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An application of semigroup theory to the coagulation-fragmentation models

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Abstract: We present the existence and uniqueness of strong solutions for the continuous coagulation-fragmentation equation with singular fragmentation and essentially bounded coagulation kernel using semigroup theory of operators. Initially, we reformulate the coupled coagulation-fragmentation problem into the semilinear abstract Cauchy problem (ACP) and consider it as the nonlinear perturbation of the linear fragmentation operator. The existence of the substochastic semigroup is proved for the pure fragmentation equation. Using the substochastic semigroup and some related results for the pure fragmentation equation, we prove the existence of global nonnegative, strong solution for the coagulation-fragmentation equation.

Key words: Coagulation, fragmentation, abstract Cauchy problem, semigroups of operators, existence & uniqueness

1. Introduction

The particle-particle interactions in a closed system bears significant importance in our daily life. As a result it is an important branch of study in several applied science sectors. In pharmaceutical engineering, food processing industry, chemical engineering and chemistry the study of crystal growth and analysis of nanoparticles bears immense significance for product development like whey powder, coffee powder, medicated tablets and capsules etc. These events of particulate processes are mathematically represented by an integro-partial differential equation, well known in the literature as coagulation-fragmentation (CF) equations [4, 5, 7, 11, 14, 15, 23, 25]. Therefore, theoretical researchers find it interesting to analyze the mathematical concepts underlying the CF equations involving different kinetic rates of particle interactions. In this article, we consider the following general nonlinear continuous CF equation

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} =& \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy - u(x,t)\int_0^\infty K(x,y)u(y,t)\,dy \\ &+ \int_x^\infty b(x,y)S(y)u(y,t)\,dy - S(x)u(x,t), \end{aligned} \tag{1.1}$$

with the initial condition

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$$u(x,0) = u_0(x) \geq 0. \tag{1.2}$$

Here, $u(x,t)$ defines the number density of particles of mass $x \geq 0$ at time $t \geq 0$. The coagulation kernel $K(x,y)$ gives the rate at which particles of mass x unite to particles of mass y to form larger particles of mass $x+y$. The function $S(x)$ denotes the rate at which a particle of mass x is selected for further disintegration. The function $b(x,y)$ is a nonnegative measurable function which describes the distribution particles of mass x produced upon by breaking up of particle of mass y. Therefore, the first and the second term of Equation (1.1) arise due to the particle aggregation, and the third and fourth term are due to particle fragmentation. It can be observed that the aggregation phenomena comprises the nonlinearity of the model (1.1), whereas, the fragmentation is a linear one.

In general, the breakage function $b(x,y)$ is considered to satisfy the following properties,

$$b(x,y) = 0, \quad \text{for all} \quad x \geq y, \quad \text{and} \quad \int_0^y x b(x,y)\, \mathrm{d}x = y, \quad \text{for all} \quad y > 0. \tag{1.3}$$

From the definition of the breakage function, the relation (1.3) follows naturally. Additionally, the second equation in (1.3) bears significant physical interpretation as it represents the local conservation of mass (or volume) during fragmentation events occurring in the closed system.

In the literature, the CF-equations have been dealt with an alternative representation of the fragmentation kernels by considering a single function $\gamma(x,y)$ to represent the multiple fragmentation events. This function $\gamma(x,y)$ denotes the rate at which a particle of mass x undergo fragmentation to produce smaller particles of mass y, and bears the following relations with the selection and breakage functions

$$\gamma(x,y) = b(y,x)S(x), \quad \text{and} \quad S(x) = \int_0^x \frac{y}{x}\gamma(x,y)\, dy. \tag{1.4}$$

Hence, the alternative characterization of the CF-equations with the multifragmentation kernel $\gamma(x,y)$ takes the following form

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,\mathrm{d}y - u(x,t)\int_0^\infty K(x,y)u(y,t)\,\mathrm{d}y \\ &\quad + \int_x^\infty \gamma(y,x)u(y,t)\,\mathrm{d}y - \int_0^x \frac{y}{x}\gamma(x,y)u(x,t)\,\mathrm{d}y. \end{aligned}\tag{1.5}$$

In analysis of a mathematical model, our primary objective is to study existence and uniqueness of the solutions. Due to its immense real life applications, the study of existence and uniqueness result for the CF equations (1.1) has become a topic of great interest. Several articles can be found in the literature where the authors have discussed the existence and uniqueness, large time behavior etc. of solutions for (1.1) by considering different form of kernels. In this regard, it is to be noted that the kernels considered in the mathematical analysis of CF equations should be physically meaningful. Theoretical works analyzing existence and uniqueness of the solutions of CF-equations are available in the articles [3, 6, 9, 10, 12, 17, 18, 22]. Due to the limited availability of the solutions in closed form, study on efficient numerical solutions have also gained much attention [13, 19, 20]. In the subsequent discussion, let us briefly review some of the notable works, which have paved the motivation towards the present study.

In the article of [16], the authors have analyzed the solution for the CF equation (1.1) with $K(x,y) = 0$ and

$$S(x) = x^{\alpha+1}, \quad b(x,y) = (\nu+2)\left[\frac{x}{y}\right]^{\nu} y^{-1} \quad \text{for} \quad \alpha < 1, \quad -2 < \nu \leq 0.$$

It is obtained by [16] that due to the above considered fragmentation rates, a rapid breakdown of the particle fragments occurs resulting in the formation of zero-size or dust particles which cause mass loss from the system. It is to be noted that the breakage function considered in the work of [16] describes the power-law fragmentation rate of particles, popularly known as the power-law kernels. These power-law kernels are widely used in different engineering experiments [1, 8]. The breakage function for power-law kernels mentioned above can also be represented in the following form [21]

$$b(x,y) = \frac{p(c+(c+1)(p-1))!}{yc!(c+(c+1)(p-2))!}\left[\frac{x}{y}\right]^{c}\left[1-\frac{x}{y}\right]^{c+(c+1)(p-2)}. \tag{1.6}$$

Here, p and c are nonnegative parameters. Physically, p denotes the number of sister particles formed, and c represents the shape factor. In particular, $p = 2$ and $c = 0$ corresponds to the well-known binary breakage function $2/y$.

1.1. State of the art

Our aim is to apply semigroup theory of operators to establish the existence and uniqueness of strong solution to (1.1) in the case where coagulation kernel satisfies

$$K(x,y) \in L_{\infty}(\mathbb{R}_+ \times \mathbb{R}_+),$$

and for a wide class of singular fragmentation kernel having bounds

$$S(x) = s_1 x^{\alpha}, \quad \text{and} \quad b(x,y) \leq \frac{C}{y}\left[\frac{x}{y}\right]^{\nu}\left[1-\frac{x}{y}\right]^{\beta}, \tag{1.7}$$

with positive constants s_1, C and $\alpha \in [1,\infty)$, $\nu > -1$, $\beta \geq 0$, and $\beta \geq \nu$ such that $b(x,y)$ obeys the mass conservation law (1.3). Similar kind of bounds on the fragmentation kernel was considered in [9, 18]. Note that for the abovementioned bounds, the multiple fragmentation kernel $\gamma(x,y)$ can be equivalently represented as

$$\gamma(x,y) \leq \frac{Cs_1}{x^{1-\alpha}}\left[\frac{y}{x}\right]^{\nu}\left[1-\frac{y}{x}\right]^{\beta}. \tag{1.8}$$

Now by substituting $\alpha = \beta = 0$ and $Cs_1 = (\nu+2)$, the above inequality yields

$$\gamma(x,y) \leq (\nu+2)\left[\frac{y}{x}\right]^{\nu} x^{-1}, \quad \nu > -1.$$

It can be noted that setting $\alpha = 0$ in the relation (1.8) indicates that the selection rate of fragmentation (1.7) is independent of the particle size. Similarly, setting $\beta = 0$ and $C = \nu + 2$ we obtain the general breakage function exhibiting power-law kinetic rate [24]. Therefore, in this study we are able to include a wider range of practically relevant daughter distribution functions which are extensively used in different industries like communition engineering, mineral processing engineering, etc.

The application of semigroup theory requires the problems (1.1) and (1.2) to be reformulated as the semilinear abstract Cauchy problem (ACP). Arguing as in [17], we write (1.2) as the following semilinear ACP

$$\left.\begin{aligned} \frac{du(t)}{dt} &= \mathcal{F}u(t) + \mathcal{C}u(t), \qquad (t \geq 0), \\ u(0) &= u_0. \end{aligned}\right\} \tag{1.9}$$

In (1.9), the fragmentation operator $\mathcal{F}$ and coagulation operator $\mathcal{C}$ are defined on suitable domains by

$$[\mathcal{F}f](x) := \int_x^\infty b(x,y)S(y)f(y)\,dy - S(x)f(x), \tag{1.10}$$

and

$$[\mathcal{C}f](x) := \frac{1}{2}\int_0^x K(x-y,y)f(x-y)f(y)\,dy - f(x)\int_0^\infty K(x,y)f(y)\,dy. \tag{1.11}$$

For a pure fragmentation problem, we study the fragmentation problem in $X_1 := L_1(\mathbb{R}_+, xdx)$ which is the space of equivalence classes of measurable, real-valued functions f such that

$$\|f\|_1 = \int_0^\infty x|f(x)|\,dx < \infty. \tag{1.12}$$

In our present analysis, to study the combined coagulation-fragmentation problem (1.9), we use a more general space $X_{0,\theta} = X_0 \cap X_\theta := L_1(\mathbb{R}_+, (1+x^\theta)dx)$, where $\theta \in \mathbb{R}_+ \cup \{0\}$.

2. Main result

The approach we adopt to establish existence and uniqueness results for the coagulation-fragmentation equation involves the application of perturbation methods from the theory of semigroups of operators. Consequently, we shall treat the initial-value problem (1.9) as a nonlinear perturbation of the linear ACP (2.1) for $\mathcal{F}$.

Throughout our analysis we have taken the following assumptions:

(A1) for all $x \geq 0$, the selection function satisfies $S(x) = s_1 x^\alpha$ where $\alpha \in [1,\infty)$ and $s_1(>0)$ is a constant;

(A2) $b(x,y) \leq \frac{C}{y}\left[\frac{x}{y}\right]^{\nu}\left[1-\frac{x}{y}\right]^{\beta}$ for $\nu > -1$, $\beta \geq 0$, $\beta \geq \nu$ and $C \leq \frac{(\nu+\beta+2)!}{(\nu+1)!\beta!}$;

(A3) $K(x,y)$ is a nonnegative, symmetric function for all $x, y \geq 0$ and satisfies $K(x,y) \in L_\infty(\mathbb{R}_+ \times \mathbb{R}_+)$.

From the assumption (A_2) it can be easily verified that there a positive constant N such that $\int_0^y b(x,y)\,dx = n(y) \leq N$, for all $y > 0$.

First we shall consider the pure fragmentation equation and prove the existence of substochastic semigroup corresponding to the fragmentation operator (1.10). Later we shall use the results to prove the existence of strong solution of the coupled coagulation-fragmentation equation.

2.1. The fragmentation equation

Putting $K(x,y) = 0$, the semilinear ACP (1.9) reduces to the linear ACP as

$$\frac{du(t)}{dt} = \mathcal{F}u(t). \tag{2.1}$$

For the convenience of our analysis, we rewrite Equation (2.1) as

$$\frac{du(t)}{dt} = \mathcal{A}u(t) + \mathcal{B}u(t), \tag{2.2}$$

where

$$[\mathcal{A}f](x) := -S(x)f(x) = -s_1 x^{\alpha} f(x), \tag{2.3}$$

and

$$[\mathcal{B}f](x) := \int_x^\infty b(x,y)S(y)f(y)\,dy \leq C s_1 \int_x^\infty \frac{1}{y}\left[\frac{x}{y}\right]^{\nu}\left[1-\frac{x}{y}\right]^{\beta} y^{\alpha} f(y)\,dy. \tag{2.4}$$

We take $D(\mathcal{F}) = D(\mathcal{A}) = \{f \in X_1; x^{\alpha} f \in X_1\}$. In the following theorem, we prove the existence of substochastic semigroup in the space X_1 by using modified Kato's perturbation theorem in L_1 setting [Corollary 5.17 of [2]].

Theorem 2.1 *Let the operators $\mathcal{A}$ and $\mathcal{B}$ be define (2.3) and (2.4), respectively. Then there exists a smallest substochastic semigroup $(S_G(t))_{t\geq 0}$ in X_1 generated by the extension $\mathcal{G}$ of $\mathcal{F} = \mathcal{A} + \mathcal{B}$.*

Proof

(i) According to the definition (2.3), $(\mathcal{A}, D(\mathcal{A}))$ generates substochastic semigroup.

(ii) Moreover, it easily follows that $D(\mathcal{B}) \supset D(\mathcal{A})$. Let us denote $D(\mathcal{A})_+$ and $D(\mathcal{B})_+$ to be the cone of the nonnegative functions in $D(\mathcal{A})$ and $D(\mathcal{B})$, respectively. Therefore, for any $f \in D(\mathcal{B})_+$, we obtain $\mathcal{B}f \geq 0$.

(iii) Also for any $f \in D(\mathcal{A})_+$, we obtain

$$\int_0^\infty x\left([\mathcal{B}f](x)\right)\,dx \le Cs_1 \int_0^\infty x \int_x^\infty \frac{1}{y}\left[\frac{x}{y}\right]^\nu \left[1-\frac{x}{y}\right]^\beta y^\alpha f(y)\,dy\,dx$$

Changing the order of integration, we have

$$\begin{aligned}\int_0^\infty x\left([\mathcal{B}f](x)\right)\,dx &\le Cs_1 \int_0^\infty \left(\int_0^y \left[\frac{x}{y}\right]^{\nu+1}\left[1-\frac{x}{y}\right]^\beta dx\right) y^\alpha f(y)\,dy\\ &= Cs_1\frac{(\nu+1)!\beta!}{(\nu+\beta+2)!}\int_0^\infty y^{\alpha+1}f(y)\,dy\\ &\le s_1\int_0^\infty yy^\alpha f(y)\,dy\\ &= -\int_0^\infty y[[\mathcal{A}f](y)]\,dy.\end{aligned}$$

This yields

$$\int_0^\infty x\left([\mathcal{A}f+\mathcal{B}f](x)\right)\,dx \le 0.$$

So all the necessary conditions of Kato's theorem are satisfied. Hence there exists a smallest substochastic semigroup say, $(S_G(t))_{t\ge0}$ in X_1 generated by the extension $\mathcal{G}$ of $\mathcal{F}$. □

The next theorem shows the invariance property of the semigroup $(S_G(t))_{t\ge0}$ over a subspace of X_1.

Theorem 2.2 *Suppose that $S(x)$ and $b(x,y)$ satisfy the assumptions $(A1-A4)$. Then the space $X_{0,\alpha} := L_1(\mathbb{R}_+,(1+x^\alpha)dx)$ is invariant under the semigroup $(S_G(t))_{t\ge0}$.*

Proof According to the definition of $X_{0,\alpha}$ and X_1, it is clear that $X_{0,\alpha}$ is embedded in X_1. Mathematically, we denote it as $X_{0,\alpha} \hookrightarrow X_1$. Further, we denote the part of $\mathcal{A}$ in $X_{0,\alpha}$ by $\mathcal{A}_\alpha$. Therefore, for $0 \le f \in D(\mathcal{A}_\alpha)$, $S(x)(1+x^\alpha)$ is integrable, yielding, of course, integrability of each component. Take $\mathcal{B}_\alpha = \mathcal{B}\mid_{D(\mathcal{A}_\alpha)}$. For $0 \le f \in D(\mathcal{A}_\alpha)$,

$$\begin{aligned}\int_0^\infty\left(\int_x^\infty b(x,y)S(y)f(y)\,dy\right)x^\alpha\,dx &= \int_0^\infty\left(\int_0^y x^\alpha b(x,y)\,dx\right)S(y)f(y)\,dy\\ &\le \int_0^\infty S(y)f(y)\left(\int_0^y xx^{\alpha-1}b(x,y)\,dx\right)\,dy\\ &\le \int_0^\infty S(y)y^{\alpha-1}f(y)\left(\int_0^y xb(x,y)\,dx\right)\,dy,\ \alpha\ge1\\ &= \int_0^\infty S(y)y^\alpha f(y)\,dy\end{aligned}$$

which gives

$$\int_0^\infty \left(\int_x^\infty b(x,y)S(y)f(y)\,dy - S(x)f(x) \right) x^\alpha\,dx \leq 0. \tag{2.5}$$

Again for $0 \leq f \in D(\mathcal{A}_\alpha)$

$$\begin{aligned}\int_0^\infty \int_x^\infty b(x,y)S(y)f(y)\,dy\,dx - \int_0^\infty S(x)f(x)\,dx & \\ &= \int_0^\infty S(y)n(y)f(y)\,dy - \int_0^\infty S(x)f(x)\,dx \\ &\leq \int_0^\infty S(y)n(y)f(y)\,dy \\ &\leq s_1 N \int_0^\infty (1+y^\alpha)\,f(y)\,dy. \end{aligned} \tag{2.6}$$

Using the inequalities (2.5) and (2.6), we obtain for any $0 \leq f \in D(\mathcal{A}_\alpha)$

$$\begin{aligned}\int_0^\infty [(\mathcal{A}_\alpha f)(x) + (\mathcal{B}_\alpha f)(x)]\,(1+x^\alpha)\,dx &\leq \int_0^\infty [(\mathcal{A}_\alpha f)(x) + (\mathcal{B}_\alpha f)(x)]\,dx \\ &\leq s_1 N \int_0^\infty (1+y^\alpha)\,f(y)\,dy = s_1 N \|f\|_{0,\alpha}. \end{aligned} \tag{2.7}$$

Hence, $X_{0,\alpha}$ is invariant under the semigroup $(S_G(t))_{t\geq 0}$. □

Moreover, relation (2.7) gives the existence of a positive semigroup $\left(S_{G_{0,\alpha}}(t)\right)_{t\geq 0}$ on $X_{0,\alpha}$. In this regard, we observe that the norm in $X_{0,\alpha}$ is stronger than X_1 and the lattice structure in X_1 is same as $X_{0,\alpha}$. Therefore, $\left(S_{G_{0,\alpha}}(t)\right)_{t\geq 0}$ is restriction of $(S_G(t))_{t\geq 0}$ to $X_{0,\alpha}$ [3].

2.2. The coagulation-fragmentation equation

We now return to the fragmentation-coagulation equation (1.1). Here we introduce the expression $\mathcal{N}$ defined by

$$\mathcal{N}[f,g](x) := \frac{1}{2}\int_0^x K(x-y,y)f(x-y)g(y)\,dy - f(x)\int_0^\infty K(x,y)g(y)\,dy,$$

for all $x > 0$ and set $\|K(x,y)\|_{L_\infty} = K_0$. We shall now prove the following proposition.

Proposition 2.3 *The expression $\mathcal{N}$ restricted to $X_{0,\alpha} \times X_{0,\alpha}$ is an $X_{0,\alpha}$ valued, continuous and bilinear operator. Denote $\mathcal{C}_{0,\alpha}(f) := \mathcal{N}[f,f]$ for $f \in X_{0,\alpha}$, then $\mathcal{C}_{0,\alpha}$ is locally Lipschitz on $X_{0,\alpha}$ and is continuously Fréchet differentiable at any point $f \in X_{0,\alpha}$. Consequently, for $0 \leq u_0 \in X_{0,\alpha}$, there exist a unique nonnegative mild solution, and for any $0 \leq u_0 \in D(\mathcal{G}_{0,\alpha})$, where $\mathcal{G}_{0,\alpha}$ is the part of $\mathcal{G}$ in $X_{0,\alpha}$, there exist a unique, nonnegative strong solution, $u(t) = U_{0,\alpha}(t,u_0)$ of*

$$\frac{du(t)}{dt} = \mathcal{G}_{0,\alpha}u(t) + \mathcal{C}_{0,\alpha}u(t), \tag{2.8}$$

defined on a maximal interval of existence $[0, \hat{T}[$ *where* $\hat{T} > 0$.

Proof For any $f, g \in X_{0,\alpha}$, we can have

$$\int_0^\infty (1+x^\alpha)\mathcal{N}[f,g](x)\,dx = \frac{1}{2}\int_0^\infty \int_0^\infty (1+(x+y)^\alpha)K(x,y)f(x)g(y)\,dx\,dy - \int_0^\infty \int_0^\infty (1+x^\alpha)K(x,y)f(x)g(y)\,dx\,dy. \tag{2.9}$$

For $\alpha \geq 1$, using the following inequality

$$(x+y)^\alpha \leq 2^\alpha(x^\alpha + y^\alpha),$$

we obtain from (2.9)

$$\int_0^\infty (1+x^\alpha)\mathcal{N}[f,g](x)\,dx \leq \left(2^{\alpha-1}-1\right)\underbrace{\int_0^\infty \int_0^\infty x^\alpha K(x,y)f(x)g(y)\,dx\,dy}_{\mathcal{I}_1} + 2^{\alpha-1}\underbrace{\int_0^\infty \int_0^\infty y^\alpha K(x,y)f(x)g(y)\,dxdy}_{\mathcal{I}_2} \tag{2.10}$$

Next we estimate $\mathcal{I}_1$:

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^\infty \int_0^\infty x^\alpha K(x,y)f(x)g(y)\,dx\,dy \\
&\leq K_0 \int_0^\infty \int_0^\infty x^\alpha f(x)g(y)\,dx\,dy \\
&\leq K_0 \int_0^\infty (1+x^\alpha)f(x)\,dx \int_0^\infty (1+x^\alpha)g(y)\,dy \\
&\leq K_0\|f\|_{0,\alpha}\|g\|_{0,\alpha}.
\end{aligned}$$

Using similar analysis as above for $\mathcal{I}_2$, we obtain

$$\mathcal{I}_2 \leq K_0\|f\|_{0,\alpha}\|g\|_{0,\alpha}.$$

Inserting the bound of $\mathcal{I}_1$ and $\mathcal{I}_2$ in (2.10), we get

$$\|\mathcal{N}[f,g]\|_{0,\alpha} \leq M\|f\|_{0,\alpha}\|g\|_{0,\alpha}, \tag{2.11}$$

for constant $M := K_0\,(2^{\alpha}-1)$. Hence $\mathcal{N}$ is a bounded bilinear operator. Next we show that $\mathcal{C}_{0,\alpha}$ is locally Lipschitz on $X_{0,\alpha}$ and is continuously Fréchet differentiable at any point $f \in X_{0,\alpha}$. The idea to prove these properties is recalled from the literature [3].

In order to show $\mathcal{C}_{0,\alpha} f$ is locally Lipschitz on $X_{0,\alpha}$, we take $f, g \in \mathcal{B}(u_0, \varsigma) := \{f \in X_{0,\alpha} : \|f - u_0\| \leq \varsigma\}$ for any $u_0 \in X_{0,\alpha}$ and $\varsigma > 0$. Then

$$\begin{aligned}\|\mathcal{C}_{0,\alpha} f - \mathcal{C}_{0,\alpha} g\|_{0,\alpha} &\leq \|\mathcal{N}[f-g, g]\|_{0,\alpha} + \|\mathcal{N}[g, f-g]\|_{0,\alpha}\\ &\leq M\|f-g]\|_{0,\alpha}\|g]\|_{0,\alpha} + M\|g]\|_{0,\alpha}\|f-g]\|_{0,\alpha}\\ &\leq 2M(\varsigma + \|u_0\|_{0,\alpha})\|f-g]\|_{0,\alpha}.\end{aligned}$$

Now to prove the Fréchet differentiability of $\mathcal{C}_{0,\alpha}$ at any point $f \in X_{0,\alpha}$, we choose the function $\mathcal{N}_f : \mathcal{B}(u_0, \varsigma) \subset X_{0,\alpha} \to X_{0,\alpha}$ such that $\mathcal{N}_f g := \mathcal{N}[f, g] + \mathcal{N}[g, f]$. Since $\mathcal{N}$ is a bounded bilinear operator, it follows that $\mathcal{N}_f$ bounded linear operator on $\mathcal{B}(u_0, \varsigma)$. For any $f \in X_{0,\alpha}$

$$\begin{aligned}\lim_{g\to 0} \frac{\|\mathcal{C}_{0,\alpha}(f+g) - \mathcal{C}_{0,\alpha} f - \mathcal{N}_f g\|_{0,\alpha}}{\|g\|_{0,\alpha}} &= \lim_{g\to 0} \frac{\|\mathcal{N}[f+g, f+g] - \mathcal{N}[f,f] - \mathcal{N}_f g\|_{0,\alpha}}{\|g\|_{0,\alpha}}\\ &= \lim_{g\to 0} \frac{\|\mathcal{N}[f,f] + \mathcal{N}[f,g] + \mathcal{N}[g,f] + \mathcal{N}[g,g] - \mathcal{N}[f,f] - \mathcal{N}_f g\|_{0,\alpha}}{\|g\|_{0,\alpha}}\\ &= \lim_{g\to 0} \frac{\|\mathcal{N}[g,g]\|_{0,\alpha}}{\|g\|_{0,\alpha}}\\ &\leq M \lim_{g\to 0} \frac{\|g\|^2_{0,\alpha}}{\|g\|_{0,\alpha}}\\ &= 0.\end{aligned}$$

Therefore, $\mathcal{C}_{0,\alpha}$ is Fréchet differentiable at any $f \in X_{0,\alpha}$ with Fréchet derivative $\mathcal{N}_f$. Also for $f, g, h \in X_{0,\alpha}$,

$$\begin{aligned}\|\mathcal{N}_f g - \mathcal{N}_h g\|_{0,\alpha} &= \|\mathcal{N}[f,g] + \mathcal{N}[g,f] - \mathcal{N}[h,g] - \mathcal{N}[g,h]\|_{0,\alpha}\\ &= \|\mathcal{N}[f-h, g] - \mathcal{N}[g, f-h]\|_{0,\alpha}\\ &\leq 2M\|g\|_{0,\alpha}\|f-h\|_{0,\alpha}.\end{aligned}$$

Therefore, $\|\mathcal{N}_f g - \mathcal{N}_h g\|_{0,\alpha} \to 0$, whenever $\|f-h\|_{0,\alpha} \to 0$. This implies that the Fréchet derivative is also continuous with respect to f. Hence the result follows. □

In the next theorem we shall prove that the solution is global in time, that is, for every $t \in [0, \infty)$.

Theorem 2.4 *The semilinear ACP (2.8) has a unique, nonnegative, globally defined strong solution for* $0 \leq u_0 \in D(A_{0,\alpha})$.

Proof Let $0 \leq u_0 \in D(A_{0,\alpha})$ and we denote

$$M_l(t) = \int_0^\infty x^l u(x,t)\,dx, \quad \text{for} \quad l \geq 0. \tag{2.12}$$

Then

$$\begin{aligned}\frac{dM_1(t)}{dt} =& \frac{1}{2}\int_0^\infty x \int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy\,dx \\ &- \int_0^\infty xu(x,t)\int_0^\infty K(x,y)u(y,t)\,dy\,dx \\ &+ \int_0^\infty x\int_x^\infty b(x,y)S(y)u(y,t)\,dy\,dx - \int_0^\infty xS(x)u(x,t)\,dx.\end{aligned}$$

Changing the order of integration of the first and the third integral in the right hand side of the above equation, we get

$$\begin{aligned}\frac{dM_1(t)}{dt} =& \frac{1}{2}\int_0^\infty \int_y^\infty xK(x-y,y)u(x-y,t)u(y,t)\,dx\,dy \\ &- \int_0^\infty xu(x,t)\int_0^\infty K(x,y)u(y,t)\,dy\,dx \\ &+ \int_0^\infty S(y)u(y,t)\int_0^y xb(x,y)\,dx\,dy - \int_0^\infty xS(x)u(x,t)\,dx.\end{aligned}$$

Taking $x - y = z$ in the first integral and using Equation (1.3) of the above equation, yields

$$\begin{aligned}\frac{dM_1(t)}{dt} =& \frac{1}{2}\int_0^\infty u(y,t)\int_0^\infty (x+y)K(x,y)u(x,t)u(y,t)\,dx\,dy \\ &- \int_0^\infty xu(x,t)\int_0^\infty K(x,y)u(y,t)\,dy\,dx \\ &+ \int_0^\infty yS(y)u(y,t)\,dy - \int_0^\infty xS(x)u(x,t)\,dx.\end{aligned}$$

Since $u(x,t)$ is a local strong solution of Equation (2.8) for any $0 \leq u_0 \in D(A_{0,\alpha})$ so all the integrals exists finitely. Moreover, using symmetric property of $K(x,y)$, we obtain

$$\frac{dM_1(t)}{dt} = 0.$$

Therefore, $M_1(t) = \bar{M}_1$ (a constant).

Again,

$$
\begin{aligned}
\frac{dM_2(t)}{dt} &= -\int_0^\infty x^2 S(x)u(x,t)\,dx + \int_0^\infty \int_x^\infty x^2 S(y)b(x,y)u(y,t)\,dy\,dx \\
&+ \frac{1}{2}\int_0^\infty \int_0^x x^2 K(x-y,y)u(x-y,t)u(y,t)\,dy\,dx \\
&- \int_0^\infty x^2 u(x,t)\int_0^\infty K(x,y)u(y,t)\,dy\,dx \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \left[(x+y)^2 - x^2 - y^2\right] K(x,y)u(x,t)u(y,t)\,dy\,dx \\
&\leq k_0 \int_0^\infty \int_0^\infty xyu(x,t)u(y,t)\,dy\,dx \\
&\leq k_0 \bar{M}_1^2
\end{aligned}
$$

Hence by integrating

$$
M_2(t) \leq \bar{M}_2 \quad \text{for all} \quad t \in [0, \hat{T}).
$$

we can proceed further in a similar way to obtain the uniform boundedness of the truncated moments, that is,

$$
M_i(t) \leq \bar{M}_i \quad \text{where,} \quad i = 1, 2, \ldots, r = \lceil \alpha \rceil. \tag{2.13}
$$

Using the continuity of $u(x,t)$, we obtain

$$
\begin{aligned}
\frac{dM_0(t)}{dt} &= -\frac{1}{2}\int_0^\infty \int_0^\infty K(x,y)u(x,t)u(y,t)\,dx\,dy + \int_0^\infty S(x)\left[n(x)-1\right]u(x,t)\,dx, \\
&\leq -\frac{k_0}{2}\int_0^\infty \int_0^\infty u(x,t)u(y,t)\,dx\,dy + s_1 N \int_0^\infty x^\alpha u(x,t)\,dx, \\
&\leq C_1 M_0(t) + C_2 \bar{M}_r,
\end{aligned} \tag{2.14}
$$

for some constants C_1, C_2 depending only on the coefficients but not on the initial value for u. Integrating (2.14), we obtain

$$
M_0(t) \leq \bar{M}_0, \quad \bar{M}_0 > 0 \quad \text{for all} \quad t \in [0, \hat{T}).
$$

To search the bound for $M_\alpha(t)$, first we notice that for any realnumber p(not necessary integer) satisfying $\beta - 1 \leq p \leq \beta$, where $\beta \in (1, \alpha]$.

$$
\begin{aligned}
M_p(t) = \int_0^\infty x^p u(x,t)\,dx &= \int_0^1 x^p u(x,t)\,dx + \int_1^\infty x^p u(x,t)\,dx \\
&\leq M_{\lceil p-1 \rceil}(t) + M_{\lceil p \rceil}(t) \\
&\leq \bar{M}_p(Constant) \quad \text{for all} \quad t \in [0, \hat{T})
\end{aligned} \tag{2.15}
$$

In particular $\beta = \alpha$ from (2.15)

$$M_\alpha(t) \leq \bar{M}_\alpha = \text{constant}, \quad \text{for all} \quad t \in [0, \hat{T}). \tag{2.16}$$

Now,

$$\|u(x,t)\|_{0,\alpha} = \int_0^\infty (1 + x^\alpha) u(x,t)\, dx = M_0(t) + M_\alpha(t).$$

Since, $M_0(t)$ and $M_\alpha(t)$ do not blow up in finite time, hence $u(x,t)$ is defined globally in time for any $u_0 \in D(A_{0,\alpha})$. □

3. Conclusion

We have obtained the existence and uniqueness of nonnegative strong solution of the coagulation-fragmentation equation with singular fragmentation kernel and essentially bounded coagulation kernel. First the existence of the substochastic semigroup corresponding to the fragmentation operator in the space X_1 is established. Then the existence of local strong solution is obtained for the combined coagulation-fragmentation equation in the space $X_{0,\alpha}$. Moreover, we have obtained the global existence of the solution. The large class of fragmentation kernels are representing power-law rates are taken from the literature as described by [21].

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