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Zero-divisor graphs of partial transformation semigroups

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Abstract: Let $\mathcal{P}_n$ be the partial transformation semigroup on $X_n = \{1, 2, \ldots, n\}$. In this paper, we find the left zero-divisors, right zero-divisors and two sided zero-divisors of $\mathcal{P}_n$, and their numbers. For $n \geq 3$, we define an undirected graph $\Gamma(\mathcal{P}_n)$ associated with $\mathcal{P}_n$ whose vertices are the two sided zero-divisors of $\mathcal{P}_n$ excluding the zero element θ of $\mathcal{P}_n$ with distinct two vertices α and β joined by an edge in case $\alpha\beta = \theta = \beta\alpha$. First, we prove that $\Gamma(\mathcal{P}_n)$ is a connected graph, and find the diameter, girth, domination number and the degrees of the all vertices of $\Gamma(\mathcal{P}_n)$. Furthermore, we give lower bounds for clique number and chromatic number of $\Gamma(\mathcal{P}_n)$.

Key words: Partial transformation semigroup, zero-divisor graph, clique number, chromatic number

1. Introduction

In 1988, the zero-divisor graphs on commutative rings were defined by Beck [2]. However, the zero element is a vertex in the zero-divisor graph within Beck's definition, later the standard definition of zero-divisor graphs on commutative rings was given by Anderson and Livingston [1]. Let R be a commutative ring and let $Z(R)$ be the set of the zero-divisors of R. The zero-divisor graph of R is defined by an undirected graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, where distinct vertices x and y of $\Gamma(R)$ are adjacent if and only if $xy = 0$. Demeyer et al. have considered this definition for commutative semigroups and they defined and found some basic properties of the zero-divisor graph of a commutative semigroup with zero [5, 6]. There are some papers about zero-divisor graphs on some classes of commutative semigroups [4, 11]. For noncommutative rings, a directed zero-divisor graph and some undirected zero-divisor graphs were defined by Redmond [9]. Suppose that R is a ring, let $Z_R(T)$ be the set of all two sided zero-divisor elements of R. Then Redmond defined an undirected zero-divisor graph $\Gamma(R)$ with vertices $Z_R(T) \setminus \{0\}$, where distinct vertices x and y are adjacent with a single edge if and only if $xy = 0 = yx$. If R is a noncommutative ring, then $\Gamma(R)$ does not need to be connected and if R is a commutative ring then $\Gamma(R)$ coincide with the standard zero-divisor graph of R. Furthermore, these definitions can be considered for noncommutative semigroups with zero. Recently, some properties of zero-divisor graphs of Catalan monoids have been researched in [12].

Suppose that $X_n = \{1, 2, \ldots, n\}$ is a finite set. Let $\mathcal{P}_n, \mathcal{T}_n$ and $\mathcal{S}_n$ be the partial transformation semigroup, the full transformation semigroup and the symmetric group on X_n, respectively. Let $\alpha \in \mathcal{P}_n$, the domain of α denoted by $\mathrm{dom}\,(\alpha)$ and image of α denoted by $\mathrm{im}(\alpha)$, moreover, codomain of α is the complement of $\mathrm{dom}\,(\alpha)$ and it is denoted by $\mathrm{codom}\,(\alpha)$.

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Let $\theta \in \mathcal{P}_n$ such that $\mathrm{dom}\,(\theta) = \emptyset$ and let α be any element of $\mathcal{P}_n$ then it is clear that $\theta\alpha = \theta = \alpha\theta$, so θ is the zero element of $\mathcal{P}_n$. Furthermore, $\mathcal{P}_n$ is a noncommutative semigroup for $n \geq 2$.

For $n \geq 2$, let $\mathcal{P}_n^* = \mathcal{P}_n \setminus \{\theta\}$. Then we define the following sets

$$\begin{aligned} L = L(\mathcal{P}_n) &= \{\alpha \in \mathcal{P}_n \mid \alpha\beta = \theta \text{ for some } \beta \in \mathcal{P}_n^*\}, \\ R = R(\mathcal{P}_n) &= \{\alpha \in \mathcal{P}_n \mid \gamma\alpha = \theta \text{ for some } \gamma \in \mathcal{P}_n^*\} \text{ and} \\ T = T(\mathcal{P}_n) &= \{\alpha \in \mathcal{P}_n \mid \alpha\beta = \theta = \gamma\alpha \text{ for some } \beta, \gamma \in \mathcal{P}_n^*\} = L \cap R \end{aligned}$$

which are called the set of left zero-divisors, right zero-divisors and two sided zero-divisors of $\mathcal{P}_n$, respectively. It is known that $|\mathcal{P}_n| = (n+1)^n$, $|\mathcal{T}_n| = n^n$ and $|\mathcal{S}_n| = n!$ for $n \in \mathbb{Z}^+$. In this paper we find the left zero-divisors, right zero-divisors and zero-divisors of $\mathcal{P}_n$, and then their numbers.

For a semigroup S with (zero) 0 if $T(S)\setminus\{0\} \neq \emptyset$ where $T(S) = \{z \in S \mid zx = 0 = yz \text{ for } x, y \in S\setminus\{0\}\}$, then we similarly define the (undirected) zero-divisor graph $\Gamma(S)$ associated with S whose the set of vertices is $T(S) \setminus \{0\}$ with distinct two vertices joined by an edge in case $xy = 0 = yx$ for some $x, y \in T(S) \setminus \{0\}$. Notice that $\theta \in T(\mathcal{P}_n)$ for all $n \geq 1$. In this paper, we prove that $\Gamma(\mathcal{P}_n)$ is a connected graph and find the diameter, girth, the degrees of all vertices, lower bounds for clique number and chromatic number of $\Gamma(\mathcal{P}_n)$ for $n \geq 3$.

For semigroup terminology see [7, 8] and for graph theoretical terminology see [10].

2. Zero-divisors of $\mathcal{P}_n$

In this section, we find the left zero-divisors, right zero-divisors and two-sided zero-divisors of $\mathcal{P}_n$ and their numbers.

Since, for any $\alpha, \beta \in \mathcal{P}_n$, it is well-known that $\mathrm{dom}\,(\alpha\beta) = [\mathrm{im}(\alpha) \cap \mathrm{dom}\,(\beta)]\alpha^{-1}$ and $\mathrm{im}(\alpha\beta) = [\mathrm{im}(\alpha) \cap \mathrm{dom}\,(\beta)]\beta$ (see Proposition 1.4.3 in [8]), we can write the following immediate result.

Lemma 2.1 *If $\alpha, \beta \in \mathcal{P}_n$, then $\alpha\beta = \theta \iff im(\alpha) \subseteq \mathrm{codom}\,(\beta)$. In particular, $\alpha^2 = \theta \iff im(\alpha) \subseteq \mathrm{codom}\,(\alpha)$.*

Lemma 2.2 *For $n \geq 1$, let L be the set of left zero-divisors of $\mathcal{P}_n$. Then we have*

$$L = \mathcal{P}_n \setminus \mathcal{S}_n \quad \textit{and} \quad |L| = (n+1)^n - n!.$$

Proof For any $\alpha \in \mathcal{P}_n \setminus \mathcal{S}_n$, notice that $X_n \setminus \mathrm{im}(\alpha) \neq \emptyset$. If we take any $\beta \in \mathcal{P}_n$ with $\mathrm{dom}\,(\beta) = X_n \setminus \mathrm{im}(\alpha)$, then it follows from Lemma 2.1 that α is a left zero divisor.

Since, for every $\alpha \in \mathcal{S}_n$, $|\mathrm{im}(\alpha\beta)| = |\mathrm{im}(\beta)|$ for all $\beta \in \mathcal{P}_n$, it follows that $\alpha\beta = \theta$ if and only if $\beta = \theta$, the zero element of $\mathcal{P}_n$. Thus, the permutations on X_n are not left zero divisors. Therefore, $|L| = |\mathcal{P}_n| - |\mathcal{S}_n| = (n+1)^n - n!$, as required. □

Since, for any $\alpha, \beta \in \mathcal{P}_n$, $\mathrm{im}(\beta\alpha) = [\mathrm{im}(\beta) \cap \mathrm{dom}\,(\alpha)]\alpha$, we also have immediate lemma.

Lemma 2.3 *For $\alpha, \beta \in \mathcal{P}_n$, $\beta\alpha = \theta$ if and only if $im(\beta) \cap \mathrm{dom}\,(\alpha) = \emptyset$.*

Lemma 2.4 *For $n \geq 1$, let R be the set of right zero-divisors of $\mathcal{P}_n$. Then we have*

$$R = \mathcal{P}_n \setminus \mathcal{T}_n \quad \textit{and} \quad |R| = (n+1)^n - n^n.$$

Proof For any $\alpha \in \mathcal{P}_n \setminus \mathcal{T}_n$, notice that $X_n \setminus \mathrm{dom}\,(\alpha) \neq \emptyset$. If we take any $\beta \in \mathcal{P}_n$ with $\mathrm{im}(\beta) = X_n \setminus \mathrm{dom}\,(\alpha)$, then it follows from Lemma 2.3 that α is a right zero divisor.

Since, for every $\alpha \in \mathcal{T}_n$, $\mathrm{dom}\,(\beta\alpha) = \mathrm{dom}\,(\beta)$ for all $\beta \in \mathcal{P}_n$, it follows that $\beta\alpha = \theta$ if and only if $\beta = \theta$. Thus, the full transformations on X_n are not right zero divisors. Therefore, $|R| = |\mathcal{P}_n| - |\mathcal{T}_n| = (n+1)^n - n^n$, as required. □

Since $\mathcal{S}_n \subseteq \mathcal{T}_n \subseteq \mathcal{P}_n$, it follows that $R \subseteq L$. Thus we have the following corollary.

Corollary 2.5 *For $n \geq 1$, let T be the set of zero-divisors of $\mathcal{P}_n$. Then we have*

$$T = R = \mathcal{P}_n \setminus \mathcal{T}_n \quad \textit{and} \quad |T| = (n+1)^n - n^n.$$

3. Zero-divisor graph of $\mathcal{P}_n$

Let $G = (V(G), E(G))$ be an undirected graph, $V(G)$ denotes vertex set of G and $E(G)$ denotes the edge set of G. If G does not have any loops and multiple edges, then G is called a simple graph. We consider simple graphs for the following definitions. Two vertices u and v of G are said to be connected if there is a path from u to v. If u and v connected for every $u, v \in V(G)$, then G is called a connected graph. Let $u, v \in V(G)$, the length of the shortest path between u and v is denoted by $d_G(u, v)$.

The diameter of G is denoted by $\mathrm{diam}(G)$ and defined by

$$\mathrm{diam}(G) = \max\{d_G(u, v) \mid u, v \in V(G)\}.$$

Let $v \in V(G)$ then the degree of a vertex is denoted by $\deg_G(v)$ and it is the number of adjacent vertices to v in G. $\Delta(G)$ shows that the maximum degree and $\delta(G)$ shows that the minimum degree among all the degrees in G.

Let $\emptyset \neq D \subseteq V(G)$, for each vertex of G if the vertex in D or the vertex is adjacent to any vertex in D then D is called a dominating set for G. The domination number of G is

$$\min\{|D| \mid D \text{ is a dominating set of } G\}$$

and this number is denoted by $\gamma(G)$. The length of the shortest cycle in G is called girth of G and it is denoted by $\mathrm{gr}(G)$, if G does not contain any cycles then its girth is defined to be infinity.

Let $\emptyset \neq C \subseteq V(G)$, if u and v are adjacent vertices for all $u, v \in C$ in G, then C is called a clique. The number of all the vertices in any maximal clique of G is called clique number of G and it is denoted by $\omega(G)$. The chromatic number of G is defined by the number of the minimum number of colours required to colour of the all vertices G with the rule no two adjacent vertices have the same colour, and it is denoted by $\chi(G)$.

Let G be a graph with n vertices, if every vertex is adjacent to each other vertices, then G is called a complete graph and it is denoted by K_n.

Let $V' \subseteq V(G)$. The (vertex) induced subgraph $G' = (V', E')$ is a subgraph of G and its vertex set is V', furthermore, its edge set consists of all of the edges in $E(G)$ that have both endpoints in V'.

In this section, we prove that $\Gamma(\mathcal{P}_n)$ is a connected graph and find the diameter, girth, domination number, the vertex degrees, and give lower bounds for clique number and chromatic number of $\Gamma(\mathcal{P}_n)$ for $n \geq 3$. In this paper, we use Γ instead of $\Gamma(\mathcal{P}_n)$ for convenience.

Let $T^* = T \setminus \{\theta\}$, then we have $V(\Gamma) = T^*$ from definitions. It follows from Lemmas 2.1 and 2.3 α is adjacent to β if and only if $\mathrm{im}(\alpha) \subseteq \mathrm{codom}\,(\beta)$ and $\mathrm{im}(\beta) \subseteq \mathrm{codom}\,(\alpha)$.

Lemma 3.1 Γ *is a connected graph for* $n \geq 3$.

Proof For $n \geq 3$, let $\gamma_i \in \mathcal{P}_n$ such that $\mathrm{dom}(\gamma_i) = \{i\}$ and $i\gamma_i = i$ for $1 \leq i \leq n$, then it is clear that $\gamma_i \in V(\Gamma)$ for $1 \leq i \leq n$. Let G be an induced subgraph of Γ induced by the vertex set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. Then it is clear that $G = K_n$. For $\alpha \in V(\Gamma) \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, there exists at least one $i \in \mathrm{codom}(\alpha)$.

Case 1: If $i \notin \mathrm{im}(\alpha)$, then $\alpha\gamma_i = \gamma_i\alpha = \theta$.

Case 2: If $i \in \mathrm{im}(\alpha)$, then $i\alpha^{-1} \neq \emptyset$. So, in this case, we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r & A_{r+1} \\ i & a_2 & \ldots & a_r & - \end{pmatrix}$$

where $\mathrm{im}(\alpha) = \{i = a_1, a_2, \ldots, a_r\}$, $A_k = a_k\alpha^{-1}$ for $1 \leq k \leq r$ and $A_{r+1} = \mathrm{codom}(\alpha)$. If $r \geq 2$, then we take

$$\beta = \begin{pmatrix} X_n \setminus \{i, a_2, \ldots, a_r\} & \{i, a_2, \ldots, a_r\} \\ i & - \end{pmatrix},$$

then we have $\beta \in V(\Gamma)$ and $\alpha\beta = \beta\alpha = \theta$. Moreover, $\beta\gamma_{a_2} = \gamma_{a_2}\beta = \theta$. So, in this case, we have a path in Γ such that $\alpha - \beta - \gamma_{a_2}$ where $\gamma_{a_2} \in V(G)$. In other case, we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 \\ i & - \end{pmatrix},$$

so, $i \in A_2$ since $i \in \mathrm{codom}(\alpha)$. There exists $j \in X_n$ such that $i \neq j$, moreover, $X_n \setminus \{i, j\} \neq \emptyset$ since $n \geq 3$. Let

$$\beta = \begin{pmatrix} X_n \setminus \{i, j\} & \{i, j\} \\ i & - \end{pmatrix}.$$

Then we have $\beta \in V(\Gamma)$ and $\alpha\beta = \beta\alpha = \theta$. Moreover, $\beta\gamma_j = \gamma_j\beta = \theta$. So, in this case, we have a path in Γ such that $\alpha - \beta - \gamma_j$ where $\gamma_j \in V(G)$. It follows that Γ is a connected graph. □

Lemma 3.2 $\mathrm{diam}(\Gamma) = 4$ *for* $n \geq 3$.

Proof Let $n \geq 3$ and $\alpha, \beta \in V(\Gamma)$. Then we have

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r & A_{r+1} \\ a_1 & a_2 & \cdots & a_r & - \end{pmatrix}, \ \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_k & B_{k+1} \\ b_1 & b_2 & \cdots & b_k & - \end{pmatrix}$$

where $a_i, b_j \in X_n$ for $1 \leq i \leq r$ and $1 \leq j \leq k$, $1 \leq r \leq n-1$ and $1 \leq k \leq n-1$, $\{A_1, A_2, \ldots, A_{r+1}\}$ is a partition of X_n and $\{B_1, B_2, \ldots, B_{k+1}\}$ is a partition of X_n. Firstly we will show that $d_\Gamma(\alpha, \beta) \leq 4$. Let $Y = \{a_1, a_2, \ldots, a_r\} \cup \{b_1, b_2, \ldots, b_k\}$.

Case 1: If $Y \neq X_n$ and $A_{r+1} \cap B_{k+1} \neq \emptyset$. Then there exists $z, t \in X_n$ such that $z \in X_n \setminus Y$ and $t \in A_{r+1} \cap B_{k+1}$. Let

$$\gamma = \begin{pmatrix} z & X_n \setminus \{z\} \\ t & - \end{pmatrix}.$$

It is clear that $\gamma \in V(\Gamma)$ and $\alpha - \gamma - \beta$ in Γ.

Case 2: If $Y = X_n$ and $A_{r+1} \cap B_{k+1} \neq \emptyset$. There exists $x, y \in X_n$ such that $x \in \{b_1, b_2, \ldots, b_k\} \setminus \{a_1, a_2, \ldots, a_r\}$ and $y \in \{a_1, a_2, \ldots, a_r\} \setminus \{b_1, b_2, \ldots, b_k\}$ since $1 \leq r \leq n-1$, $1 \leq k \leq n-1$ and $Y = X_n$. So we have $x \neq y$ and there exists $z \in X_n$ such that $z \in X_n \setminus \{x, y\}$ since $n \geq 3$. Let $t \in A_{r+1} \cap B_{k+1}$. Let

$$\lambda_1 = \begin{pmatrix} x & X_n \setminus \{x\} \\ t & - \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} y & X_n \setminus \{y\} \\ t & - \end{pmatrix}, \ \lambda_3 = \begin{pmatrix} X_n \setminus \{t\} & t \\ z & - \end{pmatrix}.$$

Then it is clear that $\lambda_1, \lambda_2, \lambda_3 \in V(\Gamma)$. If $t \notin \{x, y\}$, then $\alpha - \lambda_1 - \lambda_2 - \beta$ in Γ. If $t \in \{x, y\}$, then $\alpha - \lambda_1 - \lambda_3 - \lambda_2 - \beta$ in Γ.

Case 3: If $Y \neq X_n$ and $A_{r+1} \cap B_{k+1} = \emptyset$. There exists $z \in X_n$ such that $z \in X_n \setminus Y$. Let $x \in A_{r+1}$, $y \in B_{k+1}$ and $t \in X_n \setminus \{z\}$. Let

$$\lambda_1 = \begin{pmatrix} z & X_n \setminus \{z\} \\ x & - \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} z & X_n \setminus \{z\} \\ y & - \end{pmatrix}, \ \lambda_3 = \begin{pmatrix} X_n \setminus \{x, y\} & \{x, y\} \\ t & - \end{pmatrix}.$$

Then it is clear that $\lambda_1, \lambda_2, \lambda_3 \in V(\Gamma)$. If $z \notin \{x, y\}$, then $\alpha - \lambda_1 - \lambda_2 - \beta$ in Γ. If $z \in \{x, y\}$, then $\alpha - \lambda_1 - \lambda_3 - \lambda_2 - \beta$ in Γ.

Case 4: If $Y = X_n$ and $A_{r+1} \cap B_{k+1} = \emptyset$. There exists $x, y \in X_n$ such that $x \in \{b_1, b_2, \ldots, b_k\} \setminus \{a_1, a_2, \ldots, a_r\}$ and $y \in \{a_1, a_2, \ldots, a_r\} \setminus \{b_1, b_2, \ldots, b_k\}$ since $1 \leq r \leq n-1$, $1 \leq k \leq n-1$ and $Y = X_n$. So we have $x \neq y$ and there exists $z \in X_n$ such that $z \in X_n \setminus \{x, y\}$ since $n \geq 3$. Let $a \in A_{r+1}$ and $b \in B_{k+1}$. Then $a \neq b$ since $A_{r+1} \cap B_{k+1} = \emptyset$. Let

$$\lambda_1 = \begin{pmatrix} x & X_n \setminus \{x\} \\ a & - \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} y & X_n \setminus \{y\} \\ b & - \end{pmatrix}.$$

Then it is clear that $\lambda_1, \lambda_2 \in V(\Gamma)$. If $x \neq b$ and $y \neq a$, then $\alpha - \lambda_1 - \lambda_2 - \beta$ in Γ. If $x = b$, then we take

$$\lambda_3 = \begin{pmatrix} X_n \setminus \{x, a\} & \{x, a\} \\ z & - \end{pmatrix}.$$

Then it is clear that $\lambda_3 \in V(\Gamma)$ and $\alpha - \lambda_1 - \lambda_3 - \lambda_2 - \beta$ in Γ. If $y = a$, then we take

$$\lambda_4 = \begin{pmatrix} X_n \setminus \{y, b\} & \{y, b\} \\ z & - \end{pmatrix}.$$

Then it is clear that $\lambda_4 \in V(\Gamma)$ and $\alpha - \lambda_1 - \lambda_4 - \lambda_2 - \beta$ in Γ.

So we have concluded that $d_\Gamma(\alpha, \beta) \leq 4$ for all cases. Let

$$\alpha_1 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 & - \end{pmatrix}, \ \beta_1 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & - \end{pmatrix}.$$

α_1 and β_1 are not adjacent vertices in Γ. Moreover, α_1 has only one adjacent vertex in Γ which is $\mu_1 = \begin{pmatrix} n & X_n \setminus \{n\} \\ n & - \end{pmatrix}$, similarly β_1 has only one adjacent vertex in Γ which is $\mu_2 = \begin{pmatrix} 1 & X_n \setminus \{1\} \\ n & - \end{pmatrix}$. So we have $\mu_1 \neq \mu_2$, moreover, μ_1 and μ_2 are not adjacent vertices in Γ, thus $d_\Gamma(\alpha_1, \beta_1) = 4$. It follows that $\operatorname{diam}(\Gamma) = 4$ for $n \geq 3$. □

Theorem 3.3 $gr(\Gamma) = 3$ *for* $n \geq 3$.

Proof First of all, since Γ is a simple graph $gr(\Gamma) \geq 3$. For $n \geq 3$, let $\gamma_i \in \mathcal{P}_n$ such that $\mathrm{dom}\,(\gamma_i) = \{i\}$ and $i\gamma_i = i$ for $1 \leq i \leq n$, then it is clear that $\gamma_i \in V(\Gamma)$ for $1 \leq i \leq n$. Then we have a cycle such that $\gamma_1 - \gamma_2 - \gamma_3 - \gamma_1$, it follows that $gr(\Gamma) = 3$ for $n \geq 3$. □

Theorem 3.4 *For* $n \geq 3$, *let* $\alpha \in V(\Gamma)$. *If* $r = |im(\alpha)|$ *and* $k = |\mathrm{codom}\,(\alpha)|$, *then*

$$deg_{\Gamma}(\alpha) = \begin{cases} (k+1)^{(n-r)} - 1 & \text{if } im(\alpha) \nsubseteq \mathrm{codom}\,(\alpha) \\ (k+1)^{(n-r)} - 2 & \text{if } im(\alpha) \subseteq \mathrm{codom}\,(\alpha). \end{cases}$$

Proof For $n \geq 3$, let $\alpha \in V(\Gamma)$, $r = |\mathrm{im}(\alpha)|$ and $k = |\mathrm{codom}\,(\alpha)|$. For any $\beta \in \mathcal{P}_n$ suppose that $\alpha\beta = \theta = \beta\alpha$. It follows from Lemma 2.1 that $\mathrm{im}(\beta) \subseteq \mathrm{codom}\,(\alpha)$ and $\mathrm{im}(\alpha) \subseteq \mathrm{codom}\,(\beta)$. If $\mathrm{im}(\alpha) \nsubseteq \mathrm{codom}\,(\alpha)$, then we have $\alpha^2 \neq \theta$, so $\beta \neq \alpha$. Moreover, we have if $x \in im(\alpha)$, then $x \in \mathrm{codom}\,(\beta)$ and if $x \in X_n \setminus \mathrm{im}(\alpha)$, then $x\beta \in \mathrm{codom}\,(\alpha)$ or $x \in \mathrm{codom}\,(\beta)$. So we have $k+1$ different choice for $\forall x \in X_n \setminus \mathrm{im}(\alpha)$ since $k = |\mathrm{codom}\,(\alpha)|$. However, if $x \in \mathrm{codom}\,(\beta)$ for $\forall x \in X_n \setminus \mathrm{im}(\alpha)$, then $\beta = \theta \notin V(\Gamma)$. It follows that, $\deg_{\Gamma}(\alpha) = (k+1)^{(n-r)} - 1$ since $|X_n \setminus \mathrm{im}(\alpha)| = n - r$. If $\mathrm{im}(\alpha) \subseteq \mathrm{codom}\,(\alpha)$, then we have $\alpha^2 = \theta$, so if we take $\beta = \alpha$, then $\alpha\beta = \theta = \beta\alpha$. So from the definition of vertex degree we have $\deg_{\Gamma}(\alpha) = ((k+1)^{(n-r)} - 1) - 1 = (k+1)^{(n-r)} - 2$. □

For $n \geq 3$, if we consider $\alpha \in V(\Gamma)$ such that $|\mathrm{im}(\alpha)| = n - 1$ and $|\mathrm{codom}\,(\alpha)| = 1$, then we have $\mathrm{im}(\alpha) \nsubseteq \mathrm{codom}\,(\alpha)$ and so $\deg_{\Gamma}(\alpha) = 1$ from Theorem 3.4. If we consider $\beta \in V(\Gamma)$ such that $|\mathrm{im}(\beta)| = r$ and $|\mathrm{codom}\,(\beta)| = k$, then it is clear that $1 \leq r \leq n-1$ and $1 \leq k \leq n-1$. For the maximum degree, we take $k = n-1$ and $r = 1$. Moreover, let $x\beta = x$ for $x \in \mathrm{dom}\,(\beta)$ then $\mathrm{im}(\beta) \nsubseteq \mathrm{codom}\,(\beta)$, so $\deg_{\Gamma}(\beta) = n^{(n-1)} - 1$. Thus we have the following immediate corollary.

Corollary 3.5 $\Delta(\Gamma) = n^{(n-1)} - 1$ *and* $\delta(\Gamma) = 1$ *for* $n \geq 3$.

Theorem 3.6 $\gamma(\Gamma) = n^2$ *for* $n \geq 3$.

Proof For $n \geq 3$, let $\alpha_{ij} \in \mathcal{P}_n$ such that $\mathrm{dom}\,(\alpha_{ij}) = \{i\}$ and $i\alpha_{ij} = j$ for $1 \leq i \leq n$, $1 \leq j \leq n$. We have $\alpha_{ij} \in V(\Gamma)$. Let

$$\mathcal{A} = \{\alpha_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}$$

then it is clear that $|\mathcal{A}| = n^2$. If $i = j$ then $\deg_{\Gamma}(\alpha_{ij}) = n^{n-1} - 1$ and if $i \neq j$ then $\deg_{\Gamma}(\alpha_{ij}) = n^{n-1} - 2$. Let $X_n \setminus \{i\} = \{i_1, i_2, \ldots, i_{n-1}\}$ with $i_1 < i_2 < \ldots < i_{n-1}$ and $X_n \setminus \{j\} = \{j_1, j_2, \ldots, j_{n-1}\}$ with $j_1 < j_2 < \ldots < j_{n-1}$. For $1 \leq i \leq n$, $1 \leq j \leq n$, let $\beta_{ij} \in \mathcal{P}_n$ such that $\mathrm{codom}\,(\beta_{ij}) = \{j\}$ and $j_k\beta_{ij} = i_k$ for $1 \leq k \leq n-1$. Then we have $\beta_{ij} \in V(\Gamma)$ and $\deg_{\Gamma}(\beta_{ij}) = 1$. Let

$$\mathcal{B} = \{\beta_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}.$$

Then we have $|\mathcal{B}| = n^2$, moreover, $\mathcal{A}$ and $\mathcal{B}$ are disjoint sets. In Γ, β_{ij} has only one adjacent vertex which is α_{ij} for $1 \leq i \leq n$, $1 \leq j \leq n$. It follows that $\gamma(\Gamma) \geq n^2$ from the pigeonhole principle. Let $\alpha \in V(\Gamma) \setminus \mathcal{A}$. Then there exist $i, j \in X_n$ such that $i \in X_n \setminus \mathrm{im}(\alpha)$ and $j \in \mathrm{codom}\,(\alpha)$. We have α and α_{ij} are adjacent vertices in Γ. Thus $\mathcal{A}$ is a dominating set for Γ. We have $\gamma(\Gamma) \geq n^2$, $\mathcal{A}$ is a dominating set for Γ and $|\mathcal{A}| = n^2$. So $\gamma(\Gamma) = n^2$ for $n \geq 3$. □

We have $\omega(\Gamma) \geq n$ from the proof of Lemma 3.1. In the following theorem, we give better lower bound for clique number of Γ for $n \geq 3$.

Theorem 3.7 *If $n \geq 3$, then $\omega(\Gamma) \geq (r+1)^{n-r} - 1$ for $1 \leq r \leq n-1$.*

Proof Let $n \geq 3$ and $A = \{x_1, x_2, \ldots, x_r\} \subseteq X_n$ for $1 \leq r \leq n-1$. Let

$$B = \{\alpha \in \mathcal{P}_n \mid A \subseteq \operatorname{codom}(\alpha) \text{ and } \emptyset \neq \operatorname{im}(\alpha) \subseteq A\}.$$

It is clear that $B \neq \emptyset$ and if $\alpha \in B$, then $\alpha \in V(\Gamma)$. Let $\alpha_1, \alpha_2 \in B$ and $\alpha_1 \neq \alpha_2$, then we have $\operatorname{im}(\alpha_1) \subseteq A \subseteq \operatorname{codom}(\alpha_2)$ and $\operatorname{im}(\alpha_2) \subseteq A \subseteq \operatorname{codom}(\alpha_1)$, so α_1 and α_2 adjacent vertices in Γ from Lemma 2.1. Let G be an induced subgraph of Γ induced by the vertex set B, then we have G is a complete graph. Moreover $|B| = (r+1)^{n-r} - 1$. Thus for $n \geq 3$, $\omega(\Gamma) \geq (r+1)^{n-r} - 1$ for $1 \leq r \leq n-1$. □

For any graph G, it is known that $\chi(G) \geq \omega(G)$ (see Corollary 6.2 in [3]), so we have the following corollary.

Corollary 3.8 *If $n \geq 3$, then $\chi(\Gamma) \geq (r+1)^{n-r} - 1$ for $1 \leq r \leq n-1$.*

Example 3.9 *Let $\Gamma = \Gamma(\mathcal{P}_5)$, then Γ is a connected graph, $|V(\Gamma)| = 4650$, diam$(\Gamma) = 4$, gr$(\Gamma) = 3$, $\Delta(\Gamma) = 624$, $\delta(\Gamma) = 1$, $\gamma(\Gamma) = 25$, $\omega(\Gamma) \geq 26$ and $\chi(\Gamma) \geq 26$. Moreover, if $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & - & 2 \end{pmatrix}$, then $\alpha \in V(\Gamma)$ and $\deg_\Gamma(\alpha) = 63$.*

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