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Indecomposable vector bundles via monads on $\mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1}$

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Abstract: The existence of monads on products of projective spaces $\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ is nontrivial. In this paper, we construct monads over the polycyclic variety $\mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1}$, we prove that cohomology vector bundle associated to these monads is simple. We also construct a monad on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$. We also study the vector bundles associated to monads and prove stability and simplicity.

Key words: Vector bundles, maximal rank, monads

1. Introduction

One of the most interesting problems in algebraic geometry deals with the existence of indecomposable rank r vector bundles on algebraic varieties. The difficulty in constructing non-splitting vector bundles on algebraic varieties increases when the difference between the rank of the vector bundle and the dimension of the variety increases.

We have the famous Hartshorne's conjecture concerning the non-existence of indecomposable rank 2 vector bundles on n-dimensional projective spaces for $n \geq 7$. The conjecture has been one of the main motivations for a great activity in the study of low rank vector bundles on projective spaces. On 3-dimensional projective spaces, there are plenty of examples of indecomposable vector bundles of rank 2. But, in spite of many efforts, in the last thirty years, very few indecomposable rank two vector bundles on n-dimensional projective spaces, $n > 3$ are known. More precisely, in positive characteristic $p \neq 2$, there are no known indecomposable rank 2 vector bundles on $\mathbf{P}^n$ for $n > 4$.

In this paper, we will construct vector bundles of low rank via monads and analyze their properties. More precisely, we construct monads on a Cartesian product of projective spaces, $\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$. We relate these monads to those constructed by Okonek and Spindler[9] and Spindler and Trautmann[11] in construction of Schwarzenberger and special instanton bundles. We prove that the cohomology vector bundle E associated to the monad on $\mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1}$ is simple and, hence, indecomposable. Finally, we construct a monad on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$.

Monads appear in many contexts within algebraic geometry, and they are very useful in construction of vector bundles with prescribed invariants like rank, determinant and chern classes, see examples [5]. A lot has been done in this respect. Monads were first introduced by Horrocks[4] who showed that all vector bundles E

2010 *AMS Mathematics Subject Classification:* 14FO5, 14J60 on $\mathbf{P}^3$ could be obtained as the cohomology bundle of a monad of the following kind:

$$0 \longrightarrow \oplus_i \mathcal{O}_{\mathbf{P}^3}(a_i) \xrightarrow{A} \oplus_j \mathcal{O}_{\mathbf{P}^3}(b_j) \xrightarrow{B} \oplus_n \mathcal{O}_{\mathbf{P}^3}(c_n) \longrightarrow 0$$

where A and B are matrices whose entries are homogeneous polynomials of degrees $b_j - a_i$ and $c_n - b_j$, respectively for some integers i, j, n.

2. Preliminaries

Definition 2.1 *Let X be a non-singular irreducible projective variety of dimension d and let $\mathscr{L}$ be an ample line bundle on X. For a torsion-free sheaf F on X we define*

1. *the degree of F relative to $\mathscr{L}$ as $\deg_{\mathscr{L}} F := c_1(F) \cdot \mathscr{L}^{d-1}$, where $c_1(F)$ is the first chern class of F*
2. *the slope of F as $\mu_{\mathscr{L}}(F) := \frac{c_1(F)\mathscr{L}^{d-1}}{\mathrm{rank}(F)}$ and*
3. *the Hilbert polynomial of F as $P_F(m) := \chi(F \otimes \mathcal{O}_X(m\mathscr{L}))$.*

Definition 2.2 *Let X be an algebraic variety and let E be a torsion-free sheaf on X. Then E is $\mathscr{L}$-stable if every subsheaf $F \hookrightarrow E$ satisfies $\mu_{\mathscr{L}}(F) < \mu_{\mathscr{L}}(E)$, where $\mathscr{L}$ is an ample line bundle.*

2.1. Hoppe's criterion over cyclic varieties

Suppose that the picard group $\mathrm{Pic}(X) \simeq \mathbb{Z}$, such varieties are called cyclic. Given a locally free sheaf (or, equivalently, a holomorphic vector bundle) $E \to X$ of rank r, there is a unique integer k_E such that $-r + 1 \leq c_1(E(-k_E)) \leq 0$. Setting $E_{norm} := E(-k_E)$, we say E is normalized if $E = E_{norm}$. Then one has the following stability criterion:

Proposition 2.3 ([3], Lemma 2.6) *Let E be a* rank *r holomorphic vector bundle over a cyclic projective variety X. If $H^0((\bigwedge^q E)_{norm}) = 0$ for $1 \leq q \leq r-1$, then E is stable. If $H^0((\bigwedge^q E)_{norm}(-1)) = 0$ for $1 \leq q \leq r-1$, then E is semistable.*

2.2. Hoppe's criterion over polycyclic varieties

Suppose that the picard group $\mathrm{Pic}(X) \simeq \mathbb{Z}^l$ where $l \geq 2$ is an integer, then X is a polycyclic variety. Given a divisor B on X, we define $\delta_{\mathscr{L}}(B) := \deg_{\mathscr{L}} \mathcal{O}_X(B)$. Then one has the following stability criterion [5], Theorem 3:

Theorem 2.4 (Generalized Hoppe criterion) *Let $G \to X$ be a holomorphic vector bundle of rank $r \geq 2$ over a polycyclic variety X equiped with a polarisation $\mathscr{L}$ if*

$$H^0(X, (\wedge^s G) \otimes \mathcal{O}_X(B)) = 0$$

for all $B \in \mathrm{Pic}(X)$ and $s \in \{1, \ldots, r-1\}$ such that $\delta_{\mathscr{L}}(B) < -s\mu_{\mathscr{L}}(G)$ then G is stable and if $\delta_{\mathscr{L}}(B) \leq -s\mu_{\mathscr{L}}(G)$, then G is semi-stable.
Conversely, if G is (semi-)stable, then

$$H^0(X, G \otimes \mathcal{O}_X(B)) = 0$$

for all $B \in \mathrm{Pic}(X)$ and all $s \in \{1, \ldots, r-1\}$ such that $\delta_{\mathscr{L}}(B) < -s\mu_{\mathscr{L}}(G)$ or $\delta_{\mathscr{L}}(B) \leq -s\mu_{\mathscr{L}}(G)$.

2.3. Hoppe's criterion over $X = \mathbf{P}^n \times \mathbf{P}^m$

Suppose the ambient space is $X = \mathbf{P}^n \times \mathbf{P}^m$, then $\operatorname{Pic}(X) \simeq \mathbb{Z}^2$. We denote by a, b the generators of $\operatorname{Pic}(X)$. Denote by $\mathcal{O}_X(a,b) := {p_1}^*\mathcal{O}_{\mathbf{P}^n}(a) \otimes {p_2}^*\mathcal{O}_{\mathbf{P}^m}(b)$, where p_1 and p_2 are natural projections from X to $\mathbf{P}^n$ and $\mathbf{P}^m$, respectively. For any line bundle $\mathscr{L} = \mathcal{O}_X(a,b)$ on X and a vector bundle E, we will write $E(a,b) = E \otimes \mathcal{O}_X(a,b)$ and $(a,b) := a[h \times \mathbf{P}^n] + b[\mathbf{P}^n \times t]$ to represent its corresponding divisor. The normalization of E on X with respect to $\mathscr{L}$ is defined as follows: Set $d = \deg_{\mathscr{L}}(\mathcal{O}_X(1,0))$, since $\deg_{\mathscr{L}}(E(-k_E,0)) = \deg_{\mathscr{L}}(E) - 2k \cdot \operatorname{rank}(E)$ there's a unique integer $k_E := \lceil \mu_{\mathscr{L}}(E)/d \rceil$ such that $1 - d.\operatorname{rank}(E) \leq \deg_{\mathscr{L}}(E(-k_E,0)) \leq 0$. The twisted bundle $E_{\mathscr{L}-norm} := E(-k_E,0)$ is called the $\mathscr{L}$-normalization of E. Finally, we define the linear functional $\delta_{\mathscr{L}}$ on $\mathbb{Z}^2$ as $\delta_{\mathscr{L}}(p_1,p_2) := \deg_{\mathscr{L}} \mathcal{O}_X(p_1,p_2)$.

Proposition 2.5 ([6], Proposition 6) *Let X be a polycyclic variety with Picard number 2, let $\mathscr{L}$ be an ample line bundle and let E be a rank $r > 1$ holomorphic vector bundle over X. If $H^0((\bigwedge^q E)_{\mathscr{L}-norm}(p_1,p_2)) = 0$ for $1 \leq q \leq r-1$ and every $(p_1,p_2) \in \mathbb{Z}^2$ such that $\delta_{\mathscr{L}} \leq 0$, then E is $\mathscr{L}$-stable.*

Definition 2.6 *A vector bundle E is said to be*

1. *decomposable if it is isomorphic to a direct sum $E_1 \oplus E_2$ of two non-zero vector bundles, otherwise E is indecomposable.*

2. *simple if its only endomorphisms are the homotheties, i.e. $Hom(E,E) = k$, which is equivalent to $h^0(X, E \otimes E^*) = 1$.*

Proposition 2.7 *Let $0 \to E \to F \to G \to 0$ be an exact sequence of vector bundles.*
Then we have the following exact sequences involving exterior and symmetric powers:

1. $0 \longrightarrow \bigwedge^q E \longrightarrow \bigwedge^q F \longrightarrow \bigwedge^{q-1} F \otimes G \longrightarrow \cdots \longrightarrow F \otimes S^{q-1}G \longrightarrow S^qG \longrightarrow 0$

2. $0 \longrightarrow S^qE \longrightarrow S^{q-1}E \otimes F \longrightarrow \cdots \longrightarrow E \otimes \bigwedge^{q-1} F \longrightarrow \bigwedge^q F \longrightarrow \bigwedge^q G \longrightarrow 0$

Theorem 2.8 (Künneth formula) *Let X and Y be projective varieties over a field k. Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X and Y, respectively. Let $\mathscr{F} \boxtimes \mathscr{G}$ denote $p_1^*(\mathscr{F}) \otimes p_2^*(\mathscr{G})$*
then $H^m(X \times Y, \mathscr{F} \boxtimes \mathscr{G}) \cong \bigoplus_{p+q=m} H^p(X,\mathscr{F}) \otimes H^q(Y,\mathscr{G})$.

Since for our case we deal with Cartesian products of projective spaces, in particular, if $X = \mathbf{P}^n \times \mathbf{P}^m$, then $H^r(X, \mathcal{O}_X(c,d)) \cong \bigoplus_{p+q=r} H^p(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(c)) \otimes H^q(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(d))$ where p, q, r, c and d are integers.

Theorem 2.9 ([10], Theorem 4.1) *Let $n \geq 1$ be an integer and consider $d \in \mathbb{Z}$. We denote by S_d the space of homogeneous polynomials of degree in $n+1$ (conventionally if $d < 0$ then $S_d = 0$). The following statements hold:*

1. *$H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) = S_d$ for all d.*

2. $H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) = 0$ *for* $1 < i < n$ *and for all* d.

3. $H^n(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \cong H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-d-n-1))$.

Lemma 2.10 ([6], Lemma 9) *Let* $X = \mathbf{P}^n \times \mathbf{P}^m$, p_1 *and* p_2 *be integers and* k *a nonnegative integer. If* $p_1 + p_2 > 0$ *then* $h^p(X, \mathcal{O}_X(-p_1, -p_2)^{\oplus k}) = 0$ *where* $0 \le p < \dim(X) - 1$.

Lemma 2.11 ([6], Lemma 10) *Let* A *and* B *be vector bundles canonically pulled back from* A' *on* $\mathbf{P}^n$ *and* B' *on* $\mathbf{P}^m$, *then*

$$H^q(\bigwedge^s(A \otimes B)) = \sum_{k_1+\cdots+k_s=q} \{\bigoplus_{i=1}^{s}(\sum_{j=0}^{s}\sum_{m=0}^{k_i} H^m(\wedge^j(A)) \otimes (H^{k_i-m}(\wedge^{s-j}(B))))\}.$$

The lemma above depends on the following facts:

$$H^q(A_1 \oplus \cdots \oplus A_s) = \sum_{k_1+\cdots+k_s=q} \{\bigoplus_{i=1}^{s} H_i^k(A_i)\}.$$

$$H^q(A \otimes B) = \sum_{m=0}^{q} H^m(A) \otimes H^{q-m}(B).$$

$$\wedge^s(A \otimes B) = \sum_{j=0}^{s} \wedge^j(A) \otimes \wedge^{s-j}(B).$$

2.4. Background on monads

Definition 2.12 *Let* X *be a nonsingular projective variety. A monad on* X *is a complex of vector bundles:*

$$M_\bullet:\ 0 \longrightarrow M_0 \xrightarrow{\alpha} M_1 \xrightarrow{\beta} M_2 \longrightarrow 0$$

with α *injective and* β *surjective. Equivalently,* $M_\bullet$ *is a monad if* α *and* β *are of maximal rank and* $\beta \circ \alpha = 0$.

Definition 2.13 *A monad as defined above has a display diagram of short exact sequences as shown below:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_0 \arrow[r] \arrow[d, equal] & \ker\beta \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_0 \arrow[r, "\alpha"'] & M_1 \arrow[r] \arrow[d, "\beta"'] & \operatorname{coker}\alpha \arrow[r] \arrow[d] & 0 \\
 & & M_2 \arrow[r, equal] \arrow[d] & M_2 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The kernel of β, $\ker\beta$ and the cokernel of α, $\operatorname{coker}(\alpha)$ for the given monad are also vector bundles and the vector bundle $E = \ker(\beta)/\operatorname{im}(\alpha)$ and is called the cohomology bundle of the monad.

Definition 2.14 *[8] Let X be a nonsingular projective variety, let $\mathscr{L}$ be a very ample line sheaf, and V, W, U be finite dimensional k-vector spaces. A linear monad on X is a complex of sheaves,*

$$0 \longrightarrow V \otimes \mathscr{L}^{-1} \xrightarrow{\alpha} W \otimes \mathcal{O}_X \xrightarrow{\beta} U \otimes \mathscr{L} \longrightarrow 0$$

where $\alpha \in Hom(V,W) \otimes H^0\mathscr{L}$ is injective and $\beta \in Hom(W,U) \otimes H^0\mathscr{L}$ is surjective.

Definition 2.15 *[8] A torsion-free sheaf E on X is said to be a linear sheaf on X if it can be represented as the cohomology sheaf of a linear monad i.e. $E = \ker(\beta)/\operatorname{im}(\alpha)$, moreover $\operatorname{rank}(E) = w - u - v$, where $w = \dim W$, $v = \dim V$ and $u = \dim U$.*

3. Main results

The goal of this section is to actually construct different types of monads by varying the ambient space X, which is a smooth projective variety and the ample line bundle $\mathscr{L}$. More precisely, we construct monads on a Cartesian product of projective spaces, $\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$. Okonek and Spindler[9] and Spindler and Trautmann[11] constructed monads and proved stability of Schwarzenberger and special instanton bundles on $\mathbf{P}^{2n+1}$; here we construct monads on $\mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1}$. We prove that the cohomology vector bundle E associated to the monad on $\mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1}$ is simple, hence, indecomposable. Finally, we construct a monad on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$. We start by recalling the existence and classification of linear monads on $\mathbf{P}^n$ given by Fløystad in [2].

Lemma 3.1 ([2], Main Theorem) *Let $k \geq 1$. There exist linear monads on $\mathbf{P}^k$ of the form:*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^k}(-1)^{\oplus a} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^k}^{\oplus b} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^k}(1)^{\oplus c} \longrightarrow 0$$

if and only if at least one of the following is fulfilled:
(1)$b \geq 2c + k - 1$, $b \geq a + c$ and
(2)$b \geq a + c + k$

Lemma 3.2 ([7], Theorem 3.2) *Let $X = \mathbf{P}^n \times \mathbf{P}^m$ and let $\mathscr{L} = \mathcal{O}_X(\rho,\sigma)$ be an ample line bundle on X. Denote by $N = h^0(\mathcal{O}_X(\rho,\sigma)) - 1$. Let α, β, γ be positive integers such that at least one of the following conditions holds*
(1)$\beta \geq 2\gamma + N - 1$, and $\beta \geq \alpha + \gamma$,
(2)$\beta \geq \alpha + \gamma + N$.

Then, there exists a linear monad on X of the form

$$0 \longrightarrow \mathcal{O}_X(-\rho,-\sigma)^{\oplus\alpha} \xrightarrow[A]{} \mathcal{O}_X^{\oplus\beta} \xrightarrow[B]{} \mathcal{O}_X(\rho,\sigma)^{\oplus\gamma} \longrightarrow 0$$

Theorem 3.3 *Let $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ and let $\mathscr{L} = \mathcal{O}_X(1,\ldots,1)$ be an ample line bundle on X. Denote by $N = h^0(\mathcal{O}_X(a_1,\ldots,a_n)) - 1$. Let α, β, γ be positive integers such that at least one of the following conditions*

holds
(1)$\beta \geq 2\gamma + N - 1$, *and* $\beta \geq \alpha + \gamma$,
(2)$\beta \geq \alpha + \gamma + N$.

Then, there exists a linear monad on X *of the form*

$$0 \longrightarrow \mathcal{O}_X(-1,\dots,-1)^{\oplus\alpha} \xrightarrow[\overline{A}]{} \mathcal{O}_X^{\oplus\beta} \xrightarrow[\overline{B}]{} \mathcal{O}_X(1,\dots,1)^{\oplus\gamma} \longrightarrow 0$$

Proof For the ample line bundle $\mathscr{L} = \mathcal{O}_X(1,\dots,1)$, we have an embedding

$$i^* : X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n} \hookrightarrow \mathbf{P}(H^0(X, \mathcal{O}_X(1,\dots,1))) \cong \mathbf{P}^N$$

such that $i^*(\mathcal{O}_X(1)) \simeq \mathscr{L}$ and where $N = \prod(a_i+1) - 1$, $i = 1,\cdots,n$.

Suppose that one of the conditions is satisfied. Then, by Lemma 3.1, there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^N}^{\oplus\beta} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma} \longrightarrow 0$$

on $\mathbf{P}^N$.

Notice that

$$A \in \operatorname{Hom}_X(\mathscr{L}^{-1^{\oplus\alpha}}, \mathcal{O}_X^{\oplus\beta}) \text{ and } B \in \operatorname{Hom}(\mathcal{O}_{\mathbf{P}^N}^{\oplus\beta}, \mathcal{O}_{\mathbf{P}^N}^{\oplus\gamma}(1)) \cong \operatorname{Hom}_X(\mathcal{O}_X^{\oplus\beta}, \mathscr{L}^{\oplus\gamma}).$$

Thus, A and B induce a monad on X:

$$0 \longrightarrow \mathscr{L}^{-1^{\oplus\alpha}} \xrightarrow{\bar{A}} \mathcal{O}_X^{\oplus\beta} \xrightarrow{\bar{B}} \mathscr{L}^{\oplus\gamma} \longrightarrow 0$$

where

$$\bar{A} \in \operatorname{Hom}(\mathcal{O}_X(-1,\dots,-1)^{\oplus\alpha}, \mathcal{O}_X^{\oplus\beta}) \text{ and } \bar{B} \in \operatorname{Hom}(\mathcal{O}_X^{\oplus\beta}, \mathcal{O}_X(1,\dots,1)^{\oplus\gamma})$$

which proves what we want. □

In particular when $a_1 = a_2 = \cdots = a_n = 1$ the above Theorem guarantees the existence of the monad,

$$0 \longrightarrow \mathcal{O}_X(-1,\dots,-1)^{\oplus\alpha} \longrightarrow \mathcal{O}_X^{\oplus\beta} \longrightarrow \mathcal{O}_X(1,\dots,1)^{\oplus\gamma} \longrightarrow 0$$

Definition 3.4 *An instanton bundle on* $\mathbf{P}^{2n+1}$ *with quantum number* k *is a rank* $2n$ *vector bundle* E *on* $\mathbf{P}^{2n+1}$ *satisfying the following properties:*

1. $c_t(E) = \frac{1}{(1-t^2)^k}$,
2. E *has the natural cohomology in the range* $-2n-1 \leq j \leq 0$, *i.e. for every* j *in that range at most one of the cohomology groups* $H^q(\mathbf{P}^{2n+1}, E(j))$ *is non-zero,*
3. E *has trivial splitting type, i.e. for a general line* $l \subset \mathbf{P}^{2n+1}$ *we have* $E|_l \cong \mathcal{O}_l^{2n}$.

Proposition 3.5 ([7], Proposition 5.6) *Any instanton bundle E on $\mathbf{P}^{2n+1}$ with quantum number k is a linear vector bundle; more precisely E is the cohomology bundle of a linear monad*

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}^{2n+1}}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}^{2n+1}} \longrightarrow U \otimes \mathcal{O}_{\mathbf{P}^{2n+1}}(1) \longrightarrow 0$$

with $\dim V = \dim U = k$ *and* $\dim W = 2k + 2n$.
Conversely, a linear vector bundle arising as the cohomology bundle of a linear monad

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}^{2n+1}}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}^{2n+1}} \longrightarrow U \otimes \mathcal{O}_{\mathbf{P}^{2n+1}}(1) \longrightarrow 0$$

with $\dim V = \dim U = k$ *and* $\dim W = 2k+2n$ *is an instanton bundle provided that it has trivial splitting type.*

Definition 3.6 ([9], Proposition 4.2)

1. *A special instanton bundle on $\mathbf{P}^{2n+1}$ of quantum number k can be defined by an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \longrightarrow S^* \longrightarrow E \longrightarrow 0$$

2. *A Schwarzenberger bundle S of rank $2n+k$ is defined by a special exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \longrightarrow S \longrightarrow 0$$

Proposition 3.7 ([1], Proposition 2.11) *Let E be a rank $2n$ bundle on $\mathbf{P}^{2n+1}$ such that E has natural cohomology in the range $-2n-1 \leq q \leq 0$ and $c_t(E) = (1-t)^{-k}$. Then E is simple.*

Theorem 3.8 ([1], Theorem 2.8) *Let a, b be integers, $0 < a \leq b$. Let T, E be vector bundles on $\mathbf{P}^m$ defined by the sequences*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^m}(-1)^{\oplus b} \longrightarrow \mathcal{O}_{\mathbf{P}^m}^{\oplus m+a+b-1} \longrightarrow T \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^m}(-1)^{\oplus a} \longrightarrow T^* \longrightarrow E \longrightarrow 0$$

then T is stable and E is simple.

Theorem 3.9 *Let n and k be positive integers and A and B be morphisms of linear forms as in*

$$B := \left(\begin{array}{ccccc|ccccc} x_0 \cdots & x_n & & & & y_0 \cdots & y_n & & & \\ & & \ddots & \ddots & & & \ddots & \ddots & & \\ & & & x_0 \cdots x_n & & & & y_0 \cdots & y_n & \end{array}\right)$$

and

$$A := \left(\begin{array}{cccc} -y_0 \cdots & -y_n & & \\ & \ddots & \ddots & \\ & & -y_0 \cdots & -y_n \\ \hline x_0 \cdots & x_n & & \\ & \ddots & \ddots & \\ & & x_0 \cdots & x_n \end{array}\right)$$

then there exists a linear monad of the form

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0$$

Proof We know that for nonnegative integers a, b, c, N satisfying
(i) $b \geq 2c + N - 1$ and
(ii) $b \geq a + c$, there exists a monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus a} \xrightarrow[\alpha]{} \mathcal{O}_{\mathbf{P}^N}^{\oplus b} \xrightarrow[\beta]{} \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus c} \longrightarrow 0$$

on setting $a = c = k$, $b = 2n + 2k$ and $N = 2n + 1$ we see that
(I) $b = 2n + 2k$ and $2c + N - 1 = 2k + 2n + 1 - 1 = 2k + 2n = b$ so (i) is true and
(II) $a + c = 2k \leq 2k + 2n = b$ so (ii) is true and hence we get the expected monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0$$

□

Theorem 3.10 *The cohomology vector bundle associated to the monad*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0$$

is a special instanton bundle of quantum number k, has rank $2n$ and is simple.

Proof The display of the monad is

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} & \longrightarrow & T & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} & \xrightarrow[\alpha]{} & \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} & \longrightarrow & \operatorname{coker}(\alpha) & \longrightarrow & 0 \\
 & & & & {\scriptstyle\beta}\downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} & = & \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

On dualizing the exact sequence

$$0 \longrightarrow T \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0$$

we get

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \longrightarrow T^* \longrightarrow 0$$

setting the conditions of Theorem 3.8, $a = b = k$ and $m = 2n + 2k$, then it follows T^* is stable and so is T from which we deduce that E^* is simple and, hence, E is simple. □

Lemma 3.11 *Let T be a vector bundle on $X = \mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1}$ defined by the sequence*

$$0 \longrightarrow T \longrightarrow \mathcal{O}_X^{\oplus 2n+2k} \longrightarrow \mathcal{O}_X(1,1)^{\oplus k} \longrightarrow 0$$

, then T is stable.

Proof We show that $H^0(X, \bigwedge^q T(-p_1,-p_2)) = 0$ for all $p_1+p_2>0$ and $1 \le q \le \mathrm{rank}(T)$.

Consider the ample line bundle $\mathscr{L} = \mathcal{O}_X(1,1) = \mathcal{O}(L)$.
Its class in $\mathrm{Pic}(X) = \langle [a \times \mathbf{P}^{2n+1}], [\mathbf{P}^{2n+1} \times b] \rangle$ corresponds to the class $1 \cdot [a \times \mathbf{P}^{2n+1}] + 1 \cdot [\mathbf{P}^{2n+1} \times b]$, where a and b are hyperplanes of $\mathbf{P}^{2n+1}$ with the intersection product induced by $a^{2n+1} = 1 = b^{2n+1}$ and $a^{2n+2} = 0 = b^{2n+2}$.

Now from the display diagram of the monad, we get

$$\begin{aligned} c_1(T) &= c_1(\mathcal{O}_X^{2n+2k}) - c_1(\mathcal{O}_X(1,1)^k) \\ &= (2n+2k)(0,0) - k(1,1) \\ &= (-k,-k) \end{aligned}$$

Since $L^{4n+2} > 0$ hence , the degree of T is:

$$\begin{aligned} \deg_{\mathscr{L}} T &= -k([a \times \mathbf{P}^{2n+1}] + [\mathbf{P}^{2n+1} \times b]) \cdot (1 \cdot [a \times \mathbf{P}^{2n+1}] + 1 \cdot [\mathbf{P}^{2n+1} \times b])^{4n+1} \\ &= -kL^{4n+2} < 0. \end{aligned}$$

Since $\deg_{\mathscr{L}} T < 0$, then $(\bigwedge^q T)_{\mathscr{L}-norm} = (\bigwedge^q T)$, and it suffices by the generalized Hoppe criterion (Proposition 2.5), to prove that $h^0(\bigwedge^q T(-p_1,-p_2)) = 0$ with $p_1+p_2 \ge 0$ and for all $1 \le q \le \mathrm{rank}(T)-1$.

Next, we twist the exact sequence

$$0 \longrightarrow T \longrightarrow \mathcal{O}_X^{\oplus 2n+2k} \longrightarrow \mathcal{O}_X(1,1)^{\oplus k} \longrightarrow 0$$

by $\mathcal{O}_X(-p_1,-p_2)$ we get,

$$0 \longrightarrow T(-p_1,-p_2) \longrightarrow \mathcal{O}_X(-p_1,-p_2)^{\oplus 2n+2k} \longrightarrow \mathcal{O}_X(1-p_1,1-p_2)^{\oplus k} \longrightarrow 0$$

and taking the exterior powers of the sequence by Proposition 2.7, we get

$$0 \longrightarrow \bigwedge^q T(-p_1,-p_2) \longrightarrow \bigwedge^q(\mathcal{O}_X(-p_1,-p_2)^{\oplus 2n+2k}) \longrightarrow \bigwedge^{q-1}(\mathcal{O}_X(1-2p_1,1-2p_2)^{\oplus 2n+2k}) \longrightarrow \cdots$$

.
Taking cohomology, we have the injection:

$$0 \longrightarrow H^0(X, \bigwedge^q T(-p_1,-p_2)) \hookrightarrow H^0(X, \bigwedge^q(\mathcal{O}_X(-p_1,-p_2)^{\oplus 2n+2k}))$$

Set $\mathscr{G} = \mathcal{O}_X(-p_1,-p_2)^{2n+2k} = \mathcal{O}_X(-p_1,-p_2) \otimes \mathcal{O}_X^{\oplus 2n+2k}$ and using Lemma 2.11 $H^0(X, \bigwedge^q \mathscr{G})$ expands into $H^0(X, \sum_{j=0}^{q} \wedge^j \mathcal{O}_X(-p_1,-p_2) \otimes \mathcal{O}_X^{\oplus 2n+2k})$ and since $p_1 + p_2 > 0$ then

$$h^0(X, \bigwedge^q (\mathcal{O}_X(-p_1,-p_2)^{\oplus 2n+2k})) = h^0(X, \bigwedge^q T(-p_1,-p_2)) = 0$$

i.e. $h^0(\bigwedge^q T(-p_1,-p_2)) = 0$ and, thus, T is stable.

$\square$

Theorem 3.12 *Let $X = \mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1}$, then the cohomology vector bundle E associated to the monad*

$$0 \longrightarrow \mathcal{O}_X(-1,-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_X^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_X(1,1)^{\oplus k} \longrightarrow 0$$

of rank $2n$ is simple.

Proof The display of the monad is

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X(-1,-1)^{\oplus k} \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(-1,-1)^{\oplus k} \arrow[r, "\alpha"'] & \mathcal{O}_X^{\oplus 2n+2k} \arrow[r] \arrow[d, "\beta"'] & Q \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_X(1,1)^{\oplus k} \arrow[r, equal] \arrow[d] & \mathcal{O}_X(1,1)^{\oplus k} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since T is stable from Lemma 3.11, we prove that the cohomology vector bundle E with rank $2n$ is simple.

The first step is to take the dual short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1,-1)^{\oplus k} \longrightarrow T \longrightarrow E \longrightarrow 0$$

to get

$$0 \longrightarrow E^* \longrightarrow T^* \longrightarrow \mathcal{O}_X(1,1)^{\oplus k} \longrightarrow 0.$$

Tensoring by E we get

$$0 \longrightarrow E \otimes E^* \longrightarrow E \otimes T^* \longrightarrow E(1,1)^k \longrightarrow 0.$$

Now taking cohomology gives:

$$0 \longrightarrow H^0(X, E \otimes E^*) \longrightarrow H^0(X, E \otimes T^*) \longrightarrow H^0(E(1,1)^k) \longrightarrow \cdots$$

which implies that

$$h^0(X, E \otimes E^*) \leq h^0(X, E \otimes T^*) \tag{3.1}$$

Now, we dualize the short exact sequence

$$0 \longrightarrow T \longrightarrow \mathcal{O}_X^{\oplus 2n+2k} \longrightarrow \mathcal{O}_X(1,1)^{\oplus k} \longrightarrow 0$$

to get

$$0 \longrightarrow \mathcal{O}_X(-1,-1)^{\oplus k} \longrightarrow \mathcal{O}_X^{\oplus 2n+2k} \longrightarrow T^* \longrightarrow 0$$

For the sake of brevity we shall use the notation $\mathcal{O}_X^a$ in place of $\mathcal{O}_X^{\oplus a}$.
Now twisting by $\mathcal{O}_X(-1,-1)$ and taking cohomology, we get

$$\begin{aligned} 0 &\longrightarrow H^0(X, \mathcal{O}_X(-2,-2)^k) \longrightarrow H^0(X, \mathcal{O}_X(-1,-1)^{2n+2k}) \longrightarrow H^0(X, T^*(-1,-1)) \longrightarrow \\ &\longrightarrow H^1(X, \mathcal{O}_X(-2,-2)^k) \longrightarrow H^1(X, \mathcal{O}_X(-1,-1)^{2n+2k}) \longrightarrow H^1(X, T^*(-1,-1)) \longrightarrow \\ &\longrightarrow H^2(X, \mathcal{O}_X(-2,-2)^k) \longrightarrow H^2(X, \mathcal{O}_X(-1,-1)^{2n+2k}) \longrightarrow H^2(X, T^*(-1,-1)) \longrightarrow \cdots \end{aligned}$$

from which we deduce $H^0(X, T^*(-1,-1)) = 0$ and $H^1(X, T^*(-1,-1)) = 0$ from Theorems 2.8 and 2.9.

Lastly, tensor the short exact sequence

$$0 \longrightarrow \mathcal{O}(-1,-1)^{\oplus k} \longrightarrow T \longrightarrow E \longrightarrow 0$$

by T^* to get

$$0 \longrightarrow T^*(-1,-1)^k \longrightarrow T \otimes T^* \longrightarrow E \otimes T^* \longrightarrow 0$$

and taking cohomology, we have

$$\begin{aligned} 0 &\longrightarrow H^0(X, T^*(-1,-1)^k) \longrightarrow H^0(X, T \otimes T^*) \longrightarrow H^0(X, E \otimes T^*) \longrightarrow \\ &\longrightarrow H^1(X, T^*(-1,-1)^k) \longrightarrow \cdots \end{aligned}$$

But $H^1(X, T^*(-1,-1)^k = 0$ for $k > 1$ from above.

so we have

$$0 \longrightarrow H^0(X, T^*(-1,-1)^k) \longrightarrow H^0(X, T \otimes T^*) \longrightarrow H^0(X, E \otimes T^*) \longrightarrow 0$$

This implies that

$$h^0(X, T \otimes T^*) \leq h^0(X, E \otimes T^*) \tag{3.2}$$

Since T is stable, then it follows that it is simple, which implies $h^0(X, T \otimes T^*) = 1$.

From (1) and now (2) and putting these together, we have

$$1 \leq h^0(X, E \otimes E^*) \leq h^0(X, E \otimes T^*) = h^0(X, T \otimes T^*) = 1$$

We have $h^0(X, E \otimes E^*) = 1$, and, therefore, E is simple.

□

Construction 3.13 *Let $\psi : X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^{11}$ be the Segre embedding defined by*

$[x_0 : x_1][y_0 : y_1][z_0 : z_1 : z_2] \hookrightarrow [a_0 : a_1 : a_2 : a_3 : a_4 : a_5 : b_0 : b_1 : b_2 : b_3 : b_4 : b_5]$.

Then, by Lemma 3.1, there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{11}}(-1)^{\alpha} \xrightarrow{A} \mathcal{O}^{\beta}_{\mathbf{P}^{11}} \xrightarrow{B} \mathcal{O}_{\mathbf{P}^{11}}(1)^{\gamma} \longrightarrow 0$$

Suppose $\beta = 2\gamma + N - 1$, and $\beta = \alpha + \gamma$, that is equality on the first condition of Lemma 3.1.

Also let $\gamma = 2$, then since $N = 11$ then $\beta = 14$ and $\alpha = 12$.

Thus, we can construct the maps A and B that establish the monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{11}}(-1)^{12} \xrightarrow{A} \mathcal{O}^{14}_{\mathbf{P}^{11}} \xrightarrow{B} \mathcal{O}_{\mathbf{P}^{11}}(1)^{2} \longrightarrow 0$$

as follows:

$$B := \left(\begin{array}{ccccccc|ccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \end{array}\right)$$

a 2 by 14 matrix and

$$A := \left(\begin{array}{cccccc|cccccc} -b_0 & -b_1 & -b_2 & -b_3 & -b_4 & -b_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_0 & -b_1 & -b_2 & -b_3 & -b_4 & -b_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b_0 & -b_1 & -b_2 & -b_3 & -b_4 & -b_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_0 & -b_1 & -b_2 & -b_3 & -b_4 & -b_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_0 & -b_1 & -b_2 & -b_3 & -b_4 & -b_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_0 & -b_1 & -b_2 & -b_3 & -b_4 & -b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -b_0 & -b_1 & -b_2 & -b_3 & -b_4 & -b_5 \\ \hline a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \end{array}\right)$$

a 14 by 12 matrix.

Notice that $BA = 0$ and A and B are of maximal rank and, hence, the monad,

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{11}}(-1)^{12} \xrightarrow[A]{} \mathcal{O}^{14}_{\mathbf{P}^{11}} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{11}}(1)^{2} \longrightarrow 0$$

Now we induce the monad

$$0 \longrightarrow \mathcal{O}_X(-1,-1,-1)^{12} \xrightarrow[\overline{A}]{} \mathcal{O}^{14}_X \xrightarrow[\overline{B}]{} \mathcal{O}_X(1,1,1)^{2} \longrightarrow 0$$

We construct $\overline{A}$ and $\overline{B}$ from A and B from the Segre map using the table:

$A, B entries$	$\overline{A}, \overline{B}, entries$
a_0	$x_0y_0z_0$
a_1	$x_0y_0z_1$
a_2	$x_0y_0z_2$
a_3	$x_0y_1z_0$
a_4	$x_0y_1z_1$
a_5	$x_0y_1z_2$
b_0	$x_1y_0z_0$
b_1	$x_1y_0z_1$
b_2	$x_1y_0z_2$
b_3	$x_1y_1z_0$
b_4	$x_1y_1z_1$
b_5	$x_1y_1z_2$

Specifically, we define two matrices $\overline{A}$ and $\overline{B}$ as follows:

$$\overline{B} = \left(\begin{array}{c|c} B_1 & B_2 \end{array} \right)$$

and

$$\overline{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Where

$$B_1 := \begin{pmatrix} x_0y_0z_0 & x_0y_0z_1 & x_0y_0z_2 & x_0y_1z_0 & x_0y_1z_1 & x_0y_1z_2 & 0 \\ 0 & x_0y_0z_0 & x_0y_0z_1 & x_0y_0z_2 & x_0y_1z_0 & x_0y_1z_1 & x_0y_1z_2 \end{pmatrix}$$

and

$$B_2 := \begin{pmatrix} x_1y_0z_0 & x_1y_0z_1 & x_1y_0z_2 & x_1y_1z_0 & x_1y_1z_1 & x_1y_1z_2 & 0 \\ 0 & x_1y_0z_0 & x_1y_0z_1 & x_1y_0z_2 & x_1y_1z_0 & x_1y_1z_1 & x_1y_1z_2 \end{pmatrix}$$

Similarly, one can construct A_1 and A_2 as follows:

A_2 a 7×12 matrix

$$\begin{bmatrix} x_0y_0z_0 & x_0y_0z_1 & x_0y_0z_2 & x_0y_1z_0 & x_0y_1z_1 & x_0y_1z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0y_0z_0 & x_0y_0z_1 & x_0y_0z_2 & x_0y_1z_0 & x_0y_1z_1 & x_0y_1z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0y_0z_0 & x_0y_0z_1 & x_0y_0z_2 & x_0y_1z_0 & x_0y_1z_1 & x_0y_1z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0y_0z_0 & x_0y_0z_1 & x_0y_0z_2 & x_0y_1z_0 & x_0y_1z_1 & x_0y_1z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0y_0z_0 & x_0y_0z_1 & x_0y_0z_2 & x_0y_1z_0 & x_0y_1z_1 & x_0y_1z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0y_0z_0 & x_0y_0z_1 & x_0y_0z_2 & x_0y_1z_0 & x_0y_1z_1 & x_0y_1z_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0y_0z_0 & x_0y_0z_1 & x_0y_0z_2 & x_0y_1z_0 & x_0y_1z_1 & x_0y_1z_2 \end{bmatrix}$$

$-A_1$ is a 7×12 matrix constructed similarly that is:

$$\begin{bmatrix} x_1y_0z_0 & x_1y_0z_1 & x_1y_0z_2 & x_1y_1z_0 & x_1y_1z_1 & x_1y_1z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1y_0z_0 & x_1y_0z_1 & x_1y_0z_2 & x_1y_1z_0 & x_1y_1z_1 & x_1y_1z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1y_0z_0 & x_1y_0z_1 & x_1y_0z_2 & x_1y_1z_0 & x_1y_1z_1 & x_1y_1z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1y_0z_0 & x_1y_0z_1 & x_1y_0z_2 & x_1y_1z_0 & x_1y_1z_1 & x_1y_1z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1y_0z_0 & x_1y_0z_1 & x_1y_0z_2 & x_1y_1z_0 & x_1y_1z_1 & x_1y_1z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1y_0z_0 & x_1y_0z_1 & x_1y_0z_2 & x_1y_1z_0 & x_1y_1z_1 & x_1y_1z_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1y_0z_0 & x_1y_0z_1 & x_1y_0z_2 & x_1y_1z_0 & x_1y_1z_1 & x_1y_1z_2 \end{bmatrix}$$

We note that

1. $\overline{B} \cdot \overline{A} = 0$ *and*

2. *The matrices* $\overline{B}$ *and* $\overline{A}$ *have maximal rank*

Hence, we get the desired monad

$$0 \longrightarrow \mathcal{O}_X(-1,-1,-1)^{12} \xrightarrow[\overline{A}]{} \mathcal{O}_X^{14} \xrightarrow[\overline{B}]{} \mathcal{O}_X(1,1,1)^2 \longrightarrow 0$$

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