Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2108 – 2125
© TÜBİTAK
doi:10.3906/mat-1912-82

Reduced limit approach to semilinear partial differential equations (PDEs) involving the fractional Laplacian with measure data

Ratan Kr. GIRI[1], **Debajyoti CHOUDHURI**[2,*]
[1]Mathematics department, Technion - Israel Institute of Technology, Haifa, Israel
[2]Department of Mathematics, National Institute of Technology Rourkela, India

Received: 22.12.2019 • **Accepted/Published Online:** 26.07.2021 • **Final Version:** 16.09.2021

Abstract: We study the following partial differential equation (PDE)

$$\begin{aligned}(-\Delta)^s u + g(x,u) &= \mu \text{ in } \Omega,\\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega,\end{aligned} \tag{0.1}$$

where $(-\Delta)^s$ is the fractional Laplacian operator, Ω is a bounded domain in $\mathbb{R}^N$ with $\partial\Omega$ being the boundary of Ω, $g(.,.)$ is a nonlinear function defined over $\Omega \times \mathbb{R}$. Let $(\mu_n)_n$ be a sequence of measure in Ω. Assume that there exists a solution u_n with data μ_n, i.e. u_n satisfies the equation (0.1) with $\mu = \mu_n$. We further assume that the sequence of measures weakly converges to μ, while $(u_n)_n$ converges to u in $L^1(\Omega)$. In general, u is not a solution to the partial differential equation in (0.1) with datum $(\mu, 0)$. However, there exists a measure $\mu^\#$ such that u is a solution of the partial differential equation with this data. $\mu^\#$ is called the reduced limit of the sequence $(\mu_n)_n$. We investigate the relation between weak limit μ and the reduced limit $\mu^\#$ and the dependence of $\mu^\#$ to the sequence $(\mu_n)_n$. A closely related problem was studied by Bhakta and Marcus [3] and then by Giri and Choudhuri [15] but for the case of a Laplacian and a general second order linear elliptic differential operator, respectively instead of a fractional Laplacian.

Key words: Fractional Laplacian, reduced limit, very weak solution, good measure, Sobolev space

1. Introduction and preliminaries

The problem we will address in this article is

$$\begin{aligned}(-\Delta)^s u + g(x,u) &= \mu \text{ in } \Omega,\\ u &= 0 \text{ on } \mathbb{R}^N \setminus \Omega,\end{aligned} \tag{1.1}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^N$, $(-\Delta)^s u = c(N,s)\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy$, $0 < s < 1$, μ is a Radon measure, and g is a given nonlinear function defined on $\Omega \times \mathbb{R}$. We also assume that the nonlinearity of g satisfies the

*Correspondence: dc.iit12@gmail.com
2010 *AMS Mathematics Subject Classification:* Primary 35J61; Secondary 35J60.

following conditions:

$$\begin{aligned} &\text{(a) } g(x,u) \in C(\mathbb{R}), \quad g(x,0)=0, \\ &\text{(b) } g(x,.) \text{ is non decreasing,} \\ &\text{(c) } g(.,u) \in L^1(\Omega, \rho^s), \end{aligned} \tag{1.2}$$

where $L^1(\Omega, \rho^s)$ denotes the weighted Lebesgue space with $\rho(x) = \text{dist}(x, \partial\Omega)$. The family of functions satisfying (1.2) will be denoted by $\mathscr{G}_0$. Observe that if $g \in \mathscr{G}_0$, then the function g^* given by $g^*(x,t) = -g(x,-t)$ is also in $\mathscr{G}_0$.

Not much evidence is found in the literature, which addresses the problem of existence of a solution to the equation (1.1) with measure data and, hence, the reader is suggested to refer to Brezis [4], which is one of the earliest attempts made in studying the non-linear equations with measure data. In fact, he considered the equation for $s=1$ of the type

$$\begin{aligned} -\Delta u + |u|^{p-1}u = f(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $p > \frac{N}{N-2}$, Ω is a bounded smooth domain in $\mathbb{R}^N$ and $0 \in \Omega$ with f a given function in $L^1(\Omega)$ or a measure. A detailed study of non-linear elliptic partial differential equations of the above type with measures can be found in Brezis et al [5]. Here, they have used the notion of *'reduced measure'*. Readers will need to refer to Marcus and Véron [20] for its richness in addressing problems concerning the existence of a solution to the nonlinear, second order elliptic equations involving measures. Some other pioneering contributions to nonlinear local operators with L^1 data or measure data, which is worth mentioning, are due to Brezis & Strauss [6], Kozhevnikova [18], Marcus & Ponce [19], Véron [23] and the references therein. The motivation for studying such problems have been discussed in the preface of [6].

Recently, a great attention has been given to study the non-linear equations involving fractional Laplacian or more general integro-differential operators, see [7–10, 16, 21] and the references therein. By using the duality approach, the authors in [17] studied the problem

$$(-\Delta)^s u = \mu \text{ in } \mathbb{R}^N, \tag{1.4}$$

where μ is a Radon measure with compact support. In [13], the authors used the sub-super solutions method to obtain the existence of solution to the problem

$$\begin{aligned} (-\Delta)^s u = f(x,u) + \mu, \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathe͗ódory function and μ is a Radosn measure in Ω. Also, it extends some results in Chen & Véron [11]. The present work draws its motivation from Bhakta and Marcus [3], Giri and Choudhuri [15], who have considered the problem (1.1) for a Laplacian and a general second order linear elliptic differential operator, respectively with g satisfying the assumptions in (1.2) $(a)-(c)$.

We denote the dual space of $C_0(\bar{\Omega}) = \{f \in C(\bar{\Omega}) : f = 0 \text{ on } \partial\Omega\}$ to be the space of finite Borel measures, which will be denoted by $\mathfrak{M}(\Omega)$ endowed with the norm

$$||\mu||_{\mathfrak{M}(\Omega)} = \int_\Omega d|\mu|.$$

We denote the space $\mathfrak{M}(\Omega,\rho^s)$ to be the set of signed Radon measures μ in Ω such that $\rho^s\mu \in \mathfrak{M}(\Omega)$ where $\rho(x) = \text{dist}\,(x, \partial\Omega)$. The norm of a measure $\mu \in \mathfrak{M}(\Omega,\rho^s)$ is defined as

$$||\mu||_{\Omega,\rho^s} = \int_\Omega \rho^s d|\mu|.$$

We denote $\mathfrak{M}(\bar{\Omega})$ to be the space of bounded Borel measures over $\bar{\Omega}$ whose elements are measures $\mu|_\Omega$ extended continuously to $\bar{\Omega}$. It is also denoted as $\mathfrak{M}(\Omega) \times \mathfrak{M}(\partial\Omega)$.

Definition 1.1 *Let $(\mu_n)_n$ be a bounded sequence of measures in $\mathfrak{M}(\Omega,\rho^s)$ and $\rho^s\mu_n$ is extended to a Borel measure in $\mathfrak{M}(\bar{\Omega})$ defined as zero on $\partial\Omega$. We say that $(\rho^s\mu_n)_n$ converges weakly in $\bar{\Omega}$ to a measure $\tau \in \mathfrak{M}(\bar{\Omega})$ if $(\mu_n)_n$ converges weakly to τ in $\mathfrak{M}(\bar{\Omega})$, i.e.*

$$\int_\Omega \varphi\rho^s d\mu_n \to \int_{\bar{\Omega}} \varphi d\tau \quad \forall\ \varphi \in C(\bar{\Omega}).$$

We denote this convergence by $\rho^s\mu_n \underset{\bar{\Omega}}{\rightharpoonup} \tau$.

We denote $\mathfrak{M}_{loc}(\Omega)$ to be the space of set functions μ on $\mathfrak{B}(\Omega) = \{E \Subset \Omega : E \text{ Borel}\}$ such that $\mu\chi_K$ is a finite measure for every compact $K \subset \Omega$.

Definition 1.2 *Let $(\mu_n)_n$ be a sequence in $\mathfrak{M}_{loc}(\Omega)$. We say that $(\mu_n)_n$ converges weakly to $\mu \in \mathfrak{M}_{loc}(\Omega)$ if it shows convergence in the distribution sense, i.e.*

$$\int_\Omega \varphi d\mu_n \to \int_\Omega \varphi d\mu \quad \forall\ \varphi \in C_c(\Omega).$$

We denote this convergence by $\mu_n \underset{d}{\rightharpoonup} \mu$ where

$$C_c(\Omega) = \{\varphi \in C(\bar{\Omega}) : supp(\varphi) \textit{ is compact in } \Omega, \varphi|_{\partial\Omega} = 0\}.$$

Definition 1.3 *Let $(\mu_n)_n$ be a bounded sequence of measures in $\mathfrak{M}(\Omega)$. We say that $(\mu_n)_n$ converges weakly in Ω to a measure $\tau \in \mathfrak{M}(\Omega)$ if $(\mu_n)_n$ converges weakly to τ in $\mathfrak{M}(\Omega)$, i.e.*

$$\int_\Omega \varphi d\mu_n \to \int_\Omega \varphi d\tau \quad \forall\ \varphi \in C_0(\bar{\Omega}).$$

We denote this convergence by $\mu_n \underset{\Omega}{\rightharpoonup} \tau$. The topology defined via this weak convergence is metrizable and a bounded sequence with respect to this topology is precompact, i.e. contains a weakly convergent subsequence, and every weakly convergent sequence is bounded.

Definition 1.4 *A sequence $(\mu_n)_n$ in $\mathfrak{M}(\Omega,\rho^s)$ converges 'weakly' to $\mu \in \mathfrak{M}(\Omega,\rho^s)$ if*

$$\int_\Omega \varphi d\mu_k \to \int_\Omega \varphi d\mu, \ \ \forall\ \varphi \in C_0(\bar{\Omega},\rho^s),$$

where $C_0(\bar{\Omega},\rho^s)=\{f\in C_0(\bar{\Omega}):\frac{f}{\rho^s}\in C(\bar{\Omega})\}$. *The weak convergence in this sense is equivalent to the weak convergence* $\rho^s\mu_k\rightharpoonup\rho^s\mu$ *in* $\mathfrak{M}(\Omega)$. *In this case, also the topology of weak convergence is metrizable, and the properties mentioned above persist. Hereafter, for the sake of simplicity, we will often use the simplified notation* $\int_\Omega f$ *instead of* $\int_\Omega f(x)dx$ *when referring to integrals when no ambiguity on the variable of integration is possible.*

Remark 1.5 *From the notions of convergence in measure defined so far, we can summarize the following.*

$$\begin{aligned}
\mu_n \underset{d}{\rightharpoonup} \mu \quad &\Leftrightarrow \quad \rho^s\mu_n \underset{d}{\rightharpoonup} \rho^s\mu,\\
\rho^s\mu_n \underset{\bar{\Omega}}{\rightharpoonup} \tau \quad &\Rightarrow \quad \rho^s\mu_n \underset{d}{\rightharpoonup} \tau\chi_\Omega,\\
\mu_n \underset{\bar{\Omega}}{\rightharpoonup} \mu \quad &\Rightarrow \quad \mu_n \underset{\Omega}{\rightharpoonup} \mu,\\
\mu_n \underset{\Omega}{\rightharpoonup} \mu \quad &\Rightarrow \quad \mu_n \underset{d}{\rightharpoonup} \mu.
\end{aligned}$$

For this and other properties of weak convergence of measures, we refer to the book by Marcus and Véron [20]. In this paper, we consider the problem (1.1) with $\mu\in\mathfrak{M}(\Omega,\rho^s)$. We begin by stating, in the language of Yang [16], the following integration by parts formula from Theorem 1.3 in [16], which has been obtained for the regional fractional Laplacian which are generators of symmetric $2s$-stable processes on a subset of $\mathbb{R}^N$ ($0<s<1$) and for $u\in L^1\left(\Omega,\frac{dx}{(1+|x|)^{N+2s}}\right)$. When restricting the integral kernel of fractional Laplacian to a subset G of R^N, we obtain the nonlocal operator Δ_G^s for $0<s<1$ (see Definition 1.6 below).

Definition 1.6 *Let* $u\in L^1\left(\Omega,\frac{dx}{(1+|x|)^{N+2s}}\right)$. *The regional fractional Laplacian* Δ_G^s *is defined by the following formula*

$$\Delta_G^s u(x) \;=\; \lim_{\epsilon\downarrow 0}\Delta_{G,\epsilon}^s u(x),\;\; x\in G, \tag{1.5}$$

$$\int_\Omega w_1(-\Delta)^s w_2 dx$$
$$=\begin{cases}\int_\Omega w_2(-\Delta)^s w_1 dx - C_{N,s}\int_{\partial\Omega} w_2F^{2-2s}w_1\ m(dx)+\int_{\partial\Omega} w_2F^{2-2s}w_1\ m(dx),\ 1/2<s<1,\\ \hfill \forall w_1,w_2\in\bigcup_{\beta\geq 2s}\mathcal{D}_\beta(\bar{\Omega})\\ \int_\Omega w_2(-\Delta)^s w_1 - C_{N,s}\int_{\partial\Omega} w_2F^{2-2s}w_1\ m(dx),\ 0<s\leq 1/2, \forall w_2\in\bigcup_{\beta\geq 2s}\mathcal{D}_\beta(\bar{\Omega}), \forall w_1\in C^2(\bar{\Omega})\end{cases}$$

where Ω is a C^2 open set in $\mathbb{R}^N$, $m(dx)$ is the notation used in [16] to denote the measure on the boundary $\partial\Omega$, $\mathcal{D}_\beta(\bar{\Omega})=\{u:u(x)=f(x)h(x)+g(x),\forall x\in\Omega,\text{for some } f,g\in C^2(\bar{\Omega})\}$,

$$h(x)=\begin{cases}\rho(x)^{\beta-1} & \forall x\in\Omega'_\delta, \beta\in(0,1)\bigcup(1,\infty)\\ \ln\rho(x) & \forall x\in\Omega'_\delta,\beta=1\end{cases}$$

with $\Omega'_\delta=\{y\in\Omega:0<\rho(y)<\delta\}$, and

$$\frac{\mathcal{A}(1,-s)}{\mathcal{A}(N,-s)}C_{N,s}=\begin{cases}C_s, & N=1\\ 2C_s\int_0^{\pi/2}\cos^s\theta d\theta, & N=2\\ \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}C_s\int_0^{\pi/2}\cos^s\theta\sin^{N-2}\theta d\theta, & N\geq 3\end{cases}$$

where

$$C_s = \begin{cases} \frac{\mathcal{A}(1,-s)}{s(s-1)} C_s \int_0^\infty \frac{|t-1|^{1-s} - \max\{t,1\}^{1-s}}{t^{2-s}} dt, & s \in (0,1) \bigcup (1,2) \\ \mathcal{A}(1,-s) \int_0^\infty \frac{\ln(\max\{t,1\}) - \ln(t-1)}{t} dt, & \alpha = 1 \end{cases}$$

and $\mathcal{A}(N,-s) = \frac{|s| 2^{s-1} \Gamma\left(\frac{N+s}{2}\right)}{\pi^{N/2} \left(1 - \frac{s}{2}\right)}$. Notice that $\mathcal{D}_\beta(\bar{\Omega}) = C^2(\bar{\Omega})$ when $\beta \geq 2$ and Ω is smooth.

We now give the following definition.

Definition 1.7 *We will define $u \in L^1(\Omega)$ to be a 'very weak solution' of the problem (1.1), if $g(x,u) \in L^1(\Omega, \rho^s)$ and u satisfies*

$$\int_\Omega (u(-\Delta)^s \varphi + g(x,u)\varphi) dx \quad = \quad \int_\Omega \varphi d\mu\,, \;\; \forall \, \varphi \in W(\mathbb{R}^N) \tag{1.6}$$

where $W(\mathbb{R}^N) = \{\varphi \in C^1_{1-2s}(\bar{\Omega}) : \varphi = 0 \text{ on } \mathbb{R}^N \setminus \Omega \text{ and } (-\Delta)^s \varphi \in L^\infty(\Omega)\}$. Here $C^1_{1-2s}(\bar{\Omega})$ refers to the collection of all those functions in $C^1_0(\bar{\Omega})$ for which $\frac{\varphi}{\rho^{1-2s}}|_\Omega$ have continuous extension to $\bar{\Omega}$. Note that there are no boundary integrals in the equation (1.6). To understand the absence of boundary integrals in (1.6), we first give a definition due to Qing-Yang Guan [16].

Definition 1.8 *Let Ω be a C^1 domain in $\mathbb{R}^N$. For $0 \leq \gamma < 2$, $u \in C^1(\Omega)$ and $x \in \partial\Omega$ define operator F^γ on $\partial\Omega$ by $F^\gamma u(\vec{x}) = -\lim\limits_{t \to 0^+} \frac{d}{dt} u(\vec{x} + t\hat{n}(\vec{x})) t^\gamma$, provided that the limit exists.*

Once again, from the Theorem 1.3 in [16], we find that the boundary integrals are absent in 1.6 since

$$\begin{aligned} F^{2-2s}\varphi &= -\lim_{t \to 0^+} \frac{d}{dt} \varphi(\vec{x} + t\hat{n}) t^{2-2s} \\ &= -\lim_{t \to 0^+} \left(\nabla\varphi(\vec{x} + t\hat{n}).\hat{n} t^{2-2s} + (2-2s) \frac{\varphi(\vec{x} + t\hat{n})}{t^{2s-1}} \right) \\ &= 0. \end{aligned}$$

We will see that every weak solution to (1.1) is also a very weak solution to it. We will call a measure to be *good* if $\exists \; u \in L^1(\Omega)$, which satisfies (1.1) in the very weak sense (1.6). The space of good measures will be denoted by $\mathfrak{M}^g(\Omega)$.

We will give an example where if $(\mu_n)_n \subset \mathfrak{M}^g(\Omega)$ and if the very weak solution corresponding to each μ_n be denoted by $u_n \in L^1(\Omega)$ of the problem (1.1) such that the sequence of measures converge in a weak sense in Ω to μ, while $(u_n)_n$ converge to u in $L^1(\Omega)$, then, in general, u is not a very weak solution to the boundary value problem (1.1) with data μ. However, if there exists a measure $\mu^\#$, such that u is a very weak solution of the boundary value problem (1.1) with this data, then this measure $\mu^\#$ will be called the '*reduced limit*' of the sequence $(\mu_n)_n$. The idea of reduced limit is in some sense related to the idea of reduced measure introduced by Brezis [5], which is the largest good measure $\leq \mu$. In this paper, we will answer similar questions with respect to sequence $(\mu_n)_n \subset \mathfrak{M}^g(\Omega)$. Let u_n be a solution of (1.1) with $\mu = \mu_n$ and suppose $u_n \to u$ in $L^1(\Omega)$. We will show that there exists a Radon measure $\mu^\#$ such that u is a very weak solution of the boundary value problem

$$(-\Delta)^s u + g(x,u) = \mu^\# \text{ in } \Omega, \qquad u = 0 \text{ on } \mathbb{R}^N \setminus \Omega, \tag{1.7}$$

where $\mu^\#$ is called the reduced limit of $(\mu_n)_n$.

In addition to this, for a given L^1 data f, we will use a well known variational technique to show existence of a weak solution in a subspace $\tilde{W}^{s,2}(\Omega) = \{v \in W^{s,2}(\mathbb{R}^N) : v = 0 \text{ in } \mathbb{R}^N \setminus \Omega, (-\Delta)^{s/2} v \in L^2(\Omega)\}$ of the fractional Sobolev space $W^{s,2}(\Omega)$, equipped with a norm, which is defined as $||v||_{\tilde{W}^{s,2}(\Omega)} = \left(\int_\Omega |(-\Delta)^{s/2} v|^2 dx\right)^{1/2}$. This seminorm is a norm here since $v = 0$ in $\mathbb{R}^N \setminus \Omega$.

The paper has been organized into three sections with Section 1 being the introduction. In Section 2, we begin by studying the semilinear boundary value problem with L^1 data and will prove certain basic lemmas and theorems. In Section 3, we continue the study by considering the nonlinear problem with measure data, and we will prove the main result of this paper.

2. Semilinear problem with $L^1(\Omega, \rho^s)$ data

In this section, we consider the semilinear boundary value problem with L^1 data, which is as follows:

$$(-\Delta)^s u + g(x,u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \mathbb{R}^N \setminus \Omega. \tag{2.1}$$

We have the following result, the proof of which has been taken from the proposition 2.4 of [11].

Lemma 2.1 *If $f \in L^1(\Omega, \rho^s)$ and suppose there exists a unique very weak solution $u \in L^1(\Omega)$ of the problem*

$$\begin{aligned} (-\Delta)^s u &= f \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{2.2}$$

, then for any $\varphi \in W(\mathbb{R}^N)$ such that $\varphi \geq 0$, we have

$$\int_\Omega u_+ (-\Delta^s) \varphi dx \leq \int_\Omega \varphi f sign_+ u dx. \tag{2.3}$$

and

$$\int_\Omega |u| (-\Delta^s) \varphi dx \leq \int_\Omega \varphi f sign(u) dx \tag{2.4}$$

where $u_+ = \max\{u, 0\}$, $sign_+ u = \chi_{\{x : u(x) \geq 0\}}$.

Proof Let $(f_n)_n$ be a sequence in $C_c^\infty(\bar{\Omega})$, which are continuously extended by zero in $\mathbb{R}^N \setminus \bar{\Omega}$, such that $||f_n - f||_{L^1(\Omega, \rho^s)} \to 0$ as $n \to \infty$. Let $(u_n)_n$ be a classical solution of

$$\begin{aligned} (-\Delta)^s u_n &= f_n, \text{ in } \Omega \\ u_n &= 0, \text{ in } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{2.5}$$

These solutions have the representation $u_n(x) = \frac{\Gamma(\frac{N}{2}-2s)}{\pi^{N/2}2^{2s}\Gamma(s)} \int_\Omega \frac{f_n(y)}{|x-y|^{N-2s}}dy$, (see [2]), which is also called as the $s-$Riesz potential. Since each f_n is a smooth function, which is compactly supported in Ω, we have u_n to be at least continuous in Ω. We also have from [2] the inequality $||u_n||_1 \leq c||f_n||_1$ for every $n \in \mathbb{N}$ and hence $u_n \in L^1(\Omega)$.
For $\delta > 0$, define an even convex function γ_δ as

$$\gamma_\delta(t) = \begin{cases} |t| - \frac{\delta}{2} & \text{if } |t| \geq \delta \\ \frac{t^2}{2\delta} & \text{if } |t| < \delta. \end{cases}$$

Then for any $t, s \in \mathbb{R}$, $|\gamma_\delta'(t)| \leq 1$, $\gamma_\delta(t) \to |t|$ and $\gamma_\delta'(t) \to \text{sign}(t)$ when $\delta \to 0^+$. Moreover, $\gamma_\delta(s) - \gamma_\delta(t) \geq \gamma_\delta'(t)(s-t)$.

For $\varphi \in W(\mathbb{R}^N)$ such that $\varphi \geq 0$ we have

$$\begin{aligned} \int_\Omega \gamma_\delta(u_n)[(-\Delta)^s\varphi]dx &= \int_\Omega [(-\Delta)^s\gamma_\delta(u_n)]\varphi dx \\ \leq \int_\Omega \gamma_\delta'(u_n)(-\Delta)^s u_n \varphi dx &= \int_\Omega \gamma_\delta'(u_n) f_n \varphi dx. \end{aligned} \tag{2.6}$$

Passing to the limit $\delta \to 0$, we obtain

$$\int_\Omega |u_n|(-\Delta)^s\varphi dx \leq \int_\Omega \text{ sign}(u_n) f_n \varphi dx \leq \int_\Omega \varphi |f_n| dx \ \forall n \in \mathbb{N}. \tag{2.7}$$

Let φ_0 be the solution of (2.2) with $f \equiv 1$. Then, from [21], $\exists\ c > 0$ such that $c^{-1} \leq \frac{\varphi_0}{\rho^s} \leq c$ in Ω. Taking $\varphi \equiv \varphi_0$ in (2.7), we get

$$\int_\Omega |u_n| dx \leq c \int_\Omega |f_n| \rho^s dx. \tag{2.8}$$

Thus,

$$\int_\Omega |u_n - u_m| dx \quad \leq \quad c \int_\Omega |f_n - f_m| \rho^s dx. \tag{2.9}$$

Therefore, $(u_n)_n$ is a Cauchy sequence in $L^1(\Omega)$, and, hence, its limit v is in $L^1(\Omega)$. Since each u_n is a classical solution to (2.5), we have

$$\begin{aligned} \int_\Omega \varphi(-\Delta)^s u_n dx &= \int_\Omega f_n \varphi \\ \Rightarrow \int_\Omega u_n(-\Delta)^s \varphi dx &= \int_\Omega f_n \frac{\varphi}{\rho^s} \rho^s dx \end{aligned} \tag{2.10}$$

$\forall \varphi \in W(\mathbb{R}^N)$. Passing to the limit $n \to \infty$ in (2.10), we obtain

$$\begin{aligned} \int_\Omega v(-\Delta)^s \varphi dx &= \int_\Omega f \frac{\varphi}{\rho^s} \rho^s dx \\ &= \int_\Omega f \varphi dx. \end{aligned} \tag{2.11}$$

Thus, $v \in L^1(\Omega)$ is a very weak solution of (2.2). In fact, $v = u$ since by our assumption, there exists a unique very weak solution of (2.2). Further, passing to the limit $n \to \infty$ in (2.7), we get $\int_\Omega |u|(-\Delta)^s \varphi dx \leq \int_\Omega f(\operatorname{sign} u)\varphi dx$. The inequality $\int_\Omega u_+(-\Delta^s)\varphi dx \leq \int_\Omega f(sign_+ u)\varphi dx$ is proved by replacing γ_δ by $\tilde{\gamma}_\delta(t) = \gamma_\delta \chi_{[0,\infty)}$. □

The following apriori estimate that has been used in the proof of the Proposition 3.1 [11] will be used here to guarantee the unicity of the solution to (2.1).

Lemma 2.2 *If $u_i \in L^1(\Omega)$ are very weak solutions of (2.1) corresponding to $f = f_i$ for $i = 1, 2$, then we have the following estimates:*

$$||u_1 - u_2||_{L^1(\Omega)} + ||g(x, u_1) - g(x, u_2)||_{L^1(\Omega,\rho^s)} \leq C||f_1 - f_2||_{L^1(\Omega,\rho^s)} \tag{2.12}$$

for some $C > 0$.

Proof Since u_1, u_2 are very weak solutions of (2.1), we have

$$\int_\Omega u_i(-\Delta^s)\varphi dx + \int_\Omega g(x, u_i)\varphi dx = \int_\Omega f_i \varphi dx$$

for all $\varphi \in W(\mathbb{R}^N)$, $i = 1, 2$. Consequently,

$$\int_\Omega (u_1 - u_2)(-\Delta^s)\varphi dx + \int_\Omega (g(x, u_1) - g(x, u_2))\varphi dx = \int_\Omega (f_1 - f_2)\varphi dx$$

for all $\varphi \in W(\mathbb{R}^N)$. This implies that $u_1 - u_2$ is a very weak solution of

$$\begin{aligned} (-\Delta)^s u &= f_1 - f_2 - g(x, u_1) + g(x, u_2) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{2.13}$$

Therefore, by Lemma 2.1, for any $\varphi \in W(\mathbb{R}^N)$, $\varphi \geq 0$

$$\int_\Omega |u_1 - u_2|(-\Delta^s)\varphi dx \leq \int_\Omega (f_1 - f_2 - g(x, u_1) + g(x, u_2)) sign(u_1 - u_2)\varphi dx \tag{2.14}$$

Let $\varphi \equiv \varphi_0$ be the same test function as used in Lemma 2.1. We use this φ in (2.14) to obtain

$$\int_\Omega |u_1 - u_2| dx \leq \int_\Omega (f_1 - f_2)\varphi_0 sign(u_1 - u_2) dx - \int_\Omega (g(x, u_1) + g(x, u_2)) sign(u_1 - u_2)\varphi_0 dx$$

This implies that

$$||u_1 - u_2||_{L^1(\Omega)} + \int_\Omega (g(x, u_1) - g(x, u_2)) sign(u_1 - u_2)\varphi_0 dx \leq \int_\Omega (f_1 - f_2)\varphi_0 sign(u_1 - u_2) dx$$

By the property of g, we have $(g(x, u_1) - g(x, u_2)) sign(u_1 - u_2) = |g(x, u_1) - g(x, u_2)|$. Thus from $c^{-1} \leq \frac{\varphi_0}{\rho^s} \leq c$ it follows that

$$||u_1 - u_2||_{L^1(\Omega)} + c^{-1}||g(x, u_1) - g(x, u_2)||_{L^1(\Omega,\rho^s)} \leq c||f_1 - f_2||_{L^1(\Omega,\rho^s)}. \tag{2.15}$$

Thus, there exists a new constant $c_1 > 0$ such that

$$||u_1 - u_2||_{L^1(\Omega)} + ||g(x,u_1) - g(x,u_2)||_{L^1(\Omega,\rho^s)} \leq c_1||f_1 - f_2||_{L^1(\Omega,\rho^s)} \tag{2.16}$$

Hence, the lemma is proved. □

The above inequality also implies that if $u \in L^1(\Omega)$ is a very weak solution of the boundary value problem 2.1, then

$$||u||_{L^1(\Omega)} + ||g(x,u)||_{L^1(\Omega,\rho^s)} \leq C||f||_{L^1(\Omega,\rho^s)} \tag{2.17}$$

Next, the following comparison result will be used wherever necessary.

Lemma 2.3 ***(Comparison of solutions)*** *Let u_1 and u_2 be the very weak solutions in $L^1(\Omega)$ of the boundary value problem (2.1) corresponding to f_1 and f_2, respectively. If $f_1 \leq f_2$, then $u_1 \leq u_2$ a.e.*

Proof By the proof of Lemma 2.2, $u_1 - u_2$ is a weak solution of the problem (2.13). By taking $\varphi \equiv \varphi_0$, a solution of (2.2) with $f \equiv 1$, and by Lemma 2.1, we have

$$\int_\Omega (u_1 - u_2)_+ (-\Delta^s)\varphi_0\, dx \leq \int_\Omega (f_1 - f_2 - g(x,u_1) + g(x,u_2)) sign_+(u_1 - u_2)\varphi_0\, dx \tag{2.18}$$

Then, from the equation (2.18), it follows that

$$0 \leq \int_\Omega (u_1 - u_2)_+ dx = \int_{\Omega_+} (u_1 - u_2) dx \tag{2.19}$$

$$\leq \int_{\Omega_+} (f_1 - f_2) sign_+(u_1 - u_2)\varphi_0\, dx + \int_{\Omega_+} (g(x,u_2) - g(x,u_1)) sign_+(u_1 - u_2)\varphi_0\, dx \tag{2.20}$$

where $\Omega_+ = \{x \in \Omega : u_1 > u_2\}$. Since the test function $\varphi_0 \geq 0$ and $f_1 \leq f_2$, hence, the first integral in R.H.S of (2.20) is less than or equal to zero. Now since $g(x,.)$ is a nondecreasing function, we have

$$\int_{\Omega_+} (g(x,u_2) - g(x,u_1)) sign_+(u_1 - u_2)\varphi_0 dx \leq 0. \tag{2.21}$$

Thus, from (2.21) we get $\int_\Omega (u_1 - u_2)_+ dx \leq 0$ which shows that $(u_1 - u_2)_+ = 0$ a.e. Therefore, $u_1 \leq u_2$ a.e. □

It is important to note a priori that a weak solution in $\tilde{W}^{s,2}(\Omega)$ of the problem (2.1) is also a very weak solution in $L^1(\Omega)$ over the test function space $W(\mathbb{R}^N)$. Here, $\tilde{W}^{s,2}(\Omega)$ consists of functions in $W^{s,2}(\mathbb{R}^N)$, which are extended by zero in $\mathbb{R}^N \setminus \Omega$ (see [12]). In order to show this, we let $u \in \tilde{W}^{s,2}(\Omega)$ be a weak solution of (2.1). Hence, u satisfies

$$\int_\Omega (-\Delta)^{s/2} u (-\Delta)^{s/2} v dx + \int_\Omega g(x,u) v \rho^s dx \;=\; \int_\Omega f v \rho^s dx \tag{2.22}$$

$\forall v \in W(\mathbb{R}^N)$, $f \in L^1(\Omega,\rho^s)$. Since $v \in W(\mathbb{R}^N)$ we have $(-\Delta)^s v \in L^\infty(\Omega)$ which is continuously embedded in $L^2(\Omega)$. By applying integration by parts (see Theorem 1.3 in [16]) once more to (2.22), we obtain

$$\int_\Omega u(-\Delta)^s v dx + \int_\Omega g(x,u) v \rho^s dx \;=\; \int_\Omega f v \rho^s dx \;\forall v \in W(\mathbb{R}^N). \tag{2.23}$$

, and, hence, every weak solution in $\tilde{W}^{s,2}(\Omega)$ is a very weak solution in $L^1(\Omega)$ over the test function space $W(\mathbb{R}^N)$.

Theorem 2.4 ***(Existence of a unique very weak solution)*** *The boundary value problem (2.1) possesses a unique very weak solution u in $L^1(\Omega)$.*

Proof We first prove the existence of a weak solution in $\tilde{W}^{s,2}(\Omega)$ with the test function space $W(\mathbb{R}^N)$ for the case $f \in L^\infty(\Omega)$. For each $n \in \mathbb{N}$, define $g_n = \min\{|g|, n\}\text{sign}(g)$. Let $G_n(x,.)$ be the primitive of $g_n(x,.)$, i.e., $G_n'(x,.) = g_n(x,.)$, such that $G_n(x,0) = 0$. Note that G_n is a nonnegative function for each n. We will show the existence of a weak solution in $\tilde{W}^{s,2}(\mathbb{R}^N)$ to the problem

$$\begin{aligned} (-\Delta)^s u + g_n(x,u) &= f \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{2.24}$$

which satisfies

$$\int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}v dx + \int_\Omega g_n(x,u)v\,\rho^s dx = \int_\Omega fv\,dx, \ \ \forall\, v \in W(\mathbb{R}^N). \tag{2.25}$$

Let us consider the functional

$$\begin{aligned} I_n(u) &= \frac{1}{2}\int_\Omega |(-\Delta)^{s/2}u|^2 dx + \int_\Omega G_n(x,u)\,\rho^s dx - \int_\Omega fu\,dx \\ &= \frac{c(N,s)}{2}[u]^2_{\tilde{W}^{s,2}(\Omega)} + \int_\Omega G_n(x,u)\,\rho^s dx - \int_\Omega fu\,dx \end{aligned}$$

defined over $\tilde{W}^{s,2}(\Omega)$. Here, $[\cdot]$ denotes the Gagliardo norm [12]. Since $u \in \tilde{W}^{s,2}(\Omega)$ which is continuously embedded in $L^2(\Omega)$ and $L^2(\Omega)$ is continuously embedded in $L^1(\Omega)$, we have

$$\begin{aligned} I_n(u) &= \frac{c(N,s)}{2}[u]^2_{\tilde{W}^{s,2}(\Omega)} + \int_\Omega G_n(x,u)\,\rho^s dx - ||u||_1 \cdot ||f||_\infty \\ &\geq \frac{c(N,s)}{2}[u]^2_{\tilde{W}^{s,2}(\Omega)} + \int_\Omega G_n(x,u)\,\rho^s dx - c_5[u]_{\tilde{W}^{s,2}(\Omega)} \cdot ||f||_\infty \\ &= \left(\frac{c(N,s)}{2}[u]_{\tilde{W}^{s,2}(\Omega)} - c_5||f||_\infty\right)[u]_{\tilde{W}^{s,2}(\Omega)} + \int_\Omega G_n(x,u)\,dx, \end{aligned}$$

where $c_5 > 0$ constant. Since G_n is a nonnegative function, it shows that $\lim\limits_{[u]_{\tilde{W}^{s,2}(\Omega)}\to\infty} \frac{I_n(u)}{[u]_{\tilde{W}^{s,2}(\Omega)}} = \infty$. Therefore, the functional $I_n(u)$ is coercive.

Now we will show that the functional $I_n(u)$ is weakly lower semi continuous. For this, let $v_m \rightharpoonup u$ weakly in $\tilde{W}^{s,2}(\Omega)$. By the nonnegativity of G_n, we get by Fatou's lemma the following inequality:

$$\int_\Omega G_n(x,u)\,\rho^s dx \leq \lim_{m\to\infty}\inf \int_\Omega G_n(v_m)\,\rho^s dx.$$

Due to the weak convergence of v_m to u in $\tilde{W}^{s,2}(\Omega)$, we have $< (-\Delta)^{s/2}v_m, (-\Delta)^{s/2}v > \to < (-\Delta)^{s/2}u, (-\Delta)^{s/2}v >$ as $n \to \infty$ for every v in $W^{-s,2}(\mathbb{R}^N)$ and, hence, for every $v \in \tilde{W}^{s,2}(\mathbb{R}^N)$. Thus, $< (-\Delta)^{s/2}v_m -$

$(-\Delta)^{s/2}u, (-\Delta)^{s/2}v >\to 0$ for every $v \in \tilde{W}^{s,2}(\Omega)$ and, hence, $< (-\Delta)^{s/2}v_m - (-\Delta)^{s/2}u, (-\Delta)^{s/2}v_m - (-\Delta)^{s/2}u >\to 0$ as $n \to \infty$. Thus, $||(-\Delta)^{s/2}v_m - (-\Delta)^{s/2}u||_2 \to 0$. Hence, $\lim_{m\to\infty}\int_\Omega |(-\Delta^{s/2})v_m|^2 dx = \int_\Omega |(-\Delta^{s/2})u|^2 dx$. So we have

$$\begin{aligned} I_n(u) &\leq \liminf_{m\to\infty}[v_m]^2_{\tilde{W}^{s,2}(\Omega)} + \liminf_{m\to\infty}\int_\Omega N_{G_n(v_m)}\,\rho^s dx - \liminf_{m\to\infty}\int_\Omega f v_m \\ &\leq \liminf_{m\to\infty} I_n(v_m). \end{aligned}$$

Thus, $I_n(u)$ is weakly lower-semi continuous and coercive. Hence, the variational problem $\min_{u\in\tilde{W}^{s,2}(\Omega)}\{I_n(u)\}$ possesses a weak solution $\tilde{u}_n \in \tilde{W}^{s,2}(\Omega)$ by a result in [24]. The minimizer $\tilde{u}_n \in \tilde{W}^{s,2}(\Omega)$ is also a weak solution of the boundary value problem (2.24). Since every weak solution is a very weak solution, so by the estimate in (2.17), the sequences $(\tilde{u}_n)_n$ and $(g_n(x,\tilde{u}_n))_n$ are bounded in $L^1(\Omega)$ and $L^1(\Omega;\rho^s)$, respectively. Assume for a moment that $f \geq 0$. We will show that $\tilde{u}_n \geq 0$ a.e. in Ω. If not, then $f \geq 0$ implies $\tilde{u}_n < 0$. As a consequence, we have $f - g(\tilde{u}_n) \geq 0$. We also have the following

$$\int_\Omega\int_\Omega \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}dydx \leq -[u^-]^2_{W^{s,2}(\Omega)}, \tag{2.26}$$

obtained from the real value inequality $(a-b)(a^- - b^-) \leq -(a^- - b^-)^2$ for every $a, b \in \mathbb{R}$. Since $\tilde{u}_n$ is a weak solution on the test function space $W(\mathbb{R}^N)$, we have

$$\int_\Omega\int_\Omega (-\Delta)^{s/2}\tilde{u}_n(-\Delta)^{s/2}v dx \quad = \quad \int_\Omega (f - g(\tilde{u}_n)v dx \tag{2.27}$$

In particular, for $v = \tilde{u}_n^- \geq 0$, we have

$$\begin{aligned} 0 \geq -[\tilde{u}_n^-]^2_{W^{s,2}(\Omega)} &\geq \int_\Omega\int_\Omega \frac{(\tilde{u}_n(x)-\tilde{u}_n(y))(\tilde{u}_n^-(x)-\tilde{u}_n^-(y))}{|x-y|^{N+2s}}dydx \text{ (by 2.26)} \\ &= \int_\Omega (-\Delta)^{s/2}\tilde{u}_n(-\Delta)^{s/2}\tilde{u}_n^- dx \\ &= \int_\Omega (f - g(x,\tilde{u}_n))\tilde{u}_n^- dx \text{ (by (2.27))} \\ &= \int_{\{x\in\Omega:\tilde{u}_n(x)<0\}} (f - g(x,\tilde{u}_n))\tilde{u}_n^- dx \geq 0. \end{aligned} \tag{2.28}$$

This shows that $[\tilde{u}_n^-]_{\tilde{W}^{s,2}(\Omega)} = 0$ and, therefore, $\tilde{u}_n^- = 0$ a.e. in Ω. Thus, $\tilde{u}_n \geq 0$ a.e. in Ω. We also have that $\tilde{u}_n$ satisfies

$$\begin{aligned} (-\Delta)^s\tilde{u}_n + g_n(\tilde{u}_n) &= f \text{ in } \Omega, \\ \tilde{u}_n &= 0 \text{ on } \mathbb{R}^N\setminus\Omega. \end{aligned} \tag{2.29}$$

in the weak sense (2.25). A slight manipulation of (2.29) gives the following

$$\begin{aligned} (-\Delta)^s\tilde{u}_n + g_{n+1}(\tilde{u}_n) &= f + g_{n+1}(\tilde{u}_n) - g_n(\tilde{u}_n) \text{ in } \Omega, \\ \tilde{u}_n &= 0 \text{ on } \mathbb{R}^N\setminus\Omega. \end{aligned} \tag{2.30}$$

If $f^* = f + g_{n+1}(x, \tilde{u}_n) - g_n(x, \tilde{u}_n)$, then, due to the monotonically nondecreasing nature of g_n, we have $f^* \geq f$ a.e. on Ω. We also have $\tilde{u}_{n+1}$, which is a solution to the problem

$$\begin{aligned} (-\Delta)^s \tilde{u}_{n+1} + g_{n+1} \circ \tilde{u}_{n+1} &= f \text{ in } \Omega, \\ \tilde{u}_{n+1} &= 0 \text{ on } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{2.31}$$

As $f^* \geq f$, hence from (2.30), (2.31) and the comparison result, in Lemma 2.3, we have $\tilde{u}_{n+1} \leq \tilde{u}_n$. Thus, $(\tilde{u}_n)_n$ is a L^1 - bounded and bounded, nonincreasing sequence and so, by the dominated convergence theorem, we have $\tilde{u}_n \to u$ in $L^1(\Omega)$. Therefore, there exists a subsequence, which we will still denote as $\tilde{u}_n$ that converges to u pointwise a.e. and hence $\tilde{u}_n \to u$ in $L^1(\Omega)$. Further,

$$\begin{aligned} g_n(x, \tilde{u}_n) &= \min\{|g(x, \tilde{u}_n(x))|, n\}\, sign(g(x, \tilde{u}_n)) \\ &= \min\{g(x, \tilde{u}_n(x)), n\}\, sign(g(x, \tilde{u}_n)) \\ &= g(x, \tilde{u}_n(x)), \ \text{ for } n \geq n_0 \geq k(x) \ a.e. \ on \ \Omega \end{aligned}$$

From $(1.2a)$, we have $g_n(x, \tilde{u}_n(x)) = g(x, \tilde{u}_n(x)) \to g(x, u(x))$ a.e. for $n \geq k(x)$. This also implies that $g_n(x, \tilde{u}_n) \to g(x, u)$ in $L^1(\Omega)$ and hence in $L^1(\Omega, \rho^s)$. Now, by [2], there exists a solution V of

$$\begin{aligned} (-\Delta)^s v &= f \text{ in } \Omega, \\ v &= 0 \text{ on } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{2.32}$$

Since $u_n \geq 0$, we have $g_n(x, \tilde{u}_n) \geq 0$. Thus,

$$\begin{aligned} &(-\Delta)^s \tilde{u}_n = f - g_n(x, \tilde{u}_n) \leq f = (-\Delta)^s v \text{ in } \Omega, \\ &\text{where } \tilde{u}_n = 0 \qquad\qquad \text{and } v = 0 \text{ on } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

Therefore, by the definition of the fractional Laplacian, we can compare the solutions from which we have $\tilde{u}_n \leq V$ and hence $g(x, \tilde{u}_n) \leq g(x, V)$. In other words, if V is a solution of (2.32), then the sequence $(g(x, \tilde{u}_n))_n$ is dominated by $g(x, V)$. Since $\tilde{u}_m \to u$ and $g(x, \tilde{u}_n) \to g(x, u)$ in $L^1(\Omega)$, hence $\int_\Omega \tilde{u}_n (-\Delta)^s \varphi dx \to \int_\Omega u (-\Delta)^s \varphi dx$ and $\int_\Omega g(x, \tilde{u}_n) \varphi dx \to \int_\Omega g(x, u) \varphi dx$ for all $\varphi \in W(\mathbb{R}^N)$. Thus, it can be concluded that $u \in L^1(\Omega)$ is a very weak solution of

$$\begin{aligned} (-\Delta)^s u + g(x, u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{2.33}$$

We now drop the condition $f \geq 0$. Let z_n be a very weak solution of (2.33) corresponding to $|f|$. Then $z_n \geq 0$ and

$$\begin{aligned} &(-\Delta)^s u_n + g_n(x, u_n) = f \leq |f| = (-\Delta)^s z_n + g_n(x, z_n) \text{ in } \Omega, \\ &u_n = 0 \qquad\qquad z_n = 0 \text{ on } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

Hence, by the Lemma 2.3, we have $u_n \leq z_n$, where u_n is a weak solution. We have $g_n(x, z_n(x)) = -g_n(x, z_n(x))$ by definition of g_n and

$$\begin{aligned}(-\Delta)^s(-\tilde{u}_n) + g_n(x, -\tilde{u}_n) &= -|f| \text{ in } \Omega,\\ -(\tilde{u}_n) &= 0 \text{ on } \mathbb{R}^N \setminus \Omega.\end{aligned} \tag{2.34}$$

Since $-|f| \leq f$, we obtain by Lemma 2.3 that $-z_n \leq u_n$. Therefore, $|u_n| \leq z_n$. Furthermore, by (2.32) the sequence $(z_n)_n$ is bounded a.e. and hence $(u_n)_n$ is also bounded a.e. in Ω. This also implies that the sequence $(u_n)_n$ is bounded in $\tilde{W}^{s,2}(\Omega)$. If not, then by coercivity, we have $0 =< I'(u_n), u_n >\geq k||u_n||_{\tilde{W}^{s,2}(\Omega)}$ and hence $0 \geq 1$ which is a contradiction. Thus, $(u_n)_n$ is also bounded in $L^1(\Omega)$ and hence, by continuous embedding of $\tilde{W}^{s,2}(\Omega)$ in $L^1(\Omega)$, there exists a subsequence such that $u_n \to u$ in $L^1(\Omega)$. So, there exists a subsequence such that $u_n(x) \to u(x)$ a.e., thereby, implying $g(x, u_n) \to g(x, u)$ in $L^1(\Omega)$ and hence in $L^1(\Omega, \rho^s)$. Therefore, u is the unique very weak solution of (2.33) for $f \in L^\infty(\Omega)$. Now suppose $f \in L^1(\Omega, \rho^s)$, then, by approximation of f with smooth functions and using (2.12), we obtain existence of a unique very weak solution for every $f \in L^1(\Omega, \rho^s)$. □

3. Semilinear problem with measure data

In this section, we prove the main result of this paper, which is as follows.

Theorem 3.1 *Assume that* $(\mu_n) \subset \mathfrak{M}^g(\bar{\Omega})$ *and* $\rho^s\mu_n \underset{\bar{\Omega}}{\rightharpoonup} \tau$. *For each* n, *let* u_n *be the corresponding solution of* (1.1) *for* $\mu = \mu_n$ *and suppose that*

$$u_n \to u \text{ in } L^1(\Omega).$$

Then,
(i) $\rho^s g(x, u_n)_n$ *converges weakly in* $\bar{\Omega}$ *and*
(ii) there exists $\mu^\# \in \mathfrak{M}(\Omega, \rho^s)$, *such that* u *is a very weak solution of*

$$\begin{aligned}(-\Delta)^s u + g(x, u) &= \mu^\# \text{ in } \Omega\\ u &= 0 \text{ on } \mathbb{R}^N \setminus \Omega.\end{aligned} \tag{3.1}$$

Furthermore, if $\mu_n \geq 0$ *for every* n, *then* $\mu^\# \geq 0$.

The measure $\mu^\#$ is called the reduced limit of the sequences of measures $(\mu_n)_n$. Before the proof of Theorem 3.1, we consider some auxilliary results.

Lemma 3.2 *Consider the boundary value problem* (1.1) *with* $g \in \mathscr{G}_0$, $\mu \in \mathfrak{M}(\Omega, \rho^s)$. *If* $u_i \in L^1(\Omega)$ *are very weak solutions corresponding to* $\mu \equiv \mu_i$ *for* $i = \{1, 2\}$, *then we have the following estimate*

$$||u_1 - u_2||_{L^1(\Omega)} + ||g(x, u_1) - g(x, u_2)||_{L^1(\Omega, \rho^s)} \leq C||\mu_1 - \mu_2||_{\mathfrak{M}(\Omega, \rho^s)} \tag{3.2}$$

Furthermore, if $\mu_1 \leq \mu_2$, *then* $u_1 \leq u_2$.

Proof The proof runs along the same lines as that of the corresponding Lemmas 2.2, 2.3 and Theorem 2.4 in the previous section. $\square$

The following result is an immediate consequence of the definition of a good measure and Lemma 3.2.

Corollary 3.3 *Assume that $\mu \in \mathfrak{M}^g(\bar{\Omega})$. Then the boundary value problem (1.1) possesses a unique weak solution in $L^1(\Omega)$.*

We now give an example to show that if $(\mu_n)_n \subset \mathfrak{M}^g(\Omega)$ and if the very weak solution corresponding to each μ_n be denoted by $u_n \in L^1(\Omega)$ of the problem (1.1) such that the sequence of measures converge in a weak sense in Ω to μ while $(u_n)_n$ converges to u in $L^1(\Omega)$. Then, in general, u is not a very weak solution to the boundary value problem (1.1) with data μ.

Example 3.4 *Consider the problem for $p = 3/2$ and $N > 6$.*

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}u + |u|^{\frac{1}{2}}u &= 1 \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{3.3}$$

We prove a theorem motivated by the Proposition 1 of [5]

Theorem 3.5 *Given any measure $\mu \in \mathfrak{M}(\Omega, \rho^s)$, let u_n be the unique very weak solution of*

$$\begin{aligned} (-\Delta)^s u_n + g_n(x, u_n) &= \mu \text{ in } \Omega \\ u_n &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{3.4}$$

where $(g_n(x,t))_n$ is a sequence of nondecreasing, real valued, continuous functions in the variable t defined over $\mathbb{R}$ such that $g_n(x,0) = 0$ and $g_n(x,t) \to g(x,t)$ $\forall t \in \mathbb{R}$. Then $u_n \downarrow u^*$ in Ω as $n \uparrow \infty$, where u^* is the largest subsolution of

$$\begin{aligned} (-\Delta)^s u + g(x, u) &= \mu \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.5}$$

Proof From the estimate in (3.2), we have

$$||g_n(x, u_n)||_{L^1(\Omega,\rho^s)} \leq C||\mu||_{\mathfrak{M}(\Omega,\rho^s)}.$$

From the estimate (3.2), we also have

$$||u_n||_{L^1(\Omega)} \leq C||\mu||_{\mathfrak{M}(\Omega,\rho^s)}. \tag{3.6}$$

Therefore, $(u_n)_n$ weakly tends to, say, u^* in $L^1(\Omega)$. By Dini's theorem, $g_n \uparrow g$ uniformly on compact sets; thus, $g_n(x, u_n) \to g(x, u^*)$ a.e. in Ω. Thus, we have $g(x, u^*) \in L^1(\Omega, \rho^s)$ and from (3.6) we have $||u^*||_{L^1(\Omega)} \leq C||\mu||_{\mathfrak{M}(\Omega,\rho^s)}$.

By Fatou's lemma, we have

$$\begin{aligned}\int_\Omega u^*(-\Delta)^s\varphi+\int_\Omega g(x,u^*)\varphi &\leq \liminf_{n\to\infty}\int_\Omega u_n(-\Delta)^s\varphi+\lim_{n\to\infty}\int_\Omega g_n(x,u_n)\varphi\\ &= \liminf_{n\to\infty}\int_\Omega u_n(-\Delta)^s\varphi+\liminf_{n\to\infty}\int_\Omega g_n(x,u_n)\varphi\\ &= \int_\Omega \varphi d\mu \ \forall\ \varphi\in W(\mathbb{R}^N) \text{ such that } \varphi\geq 0. \end{aligned} \tag{3.7}$$

Hence, u^* is a subsolution of (3.5). We now claim that u^* is the largest subsolution to (3.5). For this, we consider any subsolution v of (3.5). Then, by the nondecreasing nature of the sequence $(g_n)_n$, we have

$$\begin{aligned}(-\Delta)^s v+g_n(x,v) &\leq (-\Delta)^s v+g(x,v)\\ &\leq \mu\\ &= (-\Delta)^s u_n+g_n(x,u_n) \text{ in the very weak sense.}\end{aligned} \tag{3.8}$$

Thus, by comparison, we have $v\leq u_n$ a.e. in Ω and as $n\to\infty$ $v\leq u^*$. □

Now coming to our example (3.3), it is evident that if we have a Dirac measure, then the largest subsolution $u^*=0$. So, we choose μ_k as a linear combination of Dirac measures, i.e. $\mu_k=k^{-1}|\Omega|\sum_{i=1}^k\delta_{a_i}$, where a_i are uniformly distributed points in Ω. It is easy to see that $h_{n,k}=\rho_n*\mu_k\overset{*}{\rightharpoonup}1$ in $\mathfrak{M}(\Omega,\rho^s)$. This is because

$$\begin{aligned}\left|\int_\Omega\varphi\rho^s[\int_\Omega\rho_n(x-y)\mu_k(y)dy-1]dx\right| &\leq ||\rho^s\varphi||_\infty\left(\int_\Omega k^{-1}|\Omega|\sum_{i=1}^k\rho_n(x-a_i)-|\Omega|\right)\\ &= 0\end{aligned}$$

for every $\varphi\in C(\bar{\Omega})$. To each $h_{n,k}$ let the solution corresponding to it be denoted by $u_{n,k}$. Then by Theorem 3.5, $u_{n,k}\to 0$ in $L^1(\Omega)$. Hence, for each k, choose $N_k>k$ sufficiently large so that $||u_{n,k}||_1<1/k$. So, we see that $h_{N_k,k}$ converges to 1 in a weak sense in Ω but $u_{N_k,k}\to 0$ and certainly 0 is not a solution to the problem in (3.3).

We will now show the existence of a reduced limit to a sequence of measures, under some conditions. We begin with the following result.

Lemma 3.6 *Let $(\mu_n)_n$ be as in Definition 1.1 and assume that $\rho^s\mu_n\underset{\bar{\Omega}}{\rightharpoonup}\tau$. Then*

$$\lim_{n\to\infty}\int_\Omega\varphi d\mu_n=\int_\Omega\varphi d\mu_{int}$$

for all $\varphi\in W(\mathbb{R}^N)$.

Proof Take $\bar{\varphi}(x) = \begin{cases} \frac{\varphi}{\rho^s}, & x \in \Omega, \\ 0, & x \in \partial\Omega \end{cases}$

Then $\bar{\varphi} \in C(\bar{\Omega})$ and using remark 1.5, we have,

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega \varphi d\mu_n &= \lim_{n\to\infty} \int_\Omega \rho^s \bar{\varphi} d\mu_n \\
&= \int_{\bar{\Omega}} \bar{\varphi} d\tau \quad [\because \rho^s \mu_n \underset{\bar{\Omega}}{\rightharpoonup} \tau] \\
&= \int_{\bar{\Omega}} \bar{\varphi} \chi_{\partial\Omega} d\tau + \int_{\bar{\Omega}} \rho^s \bar{\varphi} d\mu_{int} \quad [\because \tau = \rho^s \mu_{int} + \tau \chi_{\partial\Omega}] \\
&= \int_{\partial\Omega} \bar{\varphi} d\tau + \int_\Omega \rho^s \bar{\varphi} d\mu_{int} \\
&= \int_\Omega \varphi d\mu_{int}
\end{aligned}$$

Hence the lemma. □

Now, consider the sequence $(\mu_n)_n \in \mathfrak{M}^g(\Omega)$ and the corresponding problem

$$\begin{aligned} (-\Delta)^s u_n + g(x, u_n) &= \mu_n \text{ in } \Omega, \\ u &= 0 \text{ on } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{3.9}$$

Theorem 3.7 *Assume that* $g \in \mathscr{G}_0$ *and* $(\mu_n)_n \in \mathfrak{M}^g(\Omega)$ *is a sequence of measures such that* $\rho^s \mu_n \underset{\bar{\Omega}}{\rightharpoonup} \mu$. *Let* $u_n \to u$ *in* $L^1(\Omega)$ *where* u_n *is a very weak solution of the problem (3.9). Then there exists a measure* $\mu^\# \in \mathfrak{M}^g(\Omega)$ *such that* u *is a very weak solution of*

$$\begin{aligned} (-\Delta)^s u + g(x, u) &= \mu^\# \textit{ in } \Omega, \\ u &= 0 \textit{ on } \mathbb{R}^N \setminus \Omega \end{aligned} \tag{3.10}$$

Proof By (3.2), we have

$$||g(x, u_n)||_{L^1(\Omega,\rho^s)} \leq C||\mu_n||_{\mathfrak{M}(\Omega,\rho^s)}$$

for some $C > 0$. Since $\rho^s \mu_n \underset{\bar{\Omega}}{\rightharpoonup} \mu$, the sequence $(\rho^s g(x, u_n))_n$ is also uniformly bounded in $L^1(\Omega)$. Since $u_n \to u$ in $L^1(\Omega)$, $\exists$ a subsequence such that $u_n \to u$ pointwise a.e. in Ω. Thus, by a consequence of the Egorov's theorem, we have $\int_\Omega \rho^s g(x, u_n))_n \varphi dx \to \int_\Omega \rho^s g(x, u))\varphi dx$ in that subsequence for each $\varphi \in W(\mathbb{R}^N)$ and hence

$$\rho^s g(x, u_n) \underset{\bar{\Omega}}{\rightharpoonup} \lambda \; .$$

This implies that $g(x, u_n) \underset{\Omega}{\rightharpoonup} \frac{\lambda}{\rho^s} \chi_\Omega = \tilde{\lambda}$. Therefore, by the lemma 3.6,

$$\int_\Omega g(x, u_n) \varphi dx = \int_\Omega \varphi d\tilde{\lambda}$$

for all $\varphi \in W(\mathbb{R}^N)$. Since u_n is a very weak solution of the problem (3.9), we have

$$\int_\Omega u_n(-\Delta)^s\varphi dx + \int_\Omega g(x,u_n)\varphi dx = \int_\Omega \varphi d\mu_n$$

for all $\varphi \in W(\mathbb{R}^N)$. Passing to the limit $n \to \infty$ we have,

$$\int_\Omega u(-\Delta)^s\varphi + \int_\Omega \varphi d\tilde{\lambda} = \int_\Omega \varphi d\mu$$

for all $\varphi \in W(\mathbb{R}^N)$. Now this equation can be written as

$$\int_\Omega u(-\Delta)^s\varphi + \int_\Omega g(x,u)\varphi dx = \int_\Omega g(x,u)\varphi dx - \int_\Omega \varphi d\tilde{\lambda} + \int_\Omega \varphi d\mu$$

for all $\varphi \in W(\mathbb{R}^N)$. This shows that u is a very weak solution of the problem (3.10), with $\mu^\# = g(x,u) - \frac{\lambda}{\rho^s}\chi_\Omega + \mu$ and hence $\mu^\# \in \mathfrak{M}^g(\Omega)$. $\square$

4. Conclusion

The nonlocal, semilinear, elliptic, boundary value problem involving a Radon measures has been studied. The existence of very weak solution may in general fail for a general measure data input. However, we proved that the boundary value problem considered here with L^1 data possesses a unique very weak solution. We studied the reduced limits of the sequences (μ_n) of measures for a nonlocal operator $(-\Delta)^s$ with a nonlinearity. Our main result (Theorem 3.1) is obtained for any $s \in (0,1)$, which extend the Theorem 4.1 of Bhakta & Marcus [3] for the case when $s = 1$ and the boudary measure $\nu = 0$ on $\partial\Omega$. Since the fractional laplacian $(-\Delta)^s$ is nonlocal in nature, it is not possible to assume a nonzero boundary measure ν to the problem (1.1) on the boundary $\partial\Omega$ of the domain.

Acknowldgment

The author (DC) thanks for the facilities received from the Department of Mathematics, National Institute of Technology Rourkela and the grant received from the Council of Scientific and Industrial Research vide grant number 25(0292)/18/EMR-II. The author (RKG) thanks for the facilities received from the the Mathematics Department, Technion-Israel Institute of Technology, Haifa.

References

[1] Bénilan Ph, Brezi H. Nonlinear problems related to the Thomas-Fermi equation. Journal of Evolution Equations 2004; 673-770.

[2] Bernard C. Regularity of solutions to the fractional Laplace equation, 2014.

[3] Bhakta M, Marcus M. Reduced limit for semilinear boundary value problems with measure date. Journal of Differential Equations 2014; 256: 2691-2710.

[4] Brezis H. Nonlinear elliptic equations involving measures. in: Contribution to Nonlinear Partial Differential Equations Bardos, C, Vol II, Pitman Adv. Pub. Program 1983; 82-89.

[5] Brezis H, Marcus M, Ponce A. Nonlinear elliptic equations with measures revisited. in: J. Bourgain, C. Kenig, S. Klainermann, Mathematical Aspects of Nonlinear Dispersive Equations, in: Annals of Mathematics Studies, 163, Princeton University Press, Princeton, NJ, 2007; 55-110.

[6] Brezis H, Strauss W.A. Semilinear second-order elliptic equations in L^1. Journal of the Mathematical Society of Japan 1973; 25: 565-590.

[7] Cabré X, Sire Y. Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates. Annales de l'Institut Henri Poincaré C, Analyse non linéaire 2014; 31(1): 23-53.

[8] Cabré X, Sire Y. Nonlinear equations for fractional Laplacians, II: Existence, uniqueness, and qualitative properties of solutions. Transactions of the American Mathematical Society 2015; 367: 911-941.

[9] Caffarelli L, Silvestre L. An extension problem related to the fractional laplacian. Communications in Partial Differential Equations 2007; 32: 1245-1260.

[10] Chen W, Li C, Ou B. Classification of solutions for an integral equation. Communications on Pure and Applied Mathematics 2006; 59: 330-343.

[11] Chen H, Véron L. Semilinear fractional elliptic equations involving measures. Journal of Differential Equations 2014; 257: 1457-1486.

[12] Nezza E Di, Palatucci G, Valdinoci E. Hitchhiker's guide to the fractional Sobolev spaces. Université de Nimes, Equipe MIPA, 2011.

[13] Fang Y, Tang De. Method of sub-super solutions for fractional elliptic equations. Discrete & Continuous Dynamical Systems - Series B 2018; 23: 3153-3165.

[14] García- Melián J, sabina de Lis J. Maximum and Comparison principle for Operators Involving the p-Laplacian. Joural of Mathematical analysis and applications 1998; 218: 49-65.

[15] Giri R Kr, Choudhuri D. A study of second order semilinear elliptic PDE involving measures. Filomat 2019; 33(8): 2489-2506.

[16] Guan Qing-Yang, Integration by parts formula for regional fractional Laplacian. Communications in Mathematical Physics 2006; 266: 289-329.

[17] Karisen K, Petitta F, Ulusoy S., A duality approach to the fractional laplacian with measure data. Publicacions Matemátiques 2011; 55: 151-161.

[18] Kozhevnikova L M. Renormalized solutions of elliptic equations with variable exponents and general measure data. Matematicheskii Sbornik 2020; 211: 83-122.

[19] Marcus M, Ponce A. Reduced limits for nonlinear equations with measures. Journal of Funcional Analalysis 2010; 258(7): 2316-2372.

[20] Marcus M, Véron L. Nonlinear Second Order Elliptic Equations Involving Measures. de Gruyter Ser. Nonlinear Analalysis and Applications 21, 2013.

[21] Ros-Oton X, Serra J. The Dirichlet problem for the fractional Laplacian: Regularity upto the boundary. Journal de Mathématiques Pures et Appliquées 2014; 101(3): 257-302.

[22] Roselli P, Sciunzi B. A Strong Comparison principle for the p-Laplacian, Proceedings of the American Mathematical Society 2007; 135(10), 3217-3224.

[23] Véron L. Elliptic Equations Involving Measures, Handbook of Partial Differential Equation: Stationary Partail Differential Equations, Edited by M. Chipot and P. Quittner Eds, Elsevier 2004, 1: 595-712.

[24] Willem M. Minimax Theorems. Progress in Nonlinear differential equations and their applications 24; 1996.