Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2084 – 2102
© TÜBİTAK
doi:10.3906/mat-2103-102

Optimization of Mayer functional in problems with discrete and differential inclusions and viability constraints

Gülseren ÇİÇEK[1,*], **Elimhan N. MAHMUDOV**[2,3]
[1]Department of Mathematics, Faculty of Science, İstanbul University, İstanbul, Turkey
[2]Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, İstanbul, Turkey
[3]Institute of Control Systems, Azerbaijan National Academy of Sciences, Baku, Azerbaijan

Received: 26.03.2021 • **Accepted/Published Online:** 18.07.2021 • **Final Version:** 16.09.2021

Abstract: This paper derives the optimality conditions for a Mayer problem with discrete and differential inclusions with viable constraints. Applying necessary and sufficient conditions of problems with geometric constraints, we prove optimality conditions for second order discrete inclusions. Using locally adjoint mapping, we derive Euler-Lagrange form conditions and transversality conditions for the optimality of the discrete approximation problem. Passing to the limit, we establish sufficient conditions to the optimal problem with viable constraints. Conditions ensuring the existence of solutions to the viability problems for differential inclusions of second order have been studied in recent years. However, optimization problems of second-order differential inclusions with viable constraints considered in this paper have not been examined yet. The results presented here are motivated by practices for optimization of various fields as the mass movement model well known in traffic balance and operations research.

Key words: Second order differential inclusions, second order discrete inclusions, Mayer problem, Dual cone, Locally adjoint mapping

1. Introduction

Recently viability theory has been intensively examined for its use in practice. This theory is a very attractive theoretical approach for the modeling of complex dynamical systems. In 1990, Aubin dealt with viability theory as an area of mathematics that studies the evolution of dynamical systems under constraints on the state [4, 5]. Conditions ensuring the existence of solutions to the viability problems for differential inclusions of second order have been studied in recent years [1–3, 6, 9–11, 15, 29, 33]. Using fixed point theory for set-valued upper semicontinuous maps, Green's functions, and upper and lower solutions, the authors in [8] determine existence results for solutions of second order dynamic inclusions. Loewen and Rockafellar [14] consider Mayer problem of optimal control whose dynamic constraint is given by a convex valued differential inclusion and state constraint is set-valued. Benchohra and Ntouyas investigate the existence of solutions on a compact interval to a three and four-point boundary value problem for second order differential inclusions in the case when the multivalued function has nonconvex values [7]. The authors in [34] study weak-stability and saddle point theorems of multiobjective optimization problems that have an infinite number of constraints.

Dealing with optimal problems of high order differential inclusions is a complex case due to the difficulties

*Correspondence: cicekgul@istanbul.edu.tr
2010 *AMS Mathematics Subject Classification:* 34A60,quasinormal 49K21;49M25;49J53

in expressing the optimality conditions. We particularly refer the reader to work of Mahmudov for a thorough study of necessary and sufficient conditions for the optimality of the higher order differential inclusions of Bolza and Mayer types [12, 16, 18–20, 22]. The authors in [25–27] study high-order necessary optimality conditions for discrete optimal control problems. Second-order necessary optimality conditions for an optimal control problem with a nonconvex cost function and state-control constraints are obtained in [31]. The work in [13] is concerned with optimality conditions for nonconvex optimization problems in reflexive Banach spaces. The work in [30] is devoted to the optimal control of discontinuous differential inclusions of the normal cone type governed by a generalized version of the Moreau sweeping process with control functions that act in both nonconvex moving sets and additive perturbations.

Mahmudov [23] investigates optimization problems of partial differential inclusions of Goursat-Darboux type and the author presents sufficient conditions for optimality both for convex and nonconvex cases. Mahmudov, in [24], considers two optimization problems for discrete elliptic inclusions and gives optimality conditions for the problems.

There are limited number of articles devoted to the optimization problem of second order differential inclusions with viable constraints. Establishing the dual problem, Mahmudov formulates the optimality conditions for a second-order viability discrete and differential inclusions with endpoint constraints [21]. In the present paper, we derive the optimality conditions for the Mayer problem of second order discrete and differential inclusions with viable constraints that have not been examined yet.The results presented here are motivated by practices for optimization of the mass movement model well known in traffic balance and operations research, for optimization of epidemiological models that are under review recently.

We arrange our paper as follows. In Section 2, we give the necessary facts and further results from the book by Mahmudov [17]; locally adjoint mapping (LAM), Hamiltonian function, propositions giving relation between locally adjoint mapping of set-valued mappings F and Ω, and relation between subdifferentials of proper convex functions.

In Section 3, optimality conditions for the second order discrete inclusion problem are derived, reducing this problem to a convex problem with geometric constraints. We obtain necessary and sufficient optimality conditions in terms of LAMs.

Using first and second order difference operators and an auxiliary multifunction, we approximate the convex problem by the discrete approximation problem in Section 4. Euler-Lagrange form conditions and transversality conditions for the discrete approximation problem are formulated applying locally adjoint mapping and the subdifferential of the target functional. Generally, there are some difficulties in constructing adjoint inclusions and viability conditions, but we achieve it by the approximation and formulation of the equivalence theorems.

In Section 5, setting $\lambda = 1$ and passing to the formal limit in conditions of the discrete-approximation problem as the discrete step $\delta \to 0$, we establish the sufficient optimality conditions for the convex optimization problem of second order differential inclusions with viable constraints. We give an example that is an application to the results obtained for the convex problem. We derive also Euler-Lagrange type conditions and transversality conditions for the nonconvex problem with viable constraints.

Let $\mathbb{R}^n$ be n-dimensional Euclidean space, $\langle x, u \rangle$ be inner product of elements $x, u \in \mathbb{R}^n$ and (x, u) be pair of x, u. Assume that $F : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is a multivalued mapping and $\varphi : \mathbb{R}^{2n} \to \mathbb{R} \cup \{+\infty\}$ is a proper single valued function, M and N are nonempty subsets of $\mathbb{R}^n$, Q is a nonempty subset of $\mathbb{R}^n \times \mathbb{R}^n$ such that $(M \times N) \cap Q \neq \emptyset$ and $dom F \cap (M \times N) \neq \emptyset$. In this paper, we derive optimality conditions for Mayer problem for second order differential inclusions with viable constraints applied to the trajectory and its derivative

$$\text{minimize} \quad \varphi\big(x(b), x'(b)\big) \tag{1.1}$$

$$x''(t) \in F\big(x(t), x'(t)\big), \text{ a.e. } t \in [a,b], \tag{1.2}$$

$$x(t) \in M,\ x'(t) \in N, \text{ a.e. } t \in [a,b] \tag{1.3}$$

$$x(a) = \alpha_0,\ x'(a) = \beta_0,\ \big(x(b), x'(b)\big) \in Q, \tag{1.4}$$

where $\alpha_0 \in M, \quad \beta_0 \in N$. The problem is to find an arc $\tilde{x}(t)$, satisfying boundary conditions (1.4), viability constraints (1.3) and second order differential inclusions (1.2) almost everywhere (a.e.) on $[a,b]$ that minimizes the Mayer functional $\varphi(x(b), x'(b))$. A feasible trajectory $x(\cdot)$ in the problem is taken to be an absolutely continuous function on time interval $[a,b]$ together with the first order derivatives for which $x''(\cdot) \in L_1^n([a,b])$.

In order to construct the optimality conditions to the problem (1.1)–(1.4), we begin with the second order discrete problem in Section 3.

2. Preliminaries

Throughout this paper, we use the notions of Mahmudov [17]. The graph of a multivalued function $F : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by $gphF = \{(x,u,v) \mid v \in F(x,u)\}$ and F is convex if its graph is a convex subset of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. The multivalued mapping F is convex closed if its graph is a convex closed set in $\mathbb{R}^{3n}$ and is convex-valued if $F(x,u)$ is a convex set for each $(x,u) \in domF = \{(x,u) \mid F(x,u) \neq \emptyset\}$. The Hamiltonian function and argmaximum set for multivalued mapping F are defined by

$$H_F(x,u,v^*) = \sup_v \{\langle v, v^* \rangle \mid v \in F(x,u)\}, v^* \in \mathbb{R}^n,$$

$$F_{Arg}(x,u;v^*) \equiv F_A(x,u;v^*) = \{v \in F(x,u) \mid \langle v, v^* \rangle = H_F(x,u,v^*)\},$$

respectively. For convex F, we set $H_F(x,u,v^*) = -\infty$ if $F(x,u) = \emptyset$.

The interior of the set $A \subseteq \mathbb{R}^{3n}$ is denoted by $\operatorname{int} A$ and the relative interior of the set A, i.e. the set of interior points of A with respect to its affine hull Aff A is denoted by $\operatorname{ri} A$. The convex cone $K_A(z_0), z_0 = (x,u,v)$ is called the cone of tangent directions at a point $z_0 \in A$ to an arbitrary A if from $\bar{z} = (\bar{x}, \bar{u}, \bar{v}) \in K_A(z_0)$ it follows that $\bar{z}$ is a tangent vector to the set A at point $z_0 \in A$, i.e. there exists such function $\Phi(\lambda) \in \mathbb{R}^{3n}$ that $z_0 + \lambda \bar{z} + \Phi(\lambda) \in A$ for sufficiently small $\lambda > 0$ and $\lambda^{-1}\Phi(\lambda) \to 0$, as $\lambda \downarrow 0$. If A is a convex set, then $K_A(z_0) = cone(A - z_0)$.

The cone of tangent directions $K_A(z_0)$ at $z_0 \in A$ is called a local tent if for any $\bar{z}_0 \in riK_A(z_0)$ there exists a convex cone $K \subseteq K_A(z_0)$ and a continuous mapping $\psi(\bar{z})$ defined in the neighborhood of the origin such that

(i) $\bar{z}_0 \in K, LinK = LinK_A(z_0)$ where $LinK$ is the linear span of K,

(ii) $\psi(\bar{z}) = \bar{z} + r(\bar{z}), r(\bar{z})\|\bar{z}\|^{-1} \to 0$ as $\bar{z} \to 0$,

(iii) $z_0 + \psi(\bar{z}) \in A, \bar{z} \in K \cap S_\varepsilon(0)$ for some $\varepsilon > 0$, where $S_\varepsilon(0)$ is the ball of radius ε .

For a convex mapping F the cone of tangent directions at a point $(x,u,v) \in gphF$ is defined by

$$K_{gphF}(x,u,v) = cone[gphF - (x,u,v)]$$
$$= \{(\bar{x},\bar{u},\bar{v}) \mid \ \bar{x} = \lambda(x_1 - x),\ \bar{u} = \lambda(u_1 - u),\ \bar{v} = \lambda(v_1 - v)\},\ \forall(x_1,u_1,v_1) \in gphF$$

and its locally adjoint mapping (LAM) at that point is

$$F^*(v^*;(x,u,v)) = \{(x^*,u^*) \mid (x^*,u^*,-v^*) \in K^*_{gphF}(x,u,v)\},$$

where $K^*_{gphF}(x,u,v)$ is the dual cone to the cone of tangent directions $K_{gphF}(x,u,v)$.

The following multivalued mapping

$$F^*(v^*;(x,u,v)) = \{(x^*,u^*) \mid H_F(x_1,u_1,v^*) - H_F(x,u,v^*) \le \langle x^*, x_1 - x\rangle + \langle u^*, u_1 - u\rangle,$$
$$\forall(x_1,u_1) \in \mathbb{R}^{2n}\}, v \in F_A(x,u,v^*)$$

is called the LAM to a nonconvex mapping F at point $(x,u,v) \in gphF$ [17].

Trajectories that comply with the constraint (1.3) are called viable. In other words, trajectories that remain within the set M for each $t \in [a,b]$ [4, 5].

The subdifferential [17, 32] of a function g at point (x_0,u_0) is the set

$$\partial g(x_0,u_0) = \{(x^*,u^*) \mid g(x,u) - g(x_0,u_0) \ge \langle x - x_0, x^*\rangle + \langle u - u_0, u^*\rangle, \forall(x,u)\}.$$

Definition 2.1 *The function $h(\cdot,x)$ is called a CUA [17] of function $g : X \to \mathbb{R}^n \cup \{\pm\infty\}$ at every fixed point $x \in \mathrm{dom} g = \{x : |g(x)| < +\infty\}$, if*

1. $h(\bar{x},x) \ge sup_{r(\cdot)} limsup_{\lambda\downarrow 0} \frac{g(x+\lambda\bar{x}+r(\lambda))-g(x)}{\lambda}$ for all $\bar{x} \neq 0$*;*
2. $h(\bar{x},x)$ *is a closed (lower semicontinuous) positively homogeneous convex function.*

Note that for the function g there exist a lot of CUAs. If function g is convex and continuous at a point x then $g(x,\bar{x})$ is a CUA for g and in this case CUA and $\partial g(x)$ overlap.

We refer to [16, Theorem 3.2] for the following Proposition that we use in Section 4.

Proposition 2.2 *Let convex set-valued mapping $\Omega(\cdot,\cdot,t) : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be defined by $\Omega(x,u) = 2u - x + \delta^2 F\left(x, \frac{u-x}{\delta}\right)$, where F is a given convex set-valued mapping. Then the following inclusions are equivalent*

1. $(x^*,u^*) \in \Omega^*(v^*;(x,u,v)),\ v \in \Omega_{Arg}(x,u;v^*)$
2. $\left(\frac{x^*+u^*-v^*}{\delta^2}, \frac{u^*-2v^*}{\delta}\right) \in F^*\left(v^*; \left(\frac{u-x}{\delta}, \frac{v-2u+x}{\delta^2}\right)\right), \frac{v-2u+x}{\delta^2} \in F_{Arg}\left(x, \frac{u-x}{\delta}; v^*\right), v^* \in \mathbb{R}^n$,

where $\Omega_{Arg}(x,u;v^*)$ is the argmaximum set for the mapping Ω.

Now let recall [16, Theorem 3.1] that we apply in the proof of Theorem 4.3.

Proposition 2.3 *Suppose $\bar{\varphi}(\cdot,\cdot)$ is a proper convex function defined by the relation $\bar{\varphi}(x,y) \equiv \varphi\left(x, \frac{y-x}{\delta}\right)$ then the following inclusions are equivalent*

a) $(\bar{x}^*, \bar{y}^*) \in \partial_{x,y}\bar{\varphi}(x,y), (x,y) \in dom\bar{\varphi}$

b) $(\bar{x}^* + \bar{y}^*, \delta\bar{y}^*) \in \partial\varphi\left(x, \frac{y-x}{\delta}\right), \left(x, \frac{y-x}{\delta}\right) \in dom\varphi$.

3. Optimality conditions for discrete-time problem

We deal with second order discrete Mayer problem

$$\text{minimize} \quad g(x_{T-1}, x_T) \tag{3.1}$$

$$x_{t+2} \in F(x_t, x_{t+1}),\ t = 0, \ldots, T-2 \tag{3.2}$$

$$x_t \in M,\ x_{t+1} - x_t \in N,\ t = 1, 2, \ldots, T-2, \tag{3.3}$$

$$x_0 = \alpha_0,\ x_1 = \alpha_1,\ (x_{T-1}, x_T - x_{T-1}) \in Q, \tag{3.4}$$

where $g(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \cup \{+\infty\}$, is a real-valued function, $F : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is a multivalued mapping and T is a fixed natural number, M and N subsets of $\mathbb{R}^n$ and $Q \subseteq \mathbb{R}^n \times \mathbb{R}^n$, $\alpha_0 \in M, \alpha_1 \in \mathbb{R}^n$, fix numbers. The feasible trajectory for the stated problem (3.1)–(3.4) is a sequence $\{x_t\}_{t=0}^T = \{x_t| \quad t = 0, 1, \ldots, T\}$.
If the multivalued function F is convex, $g(\cdot,\cdot)$ is a convex proper function and sets in the problem are convex sets then the discrete problem (3.1)–(3.4) is said to be convex.

Definition 3.1 *[17] If one of the following cases for points $x_t^0 \in \mathbb{R}^n$ is fulfilled*

(i) $(x_t^0, x_{t+1}^0, x_{t+2}^0) \in ri(gphF)$,

(ii) $(x_t^0, x_{t+1}^0, x_{t+2}^0) \in int(gphF)$, $t = 0, \ldots T-2$ *(with the possible exception of one fixed t_0),*

and $g(\cdot,\cdot)$ is continuous at (x_t^0, x_{t+1}^0) , we say that the regularity condition for the convex problem (3.1)–(3.4) is satisfied.

Firstly, we consider convex problem (3.1)–(3.4). Let us introduce a vector $p = (x_0, x_1, \ldots, x_T) \in \mathbb{R}^{n(T+1)}$ and define the following convex sets in the space $\mathbb{R}^{n(T+1)}$

$$S_t = \{p = (x_0, x_1, \ldots, x_T) \mid (x_t, x_{t+1}, x_{t+2}) \in gphF\}, t = 0, 1, \ldots, T-2,$$

$$\tilde{R}_t = \{p = (x_0, x_1, \ldots, x_T) \mid x_t \in M,\ x_{t+1} - x_t \in N\},\ t = 1, \ldots, T-2,$$

$$\Phi_0 = \{p = (x_0, \ldots, x_T) \mid x_0 = \alpha_0\}, \quad \Phi_1 = \{p = (x_0, \ldots, x_T) \mid x_1 = \alpha_1\}$$

$$\text{and } \Phi_T = \{p = (x_0, \ldots, x_T) \mid (x_{T-1}, x_T - x_{T-1}) \in Q\}.$$

Using the definition of dual cone, we should compute the dual of the cones of tangent directions $K_{\tilde{R}_t}(p)$ and $K_{\Phi_T}(p)$. Dual cones $K^*_{S_t}(p)$ are verified according to Mahmudov in [16]

$$K^*_{S_t}(p) = \{p^* = (x_0^*, \ldots, x_T^*) \mid (x_t^*, x_{t+1}^*, x_{t+2}^*) \in K^*_{gphF}(x_t, x_{t+1}, x_{t+2}),$$
$$x_k^* = 0,\ k \neq t, t+1, t+2\}, t = 0, 1, \ldots, T-2,$$

where $K^*_{gphF}(x_t, x_{t+1}, x_{t+2})$, is dual to the cones of tangent directions $K_{gphF}(x_t, x_{t+1}, x_{t+2})$, $(x_t, x_{t+1}, x_{t+2}) \in gphF$.
Also, it is easy to compute $K^*_{\Phi_0}(p) = \{p^* = (x^*_0, \dots, x^*_T) \mid x^*_0 \in \mathbb{R}^n,\ x^*_t = 0,\ t = 1, \dots, T\}$
and $K^*_{\Phi_1}(p) = \{p^* = (x^*_0, \dots, x^*_T) \mid x^*_1 \in \mathbb{R}^n,\ x^*_t = 0,\ t \neq 1\}$.
The following Lemma establishes dual cones $K^*_{\tilde{R}_t}$, $t = 1, \dots, T-2$ using the dual cones K^*_M and K^*_N of the cones of tangent directions K_M and K_N, respectively.

Lemma 3.2 *Let $p \in \tilde{R}_t$, $t = 1, \dots, T-2$, be given. Then the cone of tangent directions to $\tilde{R}_t$ at p and its dual cone are given by*

$$K_{\tilde{R}_t}(p) = \{\overline{p} = (\overline{x}_0, \dots, \overline{x}_t, \overline{x}_{t+1}, \dots \overline{x}_T) \mid \overline{x}_t \in K_M(x_t),\ \overline{x}_{t+1} - \overline{x}_t \in K_N(x_{t+1} - x_t), \overline{x}_k \in \mathbb{R}^n, k \neq t, t+1\},$$
$$t = 1, 2, \dots, T-2,$$
$$K^*_{\tilde{R}_t}(p) = \{p^* = (x^*_0, \dots, x^*_T) \mid\ x^*_t + x^*_{t+1} \in K^*_M(x_t),\ x^*_{t+1} \in K^*_N(x_{t+1} - x_t),\ x^*_k = 0, k \neq t, t+1\},$$
$$t = 1, 2, \dots, T-2,$$

respectively.

Proof For any $t = 1, 2, \dots, T-2$ let us first compute $K_{\tilde{R}_t}(p)$, the cone of tangent directions to $\tilde{R}_t$ at the given point p. By the definition of cone of tangent directions, $K_{\tilde{R}_t}(p)$ consists of points $\overline{p} = (\overline{x}_0, \dots, \overline{x}_T)$ such that $p + \lambda\overline{p} \in \tilde{R}_t$ holds for sufficiently small $\lambda > 0$. Then from the formula of $\tilde{R}_t$ the last inclusion is analogous to $(x_{t+1} + \lambda\overline{x}_{t+1}) - (x_t + \lambda\overline{x}_t) \in N$, $(x_t + \lambda\overline{x}_t) \in M$, satisfied for sufficiently small $\lambda > 0$, and x_k arbitrary for $k \neq t, t+1$. Therefore, by inclusions $(x_{t+1} - x_t) + \lambda(\overline{x}_{t+1} - \overline{x}_t) \in N$ and $(x_t + \lambda\overline{x}_t) \in M$, we obtain $\overline{x}_{t+1} - \overline{x}_t \in K_N(x_{t+1} - x_t)$ and $\overline{x}_t \in K_M(x_t)$, respectively and since x_k are arbitrary for $k \neq t, t+1$ then $\overline{x}_k \in \mathbb{R}^n, k \neq t, t+1$. Thus, we have

$$K_{\tilde{R}_t}(p) = \{\overline{p} = (\overline{x}_0, \dots, \overline{x}_t, \overline{x}_{t+1}, \dots \overline{x}_T) \mid \overline{x}_t \in K_M(x_t),\ \overline{x}_{t+1} - \overline{x}_t \in K_N(x_{t+1} - x_t),\ \overline{x}_k \in \mathbb{R}^n, k \neq t, t+1\},$$
$$t = 1, 2, \dots, T-2.$$

On the other hand, by the definition of dual cone, $p^* \in K^*_{\tilde{R}_t}(p)$ if and only if

$$\langle p^*, \overline{p}\rangle = \sum_{k=0}^{T} \langle x^*_k, \overline{x}_k\rangle \geq 0,\ \forall \overline{p} = (\overline{x}_0, \dots, \overline{x}_t, \overline{x}_{t+1}, \dots, \overline{x}_T) \in K_{\tilde{R}_t}(p),\ t = 1, 2, \dots, T-2$$

or equivalently if and only if for all $\overline{x}_t \in K_M(x_t)$, $\overline{x}_{t+1} - \overline{x}_t \in K_N(x_{t+1} - x_t)$ and $\overline{x}_k$ arbitrary for $k \neq t, t+1$

$$\langle x^*_0, \overline{x}_0\rangle + \dots + \langle x^*_t, \overline{x}_t\rangle + \langle x^*_{t+1}, \overline{x}_{t+1}\rangle + \dots + \langle x^*_T, \overline{x}_T\rangle \geq 0$$

holds. From the arbitrariness of components $\overline{x}_k$, $k \neq t, t+1$, last inequality reduces to $\langle x^*_t, \overline{x}_t\rangle + \langle x^*_{t+1}, \overline{x}_{t+1}\rangle \geq 0$ and after some rearrangements, we obtain inequality

$$\langle x^*_t + x^*_{t+1}, \overline{x}_t\rangle + \langle x^*_{t+1}, \overline{x}_{t+1} - \overline{x}_t\rangle \geq 0, \tag{3.5}$$

satisfied for all $\overline{x}_t \in K_M(x_t)$, $\overline{x}_{t+1} - \overline{x}_t \in K_N(x_{t+1} - x_t)$. Putting $\overline{x}_{t+1} - \overline{x}_t = 0$, inequality (3.5) gives us $\langle x^*_t + x^*_{t+1}, \overline{x}_t\rangle \geq 0$ that is satisfied for each $\overline{x}_t \in K_M(x_t)$, provided that $x^*_t + x^*_{t+1} \in K^*_M(x_t)$.

Similarly, since $x_t \in M$, then inequality (3.5) holds also at $\overline{x}_t = 0 \in K_M(x_t)$. Thus, we have $x^*_{t+1} \in K^*_N(x_{t+1} - x_t)$. Therefore, we obtain the desired relation for the dual cone to the cone $K_{\tilde{R}_t}(p)$. □

Lemma 3.3 *Suppose $p \in \Phi_T$ is given. The cone of tangent directions to the set $\Phi_T = \{p = (x_0, \dots, x_T) \mid (x_{T-1}, x_T - x_{T-1}) \in Q\}$ at p and its dual are defined by*

$$K_{\Phi_T}(p) = \{\overline{p} = (\overline{x}_0, \dots, \overline{x}_{T-1}, \overline{x}_T) \mid (\overline{x}_{T-1}, \overline{x}_T - \overline{x}_{T-1}) \in K_Q(x_{T-1}, x_T - x_{T-1}),\ \overline{x}_k \in \mathbb{R}^n,\ k \neq T-1, T\}$$

and

$$K^*_{\Phi_T}(p) = \{p^* = (x^*_0, \dots, x^*_{T-1}, x^*_T) \mid (x^*_{T-1} + x^*_T, x^*_T) \in K^*_Q(x_{T-1}, x_T - x_{T-1}),\ x^*_k = 0,\ k \neq T-1, T\},$$

respectively.

Proof Let us take $\overline{p} = (\overline{x}_0, \dots \overline{x}_{T-1}, \overline{x}_T) \in K_{\Phi_T}(p)$ then from the definition of tangent cones, we obtain $p + \lambda\overline{p} \in \Phi_T$ for sufficiently small $\lambda > 0$. By the definition of the set Φ_T, the last inclusion is equivalent to $(x_{T-1} + \lambda\overline{x}_{T-1}, (x_T + \lambda\overline{x}_T) - (x_{T-1} + \lambda\overline{x}_{T-1})) \in Q$, satisfied for sufficiently small $\lambda > 0$. Therefore, it is not hard to see that $(\overline{x}_{T-1}, \overline{x}_T - \overline{x}_{T-1}) \in K_Q(x_{T-1}, x_T - x_{T-1})$ and $\overline{x}_k$ arbitrary for $k \neq T-1, T$, and as a result, we have the demanded result for the cone of tangent directions K_{Φ_T}.
On the other hand, $p^* \in K^*_{\Phi_T}(p)$ if and only if

$$\langle p^*, \overline{p}\rangle = \sum_{k=0}^{T} \langle x^*_k, \overline{x}_k\rangle \geq 0 \text{ for } \forall \overline{p} \in K_{\Phi_T}(p),$$

or similarly $\langle x^*_0, \overline{x}_0\rangle + \dots + \langle x^*_{T-1}, \overline{x}_{T-1}\rangle + \langle x^*_T, \overline{x}_T\rangle \geq 0$ satisfied for all $(\overline{x}_{T-1}, \overline{x}_T - \overline{x}_{T-1}) \in K_Q(x_{T-1}, x_T - x_{T-1})$ and $\overline{x}_k \in \mathbb{R}^n, k \neq T-1, T$.
From the arbitrariness of components $\overline{x}_k$, $k \neq T-1, T$ it is evident that $x^*_k = 0$ for $k \neq T-1, T$, so that the last inequality reduces to the following one

$$\langle x^*_{T-1}, \overline{x}_{T-1}\rangle + \langle x^*_T, \overline{x}_T\rangle \geq 0,$$

which is equivalent to $\langle x^*_{T-1} + x^*_T, \overline{x}_{T-1}\rangle + \langle x^*_T, \overline{x}_T - \overline{x}_{T-1}\rangle \geq 0$ that holds for all $(\overline{x}_{T-1}, \overline{x}_T - \overline{x}_{T-1}) \in K_Q(x_{T-1}, x_T - x_{T-1})$. Thus, dual cone to the cone $K_{\Phi_T}(p)$ is obtained as demanded. □

We give the necessary and sufficient conditions for the problem (3.1)–(3.4) in the sense of the terminology of first order discrete inclusions [17, 28, 32].

Theorem 3.4 *Let F be convex mapping, $g(\cdot, \cdot)$ be convex continuous function at the points of some feasible trajectory $\{x_t\}_{t=0}^T$, sets M, N, and Q be convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, respectively. Then $\{\tilde{x}_t\}_{t=0}^T$ is an optimal trajectory of the problem (3.1)–(3.4) if there exist a number $\lambda \in \{0, 1\}$ and vectors $\{x^*_t, u^*_t, \xi^*_t, \eta^*_t\}, t = 1, \dots, T-2, x^*_T, \eta^*_{T-1}, \xi^*_{T-1}$, simultaneously not all equal to zero satisfying the discrete Euler-Lagrange and transversality inclusions*

(i) $(x^*_t - u^*_t - \eta^*_t, u^*_{t+1} - \xi^*_t) \in F^*(x^*_{t+2}; (\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2}))$,
$(\eta^*_t + \xi^*_t, \xi^*_t) \in K^*_M(\tilde{x}_t) \times K^*_N(\tilde{x}_{t+1} - \tilde{x}_t),\ t = 1, \dots, T-2$

(ii) $(-x^*_{T-1} + u^*_{T-1} + \eta^*_{T-1}, -x^*_T + \xi^*_{T-1}) \in \lambda\partial g(\tilde{x}_{T-1}, \tilde{x}_T)$,

respectively, where $(\eta^*_{T-1}+\xi^*_{T-1},\xi^*_{T-1})\in K^*_Q(\tilde{x}_{T-1},\tilde{x}_T-\tilde{x}_{T-1})$.
And if the regularity condition is satisfied, these conditions are sufficient for the optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^T$.

Proof Denoting $g(x_{T-1},x_T)$ by $f(p)$, we will reduce this problem to the problem with geometric constraints. It can be easily seen that our basic problem (3.1)–(3.4) is equivalent to the following one

$$\text{minimize } f(p) \text{ subject to } Z=(\bigcap_{t=0}^{T-2} S_t)\cap(\bigcap_{t=1}^{T-2}\tilde{R}_t)\cap\Phi_0\cap\Phi_1\cap\Phi_T, \tag{3.6}$$

where Z is a convex set.

By the hypothesis of the theorem, $\{\tilde{x}_t\}_{t=0}^T$ is an optimal trajectory, consequently, $\tilde{p}=(\tilde{x}_0,\dots,\tilde{x}_T)$ is a solution of the problem (3.6). The result taken from [17, Theorem 3.4] provides necessary optimality conditions for the convex mathematical programming problem (3.6). According to this theorem, there exist vectors $p^*(t)\in K^*_{S_t}(\tilde{p}), t=0,1,\dots,T-2,$ $\hat{p}^*(t)\in K_{\tilde{R}_t}(\tilde{p}),$ $t=1,2,\dots,T-2,$ $p^*_a\in K^*_{\Phi_0},$ $p^*_b\in K^*_{\Phi_1}$ and $\hat{p}^*(T)\in K^*_{\Phi_T}(\tilde{p})$ not all zero, and number $\lambda\in\{0,1\}$, such that

$$\lambda p^{0*}=\sum_{t=0}^{T-2}p^*(t)+\sum_{t=0}^{T-2}\hat{p}^*(t)+p^*_a+p^*_b+\hat{p}^*(T),\ p^{0*}\in\partial_p f(\tilde{p}). \tag{3.7}$$

From the definition of the function f, it is easy to see that vector $p^{0*}\in\partial_p f(\tilde{p})$ has a form $p^{0*}=(0,\dots,0,\bar{x}^*_{T-1},\bar{x}^*_T)$, where $(\bar{x}^*_{T-1},\bar{x}^*_T)\in\partial g(\tilde{x}_{T-1},\tilde{x}_T)$ and for $t=0,1,\dots,T-2,\bar{x}^*_t=0$ by the fact that $g(\tilde{x}_t,\tilde{x}_{t+1})=0$ for $t=0,1,\dots,T-2$. By the formula given for the dual cones $K^*_{S_t}$, by Lemmas 3.2 and 3.3, we have $p^*(t)=(0,\dots,0,x^*_t(t),x^*_{t+1}(t),x^*_{t+2}(t),0,\dots,0)$,
$\hat{p}^*(t)=(0,\dots,0,\hat{x}^*_t(t),\hat{x}^*_{t+1}(t),0,\dots,0),\quad t=1,\dots,T-2,$ and
$\hat{p}^*(T)=(0,\dots,0,\hat{x}^*_{T-1}(T-1),\hat{x}^*_T(T-1))$, $p^*_a=(x^*_a,0,\dots,0)$, $p^*_b=(0,x^*_b,0,\dots,0)$ where

$$(x^*_t(t),x^*_{t+1}(t),x^*_{t+2}(t))\in K^*_{gphF}(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+1}),\ t=0,1,\dots,T-2, \tag{3.8}$$

$$\hat{x}^*_t(t)+\hat{x}^*_{t+1}(t)\in K^*_M(\tilde{x}_t),\ \hat{x}^*_{t+1}(t)\in K^*_N(\tilde{x}_{t+1}-\tilde{x}_t),\ t=1,\dots,T-2,$$
$$\text{and } \big(\hat{x}^*_{T-1}(T-1)+\hat{x}^*_T(T-1),\hat{x}^*_T(T-1)\big)\in K^*_Q(\tilde{x}_{T-1},\tilde{x}_T-\tilde{x}_{T-1}),$$
$$x^*_a\in\mathbb{R}^n,\ x^*_b\in\mathbb{R}^n,$$

respectively.

Now, using the component-wise representation of (3.7), we deduce that

$$\begin{aligned}
0&=x^*_0(0)+x^*_a\\
0&=x^*_1(1)+x^*_1(0)+\hat{x}^*_1(1)+x^*_b\\
0&=x^*_t(t)+x^*_t(t-1)+x^*_t(t-2)+\hat{x}^*_t(t)+\hat{x}^*_t(t-1),\quad t=2,\dots,T-2.
\end{aligned} \tag{3.9}$$

By the definition of locally adjoint mapping (LAM) and from inclusion (3.8), we derive that

$$(x^*_t(t),x^*_{t+1}(t))\in F^*(-x^*_{t+2}(t);(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2})),\quad t=1,2,\dots,T-2.$$

Let us denote the sum of $x^*_{t+1}(t) + \hat{x}^*_{t+1}(t)$ briefly by u^*_{t+1}, and take $-x^*_{t+2}(t) \equiv x^*_{t+2}$, $\hat{x}^*_{t+1}(t) \equiv \xi^*_t$ and $\hat{x}^*_t(t) \equiv \eta^*_t$, $t = 0, 1, \ldots, T-2$, respectively, in (3.9), then we obtain by (3.8) and last inclusion that for $t = 2, 3, \ldots, T-2$

$$(x^*_t - u^*_t - \eta^*_t, u^*_{t+1} - \xi^*_t) \in F^*(x^*_{t+2}; (\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+1})), \tag{3.10}$$

where $(\eta^*_t + \xi^*_t, \xi^*_t) \in K^*_M(\tilde{x}_t) \times K^*_N(\tilde{x}_{t+1} - \tilde{x}_t), \quad t = 1, 2, \ldots, T-2.$

Therefore, we derive the inclusions in (i). Finally, by the component-wise representation of (3.7) for $t = T-1$ and $t = T$, we have

$$\lambda \bar{x}^*_{T-1} = x^*_{T-1}(T-2) + x^*_{T-1}(T-3) + \hat{x}^*_{T-1}(T-2) + \hat{x}^*_{T-1}(T-1)$$
$$\lambda \bar{x}^*_T = x^*_T(T-2) + \hat{x}^*_T(T-1),$$

where $\hat{x}^*_{T-1}(T-2) \in K^*_N(\tilde{x}_{T-1} - \tilde{x}_{T-2})$, $\big(\hat{x}^*_{T-1}(T-1) + \hat{x}^*_T(T-1), \hat{x}^*_T(T-1)\big) \in K^*_Q(\tilde{x}_{T-1}, \tilde{x}_T - \tilde{x}_{T-1})$. If we denote $\hat{x}^*_{T-1}(T-1) = \eta^*_{T-1}$, $\hat{x}^*_T(T-1) = \xi^*_{T-1}, x^*_{T-1}(T-2) + \hat{x}^*_{T-1}(T-2) = u^*_{T-1}, x^*_{T-1}(T-3) = -x^*_{T-1}$ and $x^*_T(T-2) = -x^*_T$, respectively, the last equations can be rewritten in accepted notations

$$\begin{aligned} \lambda \bar{x}^*_{T-1} &= u^*_{T-1} - x^*_{T-1} + \eta^*_{T-1} \\ \lambda \bar{x}^*_T &= -x^*_T + \xi^*_{T-1}. \end{aligned} \tag{3.11}$$

Since $(\bar{x}^*_{T-1}, \bar{x}^*_T) \in \partial g(\tilde{x}_{T-1}, \tilde{x}_T)$, then by (3.11), we deduce that the inclusion (ii) follows, where $(\eta^*_{T-1} + \xi^*_{T-1}, \xi^*_{T-1}) \in K^*_Q(\tilde{x}_{T-1}, \tilde{x}_T - \tilde{x}_{T-1})$. Thus, we complete the first part of the proof of the theorem.

As for the sufficiency of the obtained conditions, it is clear that by [17, Theorem 3.3] under the regularity condition, the representation (3.7) holds with parameter $\lambda = 1$ for the point $p^{0*} \in \partial_p f(\tilde{p}) \cap K^*_Z(\tilde{p})$. □

Conjecture 3.5 *Let $\tilde{x}_t$ be points of the optimal trajectory $\{\tilde{x}_t\}_{t=0}^T$. Suppose that the cones of tangent directions $K_{gphF}(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2})$ to the graph of the mapping F in problem (3.1)–(3.4) are local tents. Suppose function $g(x_{T-1}, x_T)$ admits a continuous convex upper approximation (CUA) $h_t(\cdot, \tilde{x}_{T-1}, \tilde{x}_T)$ [17] at the point $(\tilde{x}_{T-1}, \tilde{x}_T)$, which ensures that the subdifferential $\partial g(\tilde{x}_{T-1}, \tilde{x}_T) = \partial h_t(0, \tilde{x}_{T-1}, \tilde{x}_T)$ is defined.*

Theorem 3.4 can be generalized to the nonconvex case; if the problem (3.1)–(3.4) is nonconvex and consequently, the mapping F is nonconvex, using the definition of a local tent we can establish the equivalence of the inclusions in [16, Theorem 3.3] for non-convex function F.

Theorem 3.6 *Suppose that for the nonconvex problem (3.1)–(3.4) Conjecture 3.5 holds. Then the necessary condition for the optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^T$ for this nonconvex problem is that there exist a number $\lambda \in \{0, 1\}$ and vectors $\{x^*_t, u^*_t, \xi^*_t, \eta^*_t\}$, simultaneously not all equal to zero, satisfying the conditions of Theorem 3.4.*

Proof In this case Conjecture 3.5 ensures the conditions of [17, Theorem 3.24] for the problem (3.6). Therefore, according to this theorem, the necessary condition is obtained as in Theorem 3.4 by starting from the relation (3.7), written for the nonconvex problem. □

4. Conditions for discrete-approximation problem

Let δ be a step on the t –axis and $x(t) \equiv x_\delta(t)$ be a grid function on a uniform grid on $[a,b]$. Let us recall first order forward and backward difference approximation operators

$$\Delta_+ x(t) = \frac{1}{\delta}[x(t+\delta) - x(t)] \equiv \Delta x(t),\ \Delta_- x(t) = \frac{1}{\delta}[x(t) - x(t-\delta)],\ t = a, a+\delta, \dots, b,$$

respectively, in special case $\Delta x(a) = \frac{1}{\delta}[x(a+\delta) - x(a)]$, $\Delta_- x(b) = \frac{1}{\delta}[x(b) - x(b-\delta)]$ and second order difference operator $\Delta^2 x(t) = \frac{1}{\delta}[\Delta x(t+\delta) - \Delta x(t)]$.

We now associate the problem (1.1)–(1.4) with the following second order discrete-approximation problem

$$\text{minimize} \quad \varphi\big(x(b-\delta), \Delta_- x(b)\big), \tag{4.1}$$

$$\Delta^2 x(t) \in F\big(x(t), \Delta x(t)\big),\ t = a, a+\delta, a+2\delta, \dots, b-2\delta, \tag{4.2}$$

$$x(t) \in M,\ \Delta x(t) \in N,\ t = a+\delta, a+2\delta, \dots, b-2\delta, \tag{4.3}$$

$$x(a) = \alpha_0,\ \Delta x(a) = \beta_0,\ (x(b-\delta), \Delta_- x(b)) \in Q. \tag{4.4}$$

Let us use the following straightforward auxiliary mapping Ω

$$\Omega(x,u) = 2u - x + \delta^2 F\left(x, \frac{u-x}{\delta}\right) \tag{4.5}$$

and function $\bar\varphi$ such that

$$\bar\varphi(x(b-\delta), x(b)) \equiv \varphi(x(b-\delta), \Delta_- x(b))$$

in order to reduce the problem (4.1)–(4.4) to a problem of the form (3.1)–(3.4). Let rewrite the problem (3.1)–(3.4) as follows

$$\text{minimize} \quad \bar\varphi\big(x(b-\delta), x(b)\big), \tag{4.6}$$

$$x(t+2\delta) \in \Omega\big(x(t), x(t+\delta)\big),\ t = a, a+\delta, a+2\delta, \dots, b-2\delta, \tag{4.7}$$

$$x(t) \in M,\ x(t+\delta) - x(t) \in \delta N,\ t = a+\delta, a+2\delta, \dots, b-2\delta, \tag{4.8}$$

$$x(a) = \alpha_0,\ x(a+\delta) = \alpha_0 + \delta\beta_0 = \alpha_1,$$

$$(\delta x(b-\delta), x(b) - x(b-\delta)) \in \delta Q. \tag{4.9}$$

By Theorem 3.4 for the optimality of the trajectory $\{\tilde x(t)\} := \{\tilde x(t) \mid t = a, a+\delta, \dots, b-\delta, b\}$, in problem (4.6)–(4.9), it is necessary that there exist a pair of vectors $\{\bar u^*(t)\}, \{\bar x^*(t)\}$ and a number $\lambda \in \{0,1\}$ not all zero, such that discrete Euler-Lagrange and transversality inclusions are fulfilled

$$\big(\bar x^*(t) - \bar u^*(t) - \bar\eta^*(t), \bar u^*(t+\delta) - \bar\xi^*(t)\big) \in \Omega^*\big(\bar x^*(t+2\delta); (\tilde x(t), \tilde x(t+\delta), \tilde x(t+2\delta))\big), \tag{4.10}$$

$$\big(\bar\eta^*(t) + \bar\xi^*(t), \bar\xi^*(t)\big) \in K^*_M\big(\tilde x(t)\big) \times K^*_{\delta N}\big(\tilde x(t+\delta) - \tilde x(t)\big), \tag{4.11}$$

$$t = a+2\delta, a+3\delta, \dots, b-2\delta$$

$$\big(-\bar x^*(b-\delta) + \bar u^*(b-\delta) + \bar\eta^*(b-\delta), -\bar x^*(b) + \bar\xi^*(b-\delta)\big) \in \lambda\partial\bar\varphi\big(\tilde x(b-\delta), \tilde x(b)\big) \tag{4.12}$$

$$(\bar\eta^*(b-\delta) + \bar\xi^*(b-\delta), \bar\xi^*(b-\delta)) \in K^*_{\delta Q}\big(\delta\tilde x(b-\delta), \tilde x(b) - \tilde x(b-\delta)\big). \tag{4.13}$$

Under the regularity condition, conditions (4.10)–(4.13) are also sufficient for the optimality of $\{\tilde{x}(t)\}$. In order to express the LAM Ω^* in (4.10) in terms of LAM F^*, we use Proposition 2.2.
Then inclusion (4.10) is equivalent to the following inclusions

$$\left(\frac{\bar{x}^*(t)-\bar{u}^*(t)+\bar{u}^*(t+\delta)-\bar{\eta}^*(t)-\bar{\xi}^*(t)-\bar{x}^*(t+2\delta)}{\delta^2},\frac{\bar{u}^*(t+\delta)-2\bar{x}^*(t+2\delta)-\bar{\xi}^*(t)}{\delta}\right)$$
$$\in F^*\big(\bar{x}^*(t+2\delta);(\tilde{x}(t),\Delta\tilde{x}(t),\Delta^2\tilde{x}(t))\big),\ t=a+\delta,a+2\delta,\dots,b-2\delta,$$
$$\Delta^2\tilde{x}(t)\in F_{Arg}\big(\tilde{x}(t),\Delta\tilde{x}(t);\bar{x}^*(t+2\delta)\big) \tag{4.14}$$

only it is taken into account that LAM is positive homogeneous on the first argument.

The objective function $\bar{\varphi}\big(x(b-\delta),x(b)\big)$ in the second order discrete approximation problem is in the form

$$\bar{\varphi}\big(x(b-\delta),x(b)\big)\equiv\varphi\big(x(b-\delta),\Delta_{-}x(b)\big). \tag{4.15}$$

Therefore, using Proposition 2.3 transversality condition (4.12) turns into inclusion in terms of subdifferential of φ,

$$\big(-\bar{x}^*(b-\delta)+\bar{u}^*(b-\delta)-\bar{x}^*(b)+\bar{\eta}^*(b-\delta)+\bar{\xi}^*(b-\delta),-\delta\bar{x}^*(b)+\delta\bar{\xi}^*(b-\delta)\big)$$
$$\in\lambda\partial\varphi\big(\tilde{x}(b-\delta),\Delta_{-}\tilde{x}(b)\big). \tag{4.16}$$

Let us now compute the dual cones to the cones of tangent directions to the sets δN and δQ, respectively.

Lemma 4.1 *Let $K_{\delta N}\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big)$ be the cone of tangent directions of the set δN at $\tilde{x}(t+\delta)-\tilde{x}(t), t=a+\delta,a+2\delta,\dots,b-2\delta$, and $K_N(\Delta\tilde{x}(t))$ be the cone of tangent directions of the set N at $\Delta\tilde{x}(t), t=a+\delta,a+2\delta,\dots,b-2\delta$, then*

$$K_{\delta N}\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big)=K_N\big(\Delta\tilde{x}(t)\big),t=a+\delta,a+2\delta,\dots,b-2\delta.$$

Also the relation between the dual cones of these cones is defined by

$$K^*_{\delta N}\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big)=K^*_N\big(\Delta\tilde{x}(t)\big),t=a+\delta,a+2\delta,\dots,b-2\delta.$$

Proof Observe, first, that for arbitrary $\bar{y}\in K_{\delta N}\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big)$, $t=a+\delta,a+2\delta,\dots,b-2\delta$, and for sufficiently small $\lambda>0$ relation $\tilde{x}(t+\delta)-\tilde{x}(t)+\lambda\bar{y}\in\delta N$ holds. So by the relation $\big(\frac{\tilde{x}(t+\delta)-\tilde{x}(t)}{\delta}\big)+\big(\frac{\lambda\bar{y}}{\delta}\big)\in N$, we obtain $\Delta\tilde{x}(t)+\frac{\lambda}{\delta}\bar{y}\in N$, $t=a+\delta,a+2\delta,\dots,b-2\delta$. Since $K_N\big(\Delta\tilde{x}(t)\big)$ is a cone and $\lambda>0$, $\delta>0$ then from the last inclusion, we have $\bar{y}\in K_N\big(\Delta\tilde{x}(t)\big)$. Consequently, $K_{\delta N}\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big)\subseteq K_N\big(\Delta\tilde{x}(t)\big),t=a+\delta,a+2\delta,\dots,b-2\delta$. Conversely, if $\bar{y}\in K_N\big(\Delta\tilde{x}(t)\big),t=a+\delta,a+2\delta,\dots,b-2\delta$, then for $\delta>0,\frac{\bar{y}}{\delta}\in K_N\big(\Delta\tilde{x}(t)\big)$, and hence $\Delta\tilde{x}(t)+\frac{\lambda\bar{y}}{\delta}\in N$. By the difference formula, we obtain $\tilde{x}(t+\delta)-\tilde{x}(t)+\lambda\bar{y}\in\delta N$. Hence, by the definition of cone of tangent directions, $\bar{y}\in K_{\delta N}\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big)$.
Therefore, $K_N\big(\Delta\tilde{x}(t)\big)\subseteq K_{\delta N}\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big),t=a+\delta,a+2\delta,\dots,b-2\delta$. That completes the first part of the proof of the theorem.
Let us now prove the relation between the dual cones. Let $y^*\in K^*_{\delta N}\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big),t=a+\delta,a+2\delta,\dots,b-2\delta$ be arbitrary, then the inequality $\langle y^*,\bar{y}\rangle\geq 0$ is satisfied for all $\bar{y}\in K_{\delta N}\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big)$,

$t = a+\delta, a+2\delta, \ldots, b-2\delta$. By the relation between cones of tangent directions obtained above, the last inequality holds for all $\bar{y} \in K_N\big(\Delta\tilde{x}(t)\big)$. Hence, it follows that $y^* \in K_N^*\big(\Delta\tilde{x}(t)\big)$. Thus, we obtain that $K_{\delta N}^*\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big) \subseteq K_N^*\big(\Delta\tilde{x}(t)\big), t = a+\delta, a+2\delta, \ldots, b-2\delta$. Going in the reverse direction, by the same way, we see that $K_N^*(\Delta\tilde{x}(t)) \subseteq K_{\delta N}^*\big(\tilde{x}(t+\delta)-\tilde{x}(t)\big)$. Thus, the relation between the dual cones follows. That completes the proof. □

Lemma 4.2 *Let $\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big) \in Q$ be given. If $K_{\delta Q}\big(\delta\tilde{x}(b-\delta), \tilde{x}(b)-\tilde{x}(b-\delta)\big)$ is the cone of tangent directions of the set δQ at $(\delta\tilde{x}(b-\delta), \tilde{x}(b)-\tilde{x}(b-\delta))$ and $K_Q\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big)$ is the cone of tangent directions of the set Q at $\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big)$, then*

$$K_{\delta Q}\big(\delta\tilde{x}(b-\delta), \tilde{x}(b)-\tilde{x}(b-\delta)\big) = K_Q\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big)$$

holds. Also relation between dual cones is defined by

$$K_{\delta Q}^*\big(\delta\tilde{x}(b-\delta), \tilde{x}(b)-\tilde{x}(b-\delta)\big) = K_Q^*\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big).$$

Proof Let $(\bar{x},\bar{y}) \in K_{\delta Q}\big(\delta\tilde{x}(b-\delta), \tilde{x}(b)-\tilde{x}(b-\delta)\big)$ then by the definition of cones of tangent directions $(\delta\tilde{x}(b-\delta)+\lambda\bar{x}, \tilde{x}(b)-\tilde{x}(b-\delta)+\lambda\bar{y}) \in \delta Q$ holds. It is equivalent to the inclusion $\left(\tilde{x}(b-\delta)+\lambda\frac{\bar{x}}{\delta}, \frac{\tilde{x}(b)-\tilde{x}(b-\delta)}{\delta}+\lambda\frac{\bar{y}}{\delta}\right) \in Q$. Then we obtain $\left(\frac{\bar{x}}{\delta}, \frac{\bar{y}}{\delta}\right) \in K_Q\left(\tilde{x}(b-\delta), \frac{\tilde{x}(b)-\tilde{x}(b-\delta)}{\delta}\right)$, and by the property of cones $(\bar{x},\bar{y}) \in K_Q\left(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\right)$.
Conversely, take $(\bar{x},\bar{y}) \in K_Q\left(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\right)$ then by the definition of cone of tangent directions $\left(\tilde{x}(b-\delta)+\lambda\bar{x}, \frac{\tilde{x}(b)-\tilde{x}(b-\delta)}{\delta}+\lambda\bar{y}\right) \in Q$. Multiplying each side of the last inclusion with δ, we obtain $\big(\delta\tilde{x}(b-\delta)+\lambda\delta\bar{x}, \tilde{x}(b)-\tilde{x}(b-\delta)+\lambda\delta\bar{y}\big) \in \delta Q$, or in other words $(\delta\bar{x},\delta\bar{y}) \in K_{\delta Q}(\delta\tilde{x}(b-\delta), \tilde{x}(b)-\tilde{x}(b-\delta))$. Thus, by the property of cones it is simply the inclusion $(\bar{x},\bar{y}) \in K_{\delta Q}\big(\delta\tilde{x}(b-\delta), \tilde{x}(b)-\tilde{x}(b-\delta)\big)$. Eventually, the relation $K_{\delta Q}\big(\delta\tilde{x}(b-\delta), \tilde{x}(b)-\tilde{x}(b-\delta)\big) = K_Q\left(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\right)$ holds.
Relation between the dual cones is similarly proved as in the proof given in Lemma 4.1. □

Theorem 4.3 *Let F be a convex function and φ be a proper function that is convex with respect to x and continuous at points of some feasible trajectory $\{x(t)\}, t = a, a+\delta, \ldots, b-\delta, b$, sets M and N be convex subsets of $\mathbb{R}^n$, and set Q be a convex subset of $\mathbb{R}^n\times\mathbb{R}^n$, respectively. Then for the optimality of the trajectory $\{\tilde{x}(t)\}$ in the discrete approximation problem (4.1)–(4.4), it is necessary that there exist a number $\lambda \in \{0,1\}$ and vectors $\{x^*(t), v^*(t), \mu^*(t), \xi^*(t)\}$ simultaneously not all equal to zero, satisfying the second order approximate Euler-Lagrange type inclusions and transversality inclusions:*

$$\big(\Delta^2x^*(t)+\Delta v^*(t-\delta)-\mu^*(t), v^*(t)-\xi^*(t)\big) \in F^*\big(x^*(t+2\delta); (\tilde{x}(t), \Delta\tilde{x}(t), \Delta^2\tilde{x}(t))\big), \tag{4.17}$$

$$\big(\mu^*(t), \xi^*(t)\big) \in K_M^*\big(\tilde{x}(t)\big) \times K_N^*\big(\Delta\tilde{x}(t)\big), \tag{4.18}$$

$$t = a+\delta, a+2\delta, \ldots, b-2\delta;$$

$$\big(v^*(b-\delta)+\Delta x^*(b-\delta)+\mu^*(b-\delta), -x^*(b)+\xi^*(b-\delta)\big) \in \lambda\partial\varphi\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big), \tag{4.19}$$

$$\big(\mu^*(b-\delta), \xi^*(b-\delta)\big) \in K_Q^*\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big) \tag{4.20}$$

$$\Delta^2\tilde{x}(t) \in F_{Arg}\big(\tilde{x}(t), \Delta\tilde{x}(t); x^*(t+2\delta)\big)$$

respectively.

Proof In relation (4.14), we obtained an inclusion for LAM F^*. Let us now denote $\bar{v}^*(t) = \frac{\bar{u}^*(t+\delta)-2\bar{x}^*(t+2\delta)}{\delta}$ and $\bar{\eta}^*(t) + \bar{\xi}^*(t) = \bar{\mu}^*(t)$ in (4.14), then it is obvious that the first and second components of the first inclusion in (4.14) may be expressed as follows

$$\frac{\bar{x}^*(t) - \bar{u}^*(t) + \bar{u}^*(t+\delta) - \bar{\eta}^*(t) - \bar{\xi}^*(t) - \bar{x}^*(t+2\delta)}{\delta^2} = \Delta^2\bar{x}^*(t) + \Delta\bar{v}^*(t-\delta) - \frac{\bar{\mu}^*(t)}{\delta^2} \tag{4.21}$$

and

$$\frac{\bar{u}^*(t+\delta) - 2\bar{x}^*(t+2\delta) - \bar{\xi}^*(t)}{\delta} = \bar{v}^*(t) - \frac{\bar{\xi}^*(t)}{\delta} \tag{4.22}$$

respectively.

Let $\frac{\bar{\mu}^*(t)}{\delta^2}$, $\frac{\bar{\xi}^*(t)}{\delta}$ and $\bar{x}^*(t), \bar{v}^*(t)$ denote again with $\mu^*(t)$, $\xi^*(t)$ and $x^*(t), v^*(t)$, respectively. Then by (4.21) and (4.22), we obtain the components of the required relation (4.17) for the LAM F^*. Note also that $\Delta^2\tilde{x}(t) \in F_{Arg}(\tilde{x}(t), \Delta\tilde{x}(t); x^*(t+2\delta))$.

On the other hand, if we rearrange (4.16) and then denote $\bar{u}^*(b-\delta) - 2\bar{x}^*(b) = \delta\bar{v}^*(b-\delta)$ and $\bar{\eta}^*(b-\delta) + \bar{\xi}^*(b-\delta) = \bar{\mu}^*(b-\delta)$, respectively, we get

$$\begin{aligned}\big(\delta\bar{v}^*(b-\delta) + \bar{x}^*(b) - \bar{x}^*(b-\delta) + \bar{\mu}^*(b-\delta), -\delta\bar{x}^*(b) + \delta\bar{\xi}^*(b-\delta)\big)& \\ \in \lambda\partial\varphi\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big).&\end{aligned} \tag{4.23}$$

Using new notations $\delta\bar{v}^*(b-\delta) = v^*(b-\delta), \delta\bar{x}^*(b-\delta) = x^*(b-\delta), \delta\bar{x}^*(b) = x^*(b), \delta\bar{\xi}^*(b-\delta) = \xi^*(b-\delta)$ and $\bar{\mu}(b-\delta) = \mu(b-\delta)$, respectively, then from (4.23) and by difference approximation relations we obtain (4.19). Moreover, the following condition

$$\big(\bar{\eta}^*(t) + \bar{\xi}^*(t), \bar{\xi}^*(t)\big) \in K_M^*\big(\tilde{x}(t)\big) \times K_N^*\big(\Delta\tilde{x}(t)\big), t = a+\delta, a+2\delta, \ldots, b-2\delta \tag{4.24}$$

is derived by the inclusion (4.11) and Lemma 4.1. Since we denote $\bar{\mu}^*(t) = \bar{\eta}^*(t) + \bar{\xi}^*(t)$ above then we see that $\bar{\mu}^*(t) \in K_M^*(\tilde{x}(\cdot))$. If we use the usual notations for $\bar{\mu}^*(t)$ and $\bar{\xi}^*(t)$, and take into account that $K_M^*\big(\tilde{x}(\cdot)\big)$ and $K_N^*\big(\Delta\tilde{x}(\cdot)\big)$ are convex cones then inclusion (4.24) becomes the desired result (4.18).

On the other hand, by (4.13) and Lemma 4.2, we have

$$\big(\bar{\eta}^*(b-\delta) + \bar{\xi}^*(b-\delta), \bar{\xi}^*(b-\delta)\big) \in K_Q^*\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big). \tag{4.25}$$

Let us put the equation $\bar{\eta}^*(b-\delta) + \bar{\xi}^*(b-\delta) = \bar{\mu}^*(b-\delta)$ into (4.25) then

$$\big(\bar{\mu}^*(b-\delta), \bar{\xi}^*(b-\delta)\big) \in K_Q^*\big(\tilde{x}(b-\delta), \Delta_-\tilde{x}(b)\big) \tag{4.26}$$

which is the inclusion (4.20), written in usual notations.

As a result, we obtain the conditions (4.17)–(4.20) of the theorem. □

Theorem 4.4 *Suppose that Condition I is satisfied for the nonconvex problem (4.1)–(4.4), then* $\{\tilde{x}(t)\}$ *is an optimal trajectory of this problem if there exist a number* $\lambda \in \{0,1\}$ *and vectors* $\{x^*(t), v^*(t), \mu^*(t), \xi^*(t)\}$ *simultaneously not all equal to zero, satisfying (4.17)–(4.20) for nonconvex case.*

5. Main results

Let us now establish the sufficient optimality conditions to the second order differential inclusions of optimal problem with viable constraints (1.1)–(1.4).

Imposing the basic qualification condition [29] $K_Q^*(\tilde{x}(b),\tilde{x}'(b))\cap[-K_{(M\times N)}^*(\tilde{x}(t),\tilde{x}'(t))]=\{0\}$ then dual cone intersection relation $K_{Q\cap(M\times N)}^*(\tilde{x}(t),\tilde{x}'(t))=K_Q^*(\tilde{x}(b),\tilde{x}'(b))+K_{(M\times N)}^*(\tilde{x}(t),\tilde{x}'(t))$ is satisfied.

Now, we are ready to give our main results.

Theorem 5.1 *Sufficient conditions for the optimality of the trajectory $\tilde{x}(t)$ in the convex problem (1.1)–(1.4) are that there exist a pair of absolutely continuous functions $\{x^*(t),v^*(t)\},t\in[a,b]$ satisfying the second order Euler-Lagrange type differential inclusion, the transversality condition at the endpoint $t=b$ and the condition ensuring that the locally adjoint mapping F^* is nonempty at a given point*

(i) $\left(x^{*\prime\prime}(t)+v^{*\prime}(t),v^*(t)\right)\in F^*\left(x^*(t);(\tilde{x}(t),\tilde{x}'(t),\tilde{x}''(t))\right)+K_M^*(\tilde{x}(t))\times K_N^*(\tilde{x}'(t))$, a.e. $t\in[a,b]$,

(ii) $\left(v^*(b)+x^{*\prime}(b),-x^*(b)\right)\in\partial\varphi(\tilde{x}(b),\tilde{x}'(b))-K_Q^*(\tilde{x}(b),\tilde{x}'(b))$,

(iii) $\tilde{x}''(t)\in F_{Arg}\left(\tilde{x}(t),\tilde{x}'(t);x^*(t)\right)$, *a.e.* $t\in[a,b]$,

where $F_{Arg}(x,u,v^*)=\{v\in F(x,u)\mid\langle v,v^*\rangle=H(x,u,v^*)\}$ is the argmaximum set for multivalued mapping F. Let also the basic qualification condition be satisfied. Here we assume $x^*(t),t\in[a,b]$ to be absolutely continuous function together with the first order derivative and $x^{*\prime\prime}(\cdot)\in L_1^n([a,b])$. Besides $v^*(t),t\in[a,b]$ is absolutely continuous and $v^{*\prime}(\cdot)\in L_1^n([a,b])$.

Proof It follows from condition (i) of the theorem that for $t\in[a,b]$

$$\left(x^{*\prime\prime}(t)+v^{*\prime}(t)-\mu^*,v^*(t)-\xi^*\right)\in\partial_{(x,v)}H\left(\tilde{x}(t),\tilde{x}'(t),x^*(t)\right),\tag{5.1}$$

where $(\mu^*,\xi^*)\in K_M^*(\tilde{x}(t))\times K_N^*(\tilde{x}'(t))$. By the definition of subdifferential of the Hamiltonian function H_F [17], inclusion (5.1) is equivalent to the inequality

$$\begin{aligned}&H_F\left(x(t),x'(t),x^*(t)\right)-H_F\left(\tilde{x}(t),\tilde{x}'(t),x^*(t)\right)\\&\le\langle x^{*\prime\prime}(t)+v^{*\prime}(t)-\mu^*,x(t)-\tilde{x}(t)\rangle+\langle v^*(t)-\xi^*,x'(t)-\tilde{x}'(t)\rangle\\&\le\langle x^{*\prime\prime}(t)+v^{*\prime}(t),x(t)-\tilde{x}(t)\rangle+\langle v^*(t),x'(t)-\tilde{x}'(t)\rangle,t\in[a,b],\end{aligned}\tag{5.2}$$

since $\langle\mu^*,x(t)-\tilde{x}(t)\rangle\ge0$ and $\langle\xi^*,x'(t)-\tilde{x}'(t)\rangle\ge0$. On the other hand, by the definition of the Hamiltonian function the inequality $H_F\left(\tilde{x}(t),\tilde{x}'(t),x^*(t)\right)\ge\langle x^*(t),\tilde{x}''(t)\rangle$ holds for all feasible solutions, then (5.2) can be rewritten as the inequality

$$\begin{aligned}\langle x^*(t),x''(t)\rangle-\langle x^*(t),\tilde{x}''(t)\rangle\le\langle x^{*\prime\prime}(t),x(t)-\tilde{x}(t)\rangle+\langle v^{*\prime}(t),x(t)-\tilde{x}(t)\rangle\\+\langle v^*(t),x'(t)-\tilde{x}'(t)\rangle.\end{aligned}$$

If we rearrange the last inequality, we derive

$$0\le\langle x^{*\prime\prime}(t),x(t)-\tilde{x}(t)\rangle-\left\langle x^*(t),\frac{d^2(x(t)-\tilde{x}(t))}{dt^2}\right\rangle+\frac{d}{dt}\langle v^*(t),x(t)-\tilde{x}(t)\rangle,t\in[a,b].\tag{5.3}$$

Also, first two inner products in (5.3) can be shown as difference of two derivatives, that is

$$\langle x^{*\prime\prime}(t), x(t)-\tilde{x}(t)\rangle - \left\langle x^*(t), \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}\right\rangle$$
$$= \frac{d}{dt}\left\langle \frac{dx^*(t)}{dt}, x(t)-\tilde{x}(t)\right\rangle - \frac{d}{dt}\left\langle x^*(t), \frac{d(x(t)-\tilde{x}(t))}{dt}\right\rangle.$$

Therefore, inequality (5.3) turns into

$$0 \le \frac{d}{dt}\left\langle \frac{dx^*(t)}{dt}, x(t)-\tilde{x}(t)\right\rangle - \frac{d}{dt}\left\langle x^*(t), \frac{d(x(t)-\tilde{x}(t))}{dt}\right\rangle + \frac{d}{dt}\langle v^*(t), x(t)-\tilde{x}(t)\rangle, t\in[a,b].$$

Integrating the last inequality over the interval $[a,b]$ and taking into account that $x(\cdot), \tilde{x}(\cdot)$ are feasible, we obtain

$$0 \le \int_a^b \left[\frac{d}{dt}\left\langle \frac{dx^*(t)}{dt}, x(t)-\tilde{x}(t)\right\rangle - \frac{d}{dt}\left\langle x^*(t), \frac{d(x(t)-\tilde{x}(t))}{dt}\right\rangle\right] dt$$
$$+\langle v^*(b), x(b)-\tilde{x}(b)\rangle - \langle v^*(a), x(a)-\tilde{x}(a)\rangle. \tag{5.4}$$

If we compute the integral on the right-hand side of (5.4), then it follows that

$$0 \le \left\langle \frac{dx^*(b)}{dt}, x(b)-\tilde{x}(b)\right\rangle - \left\langle \frac{dx^*(a)}{dt}, x(a)-\tilde{x}(a)\right\rangle + \left\langle x^*(a), \frac{dx(a)}{dt} - \frac{d\tilde{x}(a)}{dt}\right\rangle$$
$$- \left\langle x^*(b), \frac{dx(b)}{dt} - \frac{d\tilde{x}(b)}{dt}\right\rangle + \langle v^*(b), x(b)-\tilde{x}(b)\rangle - \langle v^*(a), x(a)-\tilde{x}(a)\rangle,$$

and hence

$$0 \le \left\langle \frac{dx^*(b)}{dt} + v^*(b), x(b)-\tilde{x}(b)\right\rangle - \left\langle x^*(b), \frac{dx(b)}{dt} - \frac{d\tilde{x}(b)}{dt}\right\rangle$$
$$+ \left\langle x^*(a), \frac{dx(a)}{dt} - \frac{d\tilde{x}(a)}{dt}\right\rangle - \left\langle \frac{dx^*(a)}{dt} + v^*(a), x(a)-\tilde{x}(a)\right\rangle. \tag{5.5}$$

On the other hand for all feasible arcs $x(t), t\in[a,b]$, transversality condition (ii) implies the relation

$$\varphi(x(b), x'(b)) - \varphi(\tilde{x}(b), \tilde{x}'(b)) \ge \left\langle \frac{dx^*(b)}{dt} + \mu_b^* + v^*(b), x(b)-\tilde{x}(b)\right\rangle$$
$$- \left\langle x^*(b) - \xi_b^*, \frac{dx(b)}{dt} - \frac{d\tilde{x}(b)}{dt}\right\rangle, \tag{5.6}$$

where $(\mu_b^*, \xi_b^*) \in K_Q^*(\tilde{x}(b), \tilde{x}'(b))$ that is

$$\langle \mu_b^*, x(b)-\tilde{x}(b)\rangle + \langle \xi_b^*, x'(b)-\tilde{x}'(b)\rangle \ge 0. \tag{5.7}$$

Therefore, from the fact that $\tilde{x}(a) = x(a) = \alpha_0$ and $\tilde{x}'(a) = x'(a) = \beta_0$ and from inequalities (5.5)–(5.7), we obtain

$$0 \le \varphi(x(b), x'(b)) - \varphi(\tilde{x}(b), \tilde{x}'(b)) - \langle \mu_b^*, x(b)-\tilde{x}(b)\rangle - \langle \xi_b^*, x'(b)-\tilde{x}'(b)\rangle$$
$$\le \varphi(x(b), x'(b)) - \varphi(\tilde{x}(b), \tilde{x}'(b)).$$

Then we conclude that $\varphi(x(b),x'(b)) \geq \varphi(\tilde{x}(b),\tilde{x}'(b))$ and hence we obtain that $\tilde{x}(t), t\in[a,b]$ is optimal. □

Example 5.2 *Suppose now that we have the so-called semilinear Mayer problem with viable constraint:*

$$\text{minimize } \varphi\big(x(b)\big), \tag{5.8}$$

$$x''(t) \in F\big(x(t)\big),\ a.e.\ t\in[a,b], F(x) = Ax + BU \tag{5.9}$$

$$x(t)\in M, \quad M = \{x \mid f(x) \leq 0\} \tag{5.10}$$

$$x(a) = \alpha_0, \quad x'(a) = \alpha_1 \tag{5.11}$$

where φ is continuously differentiable convex function, f is a continuous convex function, such that there exists x_1, satisfying $f(x_1)<0$, A and B are $n\times n$ and $n\times r$ continuous matrices, respectively, U is a convex closed subset of $\mathbb{R}^r$; $\alpha_j, j=0,1$ are fixed vectors. The problem is to find a controlling parameter $\tilde{u}(t)\in U$ such that arc $\tilde{x}(t)$ corresponding to it minimizes $\varphi(x(b))$. In fact, this is optimization of Cauchy problem for "semilinear" differential inclusions with viable constraints. The controlling parameter $u(\cdot)$ is called admissible if it takes only values in the given control set U which is nonempty, convex, closed set.

Corollary 5.3 *An arc $\tilde{x}(t)$ corresponding to the controlling parameter $\tilde{u}(t)$ is a solution to the problem (5.8)–(5.11) if there exists an absolutely continuous function $x^*(t)$ together with the second order derivatives, satisfying the following Euler-Lagrange type differential equation, the transversality condition at a point $t=b$:*

$$\begin{aligned}
&x^{*''}(t) \in A^*x^*(t) - cone\partial f\big(\tilde{x}(t)\big),\ a.e.\ t\in[a,b],\\
&\frac{dx^*(b)}{dt} = \varphi'\big(\tilde{x}(b)\big),\\
&\Big\langle B\tilde{u}(t), x^*(t)\Big\rangle = \sup_{u\in U}\Big\langle Bu, x^*(t)\Big\rangle.
\end{aligned}$$

Proof Obviously, the Hamiltonian is as follows

$$H_F(x,v^*) = \Big\langle Ax, v^*\Big\rangle + \sup_{u\in U}\Big\langle Bu, v^*\Big\rangle.$$

Hence,

$$F^*(v^*;(x,\tilde{v}),t) = \partial_x H_F(x,v^*) = A^*v^*,\ \tilde{v}\in F_{Arg}(x,v^*),\ \tilde{v} = Ax + B\tilde{u},$$

where the argmaximum inclusion $\tilde{v}\in F_{Arg}(x,v^*)$ means that $\langle B\tilde{u},v^*\rangle = \sup_{u\in U}\langle Bu,v^*\rangle$ and $F^*(v^*;(x,\tilde{v}),t)\neq\varnothing$. In this case, it is clear that $K_M^*(\tilde{x}(t)) = -cone\partial f(\tilde{x}(t))$ [17]. On the other hand, since $Q=\mathbb{R}^{2n}$, it follows that $K_Q^*(\tilde{x}(b),\tilde{x}'(b)) = \{0\}\times\{0\}$. Then by Theorem 5.1, we obtain

$$\begin{aligned}
&x^{*''}(t) \in A^*x^*(t) - cone\partial f\big(\tilde{x}(t)\big),\ \tilde{x}''(t)\in F_A\big(\tilde{x}(t),x^*(t)\big),\\
&\Big\langle B\tilde{u}(t), x^*(t)\Big\rangle = \sup_{u\in U}\Big\langle Bu, x^*(t)\Big\rangle.
\end{aligned}$$

□

Theorem 5.4 *Let us consider the nonconvex problem, that is problem (1.1)–(1.4) where the function $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$ is nonconvex with respect to x, and the mapping F is nonconvex. Let K_M^*, K_N^*, K_Q^* be dual cones of cones of tangent directions to non-convex sets M, N, Q, respectively, and yield the basic qualification condition. Then for the optimality of arc $\tilde{x}(t), t \in [a,b]$, among all feasible solutions of the problem (1.1)–(1.4) it is sufficient that there exists a pair of absolutely continuous functions $x^*(t), v^*(t), t \in [a,b]$, satisfying Euler-Lagrange type conditions and transversality conditions:*

(i) $\left(\dfrac{d^2x^*(t)}{dt^2} + \dfrac{dv^*(t)}{dt}, v^*(t)\right) \in F^*\big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t))\big) + K_M^*\big(\tilde{x}(t)\big) \times K_N^*\big(\tilde{x}'(t)\big),$

(ii) $\varphi(x,v) - \varphi(\tilde{x}(b), \tilde{x}'(b)) \geq \left\langle \dfrac{dx^*(b)}{dt} + v^*(b) + \mu_b^*, x - \tilde{x}(b)\right\rangle + \left\langle -x^*(b) + \xi_b^*, v - \dfrac{d\tilde{x}(b)}{dt}\right\rangle,$
$\forall (x,v) \in \mathbb{R}^{2n},$

(iii) $(\mu_b^*, \xi_b^*) \in K_Q^*\big(\tilde{x}(b), \tilde{x}'(b)\big),$

(iv) $\left\langle x^*(t), \dfrac{d^2\tilde{x}(t)}{dt^2}\right\rangle = H_F\big(\tilde{x}(t), \tilde{x}'(t), x^*(t)\big), a.e. t \in [a,b].$

Proof By condition (i) and definition of LAM in the nonconvex case (see Section 1)

$$\begin{aligned} H_F\big(x(t), x'(t), x^*(t)\big) - H_F\big(\tilde{x}(t), \tilde{x}'(t), x^*(t)\big) &\leq \langle x^{*\prime\prime}(t) + v^{*\prime}(t) - \mu^*, x(t) - \tilde{x}(t)\rangle \\ &+ \langle v^*(t) - \xi^*, x'(t) - \tilde{x}'(t)\rangle \end{aligned}$$

or similarly

$$\begin{aligned} \langle x^*(t), x''(t)\rangle - \langle x^*(t), \tilde{x}''(t)\rangle &\leq \langle x^{*\prime\prime}(t), x(t) - \tilde{x}(t)\rangle + \langle v^{*\prime}(t), x(t) - \tilde{x}(t)\rangle - \langle \mu^*, x(t) - \tilde{x}(t)\rangle \\ &+ \langle v^*(t), x'(t) - \tilde{x}'(t)\rangle - \langle \xi^*, x'(t) - \tilde{x}'(t)\rangle, \end{aligned}$$

where $(\mu^*, \xi^*) \in K_M^*\big(\tilde{x}(t)\big) \times K_N^*\big(\tilde{x}'(t)\big)$. Since by the definition of dual cone we have $\langle \mu^*, x(t) - \tilde{x}(t)\rangle \geq 0$ and $\langle \xi^*, x'(t) - \tilde{x}'(t)\rangle \geq 0$ then we can rewrite the last inclusion

$$\begin{aligned} 0 \leq \langle -x^*(t), x''(t) - \tilde{x}''(t)\rangle + \langle x^{*\prime\prime}(t), x(t) - \tilde{x}(t)\rangle + \langle v^{*\prime}(t), x(t) - \tilde{x}(t)\rangle \\ + \langle v^*(t), x'(t) - \tilde{x}'(t)\rangle. \end{aligned} \tag{5.12}$$

From the inequality (5.12) is justified (5.3). Thus, the continuation of the proof of the theorem is similar to the one for Theorem 5.1. □

Acknowledgment

The authors are grateful to the editors of the Turkish Journal of Mathematics, the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] Agarwal RP, Bohner M, Li T, Zhang C. A new approach in the study of oscillatory behavior of even-order neutral delay differential equations. Applied Mathematics and Computation 2013; 225: 787-794.

[2] Aitalioubrahim M, Sajid S. Second order viability result in Banach Spaces. Discussiones Mathematicae Differential Inclusions, Control and Optimization 2010; 30 (1): 5-21.

[3] Akinwumi TO, Ayoola EO. Viable solutions of lower semicontinuous quantum stochastic differential inclusions. Analysis and Mathematical Physics 2021; 11 (1): 1-16.

[4] Aubin JP, Cellina A. Differential Inclusions, Set-valued Maps and Viability Theory. New York, NY, USA: Springer-Verlag, 1984.

[5] Aubin JP. Viability Theory. Boston, MA, USA: Birhauser, 1991.

[6] Auslender A, Mechler J. Second order viability problems for differential inclusions. Journal of Mathematical Analysis and Applications 1994; 181 (1): 205-218.

[7] Benchohra M, Ntouyas SK. On three and four point boundary value problems for second order differential inclusions. Miskolc Mathematical Notes 2001; 2 (2): 93-101.

[8] Bohner M, Tisdell CC. Second order dynamic inclusions. Journal of Nonlinear Mathematical Physics 2005; 12 (2): 36–45.

[9] Cernea A. On the existence of viable solutions for a class of second order differential inclusions. Discussiones Mathematicae Differential Inclusions Control and Optimization 2002; 22 (1): 67-78.

[10] Cornet B, Haddad G. Theoremes de viabilite pour les inclusions differentielles du second ordre. Israel Journal of Mathematics 1987; 57 (2): 225-238 (in French).

[11] Demir Sağlam S, Mahmudov EN. Polyhedral optimization of second-order discrete and differential inclusions with delay. Turkish Journal of Mathematics 2021; 45 (1): 244-263.

[12] Demir Sağlam S, Mahmudov EN. Optimality conditions for higher order polyhedral discrete and differential inclusions. Filomat 2020; 34 (13): 4533-4553.

[13] Hassani S, Mammadov M, Jamshidi M. Optimality conditions via weak subdifferentials in reflexive Banach spaces. Turkish Journal of Mathematics 2017; 41 (1): 1-8.

[14] Loewen PD, Rockefellar RT. Optimal control of unbounded differential inclusions. SIAM Journal on Control Optimization 1994; 32 (2): 442-470.

[15] Lupulescu V. A viability result for second-order differential inclusions. Electronic Journal of Differential Equations 2002; 76: 1-12.

[16] Mahmudov EN. Optimization of second order discrete approximation inclusions. Numerical Functional Analysis and Optimization 2015; 36 (5): 624-643.

[17] Mahmudov EN. Approximation and Optimization of Discrete and Differential Inclusions. Boston, MA, USA: Elsevier, 2011.

[18] Mahmudov EN. Optimization of Mayer Problem with Sturm–Liouville-Type differential inclusions. Journal of Optimization Theory and Applications (JOTA) 2018; 177 (2): 345-375.

[19] Mahmudov EN. Optimal control of second order delay-discrete and delay-differential inclusions with state constraints. Evolution Equations and Control Theory (EECT) 2018; 7 (3): 501-529.

[20] Mahmudov EN. Optimization of fourth order Sturm-Liouville Type differential inclusions with initial point constraints. Journal of Industrial and Management Optimization (JIMO) 2020; 16 (1): 169-187.

[21] Mahmudov EN. Second-order viability problems for differential inclusions with endpoint constraint and duality. Applicable Analysis DOI: 10.1080/00036811.2020.1773444

[22] Mahmudov EN. Optimal control of higher order differential inclusions with functional constraints. European Series in Applied and Industrial Mathematics (ESAIM): Control, Optimization and Calculus of Variations 2020; 26: 1-37.

[23] Mahmudov EN. On duality in problems of optimal control described by convex differential inclusions of Goursat–Darboux type. Journal of Mathematical Analysis and Applications 2005; 307 (2): 628-640.

[24] Mahmudov EN. Necessary and sufficient conditions for discrete and differential inclusions of elliptic type. Journal of Mathematical Analysis and Applications 2006; 323 (2): 768-789.

[25] Mardanov MJ, Melikov TK. A new discrete analogue of Pontryagin's Maximum Principle. Doklady Mathematics 2018; 98: 549–551.

[26] Mardanov MJ, Melikov TK. A method for studying the optimality of controls in discrete systems. Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan 2014; 40 (2): 5-13.

[27] Mardanov MJ, Melikov TK, Mahmudov NI. On necessary optimality conditions in discrete control systems. International Journal of Control 2015; 88 (10): 2097-2106.

[28] Mordukhovich BS. Variational Analysis and Generalized Differentiation, I: Basic Theory; II: Applications. Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. 330 and 331. Berlin, Germany: Springer, 2006.

[29] Mordukhovich BS, Nam NM. An easy path to convex analysis and applications. Synthesis Lectures on Mathematics and Statistics, Morgan and Claypool Publishers series 2014.

[30] Mordukhovich BS, Cao TH. Optimal control of a nonconvex perturbed sweeping process. Journal of Differential Equations 2019; 266: 1003–1050.

[31] Nguyen T, Le T. Second-order necessary optimality conditions for an optimal control problem. Taiwanese Journal of Mathematics 2020; 24 (1): 225-264.

[32] Rockafellar RT. Convex Analysis. Princeton, NJ, USA: Princeton University Press, 1997.

[33] Smirnov GV. Introduction to the Theory of Differential Inclusions. Providence, RI, USA: American Mathematical Society, 2001.

[34] Son TQ, Wen CF. Weak-stability and saddle point theorems for a multiobjective optimization problem with an infinite number of constraints. Turkish Journal of Mathematics 2019; 43 (4): 1953-1966.