Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2021) 45: 1903 – 1913
© TÜBİTAK
doi:10.3906/mat-2102-42

A contiguous extension of Dixon's theorem for a terminating ${}_4F_3(1)$ series with applications

Mohammad Idris QURESHI[1], **Richard Bruce PARIS**[2], **Shakir Hussain MALIK**[1,*]
[1]Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi, India
[2]Division of Computing and Mathematics, Abertay University, Dundee, UK

Received: 09.02.2021 • **Accepted/Published Online:** 11.06.2021 • **Final Version:** 16.09.2021

Abstract: We derive a summation formula for the terminating hypergeometric series

$$ {}_4F_3\left[\begin{matrix} -m, a, b, 1+c \\ 1+a+m, 1+a-b, c \end{matrix}; 1\right], $$

where m denotes a nonnegative integer. Using this summation formula, we establish a reduction formula for the Srivastava–Daoust double hypergeometric function with arguments z and $-z$. Special cases of this reduction formula lead to several reduction formulas for the hypergeometric functions ${}_{p+1}F_p$ with quadratic arguments when $p = 2, 3$ and 4 by employing series rearrangement techniques. A general double series identity involving a bounded sequence of arbitrary complex numbers is also given.

Key words: Hypergeometric summation theorems, Srivastava–Daoust double hypergeometric function, bounded sequence, series rearrangement technique

1. Introduction

In our investigations, we shall use the following standard notation: $\mathbf{N} := \{1, 2, 3, \cdots\}$; $\mathbf{N}_0 := \mathbf{N} \bigcup \{0\}$; $\mathbf{Z}_0^- := \mathbf{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}$. The symbols $\mathbf{C}, \mathbf{R}, \mathbf{N}, \mathbf{Z}$ denote the sets of complex numbers, real numbers, natural numbers and integers, respectively. The well-known Pochhammer symbol (or the shifted factorial) is given by $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$, it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the gamma quotient exists.

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$ is accomplished by introducing an arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$ {}_pF_q\left[\begin{matrix} (\alpha_p) \\ (\beta_q) \end{matrix}; z\right] = {}_pF_q\left[\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n} \frac{z^n}{n!} \quad (1.1) $$

is known as the generalized hypergeometric function. Here p and q are nonnegative integers, the variable $z \in \mathbf{C}$ and we write $(\alpha_p) = (\alpha_1, \alpha_2, \ldots, \alpha_p)$. The numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator

*Correspondence: malikshakir774@gmail.com
2010 *AMS Mathematics Subject Classification:* Primary 33C05, Secondary 33C20

parameters $\beta_1, \beta_2, \ldots, \beta_q$ can, in general, take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \ldots, \quad (j = 1, 2, \ldots, q).$$

Assuming that none of the numerator and denominator parameters is zero or a negative integer, the ${}_pF_q(z)$ function defined by Equation (1.1) converges for $|z| < \infty$ $(p \leq q)$, $|z| < 1$ $(p = q+1)$ and $|z| = 1$ $(p = q+1$ and $\Re(s) > 0)$, where s is the parametric excess defined by

$$s = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j. \tag{1.2}$$

In an earlier paper [13, p.199], Srivastava and Daoust defined a generalization of the Kampé de Fériet function [2, p.150] by means of the double hypergeometric series (see also [14, 15]):

$$F^{A:\ B;\ B'}_{C:\ D;\ D'}\left[\begin{array}{c} [(\alpha_A):\vartheta,\varphi]:[(\beta_B):\psi];[(\beta'_{B'}):\psi']; \\ [(\gamma_C):\xi,\varepsilon]:[(\delta_D):\eta];[(\delta'_{D'}):\eta']; \end{array} x, y\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{A}(\alpha_j)_{m\vartheta_j+n\varphi_j} \prod_{j=1}^{B}(\beta_j)_{m\psi_j} \prod_{j=1}^{B'}(\beta'_j)_{n\psi'_j}}{\prod_{j=1}^{C}(\gamma_j)_{m\xi_j+n\varepsilon_j} \prod_{j=1}^{D}(\delta_j)_{m\eta_j} \prod_{j=1}^{D'}(\delta'_j)_{n\eta'_j}} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.3}$$

where the coefficients

$$\left\{ \begin{array}{l} \vartheta_1, \ldots, \vartheta_A;\ \varphi_1, \ldots, \varphi_A;\ \psi_1, \ldots, \psi_B;\ \psi'_1, \ldots, \psi'_{B'};\ \xi_1, \ldots, \xi_C; \\ \varepsilon_1, \ldots, \varepsilon_C;\ \eta_1, \ldots, \eta_D;\ \eta'_1, \ldots, \eta'_{D'} \end{array} \right.$$

are real and positive. The double power series in (1.3) converges for all complex values of x and y when $\Delta_1 > 0$, $\Delta_2 > 0$; for suitably constrained values of $|x|$ and $|y|$ when $\Delta_1 = \Delta_2 = 0$; and diverges (except in the trivial case $x = y = 0$) when $\Delta_1 < 0$, $\Delta_2 < 0$, where

$$\Delta_1 = 1 + \sum_{j=1}^{C} \xi_j + \sum_{j=1}^{D} \eta_j - \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{B} \psi_j,$$

$$\Delta_2 = 1 + \sum_{j=1}^{C} \varepsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^{A} \varphi_j - \sum_{j=1}^{B'} \psi'_j.$$

Motivated by the studies of Miller [4, 5], Miller and Paris [6–8], Miller and Srivastava [9], we obtain a summation formula for a terminating series ${}_4F_3(1)$ in Section 3. In Section 4, this summation formula is used to derive a reduction formula for the Srivastava–Daoust double hypergeometric function defined in (1.3) with arguments z and $-z$. The consideration of special cases of this last result enables a few reduction formulas for the generalised hypergeometric function ${}_{p+1}F_p$ $(p = 2, 3, 4)$ with quadratic arguments to be deduced using a series rearrangement technique. In the final section, we specify a general double-series identity involving a bounded sequence of complex numbers.

It should be observed that throughout we tacitly exclude any values of the parameters and arguments in Sections 3 to 5 leading to results that do not make sense.

2. Preliminaries

In this section we present some preliminary results necessary for our investigation. First, we state Cauchy's double series identity [11, p. 56], [16, p. 100]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Theta(m,n)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\Theta(m-n,n)\ , \tag{2.1}$$

provided that the associated double series are absolutely convergent.

Our second result is Dixon's theorem [10, p. 535, Entry 21]:

$${}_3F_2\left[\begin{matrix}a,\ \ b,\ \ c\\ 1+a-b,1+a-c\end{matrix};1\right]=\frac{\Gamma(1+a-b)\ \Gamma(1+a-c)\ \Gamma(1+\frac{1}{2}a)\ \Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\ \Gamma(1+a)\ \Gamma(1+a-b-c)}, \tag{2.2}$$

where $\Re(a-2b-2c)>-2$ and $1+a-b,\ 1+a-c\in\mathbf{C}\backslash\mathbf{Z}_0^-$. When $c=-m$ in (2.2), the terminating form of Dixon's theorem is given by

$${}_3F_2\left[\begin{matrix}-m,\ \ a,\ \ b\\ 1+a+m,1+a-b\end{matrix};1\right]=\frac{(1+a)_m\ (1+\frac{1}{2}a-b)_m}{(1+a-b)_m\ (1+\frac{1}{2}a)_m}, \tag{2.3}$$

where $a,\ b,\ 1+a-b\in\mathbf{C}\backslash\mathbf{Z}_0^-$ and $m\in\mathbf{N}_0$.

The contiguous extension of Dixon's theorem is [10, p. 535, Entry 22] (see also [3, p. 13, Eq.(4.7)])

$${}_3F_2\left[\begin{matrix}a,\ \ b,\ \ c\\ 2+a-b,2+a-c\end{matrix};1\right]=\frac{\Gamma(2+a-b)\Gamma(2+a-c)}{2(b-1)(c-1)\Gamma(a)\Gamma(2+a-b-c)}\times$$

$$\times\left\{\frac{\Gamma(\frac{1}{2}a)\ \Gamma(2+\frac{1}{2}a-b-c)}{\Gamma(1+\frac{1}{2}a-b)\ \Gamma(1+\frac{1}{2}a-c)}-\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\ \Gamma(\frac{5}{2}+\frac{1}{2}a-b-c)}{\Gamma(\frac{3}{2}+\frac{1}{2}a-b)\ \Gamma(\frac{3}{2}+\frac{1}{2}a-c)}\right\}, \tag{2.4}$$

where $\Re(a-2b-2c)>-4$ and $2+a-b,\ 2+a-c\in\mathbf{C}\backslash\mathbf{Z}_0^-$ and $b\neq 1,\ c\neq 1$. When $c=-m$, the terminating contiguous form of (2.4) is given by

$${}_3F_2\left[\begin{matrix}-m,\ \ a,\ \ b\\ 2+a+m,2+a-b\end{matrix};1\right]=\frac{\Gamma(2+a-b)\Gamma(2+a+m)}{2(b-1)(-m-1)\Gamma(a)\Gamma(2+a-b+m)}\times$$

$$\times\left\{\frac{\Gamma(\frac{1}{2}a)\ \Gamma(2+\frac{1}{2}a-b+m)}{\Gamma(1+\frac{1}{2}a-b)\ \Gamma(1+\frac{1}{2}a+m)}-\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\ \Gamma(\frac{5}{2}+\frac{1}{2}a-b+m)}{\Gamma(\frac{3}{2}+\frac{1}{2}a-b)\ \Gamma(\frac{3}{2}+\frac{1}{2}a+m)}\right\}$$

$${}_3F_2\left[\begin{matrix}-m,\ \ a,\ \ b\\ 2+a+m,2+a-b\end{matrix};1\right]=\frac{\Gamma(2+a+m)\ \Gamma(2+a)}{2(1-b)(m+1)\ \Gamma(2+a)\ \Gamma(a)(2+a-b)_m}\times$$

$$\times\left\{\frac{\Gamma(\frac{1}{2}a)\ \Gamma(2+\frac{1}{2}a-b+m)\ \Gamma(1+\frac{1}{2}a)\ \Gamma(2+\frac{1}{2}a-b)}{\Gamma(1+\frac{1}{2}a-b)\ \Gamma(1+\frac{1}{2}a+m)\ \Gamma(1+\frac{1}{2}a)\ \Gamma(2+\frac{1}{2}a-b)}-\right.$$

$$\left.-\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\ \Gamma(\frac{5}{2}+\frac{1}{2}a-b+m)\ \Gamma(\frac{5}{2}+\frac{1}{2}a-b)\ \Gamma(\frac{3}{2}+\frac{1}{2}a)}{\Gamma(\frac{3}{2}+\frac{1}{2}a-b)\ \Gamma(\frac{3}{2}+\frac{1}{2}a+m)\ \Gamma(\frac{5}{2}+\frac{1}{2}a-b)\ \Gamma(\frac{3}{2}+\frac{1}{2}a)}\right\}$$

$${}_3F_2\left[\begin{matrix}-m,\ \ a,\ \ b\\ 2+a+m,2+a-b\end{matrix};1\right]=\frac{(2+a)_m\ (a)_2}{2(1-b)(m+1)(2+a-b)_m}\times$$

$$\times\left\{\frac{(2+\frac{1}{2}a-b)_m\ \Gamma(\frac{1}{2}a)\ \Gamma(1+\frac{1}{2}a-b+1)}{(1+\frac{1}{2}a)_m\ \Gamma(1+\frac{1}{2}a-b)\ \Gamma(1+\frac{1}{2}a)}-\frac{(\frac{5}{2}+\frac{1}{2}a-b)_m\ \Gamma(\frac{1}{2}a+\frac{1}{2})\ \Gamma(\frac{3}{2}+\frac{1}{2}a-b+1)}{(\frac{3}{2}+\frac{1}{2}a)_m\ \Gamma(\frac{3}{2}+\frac{1}{2}a-b)\ \Gamma(\frac{1}{2}+\frac{1}{2}a+1)}\right\}$$

Using the identity $\Gamma(z+1)=z\Gamma(z)$ (see [11]), after simplification we can write

$${}_3F_2\left[\begin{matrix}-m,\ \ a,\ \ b\\ 2+a+m,2+a-b\end{matrix};1\right]=\frac{a(1+a)\ (2+a)_m}{2(1-b)(m+1)(2+a-b)_m}\times$$

$$\times\left\{\frac{(a-2b+2)(2+\frac{1}{2}a-b)_m}{a(1+\frac{1}{2}a)_m}-\frac{(a-2b+3)(\frac{5}{2}+\frac{1}{2}a-b)_m}{(1+a)(\frac{3}{2}+\frac{1}{2}a)_m}\right\}, \tag{2.5}$$

where $a,\ b,\ 2+a-b\in\mathbf{C}\backslash\mathbf{Z}_0^-$, $b\neq 1$ and $m\in\mathbf{N}_0$. The summation formulas (2.3) and (2.5) will play an important role in our subsequent analysis.

We have the closed-form evaluations of the Gauss hypergeometric function (see [1, p. 185, Ex. (39)], [11, p. 70, Ex. (10)], [12, p.19, Eq.(1.5.20)]):

$${}_2F_1\left[\begin{matrix}\alpha,\alpha-\frac{1}{2}\\ 2\alpha\end{matrix};z\right]=\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\alpha-1}, \tag{2.6}$$

and

$${}_2F_1\left[\begin{matrix}\alpha,\alpha+\frac{1}{2}\\ 2\alpha\end{matrix};z\right]=\frac{1}{\sqrt{(1-z)}}\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\alpha-1}, \tag{2.7}$$

where $2\alpha\in\mathbf{C}\backslash\mathbf{Z}_0^-$ and $|\arg(1-z)|<\pi$. These last two results enable us to obtain the following lemma:

Lemma 2.1 *We have the closed-form evaluation of Clausen's function given by*

$${}_3F_2\left[\begin{matrix}\alpha+1,\beta,\beta-\frac{1}{2}\\ \alpha,2\beta\end{matrix};z\right]=\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\beta-1}\left[1+\frac{(2\beta-1)z}{2\alpha\{1-z+\sqrt{(1-z)}\}}\right], \tag{2.8}$$

where $\alpha,\ 2\beta\in\mathbf{C}\backslash\mathbf{Z}_0^-$ and $|\arg(1-z)|<\pi$.

Proof: We have

$${}_3F_2\left[\begin{matrix}\alpha+1,\beta,\beta-\frac{1}{2}\\ \alpha,2\beta\end{matrix};z\right]=\sum_{r=0}^{\infty}\frac{(\beta)_r\ (\beta-\frac{1}{2})_r\ z^r}{(2\beta)_r\ r!}\left(1+\frac{r}{\alpha}\right)$$

$$={}_2F_1\left[\begin{matrix}\beta,\beta-\frac{1}{2}\\ 2\beta\end{matrix};z\right]+\frac{(2\beta-1)z}{4\alpha}\ {}_2F_1\left[\begin{matrix}\beta+\frac{1}{2},\beta+1\\ 2\beta+1\end{matrix};z\right]. \tag{2.9}$$

Using the closed forms (2.6) and (2.7) in the right-hand side of (2.9), we obtain after some simplification the required result (2.8).

3. A summation formula

In this section, we derive a summation formula for a terminating ${}_4F_3$ series with positive unit argument, which we believe is not in the literature. This takes the following form:

Theorem 3.1 *The following result holds true:*

$$ {}_4F_3\left[\begin{matrix}-m, a, b, 1+c\\ 1+a+m, 1+a-b, c\end{matrix}; 1\right] = \frac{(1+a)_m}{(1+a-b)_m}\left\{\left(1-\frac{a}{2c}\right)\frac{(1+\frac{1}{2}a-b)_m}{(1+\frac{1}{2}a)_m} + \left(\frac{a}{2c}\right)\frac{(\frac{1}{2}+\frac{1}{2}a-b)_m}{(\frac{1}{2}+\frac{1}{2}a)_m}\right\}, \tag{3.1}$$

where $m \in \mathbf{N}_0$ *and* $a,\ b,\ c,\ 1+a-b \in \mathbf{C}\backslash\mathbf{Z}_0^-$.

Proof. Let

$$H := {}_4F_3\left[\begin{matrix}-m, a, b, 1+c\\ 1+a+m, 1+a-b, c\end{matrix}; 1\right] = \sum_{r=0}^{m}\frac{(-m)_r(a)_r(b)_r(1+c)_r}{(1+a+m)_r(1+a-b)_r(c)_r r!}$$

$$= \sum_{r=0}^{m}\frac{(-m)_r(a)_r(b)_r}{(1+a+m)_r(1+a-b)_r r!}\left(1+\frac{r}{c}\right)$$

$$= {}_3F_2\left[\begin{matrix}-m, a, b\\ 1+a+m, 1+a-b\end{matrix}; 1\right] + \frac{1}{c}\sum_{r=1}^{m}\frac{(-m)_r(a)_r(b)_r}{(1+a+m)_r(1+a-b)_r(r-1)!}. \tag{3.2}$$

Replacing r by $r+1$, we obtain the second term on the right-hand side of (3.2) in the form

$$\frac{1}{c}\sum_{r=0}^{m-1}\frac{(-m)_{r+1}(a)_{r+1}(b)_{r+1}}{(1+a+m)_{r+1}(1+a-b)_{r+1}r!} = -\frac{mab}{c(1+a+m)(1+a-b)}\sum_{r=0}^{m-1}\frac{(-m+1)_r(a+1)_r(b+1)_r}{(2+a+m)_r(2+a-b)_r r!}.$$

Identification of this last sum as the ${}_3F_2(1)$ series with parameters augmented by unity then leads to the result

$$H = {}_3F_2\left[\begin{matrix}-m, a, b\\ 1+a+m, 1+a-b\end{matrix}; 1\right] - \frac{mab}{c(1+a+m)(1+a-b)}{}_3F_2\left[\begin{matrix}-(m-1), a+1, b+1\\ 2+a+m, 2+a-b\end{matrix}; 1\right]. \tag{3.3}$$

Use of the results stated in (2.3) and (2.5) in the first and second hypergeometric series on the right-hand side of (3.3), then leads to

$$H = \frac{(1+a)_m(1+\frac{1}{2}a-b)_m}{(1+a-b)_m(1+\frac{1}{2}a)_m} + \frac{(a)_{m+2}}{2c(1+a+m)(1+a-b)_m}\times$$

$$\times\left\{\frac{(1+a-2b)(\frac{3}{2}+\frac{1}{2}a-b)_{m-1}}{(1+a)(\frac{3}{2}+\frac{1}{2}a)_{m-1}} - \frac{(2+a-2b)(2+\frac{1}{2}a-b)_{m-1}}{(2+a)(2+\frac{1}{2}a)_{m-1}}\right\}.$$

Finally, employing the fact that $(\alpha)_{m-1} = (\alpha-1)_m/(\alpha-1)$ and after some straightforward simplification, we obtain the required result (3.1).

Corollary 1. If we set $c = \frac{1}{2}a$ in (3.1) then we recover the known summation formula

$${}_4F_3\left[\begin{matrix}-m, a, b, 1+\frac{1}{2}a\\ \frac{1}{2}a, 1+a+m, 1+a-b\end{matrix}; 1\right] = \frac{(1+a)_m(\frac{1}{2}+\frac{1}{2}a-b)_m}{(\frac{1}{2}+\frac{1}{2}a)_m(1+a-b)_m} \tag{3.4}$$

recorded in [10, p. 556, Entry 29], [12, p. 245. III.26] and [1, p. 182, Ex. 25(a)].

Corollary 2. If we set $c = \frac{1}{2}b$ in (3.1) then we obtain the summation formula

$$
{}_4F_3\left[\begin{matrix}-m, a, b, 1+\frac{1}{2}b\\ \frac{1}{2}b, 1+a+m, 1+a-b\end{matrix};1\right]
$$

$$
= \frac{(1+a)_m}{(1+a-b)_m}\left\{\left(1-\frac{a}{b}\right)\frac{(1+\frac{1}{2}a-b)_m}{(1+\frac{1}{2}a)_m}+\left(\frac{a}{b}\right)\frac{(\frac{1}{2}+\frac{1}{2}a-b)_m}{(\frac{1}{2}+\frac{1}{2}a)_m}\right\}. \tag{3.5}
$$

4. An application of Theorem 3.1 to the Srivastava–Daoust function

Here we establish a result concerning the reducibility of the Srivastava–Daoust double hypergeometric function defined in (1.3) given in the following theorem:

Theorem 4.1 *The following result holds true:*

$$
F^{A+1:\ 0;\ 3}_{B+1:\ 0;\ 2}\left[\begin{matrix}[(a_A):1,1],[1+\alpha:1,1]:\text{—};[\alpha:1],[\beta:1],[1+\gamma:1];\\ [(b_B):1,1],[1+\alpha:1,2]:\text{—};[1+\alpha-\beta:1],[\gamma;1];\end{matrix}\ z,-z\right]
$$

$$
= \left(1-\frac{\alpha}{2\gamma}\right)\ {}_{A+2}F_{B+2}\left[\begin{matrix}(a_A),1+\alpha,1+\frac{1}{2}\alpha-\beta\\ (b_B),1+\alpha-\beta,1+\frac{1}{2}\alpha\end{matrix};z\right]
$$

$$
+\left(\frac{\alpha}{2\gamma}\right)\ {}_{A+2}F_{B+2}\left[\begin{matrix}(a_A),1+\alpha,\frac{1}{2}+\frac{1}{2}\alpha-\beta\\ (b_B),1+\alpha-\beta,\frac{1}{2}+\frac{1}{2}\alpha\end{matrix};z\right], \tag{4.1}
$$

where $b_1, b_2, \ldots, b_B, \alpha, \beta, \gamma, 1+\alpha-\beta \in \mathbf{C}\backslash\mathbf{Z}_0^-$. *When* $A \leq B$ *both sides of (4.1) are convergent for* $|z| < \infty$, *but when* $A = B+1$ *the two sides are convergent for suitably constrained values of* $|z|$.

Proof: Let

$$
F := F^{A+1:\ 0;\ 3}_{B+1:\ 0;\ 2}\left[\begin{matrix}[(a_A):1,1],[1+\alpha:1,1]:\text{—};[\alpha:1],[\beta:1],[1+\gamma:1];\\ [(b_B):1,1],[1+\alpha:1,2]:\text{—};[1+\alpha-\beta:1],[\gamma;1];\end{matrix}\ z,-z\right]
$$

$$
= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{m+n}\ldots(a_A)_{m+n}(1+\alpha)_{m+n}(\alpha)_n(\beta)_n(1+\gamma)_n z^m(-z)^n}{(b_1)_{m+n}\ldots(b_B)_{m+n}(1+\alpha)_{m+2n}(1+\alpha-\beta)_n(\gamma)_n m!n!}
$$

$$
= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{m+n}\ldots(a_A)_{m+n}(\alpha)_n(\beta)_n(1+\gamma)_n(-1)^n z^{m+n}}{(b_1)_{m+n}\ldots(b_B)_{m+n}(1+\alpha+m+n)_n(1+\alpha-\beta)_n(\gamma)_n m!n!}. \tag{4.2}
$$

Replacing m by $m-n$ in (4.2), we find upon application of (2.1) that

$$
\begin{aligned}
F &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(a_1)_m\ldots(a_A)_m(\alpha)_n(\beta)_n(1+\gamma)_n(-1)^n z^m}{(b_1)_m\ldots(b_B)_m(1+\alpha+m)_n(1+\alpha-\beta)_n(\gamma)_n(m-n)!n!}\\
&= \sum_{m=0}^{\infty}\frac{(a_1)_m\ldots(a_A)_m}{(b_1)_m\ldots(b_B)_m}\frac{z^m}{m!}\sum_{n=0}^{m}\frac{(-m)_n(\alpha)_n(\beta)_n(1+\gamma)_n}{(1+\alpha+m)_n(1+\alpha-\beta)_n(\gamma)_n n!}\\
&= \sum_{m=0}^{\infty}\frac{(a_1)_m\ldots(a_A)_m}{(b_1)_m\ldots(b_B)_m}\frac{z^m}{m!}\ {}_4F_3\left[\begin{matrix}-m,\alpha,\beta,1+\gamma\\ 1+\alpha+m,1+\alpha-\beta,\gamma\end{matrix};1\right].
\end{aligned}
$$

Finally, employing the summation formula (3.1), we arrive at the right-hand side of (4.1) after some routine simplification.

Corollary 3. If we take $\gamma = \frac{1}{2}\alpha$ in (4.1) we obtain another reduction formula:

$$F^{A+1:\ 0;\ 3}_{B+1:\ 0;\ 2}\left[\begin{matrix}[(a_A):1,1],[1+\alpha:1,1]:\text{—};[\alpha:1],[\beta:1],[1+\frac{1}{2}\alpha:1];\\ [(b_B):1,1],[1+\alpha:1,2]:\text{—};[1+\alpha-\beta:1],[\frac{1}{2}\alpha;1];\end{matrix}\ z,-z\right]$$

$$= {}_{A+2}F_{B+2}\left[\begin{matrix}(a_A),1+\alpha,\frac{1}{2}+\frac{1}{2}\alpha-\beta\\ (b_B),1+\alpha-\beta,\frac{1}{2}+\frac{1}{2}\alpha\end{matrix};z\right], \tag{4.3}$$

where $b_1, b_2, \ldots, b_B, \alpha, \beta, 1+\alpha-\beta \in \mathbf{C}\backslash\mathbf{Z}_0^-$. When $A \le B$ both sides of (4.3) converge for $|z| < \infty$, but when $A = B+1$ both sides converge for suitably constrained values of $|z|$.

In the following corollaries we present some cases where the Srivastava–Daoust function in (4.1) reduces to a generalised hypergeometric function with a quadratic argument which can be expressed in terms of lower-order hypergeometric functions with linear argument. At this point it will be convenient to introduce the variable

$$Z := \frac{z}{(1+\sqrt{1-z})^2} = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}.$$

Corollary 4. In (4.1) put $A = 2,\ B = 1,\ a_1 = \frac{1}{2}+\frac{1}{2}\alpha,\ a_2 = 1+\frac{1}{2}\alpha,\ b_1 = 1+\alpha$ to yield:

$$\frac{1}{\sqrt{1-z}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{\alpha} {}_3F_2\left[\begin{matrix}\alpha,\beta,1+\gamma\\ 1+\alpha-\beta,\gamma\end{matrix};-Z\right]$$

$$= \left(1-\frac{\alpha}{2\gamma}\right){}_2F_1\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}\alpha,1+\frac{1}{2}\alpha-\beta\\ 1+\alpha-\beta\end{matrix};z\right] + \left(\frac{\alpha}{2\gamma}\right){}_2F_1\left[\begin{matrix}1+\frac{1}{2}\alpha,\frac{1}{2}+\frac{1}{2}\alpha-\beta\\ 1+\alpha-\beta\end{matrix};z\right], \tag{4.4}$$

where $|Z| < 1,\ |z| < 1$ and $\alpha, \beta, \gamma, 1+\alpha-\beta \in \mathbf{C}\backslash\mathbf{Z}_0^-$.

Proof: With the stated parameter values the left-hand side of (4.1) takes the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}+\frac{1}{2}\alpha)_{m+n}(1+\frac{1}{2}\alpha)_{m+n}(\alpha)_n(\beta)_n(1+\gamma)_n z^m(-z)^n}{(1+\alpha)_{m+2n}(1+\alpha-\beta)_n(\gamma)_n m!n!}.$$

Using the identities $(a)_{m+n} = (a)_n(a+n)_m$ and $(a)_{2n} = 2^{2n}(\frac{1}{2}a)_n(\frac{1}{2}+\frac{1}{2}a)_n$ (see [11]), we can write the above double sum as

$$\sum_{n=0}^{\infty}\frac{(\frac{1}{2}+\frac{1}{2}\alpha)_n(1+\frac{1}{2}\alpha)_n(\alpha)_n(\beta)_n(1+\gamma)_n(-z)^n}{(1+\alpha)_{2n}(1+\alpha-\beta)_n(\gamma)_n n!}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}+\frac{1}{2}\alpha+n)_m(1+\frac{1}{2}\alpha+n)_m z^m}{(1+\alpha+2n)_m m!}$$

$$= \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(1+\gamma)_n(-z)^n}{2^{2n}(1+\alpha-\beta)_n(\gamma)_n n!}{}_2F_1\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}\alpha+n,1+\frac{1}{2}\alpha+n\\ 1+\alpha+2n\end{matrix};z\right]$$

$$= \frac{1}{\sqrt{1-z}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(1+\gamma)_n}{(1+\alpha-\beta)_n(\gamma)_n n!}\left(\frac{-z}{(1+\sqrt{1-z})^2}\right)^n$$

upon use of (2.7). Identification of the sum over n as a ${}_3F_2$ function and simplification of the right-hand side of (4.1) then yields the result stated in (4.4).

Corollary 5. In (4.1) put $A=2,\ B=1,\ a_1=\frac{1}{2}+\frac{1}{2}\alpha,\ a_2=\frac{1}{2}\alpha,\ b_1=1+\alpha$ to yield upon application of (2.6):

$$\left(\frac{2}{1+\sqrt{1-z}}\right)^{\alpha}{}_4F_3\left[\begin{matrix}\frac{1}{2}\alpha,\alpha,\beta,1+\gamma\\ 1+\frac{1}{2}\alpha,1+\alpha-\beta,\gamma\end{matrix};-Z\right]$$

$$=\left(1-\frac{\alpha}{2\gamma}\right){}_3F_2\left[\begin{matrix}\frac{1}{2}\alpha,\frac{1}{2}+\frac{1}{2}\alpha,1+\frac{1}{2}\alpha-\beta\\ 1+\frac{1}{2}\alpha,1+\alpha-\beta\end{matrix};z\right]+\left(\frac{\alpha}{2\gamma}\right){}_2F_1\left[\begin{matrix}\frac{1}{2}\alpha,\frac{1}{2}+\frac{1}{2}\alpha-\beta\\ 1+\alpha-\beta\end{matrix};z\right], \tag{4.5}$$

where $|Z|<1,\ |z|<1$ and $\alpha,\beta,\gamma,1+\frac{1}{2}\alpha,1+\alpha-\beta\in\mathbf{C}\backslash\mathbf{Z}_0^-$.

Corollary 6. In (4.1) put $A=3,\ B=2,\ a_1=2\alpha,\ a_2=\frac{1}{2}+\frac{1}{2}\alpha,\ a_3=\frac{1}{2}\alpha,\ b_1=2\alpha-1,\ b_2=1+\alpha$ to yield upon application of (2.8):

$$\left(\frac{2}{1+\sqrt{1-z}}\right)^{\alpha}\Bigg\{{}_5F_4\left[\begin{matrix}\frac{1}{2}\alpha,\alpha,2\alpha,\beta,1+\gamma\\ 2\alpha-1,1+\frac{1}{2}\alpha,1+\alpha-\beta,\gamma\end{matrix};-Z\right]$$

$$+\left(\frac{\alpha z}{2(2\alpha-1)(1-z+\sqrt{1-z})}\right){}_3F_2\left[\begin{matrix}\alpha,\beta,1+\gamma\\ 1+\alpha-\beta,\gamma\end{matrix};-Z\right]\Bigg\}$$

$$=\left(1-\frac{\alpha}{2\gamma}\right){}_4F_3\left[\begin{matrix}\frac{1}{2}\alpha,\frac{1}{2}+\frac{1}{2}\alpha,2\alpha,1+\frac{1}{2}\alpha-\beta\\ 2\alpha-1,1+\frac{1}{2}\alpha,1+\alpha-\beta\end{matrix};z\right]+\left(\frac{\alpha}{2\gamma}\right){}_3F_2\left[\begin{matrix}\frac{1}{2}\alpha,2\alpha,\frac{1}{2}+\frac{1}{2}\alpha-\beta\\ 2\alpha-1,1+\alpha-\beta\end{matrix};z\right], \tag{4.6}$$

where $|Z|<1,\ |z|<1$ and $\alpha,\beta,\gamma,2\alpha-1,1+\frac{1}{2}\alpha,1+\alpha-\beta\in\mathbf{C}\backslash\mathbf{Z}_0^-$.

The manipulation of the Srivastava–Daoust function in Corollaries 5 and 6 is similar to that in Corollary 4 and so will be omitted. Corollaries 4–6 have been derived on the assumption that $|Z|<1,\ |z|<1$. However, these results may be extended by analytic continuation to all $z\in\mathbf{C}$ such that $|\arg(1-z)|<\pi$ and $z\neq 1$ in (4.4), (4.6) and $z=1$ in (4.5) (since the parametric excess (see (1.2)) of the hypergeometric functions on the right-hand sides is $s=-\frac{1}{2}$ and $s=\frac{1}{2}$, respectively).

5. A second application of Theorem 3.1 to a general double series

Theorem 5.1. *Let $\{\Phi(p)\}_{p=1}^{\infty}$ be a bounded sequence of essentially arbitrary numbers (real or complex) such that $\Phi(0)\neq 0$. Then, the following general double-series identity holds true:*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m+n)\frac{(1+\alpha)_{m+n}\ (\alpha)_n\ (\beta)_n\ (1+\gamma)_n\ z^m\ (-z)^n}{(1+\alpha)_{m+2n}\ (1+\alpha-\beta)_n\ (\gamma)_n\ m!\ n!}$$

$$=\left(1-\frac{\alpha}{2\gamma}\right)\sum_{m=0}^{\infty}\Phi(m)\frac{(1+\alpha)_m\ (1+\frac{1}{2}\alpha-\beta)_m\ z^m}{(1+\alpha-\beta)_m\ (1+\frac{1}{2}\alpha)_m\ m!}+\left(\frac{\alpha}{2\gamma}\right)\sum_{m=0}^{\infty}\Phi(m)\frac{(1+\alpha)_m\ (\frac{1}{2}+\frac{1}{2}\alpha-\beta)_m\ z^m}{(1+\alpha-\beta)_m\ (\frac{1}{2}+\frac{1}{2}\alpha)_m\ m!}, \tag{5.1}$$

where $1+\alpha,\ \beta,\ \gamma,\ 1+\alpha-\beta\in\mathbf{C}\backslash\mathbf{Z}_0^-$ and provided that the infinite series occurring on both sides of (5.1) are absolutely convergent.

Proof : Let

$$G := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m+n)\frac{(1+\alpha)_{m+n}\ (\alpha)_n\ (\beta)_n\ (1+\gamma)_n\ z^m\ (-z)^n}{(1+\alpha)_{m+2n}\ (1+\alpha-\beta)_n\ (\gamma)_n\ m!\ n!}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m+n)\frac{(\alpha)_n\ (\beta)_n\ (1+\gamma)_n\ (-1)^n\ z^{m+n}}{(1+\alpha+m+n)_n\ (1+\alpha-\beta)_n\ (\gamma)_n\ m!\ n!}. \tag{5.2}$$

Replacing m by $m-n$ in (5.2) and making use of (2.1), we obtain

$$\begin{aligned} G &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\Phi(m)\frac{(\alpha)_n\ (\beta)_n\ (1+\gamma)_n\ (-1)^n\ (z)^m}{(1+\alpha+m)_n\ (1+\alpha-\beta)_n\ (\gamma)_n\ (m-n)!\ n!} \\ &= \sum_{m=0}^{\infty}\Phi(m)\frac{z^m}{m!}\sum_{n=0}^{m}\frac{(-m)_n\ (\alpha)_n\ (\beta)_n\ (1+\gamma)_n}{(1+\alpha+m)_n\ (1+\alpha-\beta)_n\ (\gamma)_n\ n!} \\ &= \sum_{m=0}^{\infty}\Phi(m)\frac{z^m}{m!}\ {}_4F_3\left[\begin{matrix}-m,\ \alpha,\ \beta,\ 1+\gamma \\ 1+\alpha+m,\ 1+\alpha-\beta,\ \gamma\end{matrix};1\right]. \end{aligned}$$

By using the summation formula (3.1), we obtain the required result (5.1).

Remark 1. All the results (2.3), (2.5), (2.8), (3.1), (3.4), (3.5), (4.4), (4.5) and (4.6) have been verified numerically by taking suitable values of the parameters and arguments given below:

Numerical proof of (2.3): Taking left-hand side of (2.3) and setting $m=3,\ a=\frac{5}{2},\ b=\frac{3}{2}$, we get

$${}_3F_2\left[\begin{matrix}-3,\ \frac{5}{2},\ \frac{3}{2} \\ \frac{13}{2},\ 2\end{matrix};1\right] = \sum_{r=0}^{3}\frac{(-3)_r(\frac{5}{2})_r(\frac{3}{2})_r}{(\frac{13}{2})_r(2)_r\ r!}$$

$$= 1+\frac{(-3)(\frac{5}{2})(\frac{3}{2})}{(\frac{13}{2})(2)}+\frac{(-3)_2(\frac{5}{2})_2(\frac{3}{2})_2}{(\frac{13}{2})_2(2)_2\ 2!}+\frac{(-3)_3(\frac{5}{2})_3(\frac{3}{2})_3}{(\frac{13}{2})_3(2)_3\ 3!}$$

$$= 1+\frac{(-3)(\frac{5}{2})(\frac{3}{2})}{(\frac{13}{2})(2)}+\frac{(-3)(-2)(\frac{5}{2})(\frac{7}{2})(\frac{3}{2})(\frac{5}{2})}{(\frac{13}{2})(\frac{15}{2})(2)(3)\ 2}+\frac{(-3)(-2)(-1)(\frac{5}{2})(\frac{7}{2})(\frac{9}{2})(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})}{(\frac{13}{2})(\frac{15}{2})(\frac{17}{2})(2)(3)(4)\ 6},$$

after simplification, we find

$${}_3F_2\left[\begin{matrix}-3,\ \frac{5}{2},\ \frac{3}{2} \\ \frac{13}{2},\ 2\end{matrix};1\right] = \frac{5929}{14144}.$$

Now taking right-hand side of (2.3) and setting $m=3,\ a=\frac{5}{2},\ b=\frac{3}{2}$, we get

$$\frac{(\frac{7}{2})_3(\frac{3}{4})_3}{(2)_3(\frac{9}{4})_3} = \frac{(\frac{7}{2})(\frac{9}{2})(\frac{11}{2})(\frac{3}{4})(\frac{7}{4})(\frac{11}{4})}{(2)(3)(4)(\frac{9}{4})(\frac{13}{4})(\frac{17}{4})} = \frac{5929}{14144}.$$

Hence L.H.S=R.H.S

Similarly, we can verify the remaining results numerically.

6. Conclusion

We conclude our present investigation by observing that several further interesting hypergeometric summation formulas for terminating series ${}_4F_3(1)$, reduction formulas for the Gaussian hypergeometric functions ${}_3F_2$, ${}_4F_3$ and ${}_5F_4$ with the argument $-Z$ and general double-series identity (which is the generalization of a reduction formula for Srivastava–Daoust double hypergeometric function with arguments z and $-z$) can be obtained in an analogous manner. Moreover, it is hoped that the results derived in this paper will find useful applications in a wide range of problems of mathematics, statistics and the physical sciences.

Acknowledgment

The authors are very thankful to the referee for their valuable suggestions and corrections to improve the paper in its present form.

References

[1] Andrews GE, Askey R, Roy R. Special Functions. Cambridge, UK: Cambridge University Press, 1999.

[2] Appell P, Kampé de Fériet J. Fonctions Hypergéométriques et Hypersphériques: Polynômes d'Hermite. Paris, France: Gauthier-Villars, 1926 (in French).

[3] Kim YS, Rakha MA, Rathie AK. Extensions of certain classical summation theorems for the series ${}_2F_1, {}_3F_2$ and ${}_4F_3$ with applications in Ramanujan's summations. International Journal of Mathematics and Mathematical Sciences 2010; 2010: 1-26. doi:10.1155/2010/309503

[4] Miller AR. A summation formula for Clausen's series ${}_3F_2(1)$ with an application to Goursat's function ${}_2F_2(x)$. Journal of Physics A: Mathematical and General 2005; 38: 3541-3545. doi: 10.1088/0305-4470/38/16/005

[5] Miller AR. Certain summation and transformation formulas for generalized hypergeometric series. Journal of Computational and Applied Mathematics 2009; 231: 964-972. doi: 10.1016/j.cam.2009.05.013

[6] Miller AR, Paris RB. Certain transformations and summations for generalized hypergeometric series with integral parameter differences. Integral Transforms and Special Functions 2011; 22 (1): 67-77. doi: 10.1080/10652469.2010.498001

[7] Miller AR, Paris RB. Euler-type transformations for the generalized hypergeometric function ${}_{r+2}F_{r+1}(x)$. Zeitschrift für Angewandte Mathematik und Physik 2011; 62: 31-45. doi: 10.1007/s00033-010-0085-0

[8] Miller AR, Paris RB. Transformation formulas for the generalized hypergeometric function with integral parameter differences. Rocky Mountain Journal of Mathematics 2013; 43 (1): 291-327. doi:10.1216/RMJ-2013-43-1-291

[9] Miller AR, Srivastava HM. Karlsson-Minton summation theorems for the generalized hypergeometric series of unit argument. Integral Transforms and Special Functions 2010; 21 (8): 603-612. doi: 10.1080/10652460903497259

[10] Prudnikov AP, Brychkov YuA, Marichev OI. Integrals and Series, Volume 3: More special functions. Moscow, Rusia: Nauka, 1986; Translated from the Russian by G. G. Gould, New York, NY, USA: Gordon and Breach Science Publishers, 1990.

[11] Rainville ED. Special Functions. New York, NY, USA: The Macmillan Company, 1960; Reprinted by Bronx, New York, NY, USA: Chelsea Publishing Company, 1971.

[12] Slater LJ. Generalized Hypergeometric Functions. Cambridge, UK: Cambridge University Press, 1966.

[13] Srivastava HM, Daoust MC. On Eulerian integrals associated with Kampé de Fériet's function. Publications De L'Institut Mathématique Nouvelle Série, tome 1969; 9 (23): 199-202.

[14] Srivastava HM, Daoust MC. Certain generalized Neumann expansions associated with the Kampé de Fériet's function. Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series A, 72-Indagationes Mathematicae 1969; 31: 449-457.

[15] Srivastava HM, Daoust MC. A note on the convergence of Kampé de Fériet's double hypergeometric series. Mathematische Nachrichten 1972; 53: 151-159. doi: 10.1002/mana.19720530114

[16] Srivastava HM, Manocha HL. A Treatise on Generating Functions. Chichester, UK: Ellis Horwood Limited Publisher, 1984.