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Moments and zero density estimates for Dirichlet L-functions

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Abstract: In this paper, we estimate the density of zeros of primitive Dirichlet L-functions in the half-plane $\Re(s) > 1/2$, under the assumption of a plausible conjecture on high moments of Dirichlet L-functions on the critical line. We conditionally improve the results of Huxley [9], Jutila [13], Heath-Brown [5, 6] and Bourgain [1].

Key words: Dirichlet L-functions, Riemann zeta-function, zero density estimate

1. Introduction

For $\sigma \geq 1/2$ and $T \geq 1$, let $N(\sigma, T)$ be the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function $\zeta(s)$ in the rectangle $\sigma \leq \beta \leq 1$, $|\gamma| \leq T$. The zero density conjecture on Riemann zeta-function asserts that the estimate

$$N(\sigma, T) \ll_\varepsilon T^{2(1-\sigma)+\varepsilon} \tag{1.1}$$

holds for any $\sigma \geq 1/2$, where ε denotes any fixed positive number. Though it has been out of reach of our techniques, there has been many researches toward this conjecture (for example, see [1, 4, 8, 11–13, 15, 16, 18]). Currently the best known result is by Bourgain [1], who proved that (1.1) holds uniformly for $25/32 = 0.78125 < \sigma \leq 1$.

There are similar results on zero density of Dirichlet L-functions. For $\sigma \geq 1/2$ and $T \geq 1$, we denote the number of zeros $\rho = \beta + i\gamma$ of the Dirichlet L-function $L(s, \chi)$ in the rectangle $\sigma \leq \beta \leq 1$, $|\gamma| \leq T$ by $N(\sigma, T, \chi)$. For $Q \geq 1$, put

$$\Sigma(Q) := \sum_{q \leq Q} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} N(\sigma, T, \chi),$$

where the asterisk means that the sum is restricted to primitive characters. Then most results on zero density of Dirichlet L-functions are of the form

$$\Sigma(Q) \ll_\varepsilon (Q^2 T^a)^{A(\sigma)(1-\sigma)+\varepsilon}. \tag{1.2}$$

It is conjectured that (1.2) holds with $a = 1$, $A(\sigma) = 2$ for the interval $1/2 \leq \sigma \leq 1$ (see [5]). Jutila [13] proved that (1.2) with $a = 1$, $A(\sigma) = 2$ holds for $21/26 \leq \sigma \leq 1$, and Heath-Brown [5] extended the range to $11/14 \leq \sigma \leq 1$. On the other hand, in the case $a = 2$, $A(\sigma) = 2$, Huxley [9] proved that (1.2) holds for

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$11/14 \leq \sigma \leq 1$, and Jutila extended the range to $7/9 \leq \sigma \leq 1$. Heath-Brown [5, 6] further improved the result and proved that (1.2) with $a = 2$, $A(\sigma) = 2$ holds for $65/82 \leq \sigma \leq 1$.

As far as the author knows, most of the recent work on zero density estimates for the Riemann zeta-function rely on the asymptotic formula

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \sim \frac{T}{2\pi^2}\log^4 T$$

by Ingham [10], and Heath-Brown's estimate [3]

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \ll_\varepsilon T^{2+\varepsilon}.$$

On the other hand, one of the key ingredients for the study of zero density of Dirichlet L-functions is the upper bound

$$\sideset{}{^*}\sum_{\chi \pmod q} \int_1^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^4 dt \ll qT\log^4 T \tag{1.3}$$

by Montgomery [15]. It seems to be obvious that one can obtain better results on zero density of Dirichlet L-functions if one assumes some hypotheses on higher moments of Dirichlet L-functions. However, there are few results arguing concretely how large one can extend the ranges that the zero density estimates are valid.

The main purpose of this paper is to investigate the ranges for which the zero density estimates

$$\Sigma(Q) \ll_\varepsilon (QT)^{A(1-\sigma)+\varepsilon}$$

holds for a fixed positive constant A, under the assumption of a conjecture on moments of Dirichlet L-functions. Throughout this paper, ε denotes any fixed positive quantity and at certain points, we shall change ε by a constant factor. The implied constants below might be dependent on ε without notice.

We introduce the following assumption on moments of Dirichlet L-functions.

Assumption 1.1 **($\mathcal{A}(p)$)** *For a fixed positive number p and $q, T \geq 1$, we have*

$$\sum_{\chi \pmod q} \int_1^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^{2p} dt \ll_{p,\varepsilon} (qT)^{1+\varepsilon}, \tag{1.4}$$

where the sum is over all Dirichlet characters modulo q.

It follows from a modification of (1.3) and an application of Hölder's inequality that $\mathcal{A}(p)$ is valid for $0 \leq p \leq 2$. In the case of the Riemann zeta-function ($q = 1$), Soundararajan [17] and Harper [9] showed that under the assumption of the Riemann hypothesis, the estimate (1.4) (precisely, in stronger forms) holds for any $p > 0$. Almost no doubt their methods will be applicable to the case of Dirichlet L-functions, under the assumption of the generalized Riemann hypothesis.

In the case of $A = 4$, we have the following result.

Theorem 1.2 *Under the assumption of $\mathcal{A}(p)$, the estimate*

$$\Sigma(Q) \ll (QT)^{4(1-\sigma)+\varepsilon}$$

holds uniformly for $T > 1$, $1 \leq Q \ll \exp(T^{\varepsilon})$, $\sigma_4(p) \leq \sigma \leq 1$, where

$$\sigma_4(2) = \frac{4}{5} = 0.8, \quad \sigma_4(3) = \frac{7}{9} = 0.7777\ldots, \quad \sigma_4(4) = \frac{139}{182} = 0.7637\ldots,$$

$$\sigma_4(5) = \frac{73}{96} = 0.7604\ldots, \quad \sigma_4(6) = \frac{425}{562} = 0.7562\ldots, \quad \sigma_4(7) = \frac{145}{192} = 0.7552\ldots,$$

$$\sigma_4(8) = \frac{951}{1262} = 0.7535\ldots, \quad \sigma_4(9) = \frac{241}{320} = 0.7531\ldots, \quad \sigma_4(10) = \frac{1789}{2378} = 0.7523\ldots.$$

Asymptotically, $\sigma_4(p) = 3/4 + 1/8p + O(p^{-2})$ as $p \to \infty$.

On the other hand, in the case of $A = 3$, we have the following.

Theorem 1.3 *Under the assumption of $\mathcal{A}(p)$, the estimate*

$$\Sigma(Q) \ll (QT)^{3(1-\sigma)+\varepsilon}$$

holds uniformly for $T > 1$, $1 \leq Q \ll \exp(T^{\varepsilon})$, $\sigma_3(p) \leq \sigma \leq 1$, where

$$\sigma_3(5) = \frac{11}{12} = 0.9166\ldots, \quad \sigma_3(7) = \frac{39}{43} = 0.9069\ldots, \quad \sigma_3(9) = \frac{152}{169} = 0.8994\ldots,$$

$$\sigma_3(10) = \frac{43}{51} = 0.8431\ldots, \quad \sigma_3(11) = \frac{293}{355} = 0.8253\ldots, \quad \sigma_3(12) = \frac{65}{82} = 0.7926\ldots.$$

Asymptotically, $\sigma_3(p) = 7/9 + 50/81p + O(p^{-2})$ as $p \to \infty$.

In Theorem 1.3, we could not obtain better numerical results for $p = 2, 3, 4, 6, 8$. For general $A > 0$, we have the following result.

Theorem 1.4 *Let $A > 0$ be an arbitrary fixed constant and let $\sigma_0 > \max\{(A+4)/3A, (3A+1)/4A\}$. Then there exists a positive number $p = p(A)$ that depends only on A such that the estimate*

$$\Sigma(Q) \ll (QT)^{A(1-\sigma)+\varepsilon}$$

holds uniformly for $T > 1$, $1 \leq Q \ll \exp(T^{\varepsilon})$ and $\sigma \geq \sigma_0$ under the assumption of $\mathcal{A}(p)$.

We also obtain conditional estimates for the zero density of the Riemann zeta-function. In this case we introduce the following assumption.

Assumption 1.5 ($\mathcal{A}_1(p)$) *For a fixed positive number p and $T \geq 1$, we have*

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2p} dt \ll_{p,\varepsilon} T^{1+\varepsilon}.$$

Theorem 1.6 *Under the assumption of $\mathcal{A}_1(p)$, the estimate*

$$\Sigma(1) \ll T^{2(1-\sigma)+\varepsilon}$$

holds uniformly for $\sigma_2(p) \le \sigma \le 1$, where

$$\sigma_2(2) = \frac{4}{5} = 0.8, \quad \sigma_2(3) = \frac{7}{9} = 0.7777\ldots, \quad \sigma_2(4) = \frac{139}{182} = 0.7637\ldots,$$

$$\sigma_2(5) = \frac{73}{96} = 0.7604\ldots, \quad \sigma_2(6) = \frac{425}{562} = 0.7562\ldots, \quad \sigma_2(7) = \frac{145}{192} = 0.7552\ldots,$$

$$\sigma_2(8) = \frac{951}{1262} = 0.7535\ldots, \quad \sigma_2(9) = \frac{241}{320} = 0.7531\ldots, \quad \sigma_2(10) = \frac{1789}{2378} = 0.7523\ldots.$$

Asymptotically, $\sigma_2(p) = 3/4 + 1/8p + O(p^{-2})$ as $p \to \infty$.

Theorem 1.6 conditionally (i.e. assuming $\mathcal{A}_1(3)$) improves the result of Bourgain [1] mentioned above. Moreover, for general $A > 0$, we have the following result.

Theorem 1.7 *Let $A > 0$ be a fixed constant and let $\sigma_0 > \max\{(A+1)/2A, 3/4\}$. Then there exists a positive number $p = p(A)$ that depends only on A such that the estimate*

$$\Sigma(1) \ll T^{A(1-\sigma)+\varepsilon}$$

holds uniformly for $\sigma_0 \le \sigma \le 1$ under the assumption of $\mathcal{A}_1(p)$.

To obtain these results, we apply three methods to estimate the frequency that Dirichlet polynomials become considerably large. In particular, we employ the Halász-Montgomery inequality [14], Huxley's reflection principle [8] and Heath-Brown's estimate for mean values of Dirichlet polynomials [9]. However, to simplify our discussion, we do not employ either the subdivision method introduced in [13] or Bourgain's ideas in [1]. The utilization of these methods will slightly improve the values of $\sigma_2(i)$ $(2 \le i \le 10)$ in Theorem 1.6 and recover the best known result of Bourgain [1] on zero density of the Riemann zeta-function, but this will not improve the limiting value of $\sigma_2(p)$.

Currently, $\mathcal{A}(p)$ is not proven for $p > 2$. However, Conrey et al. [2] obtained an asymptotic formula

$$\sum_{q \le Q} \sideset{}{^\flat}\sum_{\chi (\mathrm{mod}\ q)} \int_{-\infty}^{\infty} \left| \Lambda\left(\frac{1}{2} + iy, \chi\right) \right|^6 dy$$

$$\sim 42 a_3 \sum_{q \le Q} \prod_{p | q} \frac{\left(1 - \frac{1}{p}\right)^5}{1 + \frac{4}{p} + \frac{1}{p^2}} \varphi^\flat(q) \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{\frac{1}{2} + iy}{2}\right) \right|^6 dy,$$

where $a_3 := \prod_p (1 - 1/p)^4 (1 + 4/p + 1/p^2)$ and $\Lambda(s, \chi)$ is the complete Dirichlet L-function defined by

$$\Lambda\left(\frac{1}{2} + s, \chi\right) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{1}{4} + \frac{s}{2}\right) L\left(\frac{1}{2} + s, \chi\right),$$

and the symbol $\flat$ means that the sum is restricted to even primitive characters. This formula might enable us to obtain the value $\sigma_4(3)$ in Theorem 1.2 unconditionally when T is relatively small. The author plans to argue the application of the above formula to the problem of zero density estimates for Dirichlet L-functions in another paper.

2. Large value estimates for Dirichlet polynomials

In this section, we recall the basics of the zero detection method. We give some upper bounds for the number of well-spaced points over which the average size of Dirichlet polynomials becomes considerably large. The settings and arguments of the former part of this section are mostly based on the ideas of Jutila [13], and they are introduced in section 11 of Ivic's book [12]. In the latter part of this section, we combine the ideas and techniques of Heath-Brown [5, 6] and Jutila [13].

2.1. The zero detection method

For $Q, T \geq 1$, let X and Y be parameters satisfying $(QT)^{\varepsilon} < X \leq Y \leq (QT)^{C}$ for some constant $C > 0$. For a primitive Dirichlet character χ with conductor at most Q, put

$$M_X(s,\chi) = \sum_{m=1}^{X} \mu(m)\chi(m)m^{-s}.$$

Let ρ be a zero of the Dirichlet L-function $L(s,\chi)$ counted by $N(\sigma,T,\chi)$. By the Mellin transform

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}ds \quad (c, x > 0),$$

we have

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L(\rho+w,\chi)M_X(\rho+w,\chi)Y^{w}\Gamma(w)dw = e^{-\frac{1}{Y}} + \sum_{m=X+1}^{\infty} a(m)\chi(m)m^{-\rho}e^{-\frac{m}{Y}}, \tag{2.1}$$

where

$$a(m) = \sum_{\substack{d|m \\ d\leq X}} \mu(d).$$

We move the line of integration in (2.1) to $\Re(\rho+w) = 1/2$. The pole of $\Gamma(w)$ at $w = 0$ is cancelled by the zero of $L(\rho+w,\chi)$. Hence if χ is not principal, we do not cross any pole, and if χ is principal (hence $L(s,\chi) = \zeta(s)$), we cross the pole at $w = 1-\rho$ of $\zeta(\rho+w)$. Since $\Gamma(w)$ is of exponential decay as $|\Im(w)| \to \infty$, only the residues associated to low lying zeros of $\zeta(s)$ might give significant contribution to (2.1), and the number of such zeros is at most $O(\log^3 T)$ (see, for example, page 271 of [12]). We denote the set of these exceptional zeros by E. By Stirling's estimation for the Gamma function, the contribution to the integral on the left-hand side of (2.1) by w such that $|\Im(w)| \geq \log^2 QT$ is also $o(1)$. Moreover, the contribution of the terms with $m > Y\log^2 Y$ on the right hand side of (2.1) is also $o(1)$. Hence if $\rho = \beta + i\gamma$ is a zero of $L(s,\chi)$ which is not contained in E, then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\log^2 QT}^{\log^2 QT} L\left(\frac{1}{2}+i\gamma+iv,\chi\right) M_X\left(\frac{1}{2}+i\gamma+iv,\chi\right)\Gamma\left(\frac{1}{2}-\beta+iv\right)Y^{\frac{1}{2}-\beta+iv}dv \\ = e^{-\frac{1}{Y}} + \sum_{X<m\leq Y\log^2 Y} a(m)\chi(m)m^{-\rho}e^{-\frac{m}{Y}} + o(1).\end{aligned} \tag{2.2}$$

Hereafter, put

$$l = \log QT. \tag{2.3}$$

Since the first term on the right hand side of (2.2) tends to 1 as $Y \to \infty$, it follows that the zeros $\rho = \beta + i\gamma$ counted by $N(\sigma, T, \chi)$, except for the elements of E, satisfy at least one of the following two conditions.
Type I:

$$\sum_{X < m \leq Y \log^2 Y} a(m)\chi(m)m^{-\rho}e^{-\frac{m}{Y}} \gg 1.$$

Type II:

$$\int_{-l^2}^{l^2} L\left(\frac{1}{2} + i\gamma + iv, \chi\right) M_X\left(\frac{1}{2} + i\gamma + iv, \chi\right) \Gamma\left(\frac{1}{2} - \beta + iv\right) Y^{\frac{1}{2} - \beta + iv} dv \gg 1. \tag{2.4}$$

We pick up a set of pairs (χ_r, ρ_r) $(\rho_r = \beta_r + i\gamma_r)$ of type I (resp. type II) characters and zeros which satisfy the condition that if $r \neq s$, $\chi_r = \chi_s$, then $|\gamma_r - \gamma_s| \geq l^4$ (hereafter we call this Condition A) and denote it by $\mathfrak{R}_1$ (resp. $\mathfrak{R}_2$), and put $|\mathfrak{R}_i| = R_i$ for $i = 1, 2$. Then

$$\Sigma(Q) \ll (R_1 + R_2 + 1)l^5.$$

2.2. Estimation of the number of type II zeros

To evaluate R_2, we need to compare the discrete and integral moments of Dirichlet L-functions on the critical line. We use the following lemma.

Lemma 2.1 *Let $\mathcal{R} = (t_r)_{r=1}^R$ be a real sequence included in the interval $[1, T]$ which satisfies the condition that $|t_r - t_s| \geq l^2$ if $r \neq s$, where l is given by (2.3). Then for $k \geq 1$, $q \ll \exp(T^\varepsilon)$ and any non-principal Dirichlet character χ modulo q, we have*

$$\sum_{r=1}^R \left|L\left(\frac{1}{2} + it_r, \chi\right)\right|^k \ll (qT)^\varepsilon \left(R + \int_1^T \left|L\left(\frac{1}{2} + it, \chi\right)\right|^k dt\right).$$

The proof of the above lemma is almost the same as that of the case of the Riemann zeta-function (see [12], Section 8). In the proof, we consider the integration of Dirichlet L-functions in the interval $[T/2 - \log^2 qT, T + \log^2 qT]$, and the condition $q \ll \exp(T^\varepsilon)$ is required to ensure that $\log^2(qT) \ll T^\varepsilon$. In the case of the Riemann zeta-function, it is known that for $k > 1$ the upper bound

$$\int_1^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^k dt \ll T^{M(k)+\varepsilon}$$

implies

$$\sum_{r=1}^R \left|\zeta\left(\frac{1}{2} + it_r\right)\right|^k \ll T^{M(k)+\varepsilon}$$

(see [12], p.199-200). For $r = 1, 2, \ldots, R_2$, we choose $v_r' \in [-l^2, l^2]$ so that

$$\left|L\left(\frac{1}{2} + i\gamma_r + iv_r', \chi_r\right)\right| = \max_{-l^2 \leq v' \leq l^2} \left|L\left(\frac{1}{2} + i(\gamma_r + v'), \chi_r\right)\right|$$

and put $t_r = \gamma_r + v_r^{'}$. By (2.4) and the following estimates

$$M_X\left(\frac{1}{2}+i\gamma+iv,\chi\right) \ll X^{\frac{1}{2}}, \quad \Gamma\left(\frac{1}{2}-\beta+iv\right) \ll 1, \quad Y^{\frac{1}{2}-\beta+iv} \ll Y^{\frac{1}{2}-\sigma} \quad (-l^2 \le v \le l^2),$$

we have

$$\left|L\left(\frac{1}{2}+it_r,\chi_r\right)\right| X^{\frac{1}{2}} Y^{\frac{1}{2}-\sigma} l^2 \gg 1$$

for $r = 1, \dots, R_2$. Raising both sides to the power of $2p$ and taking the sum over r, we have

$$R_2 \ll l^{4p} X^p Y^{p(1-2\sigma)} \sum_{r=1}^{R_2} \left|L\left(\frac{1}{2}+it_r,\chi_r\right)\right|^{2p}. \tag{2.5}$$

Let $\eta_1, \dots, \eta_{R_3}$ be the representative of the set $\{\chi_r\}_{r=1}^{R_2}$ and $t_j^{(\nu)}$ $(j = 1, \dots, R_\nu)$ be the imaginary part of zeros of $L(s, \eta_\nu)$ for $\nu = 1, \dots, R_3$ considered above. Hence it follows that $\sum_{\nu=1}^{R_3} R_\nu = R_2$. By applying Lemma 2.1, assumption $\mathcal{A}(p)$ and the trivial estimate

$$R_2 \ll \sum_{q\le Q} \sideset{}{^*}\sum_{\chi \pmod q} N\left(\frac{1}{2}, T, \chi\right) \ll (Q^2T)^{1+\varepsilon},$$

we have

$$\begin{aligned}
\sum_{r=1}^{R_2} \left|L\left(\frac{1}{2}+it_r,\chi_r\right)\right|^{2p} &= \sum_{\nu=1}^{R_3}\sum_{j=1}^{R_\nu} \left|L\left(\frac{1}{2}+it_j^{(\nu)},\eta_\nu\right)\right|^{2p} \\
&\ll \sum_{\nu=1}^{R_3} (qT)^{\varepsilon} \left(R_\nu + \int_1^T \left|L\left(\frac{1}{2}+it,\eta_\nu\right)\right|^{2p} dt\right) \\
&\ll (qT)^{\varepsilon} \left(R_2 + \sum_{q\le Q} (qT)^{1+\varepsilon}\right) \\
&\ll (Q^2T)^{1+\varepsilon}.
\end{aligned}$$

Hence by (2.5), the estimate

$$R_2 \ll X^p Y^{p(1-2\sigma)} (Q^2T)^{1+\varepsilon}$$

holds if $Q \ll \exp(T^{\varepsilon})$. We set

$$X = (QT)^{\varepsilon}. \tag{2.6}$$

Hence

$$R_2 \ll Y^{p(1-2\sigma)} (Q^2T)^{1+\varepsilon}. \tag{2.7}$$

2.3. Estimation of the number of type I zeros

Next, we estimate R_1 with X given by (2.6). Put

$$b(m) = a(m) e^{-\frac{m}{Y}}.$$

Then

$$|b(m)| \ll d(m)e^{-\frac{m}{Y}} \ll (QT)^{\varepsilon}$$

for $X < m \leq Y^{1+\varepsilon}$. We need to evaluate the number of pairs (χ, ρ) ($\Re(\rho) = \beta \geq \sigma$) which satisfies Condition A and

$$\sum_{X<m\leq Y\log^2 Y} b(m)\chi(m)m^{-\rho} \gg 1. \tag{2.8}$$

We decompose the interval $(X, Y\log^2 Y]$ into $O(l)$ subintervals of the form $(2^{n-1}Y, 2^nY]$ $(n \in \mathbb{Z})$. By (2.8), there exists an absolute constant $c > 0$ for which at least one of $n \in \mathbb{Z}$,

$$\left|\sum_{2^{n-1}Y<m\leq 2^nY} b(m)\chi(m)m^{-\rho}\right| \geq \frac{1}{cl}. \tag{2.9}$$

Hereafter, we denote the number of pairs (χ, ρ) which satisfy Condition A and (2.9) by $R = R_n$. Since the number of n giving the above subintervals is $O(l)$, it follows that $R_1 \ll l \sup_n R_n$.

Let U be a parameter which satisfies $U \ll (QT)^C$ and

$$(l^2Y)^a \leq U < (l^2Y)^{a+1}$$

for some positive integer a. Next, we choose a positive integer b so that

$$(2^nY)^b \leq U < (2^nY)^{b+1}$$

holds. Raising both sides of (2.9) to the power b, we have

$$\left|\sum_{KW<m\leq W} c(m)\chi(m)m^{-\rho}\right| \geq V,$$

where

$$W = (2^nY)^b,\ K = 2^{-b},\ V = (cl)^{-b},$$

and the coefficient $c(m)$ is bounded as

$$|c(m)| \leq \sum_{m_1\cdots m_b=m} d(m_1)\cdots d(m_b) \leq d(m)^{2b}.$$

Since $2^nY \leq l^2Y$, it follows that $b \geq a$. Since $2^nY \geq (QT)^{\varepsilon}$, we have $(QT)^{\varepsilon b} \ll (QT)^C$. Hence $b \leq C/\varepsilon \ll 1$. Therefore, it follows that $K \gg 1$, $V \gg (QT)^{-\varepsilon}$, $c(m) \ll (QT)^{\varepsilon}$. Moreover,

$$U^{\frac{a}{a+1}} \leq U^{\frac{b}{b+1}} < (2^nY)^b = W \leq U. \tag{2.10}$$

Put

$$h = \log^2(QT),\ e_n = e^{-(\frac{n}{W})^h} - e^{-(\frac{n}{KW})^h},\ B = KW$$

and

$$H(s,\chi) = \sum_{n=1}^{\infty} e_n\chi(n)n^{-s}.$$

By Halász-Montgomery inequality (see [15], Lemma 1.7), it follows that

$$R^2V^2 \ll GRW + G \sum_{\substack{r,s\le R \\ r\ne s}} \left| H(\rho_r + \overline{\rho}_s - 2\sigma, \chi_r\overline{\chi}_s) \right|, \tag{2.11}$$

where

$$G := \sum_{KW<m\le W} |c(m)|^2 m^{-2\sigma} \ll W^{1-2\sigma}(QT)^{\varepsilon}. \tag{2.12}$$

To bound $H(s,\chi)$, we use the following lemma.

Lemma 2.2 ([13], Lemma 1) *Let χ be a Dirichlet character modulo q for $q \le Q$. Let $0 \le \sigma \le 1$, $|t| \le T$, $s = \sigma + it$. If χ is a principal character, we additionally assume that $|t| \ge h^2$. Then for $B \le qT$ and $q(|t|+h^3)(\pi B)^{-1} \le M \le (qT)^2$, we have*

$$H(s,\chi) \ll 1 + B^{\frac{1}{2}} Q^{\varepsilon} \int_{-h^2}^{h^2} \left| \sum_{n=1}^{M} \overline{\chi}(n) n^{-\frac{1}{2}+i(t+\tau)} \right| d\tau. \tag{2.13}$$

As Heath-Brown pointed out in his paper ([5], p.237), the conditions $B \le qT$ and $M \le (qT)^2$ are unnecessary. We put $B = KW$ in (2.13). Then

$$H(\rho_r + \overline{\rho}_s - 2\sigma, \chi_r\overline{\chi}_s) \ll D^{\varepsilon} \left(1 + W^{\frac{1}{2}} \int_{-h^2}^{h^2} \left| \sum_{n=1}^{M} \overline{\chi}_r\chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right| d\tau \right) \tag{2.14}$$

for

$$D = Q^2T, \quad M = \frac{h^3 D}{KW}. \tag{2.15}$$

We decompose the sum over n into the blocks $M/2^q < n \le M/2^{q-1}$, $1 \le q \ll \log M$. Put

$$N = \frac{M}{2^q}, \tag{2.16}$$

$$\Sigma(\tau) = \Sigma_q(\tau) = \sum_{r,s=1}^{R} \left| \sum_{N<n\le 2N} \overline{\chi}_r\chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right|. \tag{2.17}$$

Then by (2.11)–(2.17), we have

$$R^2V^2 \ll \left(GRW + GW^{\frac{1}{2}} \int_{-h^2}^{h^2} \Sigma(\tau) d\tau \right) D^{\varepsilon} \tag{2.18}$$

for some positive integer q. By Hölder's inequality, for any positive integer k, we have

$$\Sigma(\tau) \le R^{2-\frac{1}{k}} \left(\sum_{r,s=1}^{R} \left| \left(\sum_{N<n\le 2N} \overline{\chi}_r\chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right)^k \right|^2 \right)^{\frac{1}{2k}}. \tag{2.19}$$

We adapt the following lemma.

Lemma 2.3 ([13], Lemma 2) *For any real sequence* $(t_r)_{r=1}^R$ *and complex sequence* $(a_n)_{n=1}^N$ *satisfying* $|a_n| \leq A$ $(n = 1, \cdots, N)$, *we have*

$$\sum_{r,s=1}^{R} \left| \sum_{n=1}^{N} a_n \overline{\chi}_r \chi_s(n) n^{-\sigma+i(t_r-t_s)} \right|^2 \leq A^2 \sum_{r,s=1}^{R} \left| \sum_{n=1}^{N} \overline{\chi}_r \chi_s(n) n^{-\sigma+i(t_r-t_s)} \right|^2.$$

By the above lemma, the sum over r, s in (2.19) is

$$\sum_{r,s=1}^{R} \left| \left(\sum_{N<n\leq 2N} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right)^k \right|^2 \leq D^\varepsilon \sum_{r,s=1}^{R} \left| \sum_{N^k<n\leq (2N)^k} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s)} \right|^2. \tag{2.20}$$

Next, put

$$I(s,\chi) = \sum_{n=1}^{\infty} \left(e^{-\frac{n}{(2N)^k}} - e^{-\frac{n}{N^k}} \right) \chi(n) n^{-s}.$$

Then, by (2.20) and Lemma 2.3 again, we have

$$\sum_{r,s=1}^{R} \left| \left(\sum_{N<n\leq 2N} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right)^k \right|^2 \ll D^\varepsilon \sum_{r,s=1}^{R} \left| I\left(\frac{1}{2} + i(\gamma_r-\gamma_s), \chi_r \overline{\chi}_s\right) \right|^2. \tag{2.21}$$

If either $\chi_r \overline{\chi}_s$ is not principal or $|t| > h$, then

$$\begin{aligned} I\left(\frac{1}{2}+it, \chi_r\overline{\chi}_s\right) &= \sum_{n=1}^{\infty} \left(e^{-\frac{n}{(2N)^k}} - e^{-\frac{n}{N^k}} \right) (\chi_r\overline{\chi}_s)(n) n^{-\frac{1}{2}-it} \\ &= \frac{1}{2\pi i} \int_{-ih}^{ih} L\left(\frac{1}{2}+w+it, \chi_r\overline{\chi}_s\right) \Gamma(w)((2N)^{kw} - N^{kw}) dw + o(1). \end{aligned} \tag{2.22}$$

Since χ_r and χ_s are primitive, if $\chi_r\overline{\chi}_s$ is principal, it follows that $\chi_r = \chi_s$. In this case, by Condition A, it follows that $|\gamma_r - \gamma_s| > h$ for $r \neq s$. Hence we can apply (2.22) with $t = \gamma_r - \gamma_s$. The contribution of the diagonal terms to (2.20) is $O(D^\varepsilon N^k R)$. Therefore, by (2.21) and (2.22), we have

$$\begin{aligned} &\sum_{r,s=1}^{R} \left| \left(\sum_{N<n\leq 2N} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right)^k \right|^2 \\ &\ll D^\varepsilon N^k R + D^\varepsilon R^2 \\ &\quad + D^\varepsilon \sum_{r\neq s} \left| \frac{1}{2\pi i} \int_{-ih}^{ih} L\left(\frac{1}{2}+w+i(\gamma_r-\gamma_s), \chi_r\overline{\chi}_s\right) \Gamma(w)((2N)^{kw} - N^{kw}) dw \right|^2 \\ &\ll D^\varepsilon N^k R + D^\varepsilon R^2 + D^\varepsilon \sum_{r\neq s} \int_{|\tau|<h} \left| L\left(\frac{1}{2}+i(\tau+\gamma_r-\gamma_s), \chi_r\overline{\chi}_s\right) \right|^2 d\tau. \end{aligned} \tag{2.23}$$

Using Hölder's inequality twice, for $p \geq 2$, we have

$$\begin{aligned}
&\sum_{r\neq s}\int_{|\tau|<h}\left|L\left(\frac{1}{2}+i(\tau+\gamma_r-\gamma_s),\chi_r\overline{\chi}_s\right)\right|^2 d\tau \\
&\quad \ll D^{\varepsilon}\sum_{r\neq s}\left(\int_{|\tau|<h}\left|L\left(\frac{1}{2}+i(\tau+\gamma_r-\gamma_s),\chi_r\overline{\chi}_s\right)\right|^p d\tau\right)^{\frac{2}{p}} \\
&\quad \ll D^{\varepsilon}R^{2-\frac{4}{p}}\left(\sum_{r\neq s}\int_{|\tau|<h}\left|L\left(\frac{1}{2}+i(\tau+\gamma_r-\gamma_s),\chi_r\overline{\chi}_s\right)\right|^p d\tau\right)^{\frac{2}{p}}.
\end{aligned}$$

For a Dirichlet character χ and an integer l, put

$$\Delta_\chi(l) = \#\{(r,s) \mid r\neq s,\ \chi_r\overline{\chi}_s=\chi,\ |\gamma_r-\gamma_s-l|<1\}.$$

Then

$$\begin{aligned}
&\sum_{r\neq s}\int_{|\tau|<h}\left|L\left(\frac{1}{2}+i(\tau+\gamma_r-\gamma_s),\chi_r\overline{\chi}_s\right)\right|^2 d\tau \\
&\quad \ll D^{\varepsilon}R^{2-\frac{4}{p}}\left(\sum_{\chi}\sum_{l\in\mathbb{Z}}\Delta_\chi(l)\int_{|\tau|<h}\left|L\left(\frac{1}{2}+i(l+\tau),\chi\right)\right|^p d\tau\right)^{\frac{2}{p}}.
\end{aligned} \tag{2.24}$$

For any fixed (χ_r,ρ_r) $(1\leq r\leq R)$, there is at most one primitive Dirichlet character χ_s which satisfies $\chi_r\overline{\chi}_s=\chi$, and the number of zeros of the Dirichlet L-function $L(s,\chi_s)$ which satisfy both $|\gamma_r-\gamma_s-l|<1$ and Condition A is at most one. Hence $\Delta_\chi(l)\ll R$. Moreover, trivially

$$\sum_{\chi}\sum_{l\in\mathbb{Z}}\Delta_\chi(l)\ll R^2.$$

Therefore,

$$\sum_{\chi}\sum_{l\in\mathbb{Z}}\Delta_\chi(l)^2\ll R^3.$$

By using this and Cauchy's inequality, we have

$$\begin{aligned}
&\sum_{\chi}\sum_{l\in\mathbb{Z}}\Delta_\chi(l)\int_{|\tau|<h}\left|L\left(\frac{1}{2}+i(l+\tau),\chi\right)\right|^p d\tau \\
&\quad \leq \int_{|\tau|<h}\left(\sum_{\chi}\sum_{l\in\mathbb{Z}}\Delta_\chi(l)^2\right)^{\frac{1}{2}}\left(\sum_{\chi}\sum_{l\in\mathbb{Z}}\left|L\left(\frac{1}{2}+i(l+\tau),\chi\right)\right|^{2p}\right)^{\frac{1}{2}} d\tau \\
&\quad \ll R^{\frac{3}{2}}\int_{|\tau|<h}\left(\sum_{\chi}\sum_{l\in\mathbb{Z}}\left|L\left(\frac{1}{2}+i(l+\tau),\chi\right)\right|^{2p}\right)^{\frac{1}{2}} d\tau.
\end{aligned}$$

By Cauchy's inequality, the right hand side is at most

$$R^{\frac{3}{2}}D^{\varepsilon}\left(\sum_{q\leq Q}\sum_{\chi\,(\mathrm{mod}\ q)}\int_{|\tau|<T}\left|L\left(\frac{1}{2}+i\tau,\chi\right)\right|^{2p} d\tau\right)^{\frac{1}{2}}.$$

Evaluating the above moment, we have

$$\sum_{\chi}\sum_{l\in\mathbb{Z}}\Delta_{\chi}(l)\int_{|\tau|<h}\left|L\left(\frac{1}{2}+i(l+\tau),\chi\right)\right|^{p}d\tau \ll R^{\frac{3}{2}}QT^{\frac{1}{2}}D^{\varepsilon} \tag{2.25}$$

under the assumption of $\mathcal{A}(p)$. Substituting (2.25) into (2.24), we have

$$\sum_{r\neq s}\int_{|\tau|<h}\left|L\left(\frac{1}{2}+i(\tau+\gamma_r-\gamma_s),\chi_r\overline{\chi}_s\right)\right|^{2}d\tau \ll D^{\varepsilon}R^{2-\frac{1}{p}}Q^{\frac{2}{p}}T^{\frac{1}{p}}. \tag{2.26}$$

By (2.23), this gives

$$\sum_{r,s=1}^{R}\left|\left(\sum_{N<n\leq 2N}\overline{\chi}_r\chi_s(n)n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)}\right)^{k}\right|^{2} \ll D^{\varepsilon}N^{k}R+D^{\varepsilon}R^{2}+D^{\varepsilon}R^{2-\frac{1}{p}}Q^{\frac{2}{p}}T^{\frac{1}{p}}.$$

We combine this with (2.19) to obtain

$$\Sigma(\tau)\ll D^{\varepsilon}N^{\frac{1}{2}}R^{2-\frac{1}{2k}}+D^{\varepsilon}R^{2}+D^{\varepsilon}R^{2-\frac{1}{2pk}}Q^{\frac{1}{pk}}T^{\frac{1}{2pk}}.$$

Substituting this into (2.18), we have

$$R^{2}V^{2}\ll(GRW+GR^{2}+GW^{\frac{1}{2}}N^{\frac{1}{2}}R^{2-\frac{1}{2k}}+GW^{\frac{1}{2}}R^{2}+GW^{\frac{1}{2}}R^{2-\frac{1}{2pk}}Q^{\frac{1}{pk}}T^{\frac{1}{2pk}})D^{\varepsilon}. \tag{2.27}$$

Recall that $V\gg D^{-\varepsilon}$, $GR^{2}=o(R^{2})$. Therefore, by (2.27) and (2.12), under the assumption of $\mathcal{A}(p)$, we have

$$\begin{aligned} R &\ll (GW+G^{2k}W^{k}N^{k}+G^{2pk}W^{pk}Q^{2}T)D^{\varepsilon} \\ &\ll (W^{2-2\sigma}+W^{3k-4k\sigma}N^{k}+W^{3pk-4pk\sigma}D)D^{\varepsilon}, \end{aligned} \tag{2.28}$$

provided

$$GW^{\frac{1}{2}}=o(1). \tag{2.29}$$

Next, for a positive integer l, put

$$J_l(s,\chi)=\sum_{n=1}^{\infty}\left(e^{-\left(\frac{n}{(2N)^{l}}\right)^{h}}-e^{-\left(\frac{n}{N^{l}}\right)^{h}}\right)\chi(n)n^{-s}.$$

By Lemma 2.3, we have

$$\sum_{r,s=1}^{R}\left|\left(\sum_{N<n\leq 2N}\overline{\chi}_r\chi_s(n)n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)}\right)^{l}\right|^{2}\ll D^{\varepsilon}N^{-l}\sum_{r,s=1}^{R}\left|J_l(i(\gamma_r-\gamma_s),\chi_r\overline{\chi}_s)\right|^{2}. \tag{2.30}$$

For $r\neq s$, by Lemma 2.2 (with $B=N^{l}$) and Cauchy's inequality, we have

$$\left|J_l(i(\gamma_r-\gamma_s),\chi_r\overline{\chi}_s)\right|^{2}\ll D^{\varepsilon}\left(1+N^{l}\int_{-h^{2}}^{h^{2}}\left|\sum_{n=1}^{P}\overline{\chi}_r\chi_s(n)n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)}\right|^{2}d\tau\right),$$

where

$$P = \frac{Dh^3}{N^l}. \tag{2.31}$$

The contribution of the terms with $r = s$ to (2.30) is trivially $O(RN^{2l})$. Put

$$S(\tau) = \sum_{r,s=1}^{R} \left| \sum_{n=1}^{P} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right|^2. \tag{2.32}$$

Then

$$\sum_{r,s=1}^{R} |J_l(i(\gamma_r-\gamma_s), \chi_r\overline{\chi}_s)|^2 \ll \left(R^2 + RN^{2l} + N^l \int_{-h^2}^{h^2} S(\tau)d\tau \right) D^{\varepsilon}. \tag{2.33}$$

We now estimate $S(\tau)$. Adapting Hölder's inequality to (2.32), for any positive integer j, we have

$$S(\tau) \le R^{2-\frac{2}{j}} \left(\sum_{r,s=1}^{R} \left| \left(\sum_{n=1}^{P} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right)^j \right|^2 \right)^{\frac{1}{j}}. \tag{2.34}$$

Since the jth power is expanded as

$$\left(\sum_{n=1}^{P} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right)^j = \sum_{n=1}^{P^j} \tilde{d}_j(n) \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)}, \quad \tilde{d}_j(n) \ll_{j,\varepsilon} n^{\varepsilon},$$

by Lemma 2.3, we have

$$\sum_{r,s=1}^{R} \left| \left(\sum_{n=1}^{P} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right)^j \right|^2 \le D^{\varepsilon} \sum_{r,s=1}^{R} \left| \sum_{n=1}^{\infty} c_n \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s)} \right|^2. \tag{2.35}$$

Here, the coefficient c_n is given by

$$c_n = e^{-\frac{n}{2P_1}} - e^{-\frac{n}{P_1}}$$

for some $P_1 \le P^j$. The contribution of the terms with $r = s$ to (2.35) is $O(RP^j)$. Hence

$$\sum_{r,s=1}^{R} \left| \left(\sum_{n=1}^{P} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)} \right)^j \right|^2 \ll D^{\varepsilon} RP^j + D^{\varepsilon} \sum_{r \ne s} \left| \sum_{n=1}^{\infty} c_n \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s)} \right|^2. \tag{2.36}$$

Condition A implies that if $r \ne s$, either $\overline{\chi}_r \chi_s \ne \chi_0$ or $|\gamma_r - \gamma_s| \ge h^2$ holds. Therefore, for $r \ne s$,

$$\begin{aligned} &\sum_{n=1}^{\infty} c_n \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s)} \\ &= \frac{1}{2\pi i} \int_{-ih}^{ih} L\left(\frac{1}{2} + w + i(\gamma_s-\gamma_r), \overline{\chi}_r \chi_s \right) \Gamma(w) \left((2P_1)^w - P_1^w \right) dw + o(1). \end{aligned}$$

By using the bound $\Gamma(w) \ll 1$ and applying Cauchy's inequality, we have

$$\left|\sum_{n=1}^{\infty} c_n \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s)}\right|^2 \ll D^{\varepsilon} \int_{|\tau|<h} \left|L\left(\frac{1}{2}+i(\gamma_s-\gamma_r+\tau), \overline{\chi}_r\chi_s\right)\right|^2 d\tau + o(1)$$

for $r \neq s$. Substituting this into (2.36), we have

$$\begin{aligned}\sum_{r,s=1}^{R} &\left|\left(\sum_{n=1}^{P} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)}\right)^j\right|^2 \\ &\ll D^{\varepsilon} R P^j + D^{\varepsilon} R^2 + D^{\varepsilon} \sum_{r\neq s} \int_{|\tau|<h} \left|L\left(\frac{1}{2}+i(\gamma_s-\gamma_r+\tau), \overline{\chi}_r\chi_s\right)\right|^2 d\tau.\end{aligned} \tag{2.37}$$

By (2.34) and (2.37),

$$S(\tau) \ll R^{2-\frac{2}{j}} D^{\varepsilon} \left(RP^j + R^2 + \sum_{r\neq s} \int_{|\tau|<h} \left|L\left(\frac{1}{2}+i(\gamma_s-\gamma_r+\tau), \overline{\chi}_r\chi_s\right)\right|^2 d\tau\right)^{\frac{1}{j}}. \tag{2.38}$$

Applying (2.26) to (2.38), we obtain

$$S(\tau) \ll \left(R^{2-\frac{1}{j}} P + R^2 + R^{2-\frac{1}{jp}} Q^{\frac{2}{jp}} T^{\frac{1}{jp}}\right) D^{\varepsilon}.$$

Therefore, by (2.33), we have

$$\sum_{r,s=1}^{R} |J_l(i(\gamma_r-\gamma_s), \chi_r\overline{\chi}_s)|^2 \ll \left(RN^{2l} + R^{2-\frac{1}{j}} N^l P + N^l R^2 + R^{2-\frac{1}{jp}} N^l Q^{\frac{2}{jp}} T^{\frac{1}{jp}}\right) D^{\varepsilon}. \tag{2.39}$$

By (2.39) and (2.30),

$$\sum_{r,s=1}^{R} \left|\left(\sum_{N<n\leq 2N} \overline{\chi}_r \chi_s(n) n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)}\right)^l\right|^2 \ll \left(RN^l + R^{2-\frac{1}{j}} P + R^2 + R^{2-\frac{1}{jp}} Q^{\frac{2}{jp}} T^{\frac{1}{jp}}\right) D^{\varepsilon}. \tag{2.40}$$

By (2.40) and (2.19) (with k replaced by l),

$$\Sigma(\tau) \ll \left(R^{2-\frac{1}{2l}} N^{\frac{1}{2}} + R^{2-\frac{1}{2jl}} P^{\frac{1}{2l}} + R^2 + R^{2-\frac{1}{2jlp}} Q^{\frac{1}{jlp}} T^{\frac{1}{2jlp}}\right) D^{\varepsilon}. \tag{2.41}$$

By (2.41) and (2.18),

$$\begin{aligned}R^2V^2 \ll \Big(&GRW + GW^{\frac{1}{2}} R^{2-\frac{1}{2l}} N^{\frac{1}{2}} + GW^{\frac{1}{2}} R^{2-\frac{1}{2jl}} P^{\frac{1}{2l}} \\ &+ GW^{\frac{1}{2}} R^2 + GW^{\frac{1}{2}} R^{2-\frac{1}{2jlp}} Q^{\frac{1}{jlp}} T^{\frac{1}{2jlp}}\Big) D^{\varepsilon}.\end{aligned} \tag{2.42}$$

By (2.42) and (2.12), (2.31), under the assumption of $\mathcal{A}(p)$, we obtain

$$\begin{aligned} R &\ll \left(GW + G^{2l}W^lN^l + G^{2jl}W^{jl}P^j + G^{2jlp}W^{jlp}Q^2T\right)D^\varepsilon \\ &\ll \left(W^{2-2\sigma} + W^{3l-4l\sigma}N^l + W^{3jl-4jl\sigma}N^{-jl}D^j + W^{3jlp-4jlp\sigma}D\right)D^\varepsilon, \end{aligned} \tag{2.43}$$

provided that (2.29) holds.

We replace l in the left hand side of (2.30) with a positive integer m and evaluate this in another way. By Lemma 2.3, for a positive integer n,

$$\sum_{r,s=1}^{R}\left|\left(\sum_{n\le P}\overline{\chi}_r\chi_s(n)n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)}\right)^n\right|^2 \ll D^\varepsilon\sum_{r,s}\left|\sum_{n\le P^n}\overline{\chi}_r\chi_s(n)n^{-\frac{1}{2}+i(\gamma_r-\gamma_s)}\right|^2, \tag{2.44}$$

where $P = Dh^3/N^m$. Here we will use the following result of Heath-Brown.

Lemma 2.4 ([6], Theorem 1) *Let $(\chi_r,\rho_r)_{r=1}^R$ satisfy Condition A. Then*

$$\sum_{r,s=1}^{R}\left|\sum_{n=1}^{N}\overline{\chi}_r\chi_s(n)n^{-\frac{1}{2}-i(\gamma_r-\gamma_s)}\right|^2 \ll (RN + R^2 + R^{\frac{5}{4}}D^{\frac{1}{2}})D^\varepsilon.$$

By choosing $N = P^n$ in the above lemma, we have

$$\sum_{r,s=1}^{R}\left|\sum_{n\le P^n}\overline{\chi}_r\chi_s(n)n^{-\frac{1}{2}+i(\gamma_r-\gamma_s)}\right|^2 \ll \left(RP^n + R^2 + R^{\frac{5}{4}}D^{\frac{1}{2}}\right)D^\varepsilon. \tag{2.45}$$

By (2.34) (with j replaced by n), (2.44) and (2.45), we have

$$S(\tau) \ll \left(R^{2-\frac{1}{n}}P + R^2 + R^{2-\frac{3}{4n}}D^{\frac{1}{2n}}\right)D^\varepsilon.$$

Together with (2.33), this gives

$$\sum_{r,s=1}^{R}|J_m(i(\gamma_r-\gamma_s+\tau),\chi_r\overline{\chi}_s)|^2 \ll \left(RN^{2m} + R^{2-\frac{1}{n}}N^mP + R^2N^m + R^{2-\frac{3}{4n}}N^mD^{\frac{1}{2n}}\right)D^\varepsilon.$$

By combining this with (2.30), we obtain

$$\sum_{r,s=1}^{R}\left|\left(\sum_{N<n\le 2N}\overline{\chi}_r\chi_s(n)n^{-\frac{1}{2}+i(\gamma_r-\gamma_s+\tau)}\right)^m\right|^2 \ll \left(RN^m + R^{2-\frac{1}{n}}P + R^2 + R^{2-\frac{3}{4n}}D^{\frac{1}{2n}}\right)D^\varepsilon.$$

Combining this with (2.19) (with k replaced by m), we have

$$\Sigma(\tau) \ll \left(R^{2-\frac{1}{2m}}N^{\frac{1}{2}} + R^{2-\frac{1}{2nm}}P^{\frac{1}{2m}} + R^2 + R^{2-\frac{3}{8nm}}D^{\frac{1}{4nm}}\right)D^\varepsilon.$$

Substituting this into (2.18), we have

$$R^2V^2 \ll \left(GRW + GW^{\frac{1}{2}}R^{2-\frac{1}{2m}}N^{\frac{1}{2}} + GW^{\frac{1}{2}}R^{2-\frac{1}{2nm}}P^{\frac{1}{2m}} + GW^{\frac{1}{2}}R^2 + GW^{\frac{1}{2}}R^{2-\frac{3}{8nm}}D^{\frac{1}{4nm}}\right)D^{\varepsilon}.$$

Therefore,

$$R \ll \left(W^{2-2\sigma} + W^{3m-4m\sigma}N^m + W^{3nm-4nm\sigma}N^{-nm}D^n + W^{4nm-\frac{16}{3}nm\sigma}D^{\frac{2}{3}}\right)D^{\varepsilon}, \tag{2.46}$$

provided that (2.29) holds.

Now we have obtained three estimates (2.28), (2.43) and (2.46) for R. Summing up, we arrive at the following proposition.

Proposition 2.5 *Let K be a positive constant and $(c(m))$ be a sequence which may be dependent on Q, T and ε but independent of χ, ρ satisfying $c(m) \ll_{\varepsilon} m^{\varepsilon}$. Let R be the number of pairs (χ, ρ) counted by $\Sigma(Q)$ which satisfies Condition A and*

$$\left|\sum_{KW<m\le W} c(m)\chi(m)m^{-\rho}\right| \ge V$$

for $V \gg D^{-\varepsilon}$, where $D = Q^2T$. Then, under the assumption of $\mathcal{A}(p)$ for any $q \le Q$, we have

$$R \ll D^{\varepsilon}\min\{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\}$$

provided

$$GW^{\frac{1}{2}} = o(1), \tag{2.47}$$

where

$$\mathcal{R}_1 = W^{2-2\sigma} + W^{3k-4k\sigma}N^k + W^{3pk-4pk\sigma}D, \tag{2.48}$$

$$\mathcal{R}_2 = W^{2-2\sigma} + W^{3l-4l\sigma}N^l + W^{3jl-4jl\sigma}N^{-jl}D^j + W^{3jlp-4jlp\sigma}D, \tag{2.49}$$

$$\mathcal{R}_3 = W^{2-2\sigma} + W^{3m-4m\sigma}N^m + W^{3nm-4nm\sigma}N^{-nm}D^n + W^{4nm-\frac{16}{3}nm\sigma}D^{\frac{2}{3}} \tag{2.50}$$

Here, N is some parameter satisfying $1 \ll N \ll D^{1+\varepsilon}/W$, and k, l, j, n, m are arbitrary positive integers.

3. The proofs of theorems

In this section we prove Theorems 1.2–1.4 and Theorems 1.6–1.7. To prove these theorems, we use the evaluation of $\mathcal{R}_1$ when N is relatively small, and use the evaluations of $\mathcal{R}_2$ and $\mathcal{R}_3$ when N is large. The consequence of the evaluation of $\mathcal{R}_2$ is more useful than that of $\mathcal{R}_3$ when p is large to a certain extent, and otherwise the consequence of the evaluation of $\mathcal{R}_3$ works better.

3.1. The setting of parameters

Put $Y = (QT)^{\alpha}$, $U = (QT)^{a\alpha+\varepsilon}$, where α is a positive constant which will be chosen later. Then by (2.10), the range of W is

$$(QT)^{\frac{a^2}{a+1}\alpha+\varepsilon} \le W < (QT)^{a\alpha+\varepsilon}. \tag{3.1}$$

We first consider the condition (2.47). By (2.12), we have

$$GW^{\frac{1}{2}} \ll W^{\frac{3}{2}-2\sigma}(QT)^{\varepsilon}.$$

Therefore (2.47) is satisfied if

$$\sigma > \frac{3}{4}.$$

Hence hereafter we assume that σ satisfies $3/4 < \sigma < 1$. Next, put

$$\mathcal{T} = QT.$$

Recall that $D = Q^2T$. Hence $D \ll \mathcal{T}$ if $Q = 1$, and $D \ll \mathcal{T}^2$ if $Q > 1$. So we put

$$\delta = \delta(Q) = \begin{cases} 1 & \text{if } Q = 1 \\ 2 & \text{if } Q > 1. \end{cases}$$

By (2.7),

$$R_2 \ll (QT)^{\alpha p(1-2\sigma)}(Q^2T)^{1+\varepsilon} \ll \mathcal{T}^{\alpha p(1-2\sigma)+\delta+\varepsilon}.$$

The inequality

$$\alpha p(1-2\sigma) + \delta \le A(1-\sigma) \tag{3.2}$$

holds for

$$\sigma \ge \frac{\alpha p + \delta - A}{2\alpha p - A}, \tag{3.3}$$

provided

$$2\alpha p - A > 0. \tag{3.4}$$

Since

$$W^{2-2\sigma} \ll (QT)^{a\alpha(2-2\sigma)+\varepsilon},$$

it follows that $W^{2-2\sigma} \ll \mathcal{T}^{A(1-\sigma)+\varepsilon}$ is valid if

$$a\alpha \le \frac{A}{2}. \tag{3.5}$$

We choose α so that the right hand side of (3.3) equals $3/4$. Hence

$$\alpha = \frac{4\delta - A}{2p}. \tag{3.6}$$

In this case, the condition (3.4) holds if $A < 2\delta$, and if $A = 2\delta$, (3.2) always holds. Hence here and below we assume that $A \le 2\delta$. Finally, put

$$a = \left\lfloor \frac{Ap}{4\delta - A} \right\rfloor, \tag{3.7}$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

3.2. Estimations of $\mathcal{R}_i$ $(i=1,2,3)$

Since a and α satisfy (3.5), $W^{2-2\sigma} \ll \mathcal{T}^{A(1-\sigma)+\varepsilon}$ always holds. By (3.1), (3.6), the range of W is

$$\mathcal{T}^{\frac{a^2(4\delta-A)}{2p(a+1)}+\varepsilon} \leq W < \mathcal{T}^{\frac{a(4\delta-A)}{2p}+\varepsilon}, \tag{3.8}$$

where a is given by (3.7). For arbitrarily fixed positive integer k, the condition $W^{3k-4k\sigma}N^k \ll \mathcal{T}^{A(1-\sigma)}$ holds if and only if $N \ll \mathcal{T}^{\frac{A}{k}(1-\sigma)-\frac{a^2(4\delta-A)}{2p(a+1)}(3-4\sigma)}$. Hence we estimate $\mathcal{R}_1$ in (2.48) in case of

$$1 \ll N \ll \mathcal{T}^{\frac{A}{k}(1-\sigma)-\frac{a^2(4\delta-A)}{2p(a+1)}(3-4\sigma)},$$

and estimate $\mathcal{R}_2$ and $\mathcal{R}_3$ by using (2.49) or (2.50) in case of

$$\mathcal{T}^{\frac{A}{k}(1-\sigma)-\frac{a^2(4\delta-A)}{2p(a+1)}(3-4\sigma)} \ll N \ll \frac{\mathcal{T}^{\delta+\varepsilon}}{W}.$$

First, we assume

$$1 \ll N \ll \mathcal{T}^{\frac{A}{k}(1-\sigma)-\frac{a^2(4\delta-A)}{2p(a+1)}(3-4\sigma)}.$$

Then, as mentioned above, $W^{3k-4k\sigma}N^k \ll \mathcal{T}^{A(1-\sigma)+\varepsilon}$ holds. Next, since

$$W^{3pk-4pk\sigma}D \ll \mathcal{T}^{\frac{a^2k(4\delta-A)}{2(a+1)}(3-4\sigma)+\delta+\varepsilon},$$

the condition

$$\frac{a^2k(4\delta-A)}{2(a+1)}(3-4\sigma)+\delta \leq A(1-\sigma)$$

holds for

$$\sigma \geq \frac{3a^2k(4\delta-A)+2(\delta-A)(a+1)}{4a^2k(4\delta-A)-2A(a+1)} =: \sigma_1(A;\delta;p;k), \tag{3.9}$$

provided

$$4a^2k(4\delta-A)-2A(a+1)>0. \tag{3.10}$$

Consequently, if $1 \ll N \ll \mathcal{T}^{\frac{A}{k}(1-\sigma)-\frac{a^2(4\delta-A)}{2p(a+1)}(3-4\sigma)}$, then the bound $\mathcal{R}_1 \ll \mathcal{T}^{A(1-\sigma)+\varepsilon}$ holds for

$$\sigma > \max\left\{\sigma_1(A;\delta;p;k), \frac{3}{4}\right\}$$

provided that (3.10) holds, where $\sigma_1(A;\delta;p;k)$ is defined by (3.9).

Next, we assume

$$\mathcal{T}^{\frac{A}{k}(1-\sigma)-\frac{a^2(4\delta-A)}{2p(a+1)}(3-4\sigma)} \ll N \ll \frac{\mathcal{T}^{\delta+\varepsilon}}{W} \tag{3.11}$$

and estimate $\mathcal{R}_2$ and $\mathcal{R}_3$. By (3.8),

$$W^{3l-4l\sigma}N^l \ll W^{3l-4l\sigma}\left(\frac{\mathcal{T}^{\delta+\varepsilon}}{W}\right)^l \ll \mathcal{T}^{\frac{a^2l(4\delta-A)}{p(a+1)}(1-2\sigma)+\delta l+\varepsilon}$$

for $l \in \mathbb{Z}_{>0}$, and the inequality

$$\frac{a^2 l(4\delta - A)}{p(a+1)}(1-2\sigma) + \delta l \le A(1-\sigma)$$

holds for

$$\sigma \ge \frac{a^2 l(4\delta - A) + p(\delta l - A)(a+1)}{2a^2 l(4\delta - A) - Ap(a+1)},$$

provided that

$$2a^2 l(4\delta - A) - Ap(a+1) > 0 \tag{3.12}$$

holds. By (3.8) and (3.11), we have

$$W^{3jl-4jl\sigma} N^{-jl} D^j \ll \mathcal{T}^{\frac{a^2 jl(4\delta-A)}{p(a+1)}(3-4\sigma) - \frac{Ajl}{k}(1-\sigma) + \delta j + \varepsilon},$$

and the inequality

$$\frac{a^2 jl(4\delta - A)}{p(a+1)}(3-4\sigma) - \frac{Ajl}{k}(1-\sigma) + \delta j \le A(1-\sigma)$$

holds for

$$\sigma \ge \frac{3a^2 jkl(4\delta - A) - p(a+1)(Ajl - \delta jk + kA)}{4a^2 jkl(4\delta - A) - Ap(a+1)(jl+k)},$$

provided

$$4a^2 jkl(4\delta - A) - Ap(a+1)(jl+k) > 0. \tag{3.13}$$

Next, by (3.8), we have

$$W^{3jlp-4jlp\sigma} D \ll \mathcal{T}^{\frac{a^2 jl(4\delta-A)}{2(a+1)}(3-4\sigma) + \delta + \varepsilon},$$

and the inequality

$$\frac{a^2 jl(4\delta - A)}{2(a+1)}(3-4\sigma) + \delta \le A(1-\sigma)$$

holds for

$$\sigma \ge \frac{3a^2 jl(4\delta - A) + 2(\delta - A)(a+1)}{4a^2 jl(4\delta - A) - 2A(a+1)},$$

provided that

$$4a^2 jl(4\delta - A) - 2A(a+1) > 0 \tag{3.14}$$

holds. Consequently, if N satisfies (3.11), the bound $\mathcal{R}_2 \ll \mathcal{T}^{A(1-\sigma)+\varepsilon}$ holds for

$$\sigma > \max\left\{\sigma_2(A;\delta;p;j,k,l), \frac{3}{4}\right\},$$

where

$$\begin{aligned}
&\sigma_2(A;\delta;p;j,k,l) \\
&:= \max\left\{\frac{a^2 l(4\delta - A) + p(\delta l - A)(a+1)}{2a^2 l(4\delta - A) - Ap(a+1)}, \frac{3a^2 jkl(4\delta - A) - p(a+1)(Ajl - \delta jk + kA)}{4a^2 jkl(4\delta - A) - Ap(a+1)(jl+k)},\right. \\
&\qquad\qquad \left.\frac{3a^2 jl(4\delta - A) + 2(\delta - A)(a+1)}{4a^2 jl(4\delta - A) - 2A(a+1)}\right\},
\end{aligned} \tag{3.15}$$

provided that (3.12), (3.13) and (3.14) hold.

Finally, we estimate $\mathcal{R}_3$. We again assume (3.11). Then for $m \in \mathbb{Z}_{>0}$,

$$W^{3m-4m\sigma}N^m \ll \mathcal{T}^{A(1-\sigma)+\varepsilon}$$

holds for

$$\sigma \geq \frac{a^2m(4\delta - A) + p(\delta m - A)(a+1)}{2a^2m(4\delta - A) - Ap(a+1)},$$

provided

$$2a^2m(4\delta - A) - Ap(a+1) > 0. \tag{3.16}$$

Next,

$$W^{3nm-4nm\sigma}N^{-nm}D^n \ll \mathcal{T}^{A(1-\sigma)+\varepsilon}$$

holds for

$$\sigma \geq \frac{3a^2knm(4\delta - A) - p(a+1)(Anm - \delta nk + kA)}{4a^2knm(4\delta - A) - Ap(a+1)(nm+k)},$$

provided

$$4a^2knm(4\delta - A) - Ap(a+1)(nm+k) > 0. \tag{3.17}$$

Finally,

$$W^{4nm-\frac{16}{3}nm\sigma}D^{\frac{2}{3}} \ll \mathcal{T}^{\frac{a^2nm(4\delta-A)}{2p(a+1)}(4-\frac{16}{3}\sigma)+\frac{2}{3}\delta},$$

and the inequality

$$\frac{a^2nm(4\delta - A)}{2p(a+1)}\left(4 - \frac{16}{3}\sigma\right) + \frac{2}{3}\delta \leq A(1-\sigma)$$

holds for

$$\sigma \geq \frac{6a^2nm(4\delta - A) + p(a+1)(2\delta - 3A)}{8a^2nm(4\delta - A) - 3Ap(a+1)},$$

provided

$$8a^2nm(4\delta - A) - 3Ap(a+1) > 0. \tag{3.18}$$

Consequently, if N satisfies (3.11), the bound $\mathcal{R}_3 \ll \mathcal{T}^{A(1-\sigma)+\varepsilon}$ holds for

$$\sigma > \max\left\{\sigma_3(A;\delta;p;k,m,n), \frac{3}{4}\right\},$$

where

$$\begin{aligned} &\sigma_3(A;\delta;p;k,m,n) \\ &:= \max\left\{\frac{a^2m(4\delta - A) + p(\delta m - A)(a+1)}{2a^2m(4\delta - A) - Ap(a+1)}, \frac{3a^2knm(4\delta - A) - p(a+1)(Anm - \delta nk + kA)}{4a^2knm(4\delta - A) - Ap(a+1)(nm+k)},\right. \\ &\qquad\qquad \left.\frac{6a^2nm(4\delta - A) + p(a+1)(2\delta - 3A)}{8a^2nm(4\delta - A) - 3Ap(a+1)}\right\}, \end{aligned} \tag{3.19}$$

provided that (3.16), (3.17) and (3.18) hold. Summing up, we arrive at the following consequence.

Proposition 3.1 *Let $N(\sigma, T, \chi)$ be the number of zeros $\rho = \beta + i\gamma$ of the Dirichlet L-function $L(s,\chi)$ in the rectangle $\sigma \le \beta \le 1$, $|\gamma| \le T$. For $1 \le Q \ll \exp(T^{\varepsilon})$, put*

$$\Sigma(Q) := \sum_{q \le Q} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} N(\sigma, T, \chi),$$

where the asterisk means that the sum is restricted to primitive characters. Put $\delta = 1$ if $Q = 1$ and $\delta = 2$ if $Q > 1$. Then, for $A \le 2\delta$ and any positive integers j, k, l, n, m satisfying the conditions (3.10), (3.12), (3.13), (3.14) (3.16), (3.17) and (3.18), the zero-density estimate

$$\Sigma(Q) \ll_{\varepsilon} \mathcal{T}^{A(1-\sigma)+\varepsilon}$$

holds uniformly for

$$\sigma > \max\left\{\frac{3}{4}, \sigma_1, \min\{\sigma_2, \sigma_3\}\right\}, \tag{3.20}$$

where $\mathcal{T} = QT$ and $\sigma_1 = \sigma_1(A;\delta;p;k)$, $\sigma_2 = \sigma_2(A;\delta;p;j,k,l)$, $\sigma_3 = \sigma_3(A;\delta;p;k,m,n)$ are given by (3.9), (3.15) and (3.19), respectively and a is defined by (3.7).

3.3. Completion of the proofs

Proof of Theorem 1.2

We let $\delta = 2$, $A = 4$. By (3.7), $a = p$. The values $\sigma_4(i)$ $(i = 2, \ldots, 10)$ are obtained by substituting $(j,k,l,n,m) =$ $(1,10,6,1,2)$, $(2,4,5,1,2)$, $(1,4,3,6,6)$, $(1,4,3,2,5)$, $(1,5,4,1,7)$, $(1,5,4,7,4)$, $(1,5,5,7,1)$, $(1,6,5,1,1)$, $(1,6,6,7,10)$ into (3.20) respectively.

For general $p > 1$, put $(j,k,l) = (1,1,1)$. Then

$$\sigma_1(4;2;p;1) = \frac{3p^2 - p - 1}{4p^2 - 2p - 2} = \frac{3}{4} + \frac{1}{8p} + O\left(\frac{1}{p^2}\right),$$

$$\sigma_2(4;2;p;1,1,1) = \max\left\{\frac{1}{2}, \frac{3}{4}, \frac{3p^2 - p - 1}{4p^2 - 2p - 2}\right\} = \frac{3p^2 - p - 1}{4p^2 - 2p - 2} = \frac{3}{4} + \frac{1}{8p} + O\left(\frac{1}{p^2}\right).$$

Hence $\Sigma(Q) \ll \mathcal{T}^{4(1-\sigma)+\varepsilon}$ holds uniformly for $\sigma > 3/4 + 1/8p + O(p^{-2})$. $\square$

Proof of Theorem 1.3

We let $\delta = 2$, $A = 3$. By (3.7), $a = \lfloor 3p/5 \rfloor$. Then the values $\sigma_3(i)$ $(i = 5, 7, 9, 10, 11, 12)$ are obtained by substituting $(j,k,l,n,m) = (2,8,2,2,6)$, $(1,10,2,9,1)$, $(1,10,2,9,1)$, $(2,5,2,6,3)$, $(1,1,2,3,3)$, $(1,1,2,2,4)$ respectively.

To deal with the case of general p, we may assume that p is divisible by 10. Then $a = 3p/5$. Put $(j,k,l) = (p/2, 1, 2)$. For these values, we have

$$\sigma_1(3;2;p;1) = \frac{27p^2 - 6p - 10}{36p^2 - 18p - 30} = \frac{3}{4} + O(p^{-1}),$$

$$\sigma_2\left(3;2;p;\frac{p}{2},1,2\right)=\max\left\{\frac{21p+5}{27p-15},\frac{21p^2-19p-15}{27p^2-24p-15},\frac{27p^3-6p-10}{36p^3-18p-30}\right\}$$
$$=\frac{21p+5}{27p-15}=\frac{7}{9}+\frac{50}{81p}+O(p^{-2}).$$

Hence $\Sigma(Q)\ll\mathcal{T}^{3(1-\sigma)+\varepsilon}$ holds uniformly for $\sigma>7/9+50/81p+O(p^{-2})$. □

Proof of Theorem 1.4
Put $\delta=2$. As $p\to\infty$, it follows that $a\sim Ap/(8-A)$, and

$$\sigma_1(A;2;p;k)\to\frac{3}{4},$$
$$\sigma_2(A;2;p;j,k,l)\to\max\left\{\frac{(A+2)l-A}{A(2l-1)},\frac{3Ajkl-Ajl+2jk-kA}{A(4jkl-jl-k)},\frac{3}{4}\right\}.$$

Put $l=2$ and take $j=k$ sufficiently large. Then the first term within the maximum equals $(A+4)/3A$ and the middle term is infinitely close to $(3A+1)/4A$. Thus the theorem is proved.
□

Proof of Theorem 1.6
We let $\delta=1$, $A=2$. By (3.7), $a=p$. The values $\sigma_4(i)$ $(i=2,\dots,10)$ are obtained by substituting $(j,k,l,n,m)=(2,4,4,1,2)$, $(1,3,1,1,2)$, $(1,3,3,30,19)$, $(1,3,3,15,13)$, $(1,4,4,1,17)$, $(1,5,4,30,15)$, $(1,6,5,17,24)$, $(1,6,5,3,1)$, $(1,7,6,10,10)$ into (3.20) respectively.

For general p, put $(j,k,l)=(1,1,1)$. Then

$$\sigma_1(2;1;p;1)=\frac{3p^2-p-1}{4p^2-2p-2}=\frac{3}{4}+\frac{1}{8p}+O\left(\frac{1}{p^2}\right),$$
$$\sigma_2(2;1;p;1,1,1)=\max\left\{\frac{1}{2},\frac{3}{4},\frac{3p^2-p-1}{4p^2-2p-2}\right\}=\frac{3p^2-p-1}{4p^2-2p-2}=\frac{3}{4}+\frac{1}{8p}+O\left(\frac{1}{p^2}\right).$$

Hence $\Sigma(1)\ll\mathcal{T}^{2(1-\sigma)+\varepsilon}$ holds uniformly for $\sigma>3/4+1/8p+O(p^{-2})$. □

Proof of Theorem 1.7
We take $\delta=1$. As $p\to\infty$, it follows that $a\sim Ap/(4-A)$, and

$$\sigma_1\to\frac{3}{4},$$
$$\sigma_2\to\max\left\{\frac{Al+l-A}{A(2l-1)},\frac{3Ajkl-Ajl+jk-kA}{A(4jkl-jl-k)},\frac{3}{4}\right\}.$$

Taking $k=l$ sufficiently large, the first term within the maximum is infinitely close to $(A+1)/2A$. Thus the theorem is proved. □

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