unordered triples.

Another similarity between these two configurations is a determining relation. More precisely, any conic on the projective plane $\mathbb{P}^2(\mathbb{C})$ is uniquely determined by 5 sufficiently general points as any circle on inverse plane is uniquely determined by 3 points. Thus, we have the collections of curves (10 conics or 15 circles) each of which contains one more point than sufficient to determine this curve.

It causes some questions about properties that could be transferred between these two types of planes, taking under considerations the types of arrangements. Of course with some additional assumptions specific for each plane (like determining relation).

We hope to conduct further research in this area in the near future. For now, we go back to the thread of this paper, namely Brianchon points of relative conics.

3.2. Brianchon points of relative conics

To establish relations between the Brianchon points of relative conics, we need to elaborate some system of description for the set of 60 Brianchon points of fixed curve. We use here some permutations from group S_6. As it was mentioned earlier, there exist exactly 60 permutations in S_6 giving distinct hexagons during relabeling 6 fixed points. In Figure 9, we give explicite a list of such permutations. We denote this subset of S_6 by $\widetilde{S_6}$. By notation $\sigma = (a, b, c, d, e, f)$ we mean that σ assigns a to 1, b to 2 etc.

Let us consider the hexagon $P_1P_2P_3P_4P_5P_6$ inscribed in a conic and let l_i for $\{1, \ldots, 6\}$ be the tangent to this conic at the point P_i. Let $\sigma \in \widetilde{S_6}$ be some relabeling of vertices of initial hexagon, in the sense that we obtain hexagon $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}P_{\sigma(4)}P_{\sigma(5)}P_{\sigma(6)}$. The Brianchon points for hexagon $P_1P_2P_3P_4P_5P_6$ are the intersection points of the following lines

$$R_{\sigma(1),\sigma(2)}R_{\sigma(4),\sigma(5)} \cap R_{\sigma(2),\sigma(3)}R_{\sigma(5),\sigma(6)} \cap R_{\sigma(3),\sigma(4)}R_{\sigma(1),\sigma(6)}$$

for each $\sigma \in \widetilde{S_6}$. Since we are interested in relations between the relative conics $C_1, \ldots, C_{10}$, we need to fix the initial order of their vertices. Without loss of generality, we may establish that starting orders as in the Table 3.2 (i.e. consequently to the order of vertices of the connected child hexagons given in the Table 3.1).

Saying precisely, for conic C_1 we read $R_{1,2}$ as P_1, $R_{4,5}$ as P_2 etc. Keeping this notation, we denote the Brianchon points for relative conic C_i by

$$B_{i,\sigma},$$

where $i \in \{1, \ldots, 10\}$ and $\sigma \in \widetilde{S_6}$ is relabeling of vertices of this conic. There are the following incidences in the set $\{B_{i,\sigma}\}$.

$$\begin{aligned}
\sigma_1 &= (1,2,3,4,5,6), & \sigma_{31} &= (2,3,1,5,4,6),\\
\sigma_2 &= (2,1,3,4,5,6), & \sigma_{32} &= (1,3,2,5,4,6),\\
\sigma_3 &= (3,1,2,4,5,6), & \sigma_{33} &= (3,1,2,5,4,6),\\
\sigma_4 &= (1,3,2,4,5,6), & \sigma_{34} &= (3,5,2,1,4,6),\\
\sigma_5 &= (2,3,1,4,5,6), & \sigma_{35} &= (5,3,2,1,4,6),\\
\sigma_6 &= (3,2,1,4,5,6), & \sigma_{36} &= (2,3,5,1,4,6),\\
\sigma_7 &= (3,2,4,1,5,6), & \sigma_{37} &= (3,2,5,1,4,6),\\
\sigma_8 &= (2,3,4,1,5,6), & \sigma_{38} &= (2,5,3,1,4,6),\\
\sigma_9 &= (4,3,2,1,5,6), & \sigma_{39} &= (1,5,3,2,4,6),\\
\sigma_{10} &= (3,4,2,1,5,6), & \sigma_{40} &= (3,1,5,2,4,6),\\
\sigma_{11} &= (2,4,3,1,5,6), & \sigma_{41} &= (1,3,5,2,4,6),\\
\sigma_{12} &= (4,2,3,1,5,6), & \sigma_{42} &= (3,5,1,2,4,6),\\
\sigma_{13} &= (4,1,3,2,5,6), & \sigma_{43} &= (1,4,5,2,3,6),\\
\sigma_{14} &= (1,4,3,2,5,6), & \sigma_{44} &= (1,5,4,2,3,6),\\
\sigma_{15} &= (3,4,1,2,5,6), & \sigma_{45} &= (1,5,2,4,3,6),\\
\sigma_{16} &= (4,3,1,2,5,6), & \sigma_{46} &= (2,1,5,4,3,6),\\
\sigma_{17} &= (1,3,4,2,5,6), & \sigma_{47} &= (1,2,5,4,3,6),\\
\sigma_{18} &= (3,1,4,2,5,6), & \sigma_{48} &= (2,5,1,4,3,6),\\
\sigma_{19} &= (2,1,4,3,5,6), & \sigma_{49} &= (2,4,1,5,3,6),\\
\sigma_{20} &= (1,2,4,3,5,6), & \sigma_{50} &= (1,2,4,5,3,6),\\
\sigma_{21} &= (4,2,1,3,5,6), & \sigma_{51} &= (2,1,4,5,3,6),\\
\sigma_{22} &= (2,4,1,3,5,6), & \sigma_{52} &= (1,4,2,5,3,6),\\
\sigma_{23} &= (1,4,2,3,5,6), & \sigma_{53} &= (2,5,4,1,3,6),\\
\sigma_{24} &= (1,5,2,3,4,6), & \sigma_{54} &= (2,4,5,1,3,6),\\
\sigma_{25} &= (2,5,1,3,4,6), & \sigma_{55} &= (1,5,4,3,2,6),\\
\sigma_{26} &= (1,2,5,3,4,6), & \sigma_{56} &= (1,4,5,3,2,6),\\
\sigma_{27} &= (2,1,5,3,4,6), & \sigma_{57} &= (1,3,5,4,2,6),\\
\sigma_{28} &= (2,1,3,5,4,6), & \sigma_{58} &= (1,5,3,4,2,6),\\
\sigma_{29} &= (1,2,3,5,4,6), & \sigma_{59} &= (1,4,3,5,2,6),\\
\sigma_{30} &= (3,2,1,5,4,6), & \sigma_{60} &= (1,3,4,5,2,6).
\end{aligned}$$

Figure 9. Permutations forming set $\widetilde{S_6}$

Table 2. Table 3.2 Initial orders of vertices.

Relative conic	Initial order of vertices of hexagon
C_1	$R_{1,2}R_{4,5}R_{1,3}R_{4,6}R_{2,3}R_{5,6}$
C_2	$R_{1,2}R_{3,5}R_{1,4}R_{3,6}R_{2,4}R_{5,6}$
C_3	$R_{1,2}R_{3,4}R_{1,5}R_{3,6}R_{2,5}R_{4,6}$
C_4	$R_{1,2}R_{3,4}R_{1,6}R_{4,5}R_{2,6}R_{3,5}$
C_5	$R_{1,3}R_{2,5}R_{1,4}R_{2,6}R_{3,4}R_{5,6}$
C_6	$R_{1,3}R_{2,6}R_{3,5}R_{2,4}R_{1,5}R_{4,6}$
C_7	$R_{1,3}R_{4,5}R_{1,6}R_{2,4}R_{3,6}R_{2,5}$
C_8	$R_{1,4}R_{2,3}R_{1,5}R_{2,6}R_{4,5}R_{3,6}$
C_9	$R_{1,4}R_{2,3}R_{1,6}R_{2,5}R_{4,6}R_{3,5}$
C_{10}	$R_{1,5}R_{2,3}R_{1,6}R_{2,4}R_{5,6}R_{3,4}$

Theorem 3.3 *Each pair of relative conics $\{C_i, C_j\}$ has six common Brianchon points and they simultaneously split into two disjoint triples each of which is a common set of Brianchon points for four distinct relative conics, namely C_i, C_j and two additional.*

Keeping the notation introduced in the Table 3.3 we can give explicitly all such quadruples of relative conics with three common Brianchon points. They are as follows:

$$\begin{array}{lll}
\{C_1, C_2, C_6, C_8\}, & \{C_1, C_4, C_6, C_{10}\}, & \{C_3, C_4, C_6, C_7\}, \\
\{C_1, C_2, C_7, C_9\}, & \{C_2, C_3, C_5, C_6\}, & \{C_3, C_4, C_8, C_9\}, \\
\{C_1, C_3, C_5, C_8\}, & \{C_2, C_3, C_9, C_{10}\}, & \{C_5, C_6, C_9, C_{10}\}, \\
\{C_1, C_3, C_7, C_{10}\}, & \{C_2, C_4, C_5, C_7\}, & \{C_5, C_7, C_8, C_{10}\}, \\
\{C_1, C_4, C_5, C_9\}, & \{C_2, C_4, C_8, C_{10}\}, & \{C_6, C_7, C_8, C_9\}.
\end{array}$$

Proof We prove this fact using some symbolic computations. Let P_1, P_2, P_3, P_4, P_5 be 5 general points. There is a unique non-degenerate conic $C : P_1P_2P_3P_4P_5$ through these points. Four of the given points can be chosen as fundamental points, i.e.

$$P_1 = (1 : 0 : 0), \quad P_2 = (0 : 1 : 0), \quad P_3 = (0 : 0 : 1), \quad P_4 = (1 : 1 : 1).$$

Taking $P_5 = (a : b : c)$ with $a, b, c \neq 0$ we obtain the following equation of C:

$$c(b-a)xy + a(c-b)yz + b(a-c)xz = 0 \tag{3.1}$$

(see details in [5], p. $79-83$). We pick $P_6 \in C$ distinct of initial five points. The coordinates of P_6 are nonzero. Indeed, if we assume that one is 0, by an equation (3.1) we get $P_6 = (0 : 0 : 0)$, what contradicts with the homogeneous coordinates. Without loss of generality, we may then assume that the first coordinate of P_6 is 1, and the second is t for some complex $t \neq 0$. By an equation (3.1) we establish the remaining one coordinate. Finally, using the properties of homogeneous coordinates, we obtain

$$P_6 = (c(b-a) + a(c-b)t : -b(a-c)t : c(b-a)t + a(c-b)t^2).$$

It gives the following equations of tangents to C:

$$\begin{aligned}
l_1 &: \quad c(b-a)y + b(a-c)z = 0,\\
l_2 &: \quad c(b-a)x + a(c-b)z = 0,\\
l_3 &: \quad b(a-c)x + a(c-b)y = 0,\\
l_4 &: \quad a(b-c)x + b(c-a)y + c(a-b)z = 0,\\
l_5 &: \quad bc(b-c)x + ac(c-a)y + ab(a-b)z = 0,\\
l_6 &: \quad abt^2(a-c)(c-b)x + (abt-act+ac-bc)^2y+\\
& \quad\; +bc(a-b)(c-a)z = 0.
\end{aligned}$$

It is sufficient to find all points $\{R_{i,j}\}$, establish the equations of conics $C_1, \dots C_{10}$ and finally find their sets of Brianchon points.

The calculations are rather complicated and we made with all with Singular. * The scripts allowing to verify these calculations, interested reader can find on our website. †

We conclude proof with list of common Brianchon points for conics $C_1, \dots, C_{10}$ ordered with respect to the permutations $\sigma \in \widetilde{S_6}$ and grouped in fours having three common Brianchon points.

$$\begin{aligned}
B_{1,\sigma_{21}} &= B_{2,\sigma_{21}} = B_{6,\sigma_{31}} = B_{8,\sigma_{31}},\\
B_{1,\sigma_{31}} &= B_{2,\sigma_{31}} = B_{6,\sigma_{49}} = B_{8,\sigma_{21}},\\
B_{1,\sigma_{49}} &= B_{2,\sigma_{49}} = B_{6,\sigma_{21}} = B_{8,\sigma_{49}},\\
\\
B_{1,\sigma_{22}} &= B_{2,\sigma_{22}} = B_{7,\sigma_{11}} = B_{9,\sigma_{31}},\\
B_{1,\sigma_{25}} &= B_{2,\sigma_{25}} = B_{7,\sigma_{38}} = B_{9,\sigma_{21}},\\
B_{1,\sigma_{42}} &= B_{2,\sigma_{42}} = B_{7,\sigma_{39}} = B_{9,\sigma_{49}},\\
\\
B_{1,\sigma_{12}} &= B_{3,\sigma_{31}} = B_{5,\sigma_{21}} = B_{8,\sigma_{12}},\\
B_{1,\sigma_{28}} &= B_{3,\sigma_{21}} = B_{5,\sigma_{31}} = B_{8,\sigma_{28}},\\
B_{1,\sigma_{58}} &= B_{3,\sigma_{49}} = B_{5,\sigma_{49}} = B_{8,\sigma_{58}},
\end{aligned}$$

*Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: Singular 4-2-0 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2019).

†http://data.up.krakow.pl

$$B_{1,\sigma_{27}} = B_{3,\sigma_{42}} = B_{7,\sigma_{36}} = B_{10,\sigma_{49}},$$
$$B_{1,\sigma_{40}} = B_{3,\sigma_{25}} = B_{7,\sigma_{41}} = B_{10,\sigma_{21}},$$
$$B_{1,\sigma_{57}} = B_{3,\sigma_{22}} = B_{7,\sigma_{54}} = B_{10,\sigma_{31}},$$

$$B_{1,\sigma_{11}} = B_{4,\sigma_{12}} = B_{5,\sigma_{22}} = B_{9,\sigma_{12}},$$
$$B_{1,\sigma_{38}} = B_{4,\sigma_{28}} = B_{5,\sigma_{25}} = B_{9,\sigma_{28}},$$
$$B_{1,\sigma_{39}} = B_{4,\sigma_{58}} = B_{5,\sigma_{42}} = B_{9,\sigma_{58}},$$

$$B_{1,\sigma_{36}} = B_{4,\sigma_{27}} = B_{6,\sigma_{22}} = B_{10,\sigma_{58}},$$
$$B_{1,\sigma_{41}} = B_{4,\sigma_{57}} = B_{6,\sigma_{25}} = B_{10,\sigma_{28}},$$
$$B_{1,\sigma_{54}} = B_{4,\sigma_{40}} = B_{6,\sigma_{42}} = B_{10,\sigma_{12}},$$

$$B_{2,\sigma_{12}} = B_{3,\sigma_{12}} = B_{5,\sigma_{12}} = B_{6,\sigma_{54}},$$
$$B_{2,\sigma_{28}} = B_{3,\sigma_{28}} = B_{5,\sigma_{28}} = B_{6,\sigma_{36}},$$
$$B_{2,\sigma_{58}} = B_{3,\sigma_{58}} = B_{5,\sigma_{58}} = B_{6,\sigma_{41}},$$

$$B_{2,\sigma_{40}} = B_{3,\sigma_{40}} = B_{9,\sigma_{22}} = B_{10,\sigma_{22}},$$
$$B_{2,\sigma_{27}} = B_{3,\sigma_{27}} = B_{9,\sigma_{42}} = B_{10,\sigma_{42}},$$
$$B_{2,\sigma_{57}} = B_{3,\sigma_{57}} = B_{9,\sigma_{25}} = B_{10,\sigma_{25}},$$

$$B_{2,\sigma_{11}} = B_{4,\sigma_{31}} = B_{5,\sigma_{11}} = B_{7,\sigma_{25}},$$
$$B_{2,\sigma_{38}} = B_{4,\sigma_{21}} = B_{5,\sigma_{38}} = B_{7,\sigma_{22}},$$
$$B_{2,\sigma_{39}} = B_{4,\sigma_{49}} = B_{5,\sigma_{39}} = B_{7,\sigma_{42}},$$

$$B_{2,\sigma_{36}} = B_{4,\sigma_{42}} = B_{8,\sigma_{42}} = B_{10,\sigma_{39}},$$
$$B_{2,\sigma_{41}} = B_{4,\sigma_{22}} = B_{8,\sigma_{25}} = B_{10,\sigma_{38}},$$
$$B_{2,\sigma_{54}} = B_{4,\sigma_{25}} = B_{8,\sigma_{22}} = B_{10,\sigma_{11}},$$

$$
\begin{aligned}
B_{3,\sigma_{11}} &= B_{4,\sigma_{54}} = B_{6,\sigma_{40}} = B_{7,\sigma_{40}},\\
B_{3,\sigma_{38}} &= B_{4,\sigma_{41}} = B_{6,\sigma_{27}} = B_{7,\sigma_{57}},\\
B_{3,\sigma_{39}} &= B_{4,\sigma_{36}} = B_{6,\sigma_{57}} = B_{7,\sigma_{27}},\\
\\
B_{3,\sigma_{36}} &= B_{4,\sigma_{39}} = B_{8,\sigma_{39}} = B_{9,\sigma_{39}},\\
B_{3,\sigma_{41}} &= B_{4,\sigma_{38}} = B_{8,\sigma_{38}} = B_{9,\sigma_{38}},\\
B_{3,\sigma_{54}} &= B_{4,\sigma_{11}} = B_{8,\sigma_{11}} = B_{9,\sigma_{11}},\\
\\
B_{5,\sigma_{27}} &= B_{6,\sigma_{38}} = B_{9,\sigma_{27}} = B_{10,\sigma_{27}},\\
B_{5,\sigma_{40}} &= B_{6,\sigma_{11}} = B_{9,\sigma_{57}} = B_{10,\sigma_{57}},\\
B_{5,\sigma_{57}} &= B_{6,\sigma_{39}} = B_{9,\sigma_{40}} = B_{10,\sigma_{40}},\\
\\
B_{5,\sigma_{36}} &= B_{7,\sigma_{49}} = B_{8,\sigma_{27}} = B_{10,\sigma_{36}},\\
B_{5,\sigma_{41}} &= B_{7,\sigma_{21}} = B_{8,\sigma_{40}} = B_{10,\sigma_{41}},\\
B_{5,\sigma_{54}} &= B_{7,\sigma_{31}} = B_{8,\sigma_{57}} = B_{10,\sigma_{54}},\\
\\
B_{6,\sigma_{12}} &= B_{7,\sigma_{12}} = B_{8,\sigma_{54}} = B_{9,\sigma_{54}},\\
B_{6,\sigma_{28}} &= B_{7,\sigma_{58}} = B_{8,\sigma_{36}} = B_{9,\sigma_{36}},\\
B_{6,\sigma_{58}} &= B_{7,\sigma_{28}} = B_{8,\sigma_{41}} = B_{9,\sigma_{41}}.
\end{aligned}
$$

□

Remark 3.4 *Incidences listed in the proof of Theorem 3.3 are rigid; however, they do not exclude the existence of some additional common Brianchon points for some relative conics. Some special arrangements of initial points P_i may cause appearing of such additional points.*

Remark 3.5 *Interesting phenomenon is the fact that the sets of Brianchon points for relative conics meet in fours (each quadruple has common three points) and simultaneously relative conics meet in fours at the points $R_{i,j}$. However, this quadruples do not coincide.*

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