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The kernel spaces and Fredholmness of truncated Toeplitz operators

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Abstract: In this paper, we study some conditions about invertible and Fredholm truncated Toeplitz operators which have unique symbols. For $f \in L^\infty$, if A_f is a Fredholm operator, then $f|_E \neq 0$ for any $E \subset \mathbb{T}$ with $|E| > 0$. Moreover ind $(A_f) = 0$. In particular, if A_f is invertible in $\mathfrak{L}(K_u^2)$, then f is invertible in L^∞. Besides, we give some results about the kernel spaces of truncated Toeplitz operators. For $f \in L^\infty$, we obtain the necessary and sufficient condition that the defect operator $I - A_f^* A_f$ of truncated Toeplitz operator A_f meeting some conditions is compact on the model space K_u^2.

Key words: Model spaces, truncated Toeplitz operators, invertible operators, Fredholm operators, defect operators

1. Introduction

A fundamental problem in the theory of linear operators is that the existence and uniqueness of the solution to the equation

$$Tx = a, \tag{1.1}$$

where T is a linear operator acting on a space H which contains x and a as elements. When H is a complex Hilbert space, the operator T is a linear operator acting on some domain $D(T)$ in H and having a range in H. It is obvious that the solution of (1.1) is unique if and only if the equation $Tx = 0$ has only the trivial solution $x = 0$. Further, if T has a closed range, then there exists a solution of (1.1) if and only if $\langle y, a\rangle = 0$, where a is any solution of $T^*y = 0$ and T^* denotes the adjoint of T. Moreover, if T is a Fredholm operator, then the solvability of Equation (1.1) for a given a is equivalent to determining whether a is orthogonal to the finite dimensional subspace ker T^*. Lastly, the space of the solutions of Equation (1.1) is finite dimensional. These results suggest the importance of investigating the Fredholm operators.

For a Hilbert space H, let $\mathfrak{L}(H)$ be the set of all bounded linear operators and $\mathfrak{LC}(H)$ be the set of all compact operators. We use ker T and ran T to denote the kernel space and range of T, respectively. The dimension of the set E is denoted by dim (E). We use clos $[E]$ to denote the closure of the set E.

Definition 1.1 *If H is a Hilbert space, then the quotient algebra $\mathfrak{L}(H)/\mathfrak{LC}(H)$ is a Banach algebra called the Calkin algebra. The natural homomorphism from $\mathfrak{L}(H)$ onto $\mathfrak{L}(H)/\mathfrak{LC}(H)$ is denoted by π. Then $T \in \mathfrak{L}(H)$ is*

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a Fredholm operator if $\pi(T)$ *is an invertible element of* $\mathfrak{L}(H)/\mathfrak{LC}(H)$. *The spectrum of* $\pi(T)$ *in* $\mathfrak{L}(H)/\mathfrak{LC}(H)$ *for* T *in* $\mathfrak{L}(H)$ *is called the essential spectrum of* T *and is denoted by* $\sigma_e(T)$. *The index of* T *is defined as* $\dim\ (T) = \dim\ \ker\ T - \dim\ \ker\ T^*$, *written as* $\mathrm{ind}\ (T)$.

The set of all Fredholm operators is invariant under compact perturbations. Namely, some properties of a Fredholm operator T can be possessed by the properties of $T + K$ for every K in $\mathfrak{LC}(H)$. The following theorem contains the usual definition of Fredholm operators.

Theorem 1.2 *(Atkinson) If* H *is a Hilbert space, then* $T \in \mathfrak{L}(H)$ *is a Fredholm operator if and only if the range of* T *is closed and the dimensions of* $\ker\ T$ *and* $\ker\ T^*$ *are both finite.*

In this paper, we study the Fredholm operators on the Hardy spaces. Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ denote the unit circle. Denoted by $L^2 = L^2(\mathbb{T}, dm)$ the Hilbert space of square integrable functions on $\mathbb{T}$ with respect to the Lebesgue measure, normalized so that the measure of the entire circle is 1. Let L^∞ be the space of the essentially bounded functions on the unit circle. The Hardy space H^2 denotes the Hilbert space of all holomorphic functions in $\mathbb{D}$ having square-summable Taylor coefficients at the origin, and it will be identified with the space of boundary functions, the subspace of L^2 consisting of functions whose Fourier coefficients with negative indices vanish. Let H^∞ denote the space of all bounded holomorphic functions in $\mathbb{D}$ and $C(\mathbb{T})$ denote the space of all continuous functions on $\mathbb{T}$.

Every function in H^2, other than the constant function 0, can be factorized into the product of an inner function and an outer function. An inner function is a function $u \in H^\infty$ such that $|u(e^{i\theta})| = 1$ almost everywhere with respect to the Lebesgue measure. The function $F \in H^2$ is an outer function if F is a cyclic vector of the unilateral shift S. That is, $\bigvee_0^\infty \{S^k F\} = H^2$. For more properties about Hardy spaces, we can refer to [17].

By Beurling's theorem [1], it is well known that the invariant subspace of the unilateral shift operator $Sf = zf$ on H^2 has the form uH^2, where u is an inner function. It is easy to check that $K_u^2 = H^2 \ominus uH^2$ is the invariant subspace of the backward shift operator S^* on H^2, which is called the model space. Let P denote the orthogonal projection from L^2 onto H^2 and P_u denote the orthogonal projection from L^2 onto K_u^2. For $\psi \in L^\infty$, the Toeplitz operator T_ψ induced by the symbol ψ is defined on H^2 by

$$T_\psi g = P(\psi g),\ g \in H^2.$$

Obviously, $T_\psi^* = T_{\overline{\psi}}$. Toeplitz operators acting on H^2 have very simple and natural matrix representation via infinite Toeplitz matrices that have constant entries on the diagonals parallel to the main one. For $\psi \in L^\infty$, Hankel operator H_ψ induced by the symbol ψ is defined on H^2 by

$$H_\psi g = (I - P)(\psi g),\ g \in H^2.$$

It is easy to check that $H_\psi^* h = P(\overline{\psi} h)$ for $h \in L^2 \ominus H^2$. The compressions of Toeplitz operators on K_u^2 are called truncated Toeplitz operators, which are defined by

$$A_\psi f = P_u(\psi f),\ f \in K_u^2.$$

The function ψ is called the symbol of A_ψ. Clearly, $A_\psi^* = A_{\overline{\psi}}$.

Truncated Toeplitz operators represent a far reaching generalization of classical Toeplitz matrices. Although particular case had appeared before in the literature, the general theory has been initiated in the seminal paper [20]. Since then, truncated Toeplitz operators have constituted an active area of research. We mention only a few relevant papers [2, 11, 18] and so on. On the operator theory level, Nagy shows that A_z is a model for a certain class of contraction operators [19]. Every contraction operator T on the Hilbert space H having defect indices $(1,1)$ and such that $\lim_{n\to\infty} T^{*n} = 0$ (SOT) is unitarily equivalent to A_z for some inner function u, where SOT denotes the strong operator topology. Thus, the research on truncated Toeplitz operators is of representative significance.

In [9], the author proves that if f is in L^∞ such that T_f is a Fredholm operator, then f is invertible in L^∞. Moreover, if $f \in H^\infty$, then T_f is invertible in $\mathfrak{L}(H^2)$ if and only if f is invertible in H^∞. In this case,

$$\sigma(T_f) = \text{clos } [G(f)(\mathbb{D})],$$

where $G(f)$ is the Gelfand transform of f. If f belongs to $C(\mathbb{T})$, then T_f is a Fredholm operator if and only if f does not vanish. In this case, ind (T_f) is equal to the negative winding number of the curve traced out by f with respect to the origin. In addition, if f is in $H^\infty + C(\mathbb{T})$, then T_f is a Fredholm operator if and only if f is invertible in $H^\infty + C(\mathbb{T})$. The Fredholm properties of Toeplitz operators have many characterizations, see [9], but there are very few results for Fredholmness of truncated Toeplitz operators.

For $f \in H^\infty$, A_f is defined on K_u^2 for some inner function u. It is well-known [6] that

$$\sigma(A_f) = \left\{ \lambda \in \mathbb{C} : \inf_{z\in\mathbb{D}} (|u(z)| + |f(z) - \lambda|) = 0 \right\} = f(\sigma(A_z)).$$

For $f \in H^\infty + C(\mathbb{T})$, in [3], we know that

$$\sigma_e(A_f) = \left\{ \lambda \in \mathbb{C} : \lim_{z\in\mathbb{D}} \inf_{|z|\to 1} (|u(z)| + |\widetilde{f}(z) - \lambda|) = 0 \right\} = f(\sigma_e(A_z)),$$

where $\widetilde{f}$ denotes the Possion integral of f. We can refer to [6] for more results that A_f is invertible in $\mathfrak{L}(K_u^2)$ for $f \in H^\infty$. In [7], the authors show that asymmetric truncated Toeplitz operators are equivalent after extension to Toeplitz operators with triangular symbols of a certain form and give some description about the kernel of asymmetric truncated Toeplitz operators with analytic. In [8], using truncated Toeplitz operators equivalence after extension to Toeplitz operators with 2×2 matrix symbols, the authors establish Fredholmness and invertibility criteria for truncated Toeplitz operators with u-separated symbols.

From the view of symbols of truncated Toeplitz operators, it is more difficult to find criteria for invertibility of truncated Toeplitz operators with nonanalytic symbols. In our paper, we characterize the Fredholm truncated Toeplitz operators by the properties of the symbol functions. In addition, from their own properties of model spaces and truncated Toeplitz operators, some description of kernel spaces of truncated Toeplitz operators are given.

The paper is organized as follows: In Section 2, we recall some necessary definitions and properties about model spaces and truncated Toeplitz operators. In Section 3, we give some results about the kernel spaces of truncated Toeplitz operators. In Section 4, under the condition that truncated Toeplitz operators have unique symbols, we study the sufficient condition or necessary condition about invertible truncated Toeplitz operators

A_f for $f \in L^\infty$. In addition, we also get the necessary and sufficient conditions about the quasinilpotent truncated Toeplitz operators and positive truncated Toeplitz operators. In Section 5, for $u(0) = 0$ and $f \in (K_u^2 + \overline{K_u^2}) \cap L^\infty$, the necessary condition is obtained for A_f to be Fredholm. In Section 6, for $f \in L^\infty$, we provide the necessary and sufficient condition that the defect operator $I - A_f^* A_f$ of truncated Toeplitz operator A_f meeting some conditions is compact on the model space K_u^2.

2. Preliminaries

In this section, we introduce some basic properties of truncated Toeplitz operators. The reproducing kernel of K_u^2 at $\lambda \in \mathbb{D}$ is the function $K_\lambda^u(z) = \frac{1-\overline{u(\lambda)}u(z)}{1-\overline{\lambda}z}$. As is well known that K_u^2 carries a natural conjugation C, antiunitary, involution operator, defined by $Cf = \overline{zf}u$, for $f \in K_u^2$. We have that

$$\widetilde{K_\lambda^u}(z) = (CK_\lambda^u)(z) = \frac{u(z) - u(\lambda)}{z - \lambda},$$

which is the conjugation reproducing kernel of K_u^2 at $\lambda \in \mathbb{D}$. That is, $\widetilde{f}(\lambda) = (Cf)(\lambda) = \langle \widetilde{K_\lambda^u}, f \rangle$ for $f \in K_u^2$. A bounded linear operator A on K_u^2 is called C-symmetric if $CAC = A^*$. Garcia and Putinar introduce some properties of C-symmetry in [12] and they show that all truncated Toeplitz operators are C-symmetric. About more complex symmetric operators can be found in [13].

Bounded truncated Toeplitz operators may be have some unbounded symbols. Sarason gave an example in [20]. Moreover, the symbols of truncated Toeplitz operators are not unique. For $f \in L^2$, Sarason in [20] proved that $A_f = 0$ if and only if $f \in uH^2 + \overline{uH^2}$. If $u(0) = 0$, then A_f has a unique symbol in $K_u^2 + \overline{K_u^2}$. In our paper, we mainly consider truncated Toeplitz operators defined on infinite dimensional model spaces which have unique symbols.

The set of all bounded truncated Toeplitz operators is denoted by $\mathfrak{T}_\mathfrak{u}$. For $a \in \mathbb{D}$, let φ_a be the Möbius transform $\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$. The Crofoot transform is the unitary operator $J: K_u^2 \to K_{\varphi_a \circ u}^2$ defined by

$$Jf = \frac{\sqrt{1-|a|^2}}{1-\overline{a}u} f.$$

It is proved in [20] that $J\mathfrak{T}_\mathfrak{u}J^* = \mathfrak{T}_{\varphi_a \circ \mathfrak{u}}$. If $u(0) = \alpha \neq 0$, then $(\varphi_\alpha \circ u)(0) = 0$ and $\mathfrak{T}_\mathfrak{u}$ is unitarily equivalent to $\mathfrak{T}_{\varphi_\alpha \circ \mathfrak{u}}$. Hence we may assume that $u(0) = 0$ when we consider the properties of truncated Toeplitz operators.

3. The kernel spaces of truncated Toeplitz operators

The kernel spaces of truncated Toeplitz operators are crucial in studying Fredholmness, but the kernel spaces of truncated Toeplitz operators are complicated. In this section, we introduce some results about kernel spaces of truncated Toeplitz operators.

Proposition 3.1 *Let u be a nonconstant inner function and K_u^2 be the model space. If v_1 and v_2 are inner functions and $f = \overline{v_1}v_2$, then*

$$\ker A_f = \overline{v_2}\left(uv_1H^2 \oplus K_{v_1}^2\right) \cap K_u^2.$$

Proof Denoted by $E=\overline{v_2}\left(uv_1H^2\oplus K^2_{v_1}\right)\cap K^2_u$. For any $g\in\ker A_f$, we have that $0=A_fg=P_u(\overline{v_1}v_2g)$. This implies that $\overline{v_1}v_2g\in uH^2+\overline{zH^2}$. There exist $h,\ \varphi\in H^2$ such that $\overline{v_1}v_2g=uh+\overline{z\varphi}$. That is,

$$v_2g=v_1uh+v_1\overline{z\varphi}.$$

By $K^2_{v_1}=v_1\overline{zH^2}\cap H^2$, we obtain that $v_1\overline{z\varphi}\in K^2_{v_1}$ and $\varphi\in K^2_{v_1}$. Since $v_1uH^2\subseteq v_1H^2$, we have that

$$v_1uH^2\perp K^2_{v_1}.$$

Thus $g\in E$ and $\ker A_f\subseteq E$.

For any $\psi\in E$, there exist $\varphi\in H^2$ and $\eta\in K^2_{v_1}$ such that $\psi=\overline{v_2}(v_1u\varphi+v_1\overline{z\eta})$. It follows that

$$A_f\psi=P_u\left(\overline{v_1}v_2\overline{v_2}(v_1u\varphi+v_1\overline{z\eta})\right)=P_u(u\varphi+\overline{z\eta})=0.$$

Thus $\psi\in\ker A_f$ and $E\subseteq\ker A_f$. The proof is completed.

□

For $h\in H^\infty$, we already know the kernel space of A_h, see in [22]. In the following, for $f\in L^\infty$ but $f\notin H^\infty$, we give some descriptions about the kernel space of A_f. We need the following preliminaries.

For $x,\ y\in H^\infty$, we use $GCD(x,\ y)$ to denote the greatest common divisor of I_x and I_y, where I_x denotes the inner part of x, which is defined up to a constant. The following lemmas come from [14].

Lemma 3.2 *If $f,\ g\in L^\infty$, then either* $\ker H^*_fH_g=\ker H_g$ *or* $\ker H^*_gH_f=\ker H_f$.

Lemma 3.3 *If f is in L^∞, then* $\ker H_f\neq\{0\}$ *if and only if f is of the form $\overline{\theta}b$, where θ is some inner function and $b\in H^\infty$ such that $GCD(\theta,\ b)$ is a constant.*

For $f\in L^\infty$, by $P_u=P-uP\overline{u}=Pu(I-P)\overline{u}$, we have that

$$A_f=P_ufP_u=Pu(I-P)\overline{u}fPu(I-P)\overline{u}=H^*_{\overline{u}}H_{\overline{u}f}H^*_{\overline{u}}H_{\overline{u}}. \tag{3.1}$$

In terms of (3.1), the kernel spaces of truncated Toeplitz operators are closely related to the kernel spaces of Hankel operators. By $P_u|_{H^2}=H^*_{\overline{u}}H_{\overline{u}}$, we get that

$$\ker A_f=\ker H^*_{\overline{u}}H_{\overline{u}f}\cap K^2_u.$$

By Lemma 3.2, we obtain that either

$$\ker H^*_{\overline{u}}H_{\overline{u}f}=\ker H_{\overline{u}f},$$

or

$$\ker H^*_{\overline{u}f}H_{\overline{u}}=\ker H_{\overline{u}}=uH^2.$$

Suppose that $\ker H^*_{\overline{u}f}H_{\overline{u}}\supsetneq\ker H_{\overline{u}}$. We have that

$$\ker A_f=\ker H^*_{\overline{u}}H_{\overline{u}f}\cap K^2_u=\ker H_{\overline{u}f}\cap K^2_u.$$

Then, by Lemma 3.3, we will get some descriptions about the kernel space of truncated Toeplitz operators. In the following, we give the necessary and sufficient condition such that $\ker H^*_{\overline{u}f}H_{\overline{u}}\supsetneq\ker H_{\overline{u}}$.

Lemma 3.4 *Let u be a nonconstant inner function and K_u^2 be the model space. If f is in L^∞, then* $\ker T_{\overline{f}} \cap K_u^2 = \{0\}$ *if and only if* $\ker H_{\overline{u}f}^* H_{\overline{u}} = \ker H_{\overline{u}} = uH^2$.

Proof Suppose that $\ker H_{\overline{u}f}^* H_{\overline{u}} = \ker H_{\overline{u}} = uH^2$. We have that

$$H_{\overline{u}f}^* H_{\overline{u}} g \neq 0, \tag{3.2}$$

for any $g \in K_u^2$. Since model spaces have an antiunitary operator C, there exists a function $\psi \in K_u^2$ such that

$$g = C\psi = u\overline{z\psi}.$$

Then

$$H_{\overline{u}f}^* H_{\overline{u}} g = H_{\overline{u}f}^* (I-P)(\overline{u}u\overline{z\psi}) = H_{\overline{u}f}^* \overline{z\psi} = P(u\overline{f}\overline{z\psi}) = T_{\overline{f}} g. \tag{3.3}$$

By (3.2), we get that $T_{\overline{f}} g \neq 0$ for any $g \in K_u^2$. Thus

$$\ker T_{\overline{f}} \cap K_u^2 = \{0\}. \tag{3.4}$$

Next we prove the necessity. It is obvious that $uH^2 \subseteq \ker H_{\overline{u}f}^* H_{\overline{u}}$. For any $h \notin uH^2$ but $h \in H^2$, there exist $\eta \neq 0 \in K_u^2$ and $\varphi \in H^2$ such that $h = \eta + u\varphi$. Then

$$H_{\overline{u}f}^* H_{\overline{u}} h = H_{\overline{u}f}^* H_{\overline{u}} (\eta + u\varphi) = H_{\overline{u}f}^* H_{\overline{u}} \eta.$$

By (3.3) and (3.4), we conclude that $H_{\overline{u}f}^* H_{\overline{u}} \eta \neq 0$. It follows that $h \notin \ker H_{\overline{u}f}^* H_{\overline{u}}$. Thus $\ker H_{\overline{u}f}^* H_{\overline{u}} \subseteq uH^2$. The proof is completed.

$\square$

Remark 3.5 *From Lemma 3.4, we have that* $\ker T_{\overline{f}} \cap K_u^2 \neq \{0\}$ *if and only if* $\ker H_{\overline{u}f}^* H_{\overline{u}} \supset \ker H_{\overline{u}} = uH^2$.

Lemma 3.6 *If u and θ are nonconstant inner functions with $GCD(u,\ \theta) = v \neq c$ and $u = vu_1$, then*

$$K_u^2 \cap \theta H^2 \subseteq vK_{u_1}^2,$$

where v and u_1 are inner functions and c is a constant.

Proof For $f \in K_u^2 \cap \theta H^2$, there exists $h \in H^2$ such that $f = \theta h$. By $u = vu_1$, we have that $K_u^2 = K_v^2 \oplus vK_{u_1}^2$. There exist $g \in K_v^2$ and $g_1 \in K_{u_1}^2$ such that

$$f = \theta h = g + vg_1. \tag{3.5}$$

By $GCD(u,\ \theta) = v$, we get that $\theta = v\theta_1$, where θ_1 is an inner function. Then, by (3.5), $v\theta_1 h - vg_1 = g \in vH^2$. By $vH^2 \perp K_v^2$, we obtain that $g = 0$ and $f = vg_1 \in vK_{u_1}^2$. The proof is completed.

$\square$

Remark 3.7 *If $GCD(u,\ \theta) = \theta$ and $u = \theta u_1$, then $K_u^2 \cap \theta H^2 = \theta K_{u_1}^2$. Furthermore, $\theta K_{u_1}^2$ is an invariant subspace of A_z defined on K_u^2.*

Proposition 3.8 *Let u be a nonconstant inner function with $u(0)=0$ and K_u^2 be the model space. Suppose that $f=f_1+\overline{f_2}\in(K_u^2+\overline{K_u^2})\bigcap L^\infty$, where f_1 and f_2 belong to K_u^2 with $f_1(0)=0$ and $f_2\neq 0$. If*

$$\ker\, T_{\overline{f}}\cap K_u^2\neq\{0\},$$

then there exist an inner function x and $\eta\in H^\infty$ with $GCD(x,\ \eta)=c$ and $GCD(x,\ u)=v\neq c$ such that $f=\overline{x}u\eta$ and $f_2\in K_x^2$. Moreover,

$$\{0\}\neq\ker\, A_f=\ker\, H_{\overline{u}f}\cap K_u^2=xH^2\cap K_u^2\subseteq vK_{u_1}^2,$$

and

$$\ker\, H_{\overline{uf}}\cap K_u^2=\{0\},$$

where v and u_1 are inner functions with $u=vu_1$ and c is some constant.

Proof By $\ker\, T_{\overline{f}}\cap K_u^2\neq\{0\}$, there exists $g\neq 0\in K_u^2$ such that $T_{\overline{f}}g=0$. That is, $\overline{f}g\in\overline{zH^2}$. There exists $y\neq 0\in H^2$ such that $\overline{f}g=\overline{zy}$. Since model spaces have an antiunitary operator C, there exists $\psi\neq 0\in K_u^2$ such that $g=C\psi=u\overline{z\psi}$. Then

$$\overline{f}g=\overline{f}u\overline{z\psi}=\overline{zy}.$$

That is, $\overline{u}f\psi=y\in H^2$. Moreover, $H_{\overline{u}f}\psi=(I-P)(\overline{u}f\psi)=0$. This implies that

$$\ker\, H_{\overline{u}f}\cap K_u^2\neq\{0\}. \tag{3.6}$$

By Remark 3.5, we have that $\ker\, H^*_{\overline{u}f}H_{\overline{u}}\supset\ker\, H_{\overline{u}}$. Then, by Lemma 3.2,

$$\ker\, H^*_{\overline{u}}H_{\overline{u}f}=\ker\, H_{\overline{u}f}. \tag{3.7}$$

In terms of (3.1), (3.6) and (3.7), we get that

$$\ker\, A_f=\ker\, H^*_{\overline{u}}H_{\overline{u}f}\cap K_u^2=\ker\, H_{\overline{u}f}\cap K_u^2\neq\{0\}.$$

By (3.6), we conclude that

$$\ker\, H_{\overline{u}f}\neq\{0\}.$$

By Lemma 3.3, there exists an inner function x and $\eta\in H^\infty$ with $GCD(x,\ \eta)=c$ such that $\ker\, H_{\overline{u}f}=xH^2$ and $\overline{u}f=\overline{x}\eta$, where c is a constant. Thus,

$$f=\overline{x}u\eta, \tag{3.8}$$

and

$$\ker\, A_f=\ker\, H_{\overline{u}f}\cap K_u^2=xH^2\cap K_u^2\neq\{0\}. \tag{3.9}$$

By Lemma 3.6, we obtain that $\ker\, A_f=xH^2\cap K_u^2\subseteq vK_{u_1}^2$. By (3.8), we have that

$$xf=u\eta. \tag{3.10}$$

In terms of $f=f_1+\overline{f_2}$, we get that $xf_1+x\overline{f_2}=u\eta$. That is, $x\overline{f_2}=u\eta-xf_1\in H^2$. Since $u(0)=0$ and $f_1(0)=0$, we obtain that

$$xz\overline{f_2}=\overline{z}u\eta-\overline{z}xf_1\in H^2.$$

This implies that $f_2 \in K_x^2$. By $f_2 \neq 0 \in K_u^2$, we have that $K_u^2 \cap K_x^2 \neq \{0\}$ and $GCD(x,\ u) = v$, where v is a nonconstant inner function.

Since $\ker A_{\overline{f}} = C(\ker A_f)$, we get that $\ker A_{\overline{f}} \neq \{0\}$. Then, by (3.1) and (3.9),

$$\ker A_{\overline{f}} = \ker H_{\overline{u}}^* H_{\overline{uf}} \cap K_u^2 = C(xH^2 \cap K_u^2).$$

For any $xh \neq 0 \in xH^2 \cap K_u^2$, we conclude that $C(xh) = u\overline{zxh}$. Then, by (3.10),

$$H_{\overline{uf}}(u\overline{zxh}) = (I-P)(\overline{fzxh}) = (I-P)(\overline{uzh\eta}) = \overline{uzh\eta} \neq 0. \tag{3.11}$$

In terms of $\ker H_{\overline{uf}} \subseteq \ker H_{\overline{u}}^* H_{\overline{uf}}$, we get that

$$\ker H_{\overline{uf}} \cap K_u^2 \subseteq \ker H_{\overline{u}}^* H_{\overline{uf}} \cap K_u^2 = C(xH^2 \cap K_u^2).$$

Thus, by (3.11), $\ker H_{\overline{uf}} \cap K_u^2 = \{0\}$. The proof is completed. □

Remark 3.9 *1. In fact, for* $f \in (K_u^2 + \overline{K_u^2}) \bigcap L^\infty$, $\ker T_{\overline{f}}$, $\ker H_{\overline{u}f}$ *and* $\ker A_f$ *have the following relationship:*

$$\ker T_{\overline{f}} \cap K_u^2 \neq \{0\} \Rightarrow \ker H_{\overline{u}f} \cap K_u^2 \neq \{0\} \Rightarrow \ker A_f \neq \{0\},$$

and

$$\ker A_f = \{0\} \Rightarrow \ker H_{\overline{u}f} \cap K_u^2 = \{0\} \Rightarrow \ker T_{\overline{f}} \cap K_u^2 = \{0\}.$$

2. In [4], it is well known that $f = g\overline{h}$ *where* $g,\ h \in H^\infty$ *if and only if* $\int_{\mathbb{T}} \log|f|dm > -\infty$. *Under conditions of Proposition 3.8, we have that* $f = \overline{x}u\eta$. *That is,* $\log|f| \in L^1$.

By (3.1), truncated Toeplitz operators are associated with Hankel operators. The truncated Toeplitz operators are compressions of Toeplitz operators. From this, the kernel spaces of them must have some relationships. We use $|E|$ to denote the Lebesgue measure of measurable set E.

Proposition 3.10 *Let* E *be a measurable subset of* $\mathbb{T}$ *with* $0 < |E| < 2\pi$. *For* $f \in L^\infty$ *with* $f \neq 0$ *and* $f|_E = 0$, *if* $v \in L^\infty$ *is invertible in* L^∞, *then the following hold.*

(1) $\ker T_{vf} = \ker T_{\overline{vf}} = \{0\}$. *In particular,* $\ker T_f = \ker T_{\overline{f}} = \{0\}$;

(2) $\ker H_{vf} = \ker H_{\overline{vf}} = \{0\}$. *In particular,* $\ker H_f = \ker H_{\overline{f}} = \{0\}$;

(3) $\ker H_{vf}^* = \ker H_{\overline{vf}}^* = \{0\}$. *In particular,* $\ker H_f^* = \ker H_{\overline{f}}^* = \{0\}$;

(4) Let u *be a nonconstant inner function and* K_u^2 *be the model space. If* $\ker A_f \neq \{0\}$, *then*

$$\ker A_f = \{g \in K_u^2 :\ H_{\overline{u}f} g \in \overline{uzH^2}\}.$$

Proof (1) For any $\varphi \in \ker T_{vf} \subseteq H^2$, we have that $0 = T_{vf}\varphi = P(vf\varphi)$. Then $vf\varphi \in \overline{zH^2}$, and there exists $h \in H^2$ such that $vf\varphi = \overline{zh}$. By $f|_E = 0$, we get that

$$v(e^{i\theta})f(e^{i\theta})\varphi(e^{i\theta}) = \overline{e^{i\theta}h(e^{i\theta})} = 0,$$

for any $e^{i\theta} \in E$. Thus $h = 0$ on E with $|E| > 0$. By F. and M. Riesz theorem, we have that $h = 0$ and $vf\varphi = 0$. Since v is invertible in L^∞, we obtain that $f\varphi = 0$. By $f \neq 0$ and $\varphi \in H^2$, we conclude that $\varphi = 0$ and $\ker T_{vf} = \{0\}$. We can get that $\ker T_{\overline{vf}} = \{0\}$ by the same way.

The proofs of (2) and (3) are similar to (1).

(4) For any $g \neq 0 \in \ker A_f$, we have that $A_f g = P_u(fg) = 0$. Suppose that $fg \in uH^2$. There exists $h \in H^2$ such that $fg = uh$. Then $\overline{u}fg = h \in H^2$. This implies that $H_{\overline{u}f} g = (I - P)(\overline{u}fg) = 0$. Since $\overline{u}$ is invertible in L^∞, we get that $\ker H_{\overline{u}f} = \{0\}$. This is a contradiction. Thus $fg \notin uH^2$. Since

$$0 = A_f g = P_u(fg) = P(fg) - uP(\overline{u}fg),$$

for $g \in \ker A_f$, there exists $x \neq 0 \in H^2$ such that $fg - uP(\overline{u}fg) = \overline{zx}$. This implies that $\overline{u}fg - P(\overline{u}fg) = \overline{uzx}$. That is, $H_{\overline{u}f} g \in \overline{uzH^2}$. Thus $\ker A_f = \{g \in K_u^2 : \ H_{\overline{u}f} g \in \overline{uzH^2}\}$. The proof is completed.

□

The Coburn theorem states that $\ker T_f = \{0\}$ or $\ker T_f^* = \{0\}$. In the following, we give the sufficient condition such that $\ker A_f = \{0\}$.

Proposition 3.11 *Let u be a nonconstant inner function and K_u^2 be the model space. If f belongs to L^∞ and $f \geq 0$ but $f \neq 0$, then* $\ker A_f = \{0\}$.

Proof For $g \in \ker A_f$, we have that $A_f g = 0$. Then

$$0 = \langle A_f g, g\rangle = \langle fg, g\rangle = \int_0^{2\pi} f|g|^2 \mathrm{d}m. \tag{3.12}$$

By $f \geq 0$, (3.12) can be written as $0 = \int_0^{2\pi} f|g|^2 \mathrm{d}m = \int_0^{2\pi} |f^{\frac{1}{2}} g|^2 \mathrm{d}m = \|f^{\frac{1}{2}} g\|^2$. This implies that $f^{\frac{1}{2}} g = 0$. By $f \neq 0$ and $g \in H^2$, we get that $g = 0$ and $\ker A_f = \{0\}$.

□

4. Invertible truncated Toeplitz operators

The invertible operators are special Fredholm operators. In this section, we introduce the invertibility of truncated Toeplitz operators. Basic definitions and properties of invertible operators can refer to [5, 9].

For the Banach algebra $\mathfrak{B}$ and $a \in \mathfrak{B}$, we use $\sigma(a)$ and $r(a)$ to denote the spectrum and spectral radius of a, respectively. In particular, for $f \in L^\infty$, the spectrum $\sigma(M_f)$ of the multiplication operator M_f is closely related to the the essential range $\mathfrak{R}(f)$ of f. The following lemma comes from Corollary 4.24 in [9].

Lemma 4.1 *If $f \in L^\infty$, then $\sigma(M_f) = \mathfrak{R}(f)$.*

For operator algebra $\mathfrak{L}(H)$, the invertibility of $T \in \mathfrak{L}(H)$ has the following property (see [9] Proposition 4.8).

Lemma 4.2 *If T is in $\mathfrak{L}(H)$, then T is invertible in $\mathfrak{L}(H)$ if and only if T is bounded below in H and has a dense range.*

In the following, we give the necessary condition for invertibility of A_f.

Proposition 4.3 *Let u be a nonconstant inner function with $u(0) = 0$ and K_u^2 be the model space. For $f \in (K_u^2 + \overline{K_u^2}) \bigcap L^\infty$, if A_f is invertible in $\mathfrak{L}(K_u^2)$, then f is invertible in L^∞.*

Proof By Lemma 4.1, we only need to show that M_f is invertible in $\mathfrak{L}(L^2)$. Since A_f is invertible in $\mathfrak{L}(K_u^2)$, there exists $\varepsilon > 0$ such that $\|A_f g\| \geq \varepsilon \|g\|$, for $g \in K_u^2$. Then, for each $n \in \mathbb{Z}$ and $g \in K_u^2$,

$$\|M_f z^n g\| = \|f z^n g\| = \|fg\| \geq \|P_u(fg)\| = \|A_f g\| \geq \varepsilon \|g\| = \varepsilon \|z^n g\|.$$

Since the set $\{z^n h : \ n \in \mathbb{Z}, \ h \in K_u^2\}$ is dense in L^2, it follows that M_f is bounded below in L^2. Similarly, since $A_{\overline{f}} = A_f^*$ is invertible in $\mathfrak{L}(K_u^2)$, we have that $M_{\overline{f}}$ is bounded below in L^2. Then $M_{\overline{f}}$ is one-to-one. Moreover,

$$\text{clos}\,[\text{ran}\ M_f] = (\text{ker}\ M_{\overline{f}})^\perp = L^2.$$

Thus, by Lemma 4.2, M_f is invertible in $\mathfrak{L}(L^2)$. The proof is completed.

□

Remark 4.4 *From Proposition 4.3, if the symbol f is not invertible in L^∞, then A_f must not be invertible in $\mathfrak{L}(K_u^2)$. Moreover, if $f|_E = 0$, then A_f is not invertible in $\mathfrak{L}(K_u^2)$, where E is a measurable subset of $\mathbb{T}$ with $0 < |E| < 2\pi$.*

By Proposition 4.3, when the truncated Toeplitz operator is invertible in $\mathfrak{L}(K_u^2)$, we have that the symbol is invertible in L^∞. In the following, we explain that the condition may be not necessary and sufficient. If f belongs to $C(\mathbb{T})$ and f is invertible in $C(\mathbb{T})$, we have that A_f is a Fredholm operator. The following lemma comes from [11].

Lemma 4.5 *Let u be an inner function. For $f, \ g \in C(\mathbb{T})$, if A_f and A_g are truncated Toeplitz operators on K_u^2, then $A_f A_g - A_{fg}$ is compact.*

Proposition 4.6 *Let u be a nonconstant inner function and K_u^2 be the model space. For $f \in C(\mathbb{T})$, if f is invertible in $C(\mathbb{T})$, then A_f is a Fredholm operator.*

Proof Since f is invertible in $C(\mathbb{T})$, there exists $g \in C(\mathbb{T})$ such that $fg = 1$. Then $A_{fg} = I$. By Lemma 4.5, we have that $I - A_f A_g = A_{fg} - A_f A_g$ and $I - A_g A_f = A_{fg} - A_g A_f$ are compact. Thus A_f is a Fredholm operator. □

Corollary 4.7 *Let u be a nonconstant inner function with $u(0) = 0$ and K_u^2 be the model space. If f is an outer function in $K_u^2 \bigcap H^\infty$, then the following are equivalent.*

(1) A_f is invertible in $\mathfrak{L}(K_u^2)$;

(2) f is invertible in L^∞;

(3) f is invertible in H^∞;

(4) T_f is invertible in $\mathfrak{L}(H^2)$.

Proof For $f \in H^\infty$, (3) is equivalent to (4). (See [9] Proposition 7.21). If f is an outer function, (3) is equivalent to (2). (See [9] Proposition 6.20). By Proposition 4.3, we only need to show that (2) $\Rightarrow$ (1). Suppose that f is invertible in L^∞. There exists $\varphi \in L^\infty$ such that $\varphi f = 1$. Since f is an outer function, we have that φ is analytic. Thus $\varphi \in H^\infty$. Then $A_f A_\varphi = A_\varphi A_f = A_{f\varphi} = I$. This implies that A_f is invertible in $\mathfrak{L}(K_u^2)$. □

Remark 4.8 *The function f is invertible in H^∞ if and only if T_f is invertible in $\mathfrak{L}(H^2)$. Hence, for $f \in H^\infty$, that T_f is invertible in $\mathfrak{L}(H^2)$ implies that A_f is invertible in $\mathfrak{L}(K_u^2)$. Moreover, $\sigma(A_f) \subseteq \sigma(T_f)$. Conversely, for $f \in H^\infty$, if A_f is invertible in $\mathfrak{L}(K_u^2)$, we may not get that T_f is invertible in $\mathfrak{L}(H^2)$. For example:*

Example 4.9 *If $f \neq c$ is in H^∞ and $u = \frac{z-a}{1-\bar{a}z}$ for $a \in \mathbb{D}$, where c is a constant. Since*

$$\sigma(A_f) = f(\sigma(A_z)) = f(a) \text{ and } \sigma(T_f) = \text{clos}\,[f(\mathbb{D})],$$

we have that

$$f(a) = \sigma(A_f) \subset \sigma(T_f) = \text{clos}\,[f(\mathbb{D})].$$

This implies that T_f may be not invertible in $\mathfrak{L}(H^2)$ when A_f is invertible in $\mathfrak{L}(K_u^2)$.

Corollary 4.10 *Let u be a nonconstant inner function with $u(0) = 0$ and K_u^2 be the model space. If f is in $(K_u^2 + \overline{K_u^2}) \bigcap L^\infty$, then $\mathfrak{R}(f) = \sigma(M_f) \subset \sigma(A_f)$.*

Proof Since $A_f - \lambda = A_{f-\lambda}$ for $\lambda \in \mathbb{C}$, by Proposition 4.3, we get that $\sigma(M_f) \subset \sigma(A_f)$. By Lemma 4.1, the proof is completed. □

Corollary 4.11 *Let u be a nonconstant inner function with $u(0) = 0$ and K_u^2 be the model space. For $f \in (K_u^2 + \overline{K_u^2}) \bigcap L^\infty$, if A_f is quasinilpotent, then $A_f = 0$.*

Proof If A_f is quasinilpotent, then $\sigma(A_f) = \{0\}$. By Corollary 4.10, it is easy to get that $A_f = 0$. □

Corollary 4.12 *Let u be a nonconstant inner function with $u(0) = 0$ and K_u^2 be the model space. If $f \in (K_u^2 + \overline{K_u^2}) \bigcap L^\infty$, then A_f is a self-adjoint operator if and only if f is a real-valued function.*

Proof By Corollary 4.10, the proof is obvious. □

Corollary 4.13 *Let u be a nonconstant inner function with $u(0) = 0$ and K_u^2 be the model space. If f is in $(K_u^2 + \overline{K_u^2}) \bigcap L^\infty$, then $\|A_f\| = \|f\|_\infty$.*

Proof By Corollary 4.10, we obtain that $r(A_f) = \sup\{|\lambda| : \lambda \in \sigma(A_f)\} \geq \sup\{|\lambda| : \lambda \in \mathfrak{R}(f)\} = \|f\|_\infty$. Since $\|A_f\| \geq r(A_f)$, we have that $\|f\|_\infty \geq \|A_f\| \geq r(A_f) \geq \|f\|_\infty$. Thus $\|A_f\| = \|f\|_\infty$. □

Corollary 4.14 *Let u be a nonconstant inner function with $u(0) = 0$ and K_u^2 be the model space. If $f \in (K_u^2 + \overline{K_u^2}) \bigcap L^\infty$, then A_f is positive if and only if f is nonnegative.*

Proof If A_f is positive, then the spectrum of A_f is nonnegative. By Corollary 4.10, we have that the essential range of f is nonnegative. Then f is nonnegative.

If f is nonnegative, then

$$\langle A_f g, g\rangle = \langle fg, g\rangle = \langle f^{1/2}g, f^{1/2}g\rangle = \|f^{1/2}g\|^2 \geq 0,$$

for $g \in K_u^2$. Thus A_f is positive. □

We discuss the necessary condition that A_f is invertible in $\mathfrak{L}(K_u^2)$. In the following, we give a sufficient condition that A_f is invertible in $\mathfrak{L}(K_u^2)$.

Proposition 4.15 *If f is invertible in L^∞ and its essential range is contained in the open right half-plane, then A_f is invertible in $\mathfrak{L}(K_u^2)$.*

Proof Since f is invertible in L^∞, we have that 0 is not in $\mathfrak{R}(f)$. If essential range of f is contained in the open right half-plane, then there exists $\delta > 0$ such that

$$\delta\mathfrak{R}(f) = \{\delta z : \ z \in \mathfrak{R}(f)\} \subseteq \{w \in \mathbb{C} : \ |w-1| < 1\}.$$

We conclude that $|\delta z - 1| < 1$. By simple calculation, we get that $\alpha\mathfrak{R}(f) - \beta = \mathfrak{R}(\alpha f - \beta)$. Then

$$\|\delta f - 1\|_\infty = \sup\{|\lambda| : \ \lambda \in \mathfrak{R}(\delta f - 1)\} = \sup\{|\delta z - 1| : \ z \in \mathfrak{R}(f)\} < 1.$$

Thus $\|I - A_{\delta f}\| < 1$, and $A_{\delta f} = \delta A_f$ is invertible in $\mathfrak{L}(K_u^2)$. The proof is completed.

□

5. Fredholm truncated Toeplitz operators

In this section, for $f \in L^\infty$, we study the necessary condition that A_f is a Fredholm operator. If M is a closed linear subspace of the Hilbert space H, for $h \in H$, the distance between h and M is defined as

$$d(h,\ M) = \inf\{\|h - m\|,\ m \in M\}.$$

The following definition and theorem comes from Definition IV.1.3 and Theorem IV.1.6 in [15], respectively.

Definition 5.1 *Let A be a linear operator with domain in normed linear space X (not necessarily dense in X) and range in normed linear space Y, and $\ker A$ is closed. The minimum modulus of A is defined by, written as $\gamma(A)$,*

$$\gamma(A) = \inf\left\{\frac{\|Ax\|}{d(x,\ \ker A)},\ x \in D(A)\right\},$$

where $0/0$ is defined to be ∞, and $D(A)$ denotes the domain of A.

Theorem 5.2 *Let X and Y be complete spaces and A be closed operator. Then A has a closed range if and only if $\gamma(A) > 0$.*

Remark 5.3 *If T is a bounded linear operator on the Hilbert space, then T is closed. Thus, Theorem 5.2 can apply to bounded linear operators.*

The following lemma can be found in [21].

Lemma 5.4 *If T is in $\mathfrak{L}(H)$ and* $\ker T = \{0\}$, *then the range of T is closed if and only if T is bounded below in H.*

Theorem 5.5 *Let u be a nonconstant inner function with $u(0) = 0$ and K_u^2 be the model space. For $f \in (K_u^2 + \overline{K_u^2}) \cap L^\infty$, if A_f is a Fredholm operator, then $f|_E \neq 0$ for any $E \subset \mathbb{T}$ with $|E| > 0$. Moreover,*

$$\text{ind } (A_f) = 0.$$

Proof First consider the case where $\ker A_f = \{0\}$. By $CA_fC = A_f^*$, we obtain that $\ker A_f^* = C(\ker A_f) = \{0\}$. Since A_f is a Fredholm operator, we have that A_f has a closed range. Then

$$\text{ran } A_f = \text{clos } [\text{ran } A_f] = (\ker A_f^*)^\perp = K_u^2.$$

Thus A_f is invertible in $\mathfrak{L}(K_u^2)$. Then, by Proposition 4.3, f is invertible in L^∞. Therefore $f|_E \neq 0$ for any $E \subset \mathbb{T}$ with $|E| > 0$.

Now consider the case where $\ker A_f \neq \{0\}$. Suppose that there exists a measurable subset $E_0 \subset T$ with $|E_0| > 0$ such that $f|_{E_0} = 0$. By Proposition 3.10, we have that

$$\ker H_{\overline{u}f} = \ker H_{\overline{\overline{u}f}} = \{0\}. \tag{5.1}$$

In addition,

$$A_f g = P_u(fg) = Pu(I-P)(\overline{u}fg) = H_{\overline{u}}^* H_{\overline{u}f} g,$$

and $H_{\overline{u}}^* \overline{zg} = P(u\overline{zg}) = Cg$ for $g \in K_u^2$. By $\overline{zH^2} = \overline{uzH^2} \oplus \overline{zK_u^2}$ and $\ker H_{\overline{u}}^* = \overline{uzH^2}$, we get that

$$\|H_{\overline{u}}^* \overline{zg}\| = \|P(u\overline{zg})\| = \|Cg\| = \|g\| = \|zg\|,$$

for $g \in K_u^2$. Thus $H_{\overline{u}}^*$ is a partial isometry. Then

$$\|H_{\overline{u}f} g\| \geq \|H_{\overline{u}}^* H_{\overline{u}f} g\| = \|A_f g\|, \tag{5.2}$$

for $g \in K_u^2$. By (5.1), we obtain that

$$d(g, \ \ker H_{\overline{u}f}) = \|g\|. \tag{5.3}$$

For $g \in K_u^2$ but $g \notin \ker A_f$, there exists a constant α with $|\alpha| = 1$ such that $\alpha g \perp \ker A_f$. Then, by the Pythagorean theorem,

$$\|\alpha g - \varphi\|^2 = \|\alpha g\|^2 + \|\varphi\|^2 = \|g\|^2 + \|\varphi\|^2,$$

for $\varphi \in \ker A_f$. Thus

$$\begin{aligned} d(\alpha g, \ \ker A_f) &= \inf\{\|\alpha g - \varphi\|, \ \varphi \in \ker A_f\} \\ &= \inf\left\{\sqrt{\|g\|^2 + \|\varphi\|^2}, \ \varphi \in \ker A_f\right\} \\ &\geq \inf\{\|g\|, \ \varphi \in \ker A_f\} \\ &= \|g\|. \end{aligned}$$

Then

$$\frac{1}{d(\alpha g,\ \ker\ A_f)} \leq \frac{1}{\|g\|}. \tag{5.4}$$

By (5.2), (5.3) and (5.4), we have that

$$\frac{\|H_{\overline{u}f}\alpha g\|}{\|\alpha g\|} \geq \frac{\|A_f \alpha g\|}{d(\alpha g,\ \ker\ A_f)},$$

for $g \in K_u^2$ but $g \notin \ker\ A_f$. Let

$$\beta = \inf\left\{\frac{\|H_{\overline{u}f}\alpha g\|}{\|\alpha g\|},\ g \in K_u^2 \textit{ and } g \notin \ker\ A_f\right\}.$$

We conclude that

$$\beta \geq \inf\left\{\frac{\|A_f g\|}{d(g,\ \ker\ A_f)},\ g \in K_u^2 \textit{ and } g \notin \ker\ A_f\right\}. \tag{5.5}$$

Since A_f is a Fredholm operator, we get that the range of A_f is closed, and the dimensions of $\ker\ A_f$ and $\ker\ A_f^*$ are finite. Then, by Theorem 5.2,

$$\gamma(A_f) = \inf\left\{\frac{\|A_f g\|}{d(g,\ \ker\ A_f)},\ g \in K_u^2\right\} > 0.$$

If $g \in \ker\ A_f$, then

$$\frac{\|A_f g\|}{d(g,\ \ker\ A_f)} = \frac{0}{0} = \infty.$$

Thus

$$\inf\left\{\frac{\|A_f g\|}{d(g,\ \ker\ A_f)},\ g \in K_u^2 \text{ and } g \notin \ker\ A_f\right\} > 0. \tag{5.6}$$

Moreover, by (5.5),

$$\beta = \inf\left\{\frac{\|H_{\overline{u}f}\alpha g\|}{\|\alpha g\|},\ g \in K_u^2 \text{ and } g \notin \ker\ A_f\right\} > 0. \tag{5.7}$$

For $g \in \ker\ A_f$, since the dimension of $\ker\ A_f$ is finite, we have that $H_{\overline{u}f}|_{\ker\ A_f}$ has a closed range. Thus, by Theorem 5.2,

$$\beta_1 = \inf\left\{\frac{\|H_{\overline{u}f}\alpha g\|}{\|\alpha g\|},\ g \in \ker\ A_f\right\} > 0. \tag{5.8}$$

By (5.7) and (5.8), we get that

$$\inf\left\{\frac{\|H_{\overline{u}f}\alpha g\|}{\|\alpha g\|},\ g \in K_u^2\right\} = \min\{\beta,\ \beta_1\} > 0.$$

Then, by Theorem 5.2, $H_{\overline{u}f}|_{K_u^2}$ has a closed range. By $\ker\ H_{\overline{u}f} = \{0\}$ and Lemma 5.4, we obtain that $H_{\overline{u}f}|_{K_u^2}$ is bounded below in K_u^2. There exists $\epsilon > 0$ such that

$$\|H_{\overline{u}f}g\| \geq \epsilon\|g\|, \tag{5.9}$$

for $g \in K_u^2$. Then

$$\|M_f z^n g\| = \|fg\| = \|\overline{u}fg\| \geq \|(I-P)(\overline{u}fg)\| = \|H_{\overline{u}f}g\| \geq c\|g\| = c\|z^n g\|, \tag{5.10}$$

for $n \in \mathbb{Z}$ and $g \in K_u^2$. Since the set $\Delta = \{z^n h : \ n \in \mathbb{Z}, \ g \in K_u^2\}$ is dense in L^2, we have that M_f is bounded below in L^2.

Since A_f is a Fredholm operator, there exist $T_1, \ T_2 \in \mathfrak{L}(K_u^2)$ and compact operators $K_1, \ K_2$ such that $A_f T_1 = I + K_1$ and $T_2 A_f = I + K_2$. By $CA_f C = A_f^*$ and $C^2 = I$, we get that $A_f^* C T_1 C = I + CK_1 C$ and $CT_2 C A_f^* = I + CK_2 C$. Since the set of all compact operators is an ideal, we get that A_f^* is a Fredholm operator. Then ran A_f^* is closed and the dimension of ker A_f^* is finite. By (5.1) and

$$A_f^* g = P_u(\overline{f}g) = Pu(I-P)(\overline{uf}g) = H_{\overline{u}}^* H_{\overline{uf}}g,$$

for $g \in K_u^2$, we conclude that M_f^* is bounded below in L^2 by the similar way. Then M_f is invertible in $\mathfrak{L}(L^2)$. By Lemma 4.1, we have that f is invertible in L^∞. This contradicts our assumption about f. Thus $f|_E \neq 0$ for $E \subset \mathbb{T}$ with $|E| > 0$. In terms of ker $A_f^* = C(\text{ker } A_f)$, we have that ind $(A_f) = 0$. The proof is completed. □

6. Compact defect operators of truncated Toeplitz operators

For a bounded linear operator T on the Hilbert space H, we call $D_T = I - T^*T$ and $D_{T^*} = I - TT^*$ the defect operators, $R_T = \overline{D_T H}$ and $R_{T^*} = \overline{D_{T^*} H}$ the defect spaces, and dim R_T and dim R_{T^*} the defect indices. For $f \in H^\infty$, the necessary and sufficient condition is obtained for $I - A_f^* A_f$ to be compact or of finite-rank in [22]. For $f \in (K_u^2 + \overline{K_u^2}) \cap L^\infty$, by Theorem 5.5, we obtain a sufficient condition for $I - A_f^* A_f$ to be compact. In following, for $f \in L^\infty$, using the known result about a finite sum of products of Toeplitz operators to be compact, see [16], we will simplify $I - A_f^* A_f$ as a finite sum of products of Toeplitz operators and give the necessary and sufficient condition that $I - A_f^* A_f$ is compact on the model space K_u^2.

For $f \in L^2$, we use f_+ and f_- to denote $P(f)$ and $(I-P)(f)$, respectively. In the following, for $f, \ g \in L^\infty$, we will frequently use the relationship:

$$T_{fg} - T_f T_g = H_{\overline{f}}^* H_g. \tag{6.1}$$

A finite sum of finite products of Toeplitz operators can be written as a finite sum of products of two Toeplitz operators. The key idea used in [16]:

$$T_f T_g T_h = T_f(T_{g_+} + T_{g_-})T_h = T_{fg_+} T_h + T_f T_{g_- h}, \tag{6.2}$$

for $f, \ g, \ h \in L^\infty$. Moreover,

$$T_f T_g T_h T_\varphi = T_f(T_{g_+} + T_{g_-})T_h T_\varphi = T_{fg_+} T_h T_\varphi + T_f T_{g_- h} T_\varphi,$$

for $f, \ g, \ h, \ \varphi \in L^\infty$. Similar to (6.2), the product of four Toeplitz operators can be written as a sum of two Toeplitz operators with (perhaps unbounded) symbols, and the decomposition is not unique.

Lemma 6.1 *Let u be a nonconstant inner function and K_u^2 be the model space. If f is in L^∞ such that f_-, f_+, $(uf)_+$, $(uf)_-$, $(uf_-(\overline{uf})_-)_-$ and $(uf_-(\overline{uf})_-)_+$ are in L^∞, then $I - A_f^* A_f = T_{1-\overline{f}f_+} + T$, where T is the finite sum of the products of two Toeplitz operators.*

Proof By $P_u = P - uP\overline{u}$, we have that

$$A_f = P_u f P_u = Pu(I-P)\overline{u}fPu(I-P)\overline{u} = H_{\overline{u}}^* H_{\overline{u}f} H_{\overline{u}}^* H_{\overline{u}}. \tag{6.3}$$

By (6.1), we obtain that

$$\begin{aligned} H_{\overline{u}}^* H_{\overline{u}f} H_{\overline{u}}^* H_{\overline{u}} &= (T_f - T_u T_{\overline{u}f})(I - T_u T_{\overline{u}}) \\ &= (T_f - T_u T_{\overline{u}f}) - (T_f - T_u T_{\overline{u}f}) T_u T_{\overline{u}} \\ &= H_{\overline{u}}^* H_{\overline{u}f} - (T_f T_u T_{\overline{u}} - T_u T_{\overline{u}f} T_u T_{\overline{u}}) \\ &= H_{\overline{u}}^* H_{\overline{u}f} - (T_{fu} T_{\overline{u}} - T_u (T_{f_+} + T_{f_-}) T_{\overline{u}}) \\ &= H_{\overline{u}}^* H_{\overline{u}f} - (T_{fu} T_{\overline{u}} - T_u T_{f_+} T_{\overline{u}} - T_u T_{f_-} T_{\overline{u}}) \\ &= H_{\overline{u}}^* H_{\overline{u}f} - (T_{fu} T_{\overline{u}} - T_{uf_+} T_{\overline{u}} - T_u T_{\overline{u}f_-}) \\ &= H_{\overline{u}}^* H_{\overline{u}f} - (T_{uf_-} T_{\overline{u}} - T_u T_{\overline{u}f_-}) \\ &= H_{\overline{u}}^* H_{\overline{u}f} - (T_{uf_-} T_{\overline{u}} - T_{f_-} + T_{f_-} - T_u T_{\overline{u}f_-}) \\ &= H_{\overline{u}}^* H_{\overline{u}f} - (-H_{\overline{uf_-}}^* H_{\overline{u}} + H_{\overline{u}}^* H_{\overline{u}f_-}) \\ &= H_{\overline{u}}^* H_{\overline{u}f_+} + H_{\overline{uf_-}}^* H_{\overline{u}}. \end{aligned} \tag{6.4}$$

In terms of (6.3), we get that

$$\begin{aligned} I - A_f^* A_f &= P_u - H_{\overline{u}}^* H_{\overline{uf}} H_{\overline{u}}^* H_{\overline{u}f} H_{\overline{u}}^* H_{\overline{u}} \\ &= I - T_u T_{\overline{u}} - H_{\overline{u}}^* H_{\overline{uf}} (H_{\overline{u}}^* H_{\overline{u}f_+} + H_{\overline{uf_-}}^* H_{\overline{u}}) \\ &= I - T_u T_{\overline{u}} - H_{\overline{u}}^* H_{\overline{uf}} H_{\overline{u}}^* H_{\overline{u}f_+} - H_{\overline{u}}^* H_{\overline{uf}} H_{\overline{uf_-}}^* H_{\overline{u}}. \end{aligned} \tag{6.5}$$

By (6.1) and the idea before Lemma 6.1, $H_{\overline{u}}^* H_{\overline{uf}} H_{\overline{u}}^* H_{\overline{u}f_+}$ and $H_{\overline{u}}^* H_{\overline{uf}} H_{\overline{uf_-}}^* H_{\overline{u}}$ can be written as a finite sum of products of two Toeplitz operators. By calculating, we conclude that

$$H_{\overline{u}}^* H_{\overline{uf}} H_{\overline{u}}^* H_{\overline{u}f_+} = T_{\overline{f}f_+} - T_u T_{\overline{uf}f_+} - T_{u(\overline{f})_-} T_{\overline{u}f_+} + T_u T_{\overline{u}f_+(\overline{f})_-}, \tag{6.6}$$

and

$$\begin{aligned} H_{\overline{u}}^* H_{\overline{uf}} H_{\overline{uf_-}}^* H_{\overline{u}} = {} & T_{\overline{f}} T_{f_-} - T_{u(\overline{uf})_+} T_{f_-} - T_u T_{(\overline{uf})_- f_-} - T_{\overline{f}(uf_-)_+} T_{\overline{u}} \\ & - T_{\overline{f}} T_{\overline{u}(uf_-)_-} + T_{u(\overline{uf})_+(uf_-)_+} T_{\overline{u}} + T_{u(\overline{uf})_+} T_{\overline{u}(uf_-)_-} \\ & + T_{u(uf_-(\overline{uf})_-)_+} T_{\overline{u}} + T_u T_{\overline{u}(uf_-(\overline{uf})_-)_-}. \end{aligned} \tag{6.7}$$

By (6.5), we have that

$$I - A_f^* A_f = T_{1-\overline{f}f_+} + T, \tag{6.8}$$

where

$$
\begin{aligned}
T = {} & T_u T_{\overline{uf}f_+} - T_u T_{\overline{u}} + T_{u(\overline{f})_-} T_{\overline{u}f_+} - T_u T_{\overline{u}f_+(\overline{f})_-} - T_{\overline{f}} T_{f_-} + T_{u(\overline{uf})_+} T_{f_-} \\
& + T_u T_{(\overline{uf})_- f_-} + T_{\overline{f}(uf_-)_+} T_{\overline{u}} + T_{\overline{f}} T_{\overline{u}(uf_-)_-} - T_{u(\overline{uf})_+(uf_-)_+} T_{\overline{u}} \\
& - T_{u(\overline{uf})_+} T_{\overline{u}(uf_-)_-} - T_{u(uf_-(\overline{uf})_-)_+} T_{\overline{u}} - T_u T_{\overline{u}(uf_-(\overline{uf})_-)_-}.
\end{aligned} \tag{6.9}
$$

□

The following theorem can be found in [10].

Theorem 6.2 *For $f_i,\ g_i,\ h$ in L^∞, $i = 1,\ 2,\ \cdots,\ n$, if $\sum_{i=1}^{n} T_{f_i} T_{g_i} - T_h$ has finite rank k, then there are analytic polynomials $A_i(z)$ and $B_i(z)$, not all of which are zero, with* $\max\{\deg\ A_i(z)\} = k$ *and* $\max\{\deg\ B_i(z)\} = k$ *such that*

$$
\sum_{i=1}^{n} A_i \overline{f_i} \in H^2 \text{ or } \sum_{i=1}^{n} B_i g_i \in H^2.
$$

In the following, we give the necessary condition that the defect operator $I - A_f^* A_f$ has finite rank on the model space K_u^2.

Theorem 6.3 *Let u be a nonconstant inner function and K_u^2 be the model space. For $f \in L^\infty$ with $f_-,\ f_+,\ (uf)_+,\ (uf)_-,\ (uf_-(\overline{uf})_-)_-$, and $(uf_-(\overline{uf})_-)_+$ in L^∞, if $I - A_f^* A_f$ has finite rank k, then there are analytic polynomials $A_i(z)$, $i = 1,\ ,\cdots,\ 7$, and $B_j(z)$, $j = 1,\ ,\cdots,\ 8$, with* $\max\{\deg\ A_i(z)\} = k$ *and* $\max\{\deg\ B_j(z)\} = k$ *such that*

$$
\begin{aligned}
& f\left(A_3 + A_5\overline{(uf_-)_+}\right) \\
+ \quad & \overline{u}\left(A_1 + A_2\overline{(\overline{f})_-} + A_4\overline{(\overline{uf})_+} + A_6\overline{(\overline{uf})_+(uf_-)_+} + A_7\overline{(uf_-(\overline{uf})_-)_+}\right) \in H^\infty,
\end{aligned}
$$

or

$$
\begin{aligned}
& f_-\left(B_5 + B_6(\overline{uf})_-\right) \\
+ \quad & \overline{u}\left(B_1\overline{f}f_+ + B_2 + B_3 f_+ + B_4 f_+(\overline{f})_- + B_7(uf_-)_- + B_8(uf_-(\overline{uf})_-)_-\right) \in H^\infty.
\end{aligned}
$$

Proof By $uf = uf_+ + uf_-$, we get that

$$
(uf)_+ = P(uf_+ + uf_-) = uf_+ + (uf_-)_+,
$$

and

$$
(uf)_- = (I - P)(uf_+ + uf_-) = (uf_-)_-.
$$

Since f is in L^∞ such that $f_-,\ f_+,\ (uf)_+,\ (uf)_-$ are in L^∞, we have that $(uf_-)_+$ and $(uf_-)_-$ are in L^∞. Since $I - A_f^* A_f$ has finite rank, by (6.8) and Theorem 6.2, there are analytic polynomials $A_i(z)$, $i = 1,\ ,\cdots,\ 7$, and $B_j(z)$, $j = 1,\ ,\cdots,\ 8$, not all of which are zero, with $\max\{\deg\ A_i(z)\} = k$ and $\max\{\deg\ B_j(z)\} = k$ such that

$$
\begin{aligned}
& A_1\overline{u} + A_2\overline{u(\overline{f})_-} + A_3 f + A_4\overline{u(\overline{uf})_+} \\
+ \quad & A_5 f\overline{(uf_-)_+} + A_6\overline{u(\overline{uf})_+(uf_-)_+} + A_7\overline{u(uf_-(\overline{uf})_-)_+} \in H^\infty,
\end{aligned}
$$

or

$$\begin{aligned} & B_1\overline{uf}f_+ + B_2\overline{u} + B_3\overline{u}f_+ + B_4\overline{u}f_+(\overline{f})_- + B_5f_- \\ + \quad & B_6(\overline{uf})_-f_- + B_7\overline{u}(uf_-)_- + B_8\overline{u}(uf_-(\overline{uf})_-)_- \in H^\infty. \end{aligned}$$

By simplifying, the proof is completed.

□

The following theorem can refer to [16].

Theorem 6.4 *A finite sum T of finite products of Toeplitz operators is a compact perturbation of a Toeplitz operator if and only if*

$$\lim_{|z|\to 1} \|T - T^*_{\varphi_z} T T_{\varphi_z}\| = 0,$$

where $\varphi_z(w) = \frac{z-w}{1-\overline{z}w}$.

In the following, we give the necessary and sufficient condition that the defect operator $I - A^*_f A_f$ is compact on the model space K^2_u.

Theorem 6.5 *Let u be a nonconstant inner function and K^2_u be the model space. If f is in L^∞ such that f_-, f_+, $(uf)_+$, $(uf)_-$, $(uf_-(\overline{uf})_-)_-$ and $(uf_-(\overline{uf})_-)_+$ are in L^∞, then $I - A^*_f A_f$ is compact if and only if*

$$\lim_{|z|\to 1} \|T - T^*_{\varphi_z} T T_{\varphi_z}\| = 0,$$

where $\varphi_z(w) = \frac{z-w}{1-\overline{z}w}$ *and T is equal to* (6.9).

Proof By (6.8), we have that $I - A^*_f A_f$ is compact if and only if $T_{1-\overline{f}f_+} + T$ is compact, where T is a finite sum of the products of two Toeplitz operators. By Theorem 6.4, the proof is completed.

□

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