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On estimation of the number of eigenvalues of the magnetic Schrödinger operator in a three-dimensional layer

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Abstract: In this paper, we study the magnetic Schrödinger operator in a three-dimensional layer. We obtain an estimate for the number of eigenvalues of this operator lying to the left of the essential spectrum threshold. We show that the magnetic Schrödinger operator to the left of the continuous spectrum threshold can have only a finite number of eigenvalues of infinite multiplicity.

Key words: Magnetic Schrödinger operator, superconductor of the second kind, critical magnetic field, eigenvalues

1. Introduction

It is known that (see [1, 4, 5, 8, 9, 12]) at a certain value of the magnetic field strength in superconducting materials of the second kind, a surface superconducting phenomenon occurs. This kind of magnetic field is called the third critical field. For superconducting materials with limited cross sections, when determining the intensity of the critical field by means of division of the unity and the localization formula of IMS (Ismagilov, Morgan, Simon, Sigal; see [2, p. 27]), using the magnetic Laplacian $\left(-i\nabla - \vec{A}\right)^2$, where $\vec{A} = (A_1, A_2, A_3)$ is the magnetic potential, various model magnetic Schrödinger operators are constructed (see [3, 8, 9, 10]) and their eigenvalues are used, which lie to the left of the essential spectrum threshold.

The goal of the paper is to study one of these models in a three-dimensional layer, to estimate the number of eigenvalues of this operator lying to the left of the continuous spectrum threshold, and, using this estimate, to obtain information on the number of such eigenvalues.

2. Main results

Let a superconductor of the second kind in the form of the layer

$$\Pi_{0,R} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \mathbb{R}^2,\ \ 0 < x_3 < R\right\},$$

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where $R > 0$ and $\mathbb{R}^j$ is the j-dimensional Euclidean space, be acted upon by the external magnetic field

$$\overrightarrow{B} = (\beta_{23}; -\beta_{13}; \beta_{12})$$

of the constant tension $b = \sqrt{\beta_{23}^2 + \beta_{13}^2 + \beta_{12}^2}$, making an angle θ $\left(0 < \theta < \dfrac{\pi}{2}\right)$ with the plane $x_3 = 0$.

Due to the gauge-invariant property of the magnetic Laplacian (see [6]), it is possible to choose the magnetic potential $\overrightarrow{A} = (A_1, A_2, A_3)$, to which corresponds the above-mentioned magnetic field $\overrightarrow{B}$, as follows:

$$\vec{A} = (0; b(x_1 \sin\theta - x_3 \cos\theta); 0).$$

Indeed, with this choice of the magnetic potential, the coordinates of the magnetic field $\overrightarrow{B}$ take the following form:

$$\beta_{23} = \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} = -b\cos\theta,$$

$$\beta_{13} = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} = 0,$$

$$\beta_{12} = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} = -b\sin\theta.$$

Hence we have

$$\left|\vec{B}\right| = \sqrt{\beta_{23}^2 + \beta_{13}^2 + \beta_{12}^2} = \sqrt{b^2\cos^2\theta + b^2\sin^2\theta} = b.$$

We introduce the operator $H_R(\theta) : L_2(\Pi_{0,R}) \to L_2(\Pi_{0,R})$ according to the formula

$$H_R(\theta)\psi(x) = -\frac{\partial^2\psi(x)}{\partial x_1^2} - \frac{\partial^2\psi(x)}{\partial x_3^2} + \left(-i\frac{\partial}{\partial x_2} - b(x_1\sin\theta - x_3\cos\theta)\right)^2 \psi(x)$$

with the domain of definition

$$D(H_R(\theta)) = \Bigg\{\psi\,(x) \in L_2(\Pi_{0,R}) : -\frac{\partial^2\psi(x)}{\partial x_1^2} - \frac{\partial^2\psi(x)}{\partial x_3^2} +$$

$$+\left(-i\frac{\partial}{\partial x_2} - b(x_1\sin\theta - x_3\cos\theta)\right)^2 \psi\,(x) \in L_2(\Pi_{0,R});$$

$$\left.\frac{\partial\psi(x)}{\partial x_3}\right|_{x_3=0} = 0,\ \psi(x_1, x_2, R) = 0\Bigg\}.$$

Lemma 2.1 *$H_R(\theta)$ is a nonnegative selfadjoint operator in the Hilbert space $L_2(\Pi_{0,R})$.*

The proof of the lemma follows from the Leinfelder–Zimader theorem on the selfadjointness of the magnetic Schrödinger operator (see [7]).

Let denote by $H_R(\theta;\xi_2)$ an operator obtained from the operator $H_R(\theta)$ using the Fourier transform with respect to the variable x_2. Obviously, the operator $H_R(\theta;\xi_2)$ is defined on the linear manifold

$$D(H_R(\theta;\xi_2)) = \left\{ \overset{\wedge}{\psi}(x_1,\xi_2,x_3) \in L_2(\overset{\wedge}{\Pi}_{0,R}) : -\frac{\partial^2 \overset{\wedge}{\psi}}{\partial x_1^2} - \right.$$

$$-\frac{\partial^2 \overset{\wedge}{\psi}}{\partial x_3^2} - (\xi_2 - b(x_1 \sin\theta - x_3\cos\theta))^2 \overset{\wedge}{\psi} \in L_2(\overset{\wedge}{\Pi}_{0,R}),$$

$$\left. \left.\frac{\partial \overset{\wedge}{\psi}(x_1,\xi_2,x_3)}{\partial x_3}\right|_{x_3=0} = 0,\ \overset{\wedge}{\psi}(x_1,\xi_2,R) = 0 \right\}$$

and acts according to the rule

$$H_R(\theta;\xi_2)\overset{\wedge}{\psi}(x_1,\xi_2,x_3) = -\frac{\partial^2 \overset{\wedge}{\psi}}{\partial x_1^2} - \frac{\partial^2 \overset{\wedge}{\psi}}{\partial x_3^2} - (\xi_2 - b(x_1\sin\theta - x_3\cos\theta))^2 \overset{\wedge}{\psi},$$

where

$$\overset{\wedge}{\psi}(x_1,\xi_2,x_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x_1,x_2,x_3) e^{i\xi_2 x_2} dx_2,$$

$$\overset{\wedge}{\Pi}_{0,R} = \left\{ (x_1,\xi_2,x_3) : x_1 \in \mathbb{R}^1, \xi_2 \in \mathbb{R}^1, 0 < x_3 < R \right\}.$$

The unitarity of the Fourier transform operator implies that the following lemma holds.

Lemma 2.2 *For any $\xi_2 \in \mathbb{R}^1$ and $\theta \in \left(0, \frac{\pi}{2}\right)$ the operator $H_R(\theta;\xi_2)$ is selfadjoint and the spectrum of the operator $H_R(\theta)$ is the union of the spectra of the operators $H_R(\theta;\xi_2)$, i.e.*

$$\sigma(H_R(\theta)) = \bigcup_{-\infty<\xi_2<+\infty} \sigma(H_R(\theta;\xi_2)).$$

Lemma 2.3 *The spectrum of the operators $H_R(\theta;\xi_2)$ does not depend on the parameter ξ_2.*

Proof Performing the linear transformation

$$\begin{cases} x_1 = \eta_1 + \frac{\xi_2}{b\sin\theta}, \\ \xi_2 = \eta_2, \\ x_3 = \eta_3 \end{cases}$$

and introducing the notation

$$\varphi(\eta) \equiv \varphi(\eta_1,\eta_2,\eta_3) = \overset{\wedge}{\psi}(\eta_1 + \frac{\eta_2}{b\sin\theta}, \eta_2, \eta_3),$$

we obtain

$$-\frac{\partial^2 \overset{\wedge}{\psi}(x_1,\xi_2,x_3)}{\partial x_1^2} - \frac{\partial^2 \overset{\wedge}{\psi}(x_1,\xi_2,x_3)}{\partial x_3^2} + (\xi_2 - b(x_1\sin\theta - x_3\cos\theta))^2 \overset{\wedge}{\psi}(x_1,\xi_2,x_3) =$$

$$= -\frac{\partial^2\varphi(\eta)}{\partial\eta_1^2} - \frac{\partial^2\varphi(\eta)}{\partial\eta_3^2} + \left\{\xi_2 - b\left[(\eta_1 + \frac{\eta_2}{b\sin\theta})\sin\theta - \eta_3\cos\theta\right]\right\}^2\varphi(\eta) =$$

$$= -\frac{\partial^2\varphi(\eta)}{\partial\eta_1^2} - \frac{\partial^2\varphi(\eta)}{\partial\eta_3^2} + b^2(\eta_3\cos\theta - \eta_1\sin\theta)^2\varphi(\eta). \tag{2.1}$$

It follows from equality (2.1), that for any $\xi_2 \in \mathbb{R}^1$ there holds the equality

$$\sigma(H_R(\theta;\xi_2)) = \sigma(H_R(\theta;0)).$$

This implies the assertion of the lemma. □

Consider in the Hilbert space $L_2(\mathrm{M}_{0,R\sqrt{b}})$ the operator $P_R^b(\theta)$ generated by the differential expression

$$l_R(\theta) = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + (y\cos\theta - x\sin\theta)^2$$

with the domain of definition

$$D(P_R^b(\theta)) =$$

$$= \left\{u(x,y) \in L_2(M_{0,R\sqrt{b}}) : l_R(\theta)u \in L_2(M_{0,R\sqrt{b}});\ \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, u(x, R\sqrt{b}) = 0\right\},$$

where $M_{0,R\sqrt{b}} = \left\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}^1, 0 < y < R\sqrt{b}\right\}$.

Lemma 2.4 *There holds the equality* $\sigma(H_R(\theta,0)) = b \times \sigma(P_R^b(\theta))$.

Proof Using the transformation

$$\begin{cases} \eta_1 = \dfrac{x}{\sqrt{b}}, \\ \eta_3 = \dfrac{y}{\sqrt{b}} \end{cases}$$

and introducing the notation

$$u(x,y) = \varphi\left(\frac{x}{\sqrt{b}}, \eta_2, \frac{y}{\sqrt{b}}\right),$$

from (2.1), we obtain

$$-\frac{\partial^2\varphi(\eta)}{\partial\eta_1^2} - \frac{\partial^2\varphi(\eta)}{\partial\eta_3^2} + b^2(\eta_3\cos\theta - \eta_1\sin\theta)^2\varphi(\eta) =$$

$$= b\left[-\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} + (y\cos\theta - x\sin\theta)^2\right]u(x,y). \tag{2.2}$$

The assertion of the lemma follows from formula (2.2). □

Lemma 2.5 *The essential spectrum of the operator* $P_R^b(\theta)$ *coincides with the semiaxis* $[1, +\infty)$.

The proof of the lemma follows from Persson's theorem (see [3, p. 273]).

Now let us estimate the number of eigenvalues of the operator $P_R^b(\theta)$ lying to the left of a unity.

Let us introduce the notation

$$N(1, P_R^b(\theta)) = \sum_{\lambda_n(P_R^b(\theta))<1} 1,$$

where $\lambda_n(P_R^b(\theta))$ is the n-th eigenvalue of the operator $P_R^b(\theta)$.

Theorem 2.6 *For any angle $\theta \in \left(0, \frac{\pi}{2}\right)$ and for any positive number R, the following inequality holds:*

$$N(1, P_R^b(\theta)) \le \frac{R\sqrt{b}}{\pi\sqrt{2}\sin\theta}\left(1 + R^2 b\cos^2\theta - \frac{\sin\theta}{\sqrt{2}}\right)^{3/2}. \tag{2.3}$$

Proof Consider three auxiliary operators A, B_y, and C defined as follows:

$$Ah(y) = -\frac{d^2h(y)}{dy^2},\ h(y) \in D(A),$$

$$D(A) = \left\{h(y) \in L_2\left(0, R\sqrt{b}\right) : -\frac{d^2h(y)}{dy^2} \in L_2\left(0, R\sqrt{b}\right),\ h'(0) = h\left(R\sqrt{b}\right) = 0\right\};$$

$$B_y g(x) = -\frac{d^2g(x)}{dx^2} + (y\cos\theta - x\sin\theta)^2 g(x), g(x) \in D(B_y),\ y \in \left(0, R\sqrt{b}\right),$$

$$D(B_y) = \Big\{g(x) \in L_2(-\infty, +\infty) :$$

$$-\frac{d^2g(x)}{dx^2} + (y\cos\theta - x\sin\theta)^2 g(x) \in L_2(-\infty, +\infty)\Big\};$$

$$C\upsilon(x) = -\frac{d^2\upsilon(x)}{dx^2} + \left(\frac{1}{2}x^2\sin^2\theta - R^2b\cos^2\theta\right)\upsilon(x),\ \upsilon(x) \in D(C),$$

$$D(C) = \Big\{\upsilon(x) \in L_2(-\infty, +\infty) :$$

$$-\frac{d^2\upsilon(x)}{dx^2} + \left(\frac{1}{2}x^2\sin^2\theta - R^2b\cos^2\theta\right)\upsilon(x) \in L_2(-\infty, +\infty)\Big\}.$$

First, let us prove that $B_y \ge C$ for any $y \in (0, R\sqrt{b})$.

We have

$$(y\cos\theta - x\sin\theta)^2 = y^2\cos^2\theta + x^2\sin^2\theta - 2xy\sin\theta\cos\theta =$$

$$= y^2\cos^2\theta + x^2\sin^2\theta - 2\left(\frac{x}{\sqrt{2}}\sin\theta \cdot \sqrt{2}y\cos\theta\right) \ge$$

$$\ge y^2\cos^2\theta + x^2\sin^2\theta - \frac{x^2}{2}\sin^2\theta - 2y^2\cos^2\theta = \frac{x^2}{2}\sin^2\theta - y^2\cos^2\theta. \tag{2.4}$$

Taking into account that the number y is on the interval $(0, R\sqrt{b})$, from (2.4), we obtain

$$(y\cos\theta - x\sin\theta)^2 \geq \frac{x^2}{2}\sin^2\theta - R^2b\cos^2\theta. \tag{2.5}$$

It follows from inequality (2.5) that $B_y \geq C$.

The minimax principle (see [13, p. 75]) implies that

$$N(1, P_R^b(\theta)) \leq N(1, A) + N(1, B_y). \tag{2.6}$$

Taking into account the inequality $B_y \geq C$ and using the minimax principle once again, we obtain

$$N(1, B_y) \leq N(1, C). \tag{2.7}$$

It follows from inequalities (2.6) and (2.7) that

$$N(1, P_R^b(\theta)) \leq N(1, A) + N(1, C). \tag{2.8}$$

Obviously, the eigenvalues of the operator A will be

$$\lambda_j(A) = \frac{\pi^2}{R^2b}\left(j + \frac{1}{2}\right)^2, \ j = 0, 1, 2, \ldots . \tag{2.9}$$

To find the eigenvalues of the operator C in the equation

$$-\frac{d^2\upsilon}{dx^2} + \left(\frac{1}{2}x^2\sin^2\theta - R^2b\cos^2\theta\right)\upsilon = \nu\upsilon, \tag{2.10}$$

where ν is the spectral parameter, we make the substitution

$$t = \frac{\sqrt{\sin\theta}}{\sqrt[4]{2}}x. \tag{2.11}$$

Given the equality

$$-\frac{d^2}{dx^2} = -\frac{d^2}{dt^2}\frac{\sin\theta}{\sqrt{2}},$$

substitution (2.11) and introducing the notation

$$T(t) = \upsilon\left(\frac{\sqrt[4]{2}}{\sqrt{\sin\theta}}t\right),$$

from (2.10), we obtain

$$-\frac{d^2T(t)}{dt^2} + t^2T(t) = \frac{\sqrt{2}}{\sin\theta}(\nu + R^2b\cos^2\theta)T(t). \tag{2.12}$$

Taking into account that the eigenvalues of the harmonic operator

$$-\frac{d^2}{dt^2} + t^2$$

are the numbers $2k+1$, $k=0,1,2,...$, from (2.12) we obtain that the eigenvalues of the operator C will be

$$\nu_k = \frac{2k+1}{\sqrt{2}}\sin\theta - R^2 b\cos^2\theta, k=0,1,2,\dots .$$

Using explicit forms (see (2.9) and (2.12)) of the eigenvalues of the operators A and C, from inequality (2.8), we obtain

$$N(1, P_R^b(\theta)) \leq$$

$$\leq card\left\{(k;j)\in\mathbb{N}_0\times\mathbb{N}_0 : \frac{2k+1}{\sqrt{2}}\sin\theta - R^2b\cos^2\theta + \frac{\pi^2}{R^2b}\left(j+\frac{1}{2}\right)^2 < 1\right\}. \tag{2.13}$$

Here, $card\, Q$ denotes the cardinality of the set Q, and $\mathbb{N}_0 = \{0\}\bigcup\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.

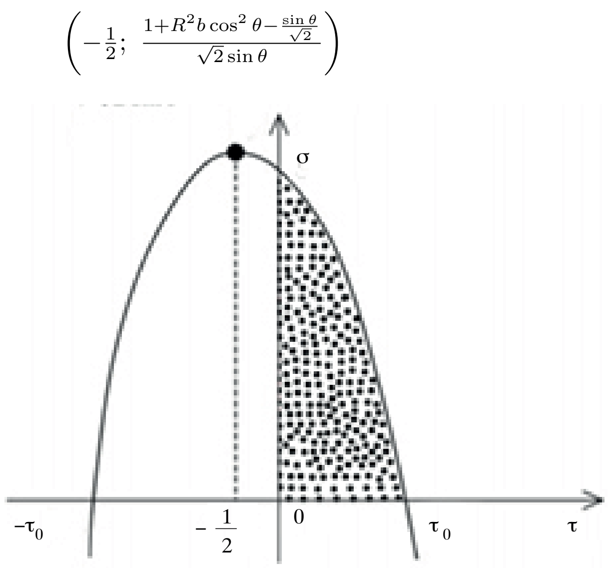

Figure. The area S.

Obviously, the pairs $(k;j)$ satisfying the inequality

$$\frac{2k+1}{\sqrt{2}}\sin\theta - R^2b\cos^2\theta + \frac{\pi^2}{R^2b}\left(j+\frac{1}{2}\right)^2 < 1,$$

are located in the first quarter of the plane (σ,τ) (see Figure) and are inside the region bounded by the straight line $\sigma=0$ and the curve

$$\sigma = -\frac{\pi^2}{\sqrt{2}R^2b\sin\theta}\left(\tau+\frac{1}{2}\right)^2 + \frac{1+R^2b\cos^2\theta - \frac{\sin\theta}{\sqrt{2}}}{\sqrt{2}\sin\theta}. \tag{2.14}$$

The curve (2.14) intersects the axis $\sigma=0$ at the points $\pm\tau_0$, where

$$\tau_0 = -\frac{1}{2} + \frac{R\sqrt{b}}{\pi}\sqrt{1+R^2b\cos^2\theta - \frac{\sin\theta}{\sqrt{2}}}. \tag{2.15}$$

Let S denote the area of the region bounded by the lines $\sigma = 0$, $\tau = 0$ and the curve (2.14), and located in the first quarter.

From inequality (2.13), we obtain

$$N(1, P_R^b(\theta)) \le S. \tag{2.16}$$

Let us calculate S:

$$S = \int_0^{\tau_0} \sigma(\tau) d\tau = \int_0^{\tau_0} \left[-\frac{\pi^2}{\sqrt{2} R^2 b \sin\theta} (\tau + \frac{1}{2})^2 + \frac{1 + R^2 b \cos^2\theta - \frac{\sin\theta}{\sqrt{2}}}{\sqrt{2}\sin\theta} \right] d\tau =$$

$$= \frac{1}{\sqrt{2}\sin\theta} \left\{ -\frac{\pi^2}{3R^2 b} \left[(\tau_0 + \frac{1}{2})^3 - \frac{1}{8} \right] + (1 + R^2 b \cos^2\theta - \frac{\sin\theta}{\sqrt{2}})\tau_0 \right\} =$$

$$= \frac{1}{\sqrt{2}\sin\theta} \left[-\frac{\pi^2}{3R^2 b} (\tau_0^3 + \frac{3}{2}\tau_0^2 + \frac{3}{4}\tau_0) + (1 + R^2 b \cos^2\theta - \frac{\sin\theta}{\sqrt{2}})\tau_0 \right] =$$

$$= \frac{\tau_0}{\sqrt{2}\sin\theta} \left[-\frac{\pi^2}{3R^2 b} (\tau_0^2 + \frac{3}{2}\tau_0) + 1 + R^2 b \cos^2\theta - \frac{\sin\theta}{\sqrt{2}} - \frac{\pi^2}{4R^2 b} \right]. \tag{2.17}$$

Using the formulas (2.15) and (2.17), we obtain

$$S \le \frac{R\sqrt{b}}{\pi\sqrt{2}\sin\theta} (1 + R^2 b \cos^2\theta - \frac{\sin\theta}{\sqrt{2}})^{3/2}. \tag{2.18}$$

From the inequalities (2.16) and (2.18), we obtain the inequality (2.3). The theorem is therefore proved. $\square$

Remark 2.7 *If in inequality (2.3), we assume* $R = \frac{2r}{\sqrt{b}}$ *and* $c = \frac{8\sqrt{2}}{\pi}$, *and take into account that*

$$\frac{1}{4} + r^2 \cos^2\theta - \frac{\sin\theta}{4\sqrt{2}} \le 1 + r^2 \cos^2\theta,$$

then

$$N(1, P_r^b(\theta)) \le \frac{cr}{\sin\theta} (1 + r^2 \cos^2\theta)^{3/2}. \tag{2.19}$$

Note that the inequality (2.19) was obtained in [11, *see inequality* (2.12)].

Remark 2.8 *If a positive number* R *satisfies the inequality*

$$R\sqrt{b}\sqrt{1 + R^2 b \cos^2\theta - \frac{\sin\theta}{\sqrt{2}}} < \frac{\pi}{2},$$

then the operator $P_R^b(\theta)$ *has no eigenvalue less than one.*

Corollary 2.9 *The operator* $P_R^b(\theta)$ *can have only a finite number of eigenvalues of finite multiplicity to the left of the essential spectrum threshold.*

Corollary 2.10 *The operator* $H_R(\theta)$ *can have only a finite number of eigenvalues of infinite multiplicity to the left of the continuous spectrum threshold.*

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