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Some new representations of Hikami's second-order mock theta function $\mathfrak{D}_5(q)$

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Abstract: In this paper, a second-order mock theta function $\mathfrak{D}_5(q)$ given by Hikami [11] is studied. By using basic hypergeometric transformation formulae, we attain some new representations of Hikami's mock theta function $\mathfrak{D}_5(q)$. Meanwhile, dual nature of bilateral series associated to mock theta function $\mathfrak{D}_5(q)$ is also discussed.

Key words: Second-order mock theta function, q-hypergeometric series, dual nature, Appell-Lerch sums

1. Introduction

Throughout this paper, we assume that $|q| < 1$ and adopt the standard notations [7]. Let $\mathbb{N}$ be the set of natural number with $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. For an indeterminate a, q-shifted factorial of order $n \in \mathbb{N}_0$ is defined by

$$(a;q)_0 := 1,$$

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), n \in \mathbb{N},$$

and

$$(a;q)_{-n} := \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, n \in \mathbb{N}.$$

For conveniences, we adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \cdots, a_m; q)_n := (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n,$$

where n is an integer or ∞.

Basic hypergeometric series (q-series) is called q-analogue of hypergeometric series. For some studies on various results of q-series and hypergeometric series, readers can refer to [1, 2, 6, 14, 15, 19, 21, 22].

The unilateral basic hypergeometric series ${}_r\varphi_s$ is defined by

$${}_r\varphi_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(q, b_1, \cdots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

where $q \neq 0$ when $r > s+1$.

The bilateral basic hypergeometric series is given by

$${}_r\psi_s\left[\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n} (-1)^{(s-r)n} q^{(s-r)\binom{n}{2}} z^n.$$

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In his letter to Hardy, Ramanujan listed 17 functions which he called mock theta functions. He defined each mock theta function as a q-series in Eulerian form and divided them into four classes: one class of third order, two of fifth order, and one of seventh order. However, he did not explain what he meant by the order of the mock theta functions. Ramanujan did not give explicit definition for mock theta functions. Nobody knows what Ramanujan had in his mind when he mentioned the "order" of the mock theta functions. Later, Ramanujan's statements were interpreted by Andrews and Hickerson [3] to mean a function $f(q)$ defined by a q-series which converges for $|q| < 1$ and satisfies the following two conditions:

(1) For every root of unity ξ, there is a theta function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $q \to \xi$ radially.

(2) There is no single theta function which works for all ξ; i.e. for every theta function $\theta(q)$ there is some root of unity ξ for which $f(q) - \theta_\xi(q)$ is unbounded as $q \to \xi$ radially.

A similar definition was given by Gordon and McIntosh (cf. [8]). For second-order mock theta functions, McIntosh considered three second-order mock theta functions and gave transformation formulas for them [16]. For more details on other mock theta functions, one may refer to [8, 10, 23]. In his work on mathematical physics and the quantum invariant of a three manifold, Hikami [11, 12] came across the q-series

$$\mathfrak{D}_5(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^n}{(q;q^2)_{n+1}} \tag{1.1}$$

and proved this function is indeed a mock theta function and called it of "2nd" order. This q-series can be rewritten as

$$\mathfrak{D}_5(q) = \frac{1}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} (q;q^2)_n^2 q^{2n}. \tag{1.2}$$

And he further showed that

$$\mathfrak{D}_5(q) = 2h_1(q) - (-q;q)_\infty^2 \omega(q), \tag{1.3}$$

where $\omega(q)$ denotes the third-order mock theta function defined by (cf. [23]):

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}$$

and $h_1(q)$ denotes the second-order mock theta function defined by (cf. [8]):

$$\begin{aligned} h_1(q) = \sum_{n=0}^{\infty} \frac{q^n(-q;q)_{2n}}{(q;q^2)_{n+1}^2} &= \frac{J_2}{J_1^2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+2)}}{1-q^{2n+1}} \\ &= \frac{J_2}{2J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+2)}}{1-q^{2n+1}}. \end{aligned} \tag{1.4}$$

Let Z be the set of integer number and recall that the Appell-Lerch sums [10] are defined by

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r \in Z} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1 - q^{r-1}xz}, \tag{1.5}$$

where

$$j(z;q) := (z, \frac{q}{z}, q; q)_\infty = \sum_{n\in Z} (-1)^n q^{\binom{n}{2}} z^n. \tag{1.6}$$

Let a and m be integers with $m \in \mathbb{N}$. Define

$$J_{a,m} := j(q^a; q^m),$$
$$J_m := J_{m,3m} = \prod_{i\geq 1}(1-q^{mi}).$$

Let $\mathbb{C}$ be the set of complex number with $\mathbb{C}^* = \mathbb{C} - \{0\}$.

In the sequel, we will use the following properties of the Appell-Lerch sums [10, 18, 24]:

$$m(x,q,z) = m(x,q,qz), \tag{1.7}$$

$$m(x,q,z) = x^{-1}m(x^{-1},q,z^{-1}), \tag{1.8}$$

$$m(x,q,z_1) - m(x,q,z_0) = \frac{z_0 J_1^3 j(z_1/z_0;q) j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}, \tag{1.9}$$

for generic $x, z_0, z_1 \in \mathbb{C}^*$. In this paper, we shall make use of the following definition of Hecke-type double sums [10].

Definition 1.1 *(cf. [10]) Let $x, y \in \mathbb{C}^*$ and define $sg(r) := 1$ for $r \geq 0$ and $sg(r) := -1$ for $r < 0$. Then*

$$\begin{aligned} f_{a,b,c}(x,y,q) : &= \sum_{sg(r)=sg(s)} sg(r)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}} \\ &= \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}, \end{aligned}$$

which is an indefinite theta series when $ac < b^2$. Here, we assume $a, c > 0$.

Jackson's transformation of ${}_2\varphi_1$ and ${}_3\varphi_2$ series [7, Eq. (III.5)], Heine's first transformation of ${}_2\varphi_1$ series ([7, Eq. (III.1)]) and transformation formula [7, E.x. 3.4] are restated respectively as follows:

$$\begin{aligned} {}_2\varphi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q, z\right] &= \frac{(\frac{abz}{c};q)_\infty}{(\frac{bz}{c};q)_\infty}\ {}_3\varphi_2\left[\begin{matrix} a, & \frac{c}{b}, & 0 \\ & c, & \frac{cq}{bz} \end{matrix}; q, q\right] \\ &\quad + \frac{(a, bz, \frac{c}{b};q)_\infty}{(c, z, \frac{c}{bz};q)_\infty}\ {}_3\varphi_2\left[\begin{matrix} z, & \frac{abz}{c}, & 0 \\ & bz, & \frac{bzq}{c} \end{matrix}; q, q\right]. \end{aligned} \tag{1.10}$$

$${}_2\varphi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q, z\right] = \frac{(b, az;q)_\infty}{(c, z;q)_\infty}\ {}_2\varphi_1\left[\begin{matrix} c/b, & z \\ & az \end{matrix}; q, b\right]. \tag{1.11}$$

and

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a,c;q)_{m+n}(b^2;q^2)_m}{(q;q)_m(q;q)_n(d;q)_{m+n}(b^2;q)_m}(-z)^m z^n \\ &= {}_4\varphi_3\left[\begin{matrix} a, & aq, & c, & cq \\ & d, & dq, & qb^2 \end{matrix}; q^2, z^2\right], \ |z| < 1. \end{aligned} \tag{1.12}$$

The following bilateral sum plays an essential role in constructing bilateral series for the mock theta function. Now, we state it as a lemma.

Lemma 1.2 ([18, Proposition 2.7]). For $a, b \neq 0$,

$$\sum_{n=-\infty}^{\infty} \frac{a^{-n-1}b^{-n}q^{n^2}}{(-\frac{1}{a};q)_{n+1}(-\frac{q}{b};q)_n} = \sum_{n=-\infty}^{\infty} (-aq;q)_n(-b;q)_{n+1}q^{n+1}$$
$$= \frac{(-aq;q)_\infty j(-b;q)}{b(q;q)_\infty(-\frac{q}{b};q)_\infty} m(\frac{a}{b}, q, -b). \tag{1.13}$$

In addition to the above notations, we also need the following bilateral series of mock theta function defined by

$$B(M;q) := \sum_{n=-\infty}^{\infty} c(n;q),$$

where $M(q) := \sum_{n=0}^{\infty} c(n;q)$ is a mock theta function and the tail of the bilateral series $B(M;q)$ is

$$\sum_{n=-\infty}^{-1} c(n;q) = D(M;q).$$

In this paper, we first make the replacement $n \to -n$ in $D(M,q)$ to obtain the new q-hypergeometric series in terms of Appell-Lerch sums. These q-series are referred to as duals of the second type [18]. Then, by using the substitution $q \to q^{-1}$ and the heuristic [10, 18]

$$m(x,q,z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$$

in $D(M,q)$, where "$\sim$" means up to the addition of a theta function, we attempt to attain the dual in terms of partial theta functions defined by Hickerson and Mortenson of dual of the second type in terms of Appell-Lerch sums of such bilateral series associated to mock theta function $\mathfrak{D}_5(q)$. However, it is regretted that we find it does not exist.

Recently, mock theta functions were studied by Hickerson and Mortenson [10]. They gave representations of mock theta functions in terms of Appell-Lerch sums. Choi [5] and Mc Laughlin[17] used bilateral basic hypergeometric series to estabish some mock theta function identities. Chen [4], Hu et al. [13] gave the Appell-Lerch sums representations of the bilateral series associated to mock theta functions, and studied their dual nature. Motivated by the abovementioned results, we study new representations of the second-order mock theta function $\mathfrak{D}_5(q)$ and the dual nature of the bilateral series associated to it in this paper.

This paper is organized as follows. In Section 2, we mainly make use of basic hypergeometric transformation formulae to obtain some new representations concerning Hikami's second mock theta function $\mathfrak{D}_5(q)$. Finally, we close this paper with dual nature of bilateral series associated to $\mathfrak{D}_5(q)$.

2. Main results and proofs

In this section, we mainly use the mentioned results in the previous section to give some new representations for the second-order mock theta function $\mathfrak{D}_5(q)$ and discuss the dual nature of the bilateral series associated to $\mathfrak{D}_5(q)$.

In Theorem 2.1 below, $\mathfrak{D}_5(q)$ is expressed as the linear sum of the second-order mock theta function $B(q)$ and the third-order mock theta function $\nu(q)$.

Theorem 2.1 *The following identity holds true:*

$$\sqrt{q}\mathfrak{D}_5(q) = B(\sqrt{q}) - \frac{J_2^2}{J_1^2}\nu(-\sqrt{q}), \tag{2.1}$$

where $B(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^n}{(q;q^2)_{n+1}}$ *and* $\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}$ *are the second-order and third-order mock theta functions (cf. [8, 10]), respectively.*

Proof Taking $a = q, b = -q^{\frac{1}{2}}, c = q^{\frac{3}{2}}, z = q^{\frac{1}{2}}$ in (1.10), we gain that

$$\sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}};q)_n q^{\frac{n}{2}}}{(q^{\frac{3}{2}};q)_n} = \frac{1}{1+q^{-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-q;q)_n q^n}{(q^3;q^2)_n} + \frac{(q^2;q^2)_\infty(-q;q)_\infty}{(1+q^{\frac{1}{2}})(q^3;q^2)_\infty(q^{\frac{1}{2}};q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{\frac{1}{2}};q)_n q^n}{(q^2;q^2)_n}.$$

Multiplying both sides of the above identity by $\frac{1}{1-q}$, we achieve that

$$\frac{1}{1-q}\sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}};q)_n q^{\frac{n}{2}}}{(q^{\frac{3}{2}};q)_n} = \frac{1}{1+q^{-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-q;q)_n q^n}{(q;q^2)_{n+1}} + \frac{(q^2;q^2)_\infty(-q;q)_\infty}{(1+q^{\frac{1}{2}})(q;q^2)_\infty(q^{\frac{1}{2}};q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{\frac{1}{2}};q)_n q^n}{(q^2;q^2)_n}.$$

Through certain simplifications, we have that

$$\sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}};q)_n q^{\frac{n}{2}}}{(q^{\frac{1}{2}};q)_{n+1}} = q^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-q;q)_n q^n}{(q;q^2)_{n+1}} + \frac{(q^2;q^2)_\infty(-q;q)_\infty}{(q;q^2)_\infty(q^{\frac{1}{2}};q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{\frac{1}{2}};q)_n q^n}{(q^2;q^2)_n}.$$

Making the substitution $q \to q^2$, we obtain that

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^n}{(q;q^2)_{n+1}} = q \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{2n}}{(q^2;q^4)_{n+1}} + \frac{(q^4;q^4)_\infty(-q^2;q^2)_\infty}{(q^2;q^4)_\infty(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(q^4;q^4)_n},$$

i.e.

$$B(q) = q\mathfrak{D}_5(q^2) + \frac{J_4^3}{J_1 J_2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(q^4;q^4)_n}. \tag{2.2}$$

Replacing q, a, b, c, z by $q^2, q, 0, -q^2, q^2$ in (1.11), respectively, we attain that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n}}{(q^4;q^4)_n} &= \frac{(q^3;q^2)_\infty}{(-q^2,q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^3;q^2)_n}\\ &= \frac{(q;q^2)_\infty}{(q^4;q^4)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q^2)_{n+1}}\\ &= \frac{(q;q^2)_\infty}{(q^4;q^4)_\infty}\nu(-q)\\ &= \frac{J_1}{J_2J_4}\nu(-q). \end{aligned}\tag{2.3}$$

Combining (2.2) with (2.3), we derive that

$$q\mathfrak{D}_5(q^2) = B(q) - \frac{J_4^2}{J_2^2}\nu(-q).$$

Making the substitution $q \to q^{1/2}$, we have that

$$\sqrt{q}\mathfrak{D}_5(q) = B(\sqrt{q}) - \frac{J_2^2}{J_1^2}\nu(-\sqrt{q}).$$

This completes the proof of the identity (2.1). □

Remark: Making use of the results of the previously mentioned and $h_1(q^2) = \frac{B(q)-B(-q)}{4q}$ ([8, P. 136]), the identity (2.1) can be reduced to $\nu(q) + q\omega(q^2) = \frac{J_4^3}{J_2^2}$ ([23, P. 71]).

In Theorem 2.2 below, we give three different representations of $\mathfrak{D}_5(q)$.

Theorem 2.2 *Each of the following identities holds true:*

$$\mathfrak{D}_5(q) = -\frac{1}{q}m(1,q^2,q^{3/2}) + \frac{J_2^2}{qJ_1^2}\left(m(q,q^6,q^{3/2}) + m(q,q^6,q^{9/2})\right)\tag{2.4}$$

$$= \frac{J_2^2}{J_1^2}\left(m(q,q^6,q^2) + m(q,q^6,q^4)\right) - \frac{J_2 j(q;q^2)}{qJ_1^2}m(1,q^2,-1) + \frac{J_2^2}{4qJ_1^2}\frac{j(-q;q^2)^3}{J_4^2}\tag{2.5}$$

$$= \frac{J_2^2}{J_1^2}\left(m(q,q^6,q^2) + m(q,q^6,q^4)\right) - \frac{J_2 j(q;q^2)}{qJ_1^2}m(1,q^2,q).\tag{2.6}$$

Proof Plugging the identity of the second-order mock theta function $B(q)$ in terms of the Appell-Lerch sums ([10, Eq. (4.2)])

$$B(q) = -q^{-1}m(1,q^4,q^3),$$

and the identity of the second-order mock theta function $\nu(q)$ in terms of the Appell-Lerch sums ([10, Eq. (4.9)])

$$\nu(q) = q^{-1}m(q^2,q^{12},-q^3) + q^{-1}m(q^2,q^{12},-q^9),$$

into (2.1), we get that

$$q^{\frac{1}{2}}\mathfrak{D}_5(q) = -q^{-\frac{1}{2}}m(1,q^2,q^{3/2}) + q^{-\frac{1}{2}}\frac{J_2^2}{J_1^2}\left(m(q,q^6,q^{3/2}) + m(q,q^6,q^{9/2})\right).$$

Multiplying both sides by $q^{\frac{-1}{2}}$, we obtain our desired result

$$\mathfrak{D}_5(q) = -\frac{1}{q}m(1,q^2,q^{3/2}) + \frac{J_2^2}{qJ_1^2}\left(m(q,q^6,q^{3/2}) + m(q,q^6,q^{9/2})\right).$$

This completes the proof of the identity (2.4).

Recall that two identities obtained by Mortenson: [18, Eq. (4.15)]

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{2n+1}(-aq,-\frac{q}{a};q^2)_n\\ &\quad= -qg(-aq,q^2) + a\frac{j(-aq;q^2)}{J_2}m(a^2,q^2,-1) - \frac{1}{2}\frac{aj(aq;q^2)^3j(a^2;q^4)}{J_4^2j(a^4;q^4)}\end{aligned} \tag{2.7}$$

and [18, Eq. (4.16)]

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{2n+1}(-aq,-\frac{q}{a};q^2)_n\\ &\quad= -qg(-aq,q^2) + a\frac{j(-aq;q^2)}{J_2}m(a^2,q^2,-a^{-1}q),\end{aligned} \tag{2.8}$$

where $g(x,q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(x;q)_{n+1}(\frac{q}{x};q)_{n+1}}$.

By using (1.7), (1.8) and $g(x,q) = -x^{-1}m(q^2x^{-3},q^3,x^2) - x^{-2}m(qx^{-3},q^3,x^2)$ (cf. [9]), the identities (2.5) and (2.6) follow from (2.7) and (2.8) with $a=-1$, respectively. $\square$

Using (2.4) and the known results, we easily deduce the following result stated by Corollary 2.3 below.

Corollary 2.3 *The following theta identity holds true:*

$$\frac{q^2J_{1,12}}{J_{3,12}J_{2,12}} - \frac{qJ_{5,12}}{J_{6,12}J_{1,12}} = \frac{J_4^4J_{4,12}}{qJ_2^2J_{12}^3}.$$

Proof In terms of (1.4) and the definition of the Appell-Lerch sums, we have that

$$h_1(q) = -\frac{1}{2q}m(1,q^2,q^3).$$

Since ([10, Eq. (4.8)])

$$\omega(q) = -q^{-1}m(q,q^6,q^2) - q^{-1}m(q,q^6,q^4), \tag{2.9}$$

then by (1.3) we get that

$$q\mathfrak{D}_5(q) = -m(1,q^2,q^3) + \frac{J_2^2}{J_1^2}\left(m(q,q^6,q^2) + m(q,q^6,q^4)\right). \tag{2.10}$$

Combining (2.4) with (2.10) and after some simplifications, we derive that

$$\begin{aligned} &m(1,q^2,q^{3/2})-m(1,q^2,q^3)\\ &\qquad=\frac{J_2^2}{J_1^2}\left(m(q,q^6,q^{3/2})+m(q,q^6,q^{9/2})-m(q,q^6,q^2)-m(q,q^6,q^4)\right).\end{aligned}$$

Making the substitution $q\to q^2$, we derive that

$$\begin{aligned} &m(1,q^4,q^3)-m(1,q^4,q^6)\\ &\qquad=\frac{J_4^2}{J_2^2}\left(m(q^2,q^{12},q^3)+m(q^2,q^{12},q^9)-m(q^2,q^{12},q^4)-m(q^2,q^{12},q^8)\right).\end{aligned}$$

By (1.9) and after some simplifications, we obtain Corollary 2.3. This completes the proof. □

In addition, the second-order mock theta function $\mathfrak{D}_5(q)$ can also be expressed in terms of the Hecke-type double sums as follows.

Theorem 2.4 *The representation of $\mathfrak{D}_5(q)$ in terms of the Hecke-type double sums is as follows:*

$$\mathfrak{D}_5(q)=\frac{J_2}{J_1^2}f_{3,2,1}(q^6,q^3,q^2).$$

Proof Recall that [18, Eq. (4.17)]:

$$\sum_{n=0}^{\infty}q^{2n+1}(-aq,-\frac{q}{a};q^2)_n=\frac{qf_{3,2,1}(q^6,-aq^3,q^2)}{J_2}.$$

Taking $a=-1$, we get that

$$\sum_{n=0}^{\infty}q^{2n+1}(q;q^2)_n^2=\frac{qf_{3,2,1}(q^6,q^3,q^2)}{J_2}.$$

Multiplying both sides of the above identity by $\frac{1}{(q;q^2)_\infty^2}$ and after some simplifications, we derive that

$$\frac{1}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}q^{2n}(q;q^2)_n^2=\frac{J_2}{J_1^2}f_{3,2,1}(q^6,q^3,q^2).$$

This completes the proof. □

It is all known that the basic hypergeometric series is a basic tool to study mock theta functions. Making use of certain transformation formula, we can derive the following result as follows.

Theorem 2.5 *The following representation of $\mathfrak{D}_5(q)$ in terms of q-hypergeometric double sums holds true:*

$$\mathfrak{D}_5(q)=\frac{J_2^2}{J_1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m q^{m+mn+n}\frac{(q;q^2)_m}{(q;q)_m(q^2;q)_{m+n}}. \tag{2.11}$$

Proof Taking $a = c = q, b = q^{\frac{1}{2}}, d = 0, z = q$ in (1.12), we get that

$$\sum_{n=0}^{\infty}(q;q^2)_n^2 q^{2n} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q;q)_{m+n}^2(q;q^2)_m(-1)^m q^{m+n}}{(q;q)_m^2(q;q)_n}.$$

Multiplying both sides of the above identity by $\frac{1}{(q;q^2)_\infty^2}$ and after certain simplifications, we obtain that

$$\begin{aligned}\frac{1}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}(q;q^2)_n^2 q^{2n} &= \frac{J_2^2}{J_1^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q;q)_{m+n}^2(q;q^2)_m(-1)^m q^{m+n}}{(q;q)_m^2(q;q)_n} \\ &= \frac{J_2^2}{J_1^2}\sum_{m=0}^{\infty}(q;q^2)_m(-q)^m {}_2\varphi_1\left[\begin{matrix} q^{m+1}, & q^{m+1} \\ & 0 \end{matrix}; q, q\right].\end{aligned}$$

By using (1.11) in the above identity, we obtain that

$$\begin{aligned}\frac{1}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}(q;q^2)_n^2 q^{2n} &= \frac{J_2^2}{J_1^2}\sum_{m=0}^{\infty}(q;q^2)_m(-q)^m\frac{J_1}{(q;q)_m(q^2;q)_m}\sum_{n=0}^{\infty}\frac{q^{(m+1)n}}{(q^{m+2};q)_n} \\ &= \frac{J_2^2}{J_1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m q^{m+mn+n}\frac{(q;q^2)_m}{(q;q)_m(q^2;q)_{m+n}}.\end{aligned}$$

Thus, we obtain our desired result (2.11). □

The bilateral series $B(\mathfrak{D}_5;q) = \frac{1}{(q;q^2)_\infty^2}\sum_{n=-\infty}^{\infty}(q;q^2)_n^2 q^{2n}$ was studied by Srivastava [20]. He derived a bilateral representation of $\mathfrak{D}_5;q)$ and established a relationship between $B(\mathfrak{D}_5;q)$ and the third-order mock theta function $B(\omega;q)$. Motivated by these developments, we study the bilateral series of $\mathfrak{D}_5(q)$ again. First, we attain the following new representation of $B(\mathfrak{D}_5;q)$ in terms of the Appell-Lerch sums by using the identity (1.13).

Theorem 2.6 *The following representation of $B(\mathfrak{D}_5;q)$ in terms of the Appell-Lerch sums holds true:*

$$B(\mathfrak{D}_5;q) = -\frac{1}{q}m(1,q^2,q), \tag{2.12}$$

where $B(\mathfrak{D}_5;q) = \frac{1}{(q;q^2)_\infty^2}\sum_{n=-\infty}^{\infty}(q;q^2)_n^2 q^{2n}$.

Proof Making the substitution $q \to q^2$ in (1.13), we get that

$$\sum_{n=-\infty}^{\infty}(-aq^2;q^2)_n(-b;q^2)_{n+1}q^{2n+2} = \frac{(-aq^2;q^2)_\infty j(-b;q^2)}{b(q^2;q^2)_\infty(-\frac{q^2}{b};q^2)_\infty}m(\frac{a}{b},q^2,-b).$$

Taking $a = b = -q^{-1}$, we attain that

$$(1-q^{-1})q^2\sum_{n=-\infty}^{\infty}(q;q^2)_n^2 q^{2n} = \frac{(q;q^2)_\infty j(q^{-1};q^2)}{-q^{-1}(q^2;q^2)_\infty(q^3;q^2)_\infty}m(1,q^2,q^{-1}).$$

By using (1.7) and after some simplifications, we derive that

$$\sum_{n=-\infty}^{\infty}(q;q^2)_n^2 q^{2n} = -\frac{j(q;q^2)}{q(q^2;q^2)_\infty} m(1,q^2,q).$$

Multiplying both sides of the last equation by $\frac{1}{(q;q^2)_\infty^2}$ and after certain simplifications, we get our desired result. This completes the proof. □

Corollary 2.7 *The following identity of the Appell-Lerch sums holds true:*

$$m(1,q^2,q) = \frac{J_2^2 J_{1,4}}{J_1^2 J_4} m(q^2,q^4,q). \tag{2.13}$$

Proof Recall [13, Eq. (42)]

$$B(\omega;q) = -\frac{1}{q}\frac{J_{1,4}}{J_4} m(q^2,q^4,q), \tag{2.14}$$

and the bilateral third-order mock theta function $B(\omega;q)$ [20, Eq. (3.5)]:

$$(q;q^2)_\infty^2 B(\mathfrak{D}_5;q) = B(\omega;q). \tag{2.15}$$

Plugging (2.14) into (2.15), we get that

$$B(\mathfrak{D}_5;q) = -\frac{J_2^2 J_{1,4}}{q J_1^2 J_4} m(q^2,q^4,q) \tag{2.16}$$

Combining (2.12) with (2.16) and after certain simplifications, we derive our desired result (2.13). This completes the proof. □

Finally, we close this paper with the dual nature of the bilateral series associated to mock theta function $\mathfrak{D}_5(q)$.

Theorem 2.8 *For the second-order mock theta function* $\mathfrak{D}_5(q)$ *, the dual of the second type in terms of Appell-Lerch sums of such mock theta function is the following,*

$$D(\mathfrak{D}_5;q) = \frac{1}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} = -\frac{J_2^2}{qJ_1^2}\left(m(q,q^6,q^2)+m(q,q^6,q^4)\right). \tag{2.17}$$

Proof

$$\begin{aligned}
D(\mathfrak{D}_5;q) &= \frac{1}{(q;q^2)_\infty^2}\sum_{n=-\infty}^{-1}(q;q^2)_n^2 q^{2n} \\
&= \frac{1}{(q;q^2)_\infty^2}\sum_{n=1}^{\infty}\frac{q^{2n^2-2n}}{(q;q^2)_n^2} \\
&= \frac{1}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} \\
&= \frac{1}{(q;q^2)_\infty^2}\omega(q).
\end{aligned}$$

Using (2.9), we obtain that

$$D(\mathfrak{D}_5;q) = -\frac{J_2^2}{qJ_1^2}\left(m(q,q^6,q^2) + m(q,q^6,q^4)\right).$$

This completes the proof. □

Remark: By using the substitution $q \to q^{-1}$ in $D(*,q)$ of the dual of the second type for the second-order mock theta function $\mathfrak{D}_5(q)$ and after computing, we find the associated Eulerian series is not absolutely convergent for $|q| < 1$, so its dual does not exist.

3. Conclusion

This paper is motivated essentially by some representations of Hikami's mock theta functions. In our present investigation, we have established some new presentations of Hikami's mock theta function $\mathfrak{D}_5(q)$. Meanwhile, we have also obtained a new representation of $B(\mathfrak{D}_5;q)$ in terms of the Appell-Lerch sums, and further discussed the dual of the bilateral series associated to the second-order mock theta function $B(\mathfrak{D}_5;q)$. Finally, we have found that the dual in terms of partial theta functions of the dual of the second type in terms of Appell-Lerch sums of such mock theta function does not exists. Mock theta functions involve some problems about mock modular forms, partition function, etc. And there are very widely applications in areas of number theory, algebra, combinatorial mathematics, etc. Therefore, research on mock theta functions will be a valuable subject. Different generalizations of $\mathfrak{D}_5(q)$ and its bilateral series will be also worth studying.

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