

2-absorbing φ - δ -primary ideals

Sanem YAVUZ¹, Serkan ONAR², Bayram Ali ERSOY^{1,*}, Ünsal TEKİR³, Suat KOÇ³

¹Department of Mathematics, Yıldız Technical University, İstanbul, Turkey

²Department of Mathematical Engineering, Yıldız Technical University, İstanbul, Turkey

³Department of Mathematics, Marmara University, İstanbul, Turkey

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Abstract: This paper aims to introduce 2-absorbing φ - δ -primary ideals over commutative rings which unify the concepts of all generalizations of 2-absorbing and 2-absorbing primary ideals. Let A be a commutative ring with a nonzero identity and $\mathcal{I}(A)$ be the set of all ideals of A . Suppose that $\delta : \mathcal{I}(A) \rightarrow \mathcal{I}(A)$ is an expansion function and $\varphi : \mathcal{I}(A) \rightarrow \mathcal{I}(A) \cup \{\emptyset\}$ is a reduction function. A proper ideal Q of A is said to be a 2-absorbing φ - δ -primary if whenever $abc \in Q - \varphi(Q)$, where $a, b, c \in R$, then either $ab \in Q$ or $ac \in \delta(Q)$ or $bc \in \delta(Q)$. Various examples, properties, and characterizations of this new class of ideals are given.

Key words: 2-absorbing ideal, 2-absorbing primary ideal, 2-absorbing φ - δ -primary ideal

1. Introduction

In this paper, we focus only on commutative rings with a nonzero identity. Let A denote such a ring and $\mathcal{I}(A)$ denote the set of all ideals of A . The concepts of prime ideals and its generalizations have a distinguished place in commutative algebra since they have some applications to other areas such as graph theory, coding theory, algebraic geometry, cryptology, and general topology. See, for instance, [1, 3, 14, 17, 22]. Among several recent generalizations of prime ideals in commutative rings, we find the following, due to Badawi [4]. A nonzero proper ideal Q of A is said to be a 2-absorbing ideal (weakly 2-absorbing ideal) if whenever $abc \in Q$ ($0 \neq abc \in Q$) for some $a, b, c \in A$, then $ab \in Q$ or $ac \in Q$ or $bc \in Q$ [4] ([5]). Afterwards, Badawi et al. introduced the concept of 2-absorbing primary (weakly 2-absorbing primary) ideals as follows: a proper ideal Q of A is said to be a 2-absorbing primary ideal (weakly 2-absorbing primary ideal) if whenever $abc \in Q$ ($0 \neq abc \in Q$) for some $a, b, c \in A$, then $ab \in Q$ or $ac \in \sqrt{Q}$ or $bc \in \sqrt{Q}$, where \sqrt{Q} denotes the radical of the ideal Q of A [8] ([10]). In a recent study, Fahid and Zhao introduced and studied the concept of 2-absorbing δ -primary ideals which unify 2-absorbing ideals and 2-absorbing primary ideals [11]. Recall from [23] that a function $\delta : \mathcal{I}(A) \rightarrow \mathcal{I}(A)$ is said to be an expansion function if $I \subseteq \delta(I)$ and $I \subseteq J$ implies that $\delta(I) \subseteq \delta(J)$ for every $I, J \in \mathcal{I}(A)$. For various examples of expansion functions, the reader may consult [23, Example 1.2]. Note that the radical operation, denoted by $\sqrt{}$, is an example of expansion function. A proper ideal Q of A is said to be a δ -primary ideal if whenever $ab \in Q$ for some $a, b \in A$, then either $a \in Q$ or $b \in \delta(Q)$ [23]. Also, Q is said to be a 2-absorbing δ -primary ideal if whenever $abc \in Q$ for some $a, b, c \in A$, then $ab \in Q$ or $ac \in \delta(Q)$ or

*Correspondence: ersoya@yildiz.edu.tr

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$bc \in \delta(Q)$ [11]. If we take δ as the radical operation, all 2-absorbing δ -primary ideals and 2-absorbing primary ideals are equivalent. On the other hand, if $\delta(I) = I$ for every $I \in \mathcal{I}(\mathcal{A})$, that is, δ is the identity function, then all 2-absorbing ideals and 2-absorbing δ -primary ideals coincide. In this point of view, 2-absorbing δ -primary ideals unify the concepts of 2-absorbing ideals and 2-absorbing primary ideals. In a very recent paper, Fahid and Badawi presented a generalization of 2-absorbing δ -primary ideals as follows: a proper ideal Q of A is said to be weakly 2-absorbing δ -primary if whenever $0 \neq abc \in Q$ for some $a, b, c \in A$ then $ab \in Q$ or $ac \in \delta(Q)$ or $bc \in \delta(Q)$ [6]. In fact, weakly 2-absorbing δ -primary ideals unify the concepts of weakly 2-absorbing ideals and weakly 2-absorbing primary ideals. For various generalizations of prime ideals and 2-absorbing ideals, we refer to [7, 9, 12, 15, 16, 19–21]. Our aim in this paper is to unify the concepts of 2-absorbing δ -primary ideals and weakly 2-absorbing δ -primary ideals and to extend their properties to the our new concept. Let $\varphi : \mathcal{I}(\mathcal{A}) \rightarrow \mathcal{I}(\mathcal{A}) \cup \{\emptyset\}$ be a function. Recall from [13] that a function φ is said to be a reduction function if $\varphi(I) \subseteq I$ and $I \subseteq J$ implies that $\varphi(I) \subseteq \varphi(J)$ for every $I, J \in \mathcal{I}(\mathcal{A})$. We call a proper ideal Q of A a 2-absorbing φ - δ -primary ideal if whenever $abc \in Q - \varphi(Q)$ for some $a, b, c \in Q$ then $ab \in Q$ or $ac \in \delta(Q)$ or $bc \in \delta(Q)$. It is clear that if we take $\varphi(I) = 0$ ($\varphi(I) = \emptyset$) for every $I \in \mathcal{I}(\mathcal{A})$, then all weakly 2-absorbing δ -primary ideals (2-absorbing δ -primary ideals) and 2-absorbing φ - δ -primary ideals coincide. Among many results in this paper, we give several characterizations of 2-absorbing φ - δ -primary ideals in commutative rings (See, Proposition 2.2, Lemma 2.6, and Theorem 2.7). Also, we investigate the stability of 2-absorbing φ - δ -primary ideals under intersection, under the radical operation, under the homomorphism, in factor rings, in localization of rings (See, Proposition 2.3, Theorem 2.5, Theorem 2.12, Corollary 2.14, Proposition 2.21). Moreover, we give the Correspondence Theorem for 2-absorbing φ - δ -primary ideals (See Theorem 2.13). Furthermore, we show that if Q is a 2-absorbing φ - δ -primary ideal that is not a 2-absorbing δ -primary ideal, then $Q^3 \subseteq \varphi(Q)$ (See, Theorem 2.18). Finally, we determine all 2-absorbing φ - δ -primary ideals in direct product of rings (See, Theorem 2.24-Theorem 2.26).

2. Characterization of 2-absorbing φ - δ -primary ideals

Throughout this section, A denotes a commutative ring with $1 \neq 0$, $\delta : \mathcal{I}(\mathcal{A}) \rightarrow \mathcal{I}(\mathcal{A})$ denote, an expansion function, and $\varphi : \mathcal{I}(\mathcal{A}) \rightarrow \mathcal{I}(\mathcal{A}) \cup \{\emptyset\}$ denotes a reduction function.

Definition 2.1 *A proper ideal Q of A is said to be a 2-absorbing φ - δ -primary ideal if $abc \in Q - \varphi(Q)$ for some $a, b, c \in A$ then $ab \in Q$ or $ac \in \delta(Q)$ or $bc \in \delta(Q)$.*

The following is our first result which can be easily verified. Hence, we omit the proof.

Proposition 2.2 *Suppose that $\delta, \gamma : \mathcal{I}(\mathcal{A}) \rightarrow \mathcal{I}(\mathcal{A})$ are two expansion functions and $\varphi, \psi : \mathcal{I}(\mathcal{A}) \rightarrow \mathcal{I}(\mathcal{A}) \cup \{\emptyset\}$ are two reduction functions. The following statements are satisfied.*

- (i) *If $\delta \leq \gamma$, then every 2-absorbing φ - δ -primary ideal is also a 2-absorbing φ - γ -primary ideal.*
- (ii) *If $\varphi \leq \psi$, then every 2-absorbing φ - δ -primary ideal is also a 2-absorbing ψ - δ -primary ideal.*
- (iii) *If $\delta \leq \gamma$ and $\varphi \leq \psi$, then every 2-absorbing φ - δ -primary ideal is also a 2-absorbing ψ - γ -primary ideal.*
- (iv) *Every 2-absorbing δ -primary ideal is also 2-absorbing φ - δ -primary ideal.*

Proposition 2.3 (i) Assume that $\{Q_i\}_{i \in \Delta}$ is a directed family of 2-absorbing φ - δ -primary ideals of A . Then, $\bigcup_{i \in \Delta} Q_i$ is a 2-absorbing φ - δ -primary ideal of A .

(ii) Assume that $\{Q_i\}_{i \in \Delta}$ is a family of 2-absorbing φ - δ -primary ideals of A , $\varphi(Q_i) = \varphi(Q_j)$ and $\delta(Q_i) = \delta(Q_j)$ for every $i, j \in \Delta$. If δ, φ have the intersection property, then $\bigcap_{i \in \Delta} Q_i$ is a 2-absorbing φ - δ -primary ideal of A .

Proof (i) Suppose that $abc \in \bigcup_{i \in \Delta} Q_i - \varphi\left(\bigcup_{i \in \Delta} Q_i\right)$ for some $a, b, c \in A$. Assume that $ac, bc \notin \delta\left(\bigcup_{i \in \Delta} Q_i\right)$.

Then for each i , we have $ac, bc \notin \delta(Q_i)$. Since $abc \in \bigcup_{i \in \Delta} Q_i - \varphi\left(\bigcup_{i \in \Delta} Q_i\right)$, we have $abc \in Q_t - \varphi(Q_t)$ for some $t \in \Delta$. Since Q_t is a 2-absorbing φ - δ -primary ideal, $ab \in Q_t \subseteq \bigcup_{i \in \Delta} Q_i$ which completes the proof.

(ii) Let $\varphi(Q_i) = I$ and $\delta(Q_i) = J$. Then first note that $\varphi\left(\bigcap_{i \in \Delta} Q_i\right) = I$ and $\delta\left(\bigcap_{i \in \Delta} Q_i\right) = J$. Let $abc \in \bigcap_{i \in \Delta} Q_i - I$ for some $a, b, c \in A$. Then for every i , we have $abc \in Q_i$. We may assume that $ac, bc \notin J$. Since Q_i is a 2-absorbing φ - δ -primary ideal of A , we conclude that $ab \in Q_i$ for each i . This implies that $ab \in \bigcap_{i \in \Delta} Q_i$. Hence, $\bigcap_{i \in \Delta} Q_i$ is a 2-absorbing φ - δ -primary ideal. \square

The condition " $\delta(Q_i) = \delta(Q_j)$ for every i, j " in Proposition 2.3 is necessary. See the following example.

Example 2.4 Let $A = \mathbb{Z}$, $Q_1 = 6\mathbb{Z}$ and $Q_2 = 15\mathbb{Z}$. Then Q_1, Q_2 are 2-absorbing φ - δ -primary ideal for every expansion function δ and every reduction function φ . Assume that $\delta(I) = I$ and $\varphi(I) = \emptyset$ for every ideal I of A . Then note that $\delta(Q_1) \neq \delta(Q_2)$ and also $Q_1 \cap Q_2 = 30\mathbb{Z} = \delta(Q_1 \cap Q_2)$. Since $2 \cdot 3 \cdot 5 \in Q_1 \cap Q_2$, $6 \notin Q_1 \cap Q_2$, $10, 15 \notin \delta(Q_1 \cap Q_2)$, it follows that $Q_1 \cap Q_2$ is not a 2-absorbing φ - δ -primary ideal.

Theorem 2.5 If Q is a 2-absorbing φ - δ -primary ideal of A such that $\sqrt{\delta(Q)} \subseteq \delta(\sqrt{Q})$ and $\sqrt{\varphi(Q)} \subseteq \varphi(\sqrt{Q})$, then \sqrt{Q} is a 2-absorbing φ - δ -primary ideal of A .

Proof Suppose that $a, b, c \in A$ such that $abc \in \sqrt{Q} - \varphi(\sqrt{Q})$ and $ab \notin \sqrt{Q}$, $ac \notin \delta(\sqrt{Q})$. Since $abc \in \sqrt{Q} - \varphi(\sqrt{Q})$, there exists a positive integer n such that $(abc)^n = a^n b^n c^n \in Q - \varphi(Q)$. As $ab \notin \sqrt{Q}$, $ac \notin \delta(\sqrt{Q})$, we have that $(ab)^n \notin Q$ and $(ac)^n \notin \delta(Q)$. On the other hand, since Q is a 2-absorbing φ - δ -primary ideal of A , we have $(bc)^n \in \delta(Q)$, that is $bc \in \sqrt{\delta(Q)} \subseteq \delta(\sqrt{Q})$. Hence, \sqrt{Q} is a 2-absorbing φ - δ -primary ideal of A . \square

Let Q be an ideal of A and J a nonempty subset of A . The residual of Q by J is denoted by $(Q : J) = \{a \in A : aJ \subseteq Q\}$. In particular, if $J = \{a\}$ is the singleton, we use $(Q : a)$ instead of $(Q : \{a\})$.

Lemma 2.6 Let Q be a proper ideal of A . The following statements are equivalent.

- (i) Q is a 2-absorbing φ - δ -primary ideal of A .
- (ii) For each $a, b \in A$ such that $ab \notin Q$, $(Q : ab) = (\varphi(Q) : ab) \cup (\delta(Q) : a) \cup (\delta(Q) : b)$.

Proof (i) \Rightarrow (ii) : Let $c \in (Q : ab)$, that is, $abc \in Q$. If $abc \in \varphi(Q)$, we have $c \in (\varphi(Q) : ab)$. Hence, assume that $abc \in Q - \varphi(Q)$. Since Q is a 2-absorbing φ - δ -primary ideal of A , we have $ac \in \delta(Q)$ or $bc \in \delta(Q)$. This

gives $c \in (\delta(Q) : a) \cup (\delta(Q) : b)$. Thus, we conclude that $(Q : ab) \subseteq (\varphi(Q) : ab) \cup (\delta(Q) : a) \cup (\delta(Q) : b)$. Since the reverse inclusion always holds, we have the equality.

(ii) \Rightarrow (i) : It is easy. □

Recall from [18] that a ring A is said to be a u -ring if whenever $I \subseteq \bigcup_{i=1}^n I_i$ for some ideals I, I_1, I_2, \dots, I_n of A , then $I \subseteq I_i$ for some $1 \leq i \leq n$. Note that Prüfer domains and principal ideal domains are some examples of u -rings.

Theorem 2.7 *Let Q be a proper ideal of a u -ring A . The following statements are equivalent.*

- (i) Q is a 2-absorbing φ - δ -primary ideal of A .
- (ii) For each $a, b \in A$ such that $ab \notin Q$, $(Q : ab) = (\varphi(Q) : ab) \cup (\delta(Q) : a) \cup (\delta(Q) : b)$.
- (iii) If $abJ \subseteq Q$ and $abJ \not\subseteq \varphi(Q)$ for some $a, b \in A$ and an ideal J of A , then $ab \in Q$ or $aJ \subseteq \delta(Q)$ or $bJ \subseteq \delta(Q)$.
- (iv) For each $a \in A$, $J \in \mathcal{I}(A)$ with $aJ \not\subseteq \delta(Q)$, $(Q : aJ) = (Q : a)$ or $(Q : aJ) = (\varphi(Q) : aJ)$ or $(Q : aJ) \subseteq (\delta(Q) : J)$.
- (v) If $aIJ \subseteq Q$ and $aIJ \not\subseteq \varphi(Q)$ for some $a \in A$ and ideals I, J of A , then $aI \subseteq Q$ or $aJ \subseteq \delta(Q)$ or $IJ \subseteq \delta(Q)$.
- (vi) For each $I, J \in \mathcal{I}(A)$ with $IJ \not\subseteq \delta(Q)$, $(Q : IJ) = (Q : I)$ or $(Q : IJ) = (\varphi(Q) : IJ)$ or $(Q : IJ) \subseteq (\delta(Q) : J)$.
- (vii) If $IJK \subseteq Q$ and $IJK \not\subseteq \varphi(Q)$ for some ideals I, J, K of A , then $IJ \subseteq Q$ or $IK \subseteq \delta(Q)$ or $JK \subseteq \delta(Q)$.

Proof (i) \Leftrightarrow (ii) : It follows from Lemma 2.6.

(ii) \Rightarrow (iii) : Let $abJ \subseteq Q$ and $abJ \not\subseteq \varphi(Q)$ for some $a, b \in A$ and an ideal J of A . If $ab \in Q$, then we are done. Assume that $ab \notin Q$. Since A is a u -ring, by (ii), we have $(Q : ab) \subseteq (\varphi(Q) : ab)$ or $(Q : ab) \subseteq (\delta(Q) : a)$ or $(Q : ab) \subseteq (\delta(Q) : b)$. Since $J \subseteq (Q : ab)$ and $J \not\subseteq (\varphi(Q) : ab)$, we conclude either $J \subseteq (\delta(Q) : a)$ or $J \subseteq (\delta(Q) : b)$ that means $aJ \subseteq \delta(Q)$ or $bJ \subseteq \delta(Q)$.

(iii) \Rightarrow (iv) : Let $aJ \not\subseteq \delta(Q)$ for some $a \in A$, $J \in \mathcal{I}(A)$. Let $b \in (Q : aJ)$. Then we have $abJ \subseteq Q$. If $ab \in Q$, then we have $b \in (Q : a)$, so assume that $ab \notin Q$. If $abJ \subseteq \varphi(Q)$, then we have $b \in (\varphi(Q) : aJ)$. Hence, assume that $abJ \not\subseteq \varphi(Q)$. Since $abJ \subseteq Q$, $abJ \not\subseteq \varphi(Q)$ and $ab \notin Q$, by (iii), we have $aJ \subseteq \delta(Q)$ or $bJ \subseteq \delta(Q)$. Thus, we have $(Q : aJ) \subseteq (Q : a) \cup (\varphi(Q) : aJ) \cup (\delta(Q) : J)$. Since A is a u -ring, we conclude that $(Q : aJ) \subseteq (Q : a)$ or $(Q : aJ) \subseteq (\varphi(Q) : aJ)$ or $(Q : aJ) \subseteq (\delta(Q) : J)$. **Case 1:** Assume that $(Q : aJ) \subseteq (Q : a)$. Since the reverse inclusion is always true, we obtain $(Q : aJ) = (Q : a)$. **Case 2:** Assume that $(Q : aJ) \subseteq (\varphi(Q) : aJ)$. As $\varphi(Q) \subseteq Q$, we have $(\varphi(Q) : aJ) \subseteq (Q : aJ)$, which implies that $(\varphi(Q) : aJ) = (Q : aJ)$. **Case 3:** If $(Q : aJ) \subseteq (\delta(Q) : J)$, then we are done. (iv) \Rightarrow (v) : Suppose that $aIJ \subseteq Q$ and $aIJ \not\subseteq \varphi(Q)$ for some $a \in A$ and ideals I, J of A . Then $I \subseteq (Q : aJ)$ and $I \not\subseteq (\varphi(Q) : aJ)$. If $aJ \subseteq \delta(Q)$, then we are done. Thus, assume that $aJ \not\subseteq \delta(Q)$. By (iv), $(Q : aJ) = (Q : a)$ or $(Q : aJ) \subseteq (\delta(Q) : J)$. Since $I \subseteq (Q : aJ)$, we have $aI \subseteq Q$ or $IJ \subseteq \delta(Q)$ which completes the proof.

(v) \Rightarrow (vi) : Let $I, J \in \mathcal{I}(A)$ with $IJ \not\subseteq \delta(Q)$. Now, take $a \in (Q : IJ)$. Then we get $aIJ \subseteq Q$. If $aIJ \subseteq \varphi(Q)$, then $a \in (\varphi(Q) : IJ)$. Now, assume that $aIJ \not\subseteq \varphi(Q)$. Then by (v), $aI \subseteq Q$ or $aJ \subseteq \delta(Q)$ which

implies that $a \in (Q : I)$ or $a \in (\delta(Q) : J)$. Thus, we conclude that $(Q : IJ) \subseteq (\varphi(Q) : IJ) \cup (Q : I) \cup (\delta(Q) : J)$. Since A is a u -ring, we get $(Q : IJ) \subseteq (\varphi(Q) : IJ)$ or $(Q : IJ) \subseteq (Q : I)$ or $(Q : IJ) \subseteq (\delta(Q) : J)$. **Case 1:** Assume that $(Q : IJ) \subseteq (\varphi(Q) : IJ)$. Since $\varphi(Q) \subseteq Q$, we have $(\varphi(Q) : IJ) \subseteq (Q : IJ)$, which implies that $(\varphi(Q) : IJ) = (Q : IJ)$. **Case 2:** Assume that $(Q : IJ) \subseteq (Q : I)$. Since $(Q : I) \subseteq (Q : IJ)$, we have $(Q : IJ) = (Q : I)$. **Case 3:** Assume that $(Q : IJ) \subseteq (\delta(Q) : J)$. Then there is nothing to prove.

(vi) \Rightarrow (vii) : Suppose that $IJK \subseteq Q$ and $IJK \not\subseteq \varphi(Q)$ for some ideals I, J, K of A . Then we have $I \subseteq (Q : JK)$ and $I \not\subseteq (\varphi(Q) : JK)$. Then by (vi), we have $JK \subseteq \delta(Q)$ or $I \subseteq (Q : JK) = (Q : J)$ or $I \subseteq (Q : JK) \subseteq (\delta(Q) : K)$, which completes the proof.

(vii) \Rightarrow (i) : It is straightforward. □

In Theorem 2.7 (iv), the containment " $(Q : aJ) \subseteq (\delta(Q) : J)$ " may be strict even if Q satisfies all axioms in Theorem 2.7. See the following example.

Example 2.8 Let $A = \mathbb{Z}$ be the ring of integers and $Q = 36\mathbb{Z}$. Assume that $\delta(Q) = \sqrt{Q}$ and $\varphi(Q) = (0)$. Then note that A is a u -ring and Q is a 2-absorbing φ - δ -primary ideal of A . Then Q satisfies all axioms in Theorem 2.7. Let $J = 3\mathbb{Z}$ and $a = 3$. Then note that $aJ = 9\mathbb{Z} \not\subseteq \delta(Q) = 6\mathbb{Z}$. Moreover, it is easy to see that $(Q : aJ) = 4\mathbb{Z} \neq (Q : a) = 12\mathbb{Z}$ and $(Q : aJ) = 4\mathbb{Z} \neq ((0) : aJ) = (0)$. However, $(Q : aJ) = 4\mathbb{Z} \subseteq (\delta(Q) : J) = 2\mathbb{Z}$.

Theorem 2.9 Let Q be a 2-absorbing φ - δ -primary ideal of A and $x \in A - Q$ such that $(\varphi(Q) : x) \subseteq \varphi(Q : x)$ and $(\delta(Q) : x) \subseteq \delta(Q : x)$. Then $(Q : x)$ is a 2-absorbing φ - δ -primary ideal of A . In particular, if J is an ideal of A with $J \not\subseteq Q$ such that $(\varphi(Q) : J) \subseteq \varphi(Q : J)$ and $(\delta(Q) : J) \subseteq \delta(Q : J)$, then $(Q : J)$ is a 2-absorbing φ - δ -primary ideal of A .

Proof Let $abc \in (Q : x) - \varphi((Q : x))$. Then $ab(cx) \in Q - \varphi(Q)$. Since Q is a 2-absorbing φ - δ -primary ideal of A , we have $ab \in Q$ or $acx \in \delta(Q)$ or $bcx \in \delta(Q)$. Since $(\delta(Q) : x) \subseteq \delta(Q : x)$, we have $ab \in (Q : x)$ or $ac \in \delta(Q : x)$ or $bc \in \delta(Q : x)$. The rest is similar. □

Let $f : A \rightarrow B$ be a ring homomorphism. Suppose that δ is an ideal expansion of $\mathcal{I}(A)$, φ is a reduction function of $\mathcal{I}(A)$ and also, γ is an ideal expansion of $\mathcal{I}(B)$, ψ is a reduction function of $\mathcal{I}(B)$. Then f is said to be a (δ, φ) - (γ, ψ) -homomorphism if $\delta(f^{-1}(J)) = f^{-1}(\gamma(J))$ and $\varphi(f^{-1}(J)) = f^{-1}(\psi(J))$ for every $J \in \mathcal{I}(B)$.

Example 2.10 Let δ be the radical operation on $\mathcal{I}(A)$ and γ the radical operation on $\mathcal{I}(B)$. Also assume that φ and ψ are empty reduction functions. Then note that every homomorphism from A to B is a (δ, φ) - (γ, ψ) -homomorphism.

Remark 2.11 Suppose that f is a (δ, φ) - (γ, ψ) -epimorphism from A to B and I is an ideal of A containing $\text{Ker}(f)$. Then $\gamma(f(I)) = f(\delta(I))$ and $\psi(f(I)) = f(\varphi(I))$. To see this, let $J = f(I)$. Then (δ, φ) - (γ, ψ) -epimorphism gives the equalities $\delta(f^{-1}(J)) = f^{-1}(\gamma(J))$ and $\varphi(f^{-1}(J)) = f^{-1}(\psi(J))$. Since $f^{-1}(J) = I$ and $f(f^{-1}(K)) = K$ for every ideal K of B , we conclude that

$$f(\delta(f^{-1}(J))) = f(\delta(I)) = f(f^{-1}(\gamma(J))) = \gamma(J) = \gamma(f(I))$$

$$f(\varphi(f^{-1}(J))) = f(\varphi(I)) = f(f^{-1}(\psi(J))) = \psi(J) = \psi(f(I)).$$

Theorem 2.12 Let $f : A \rightarrow B$ be a (δ, φ) - (γ, ψ) -homomorphism, where δ is an ideal expansion of $\mathcal{I}(A)$, φ is a reduction function of $\mathcal{I}(A)$, and also, γ is an ideal expansion of $\mathcal{I}(B)$, ψ is a reduction function of $\mathcal{I}(B)$. Then the following statements are satisfied.

(i) If J is a 2-absorbing ψ - γ -primary ideal of B , then $f^{-1}(J)$ is a 2-absorbing φ - δ -primary ideal of A .

(ii) If Q is a 2-absorbing φ - δ -primary ideal of A containing $\text{Ker}(f)$ and f is surjective, then $f(Q)$ is a 2-absorbing ψ - γ -primary ideal of B .

Proof (i) : Let J be a 2-absorbing ψ - γ -primary ideal of B . Choose $a, b, c \in A$ such that $abc \in f^{-1}(J) - \varphi(f^{-1}(J))$. Then we have $f(a)f(b)f(c) \in J - \psi(J)$. Since J is a 2-absorbing ψ - γ -primary ideal of B , we conclude that $f(a)f(b) \in J$ or $f(a)f(c) \in \gamma(J)$ or $f(b)f(c) \in \gamma(J)$, which implies that $ab \in f^{-1}(J)$ or $ac \in f^{-1}(\gamma(J)) = \delta(f^{-1}(J))$ or $bc \in f^{-1}(\gamma(J)) = \delta(f^{-1}(J))$. Hence, $f^{-1}(J)$ is a 2-absorbing φ - δ -primary ideal of A .

(ii) : Let Q be a 2-absorbing φ - δ -primary ideal of A containing $\text{Ker}(f)$ and f surjective $xyz \in f(Q) - \psi(f(Q))$ for some $x, y, z \in B$. Since f is surjective, we can write $f(a) = x, f(b) = y, f(c) = z$ for some $a, b, c \in A$. This implies that $f(abc) = xyz \in f(Q)$. Since $\text{Ker}(f) \subseteq Q$, we conclude that $abc \in Q$. Moreover, by Remark 2.11, $abc \in Q - \varphi(Q)$. As Q is a 2-absorbing φ - δ -primary ideal of A , we have $ab \in Q$ or $ac \in \delta(Q)$ or $bc \in \delta(Q)$. Thus, we conclude that $xy = f(ab) \in f(Q)$ or $xz = f(ac) \in f(\delta(Q)) = \gamma(f(Q))$ or $yz \in \gamma(f(Q))$. Therefore, $f(Q)$ is a 2-absorbing ψ - γ -primary ideal of B . \square

From the previous theorem, we obtain the following correspondence theorem for 2-absorbing φ - δ -primary ideals.

Theorem 2.13 (Correspondence theorem) Let $f : A \rightarrow B$ be a (δ, φ) - (γ, ψ) -epimorphism, where δ is an ideal expansion of $\mathcal{I}(A)$, φ is a reduction function of $\mathcal{I}(A)$ and also, γ is an ideal expansion of $\mathcal{I}(B)$, ψ is a reduction function of $\mathcal{I}(B)$. Then f induces to a one-to-one correspondence (that preserves inclusion) between the 2-absorbing φ - δ -primary ideals of A containing $\text{Ker}(f)$ and the 2-absorbing ψ - γ -primary ideals of B in such a way that if Q is a 2-absorbing φ - δ -primary ideal of A containing $\text{Ker}(f)$ then $f(Q)$ is the corresponding 2-absorbing ψ - γ -primary ideal of B , and J is a 2-absorbing ψ - γ -primary ideal of B , then $f^{-1}(J)$ is the corresponding 2-absorbing φ - δ -primary ideal of A .

Assume that δ is an expansion function of $\mathcal{I}(A)$, φ is a reduction function of $\mathcal{I}(A)$, and J is an ideal of A . Then $\delta_q : \mathcal{I}(A/J) \rightarrow \mathcal{I}(A/J)$ defined by $\delta_q(I/J) = \delta(I)/J$, and $\varphi_q : \mathcal{I}(A/J) \rightarrow \mathcal{I}(A/J) \cup \{\emptyset\}$ defined by $\varphi_q(I/J) = \varphi(I)/J$ are expansion and reduction functions, respectively.

Theorem 2.14 (i) Let J and Q be two proper ideals of A such that $J \subseteq \varphi(Q) \subseteq Q$. Then Q is a 2-absorbing φ - δ -primary ideal of A if and only if Q/J is a 2-absorbing φ_q - δ_q -primary ideal of A/J .

(ii) Let Q be a proper ideal of A . Then Q is a 2-absorbing φ - δ -primary ideal of A if and only if $Q/\varphi(Q)$ is a weakly 2-absorbing δ_q -primary ideal of $A/\varphi(Q)$.

(iii) Let Q be a proper ideal of A . Then Q is a φ - δ -primary ideal of A if and only if $Q/\varphi(Q)$ is a weakly δ_q -primary ideal of $A/\varphi(Q)$.

Proof (i) : Let Q be a 2-absorbing φ - δ -primary ideal of A and $(a + J)(b + J)(c + J) \in Q/J - \varphi_q(Q/J)$ for some $a, b, c \in A$. Then we have $abc \in Q - \varphi(Q)$. As Q is a 2-absorbing φ - δ -primary ideal, we conclude that

$ab \in Q$ or $ac \in \delta(Q)$ or $bc \in \delta(Q)$. This implies that $(a + J)(b + J) \in Q/J$ or $(a + J)(c + J) \in \delta_q(Q/J)$ or $(b + J)(c + J) \in \delta_q(Q/J)$. Thus, Q/J is a 2-absorbing φ_q - δ_q -primary ideal of A/J .

Conversely, assume that Q/J is a 2-absorbing φ_q - δ_q -primary ideal of A/J . Let $abc \in Q - \varphi(Q)$ for some $a, b, c \in A$. Then we have $(a + J)(b + J)(c + J) \in Q/J - \varphi_q(Q/J)$ because $J \subseteq \varphi(Q)$. Since Q/J is a 2-absorbing φ_q - δ_q -primary ideal, we have $(a + J)(b + J) \in Q/J$ or $(a + J)(c + J) \in \delta_q(Q/J) = \delta(Q)/J$ or $(b + J)(c + J) \in \delta(Q)/J$. Thus, we conclude that $ab \in Q$ or $ac \in \delta(Q)$ or $bc \in \delta(Q)$, which completes the proof.

(ii) : Let Q be a 2-absorbing φ - δ -primary ideal of A and $0_{A/\varphi(Q)} \neq (a + \varphi(Q))(b + \varphi(Q))(c + \varphi(Q)) \in Q/\varphi(Q)$ for some $a, b, c \in A$. Then we have $abc \in Q - \varphi(Q)$. This implies that $ab \in Q$ or $ac \in \delta(Q)$ or $bc \in \delta(Q)$. Then we conclude that $(a + \varphi(Q))(b + \varphi(Q)) \in Q/\varphi(Q)$ or $(a + \varphi(Q))(c + \varphi(Q)) \in \delta_q(Q/\varphi(Q))$ or $(b + \varphi(Q))(c + \varphi(Q)) \in \delta_q(Q/\varphi(Q))$. Hence, $Q/\varphi(Q)$ is a weakly 2-absorbing δ_q -primary ideal of $A/\varphi(Q)$. Conversely, assume that $Q/\varphi(Q)$ is a weakly 2-absorbing δ_q -primary ideal of $A/\varphi(Q)$ and let $abc \in Q - \varphi(Q)$ for some $a, b, c \in A$. This gives $0_{A/\varphi(Q)} \neq (a + \varphi(Q))(b + \varphi(Q))(c + \varphi(Q)) \in Q/\varphi(Q)$. Then by assumption, we conclude that $(a + \varphi(Q))(b + \varphi(Q)) \in Q/\varphi(Q)$ or $(a + \varphi(Q))(c + \varphi(Q)) \in \delta_q(Q/\varphi(Q))$ or $(b + \varphi(Q))(c + \varphi(Q)) \in \delta_q(Q/\varphi(Q))$. Since $\delta_q(Q/\varphi(Q)) = \delta(Q)/\varphi(Q)$, we have $ab \in Q$ or $ac \in \delta(Q)$ or $bc \in \delta(Q)$. Therefore, Q is a 2-absorbing φ - δ -primary ideal of A .

(iii) : It is similar to (ii). □

Let Q be a proper ideal of A . Recall from [13] that Q is said to be a φ - δ -primary ideal if whenever $ab \in Q - \varphi(Q)$ for some $a, b \in A$ then $a \in Q$ or $b \in \delta(Q)$. If $\varphi(Q) = 0$, then φ - δ -primary ideal is just called a weakly δ -primary ideal.

Recall from [6] that if an ideal Q of A is a weakly 2-absorbing δ -primary (weakly δ -primary) that is not a 2-absorbing δ -primary (δ -primary) ideal of A , then there exist $a, b, c \in A$ ($a, b \in A$) such that $abc = 0$, $ab \notin Q$ and $ac, bc \notin \delta(Q)$ ($ab = 0$, $a \notin Q$ and $b \notin \delta(Q)$). In this case, (a, b, c) ((a, b)) is said to be a δ -triple zero (δ -twin zero) of Q .

Remark 2.15 (1) Suppose that Q is a φ - δ -primary ideal of A that is not δ -primary. Then there exist $a, b \in A$ such that $ab \in \varphi(Q)$, $a \notin Q$ and $b \notin \delta(Q)$. In this case, we say that (a, b) is a φ - δ -twin zero of Q .

(2) Suppose that Q is a 2-absorbing φ - δ -primary ideal of A that is not a 2-absorbing δ -primary. Then there exist $a, b, c \in A$ such that $abc \in \varphi(Q)$, $ab \notin Q$ and $ac, bc \notin \delta(Q)$. In this case, we say that (a, b, c) is a φ - δ -triple zero of Q .

Lemma 2.16 Let Q be a 2-absorbing φ - δ -primary ideal of A and $a, b, c \in A$. The following statements are equivalent.

(i) (a, b, c) is a φ - δ -triple zero of Q .

(ii) $(a + \varphi(Q), b + \varphi(Q), c + \varphi(Q))$ is a δ_q -triple zero of $Q/\varphi(Q)$.

Proof (i) \Rightarrow (ii) : Suppose that (a, b, c) is a φ - δ -triple zero of Q . Then $abc \in \varphi(Q)$, $ab \notin Q$ and $ac, bc \notin \delta(Q)$. This implies that $(a + \varphi(Q))(b + \varphi(Q))(c + \varphi(Q)) = 0_{Q/\varphi(Q)}$, $(a + \varphi(Q))(b + \varphi(Q)) \notin Q/\varphi(Q)$ and $(a + \varphi(Q))(c + \varphi(Q)), (b + \varphi(Q))(c + \varphi(Q))$ are not in $\delta(Q)/\varphi(Q) = \delta_q(Q/\varphi(Q))$. Thus, $(a + \varphi(Q), b + \varphi(Q), c + \varphi(Q))$ is a δ_q -triple zero of $Q/\varphi(Q)$.

(ii) \Rightarrow (i) : Suppose that $(a + \varphi(Q), b + \varphi(Q), c + \varphi(Q))$ is a δ_q -triple zero of $Q/\varphi(Q)$. This implies that $(a + \varphi(Q))(b + \varphi(Q))(c + \varphi(Q)) = 0_{Q/\varphi(Q)}$, $(a + \varphi(Q))(b + \varphi(Q)) \notin Q/\varphi(Q)$ and $(a + \varphi(Q))(c + \varphi(Q)), (b + \varphi(Q))(c + \varphi(Q))$ are not in $\delta(Q)/\varphi(Q) = \delta_q(Q/\varphi(Q))$. Thus, (a, b, c) is a φ - δ -triple zero of Q .

$\varphi(Q))(c + \varphi(Q))$ are not in $\delta(Q)/\varphi(Q) = \delta_q(Q/\varphi(Q))$. Thus, $abc \in \varphi(Q)$, $ab \notin Q$ and $ac, bc \notin \delta(Q)$. Hence, (a, b, c) is a φ - δ -triple zero of Q . \square

The following result can be proved similar to the previous lemma. Hence, we omit the proof.

Lemma 2.17 *Let Q be a φ - δ -primary ideal of A and $a, b \in A$. The following statements are equivalent.*

- (i) (a, b) is a φ - δ -twin zero of Q .
- (ii) $(a + \varphi(Q), b + \varphi(Q))$ is a δ_q -twin zero of $Q/\varphi(Q)$.

Theorem 2.18 *Suppose that Q is a 2-absorbing φ - δ -primary ideal of A and (a, b, c) is a φ - δ -triple zero of Q for some $a, b, c \in A$. Then,*

- (i) $abQ, acQ, bcQ \subseteq \varphi(Q)$.
- (ii) $aQ^2, bQ^2, cQ^2 \subseteq \varphi(Q)$.
- (iii) $Q^3 \subseteq \varphi(Q)$.

Proof Suppose that (a, b, c) is a φ - δ -triple zero of Q and Q is a 2-absorbing φ - δ -primary ideal of A . Then by Theorem 2.14 and Lemma 2.16, $Q/\varphi(Q)$ is a weakly 2-absorbing δ_q -primary ideal and $(a + \varphi(Q), b + \varphi(Q), c + \varphi(Q))$ is a δ_q -triple zero of $Q/\varphi(Q)$. Also note that $Q/\varphi(Q)$ is not a 2-absorbing δ_q -primary ideal since Q is not a 2-absorbing δ -primary ideal. Then by [6, Theorem 2.8], $(a + \varphi(Q))(b + \varphi(Q))(Q/\varphi(Q)) = 0_{Q/\varphi(Q)}$, $(a + \varphi(Q))(c + \varphi(Q))(Q/\varphi(Q)) = 0_{Q/\varphi(Q)}$ and $(b + \varphi(Q))(c + \varphi(Q))(Q/\varphi(Q)) = 0_{Q/\varphi(Q)}$. This implies that $abQ, acQ, bcQ \subseteq \varphi(Q)$. On the other hand, again by [6, Theorem 2.8], $(a + \varphi(Q))(Q/\varphi(Q))^2 = 0_{Q/\varphi(Q)}$, $(b + \varphi(Q))(Q/\varphi(Q))^2 = 0_{Q/\varphi(Q)}$ and $(c + \varphi(Q))(Q/\varphi(Q))^2 = 0_{Q/\varphi(Q)}$. This yields that $aQ^2, bQ^2, cQ^2 \subseteq \varphi(Q)$. The statement (iii) follows from [6, Theorem 2.9]. \square

Theorem 2.19 *Suppose that Q is a φ - δ -primary ideal of A and (a, b) is a φ - δ -twin zero of Q for some $a, b \in A$. Then, $aQ, bQ \subseteq \varphi(Q)$. In this case, $Q^2 \subseteq \varphi(Q)$.*

Proof It follows from Theorem 2.14, Lemma 2.17, [6, Theorem 2.8], and [6, Theorem 2.9]. \square

Corollary 2.20 (i) *Suppose that Q is a 2-absorbing φ - δ -primary ideal of A with $Q^3 \not\subseteq \varphi(Q)$. Then Q is a 2-absorbing δ -primary ideal of A .*

(ii) *Suppose that Q is a 2-absorbing φ - δ -primary ideal of A that is not a 2-absorbing δ -primary ideal of A . Then $\sqrt{Q} = \sqrt{\varphi(Q)}$.*

Proof (i) : Suppose that Q is a 2-absorbing φ - δ -primary ideal of A with $Q^3 \not\subseteq \varphi(Q)$. Now, we will show that Q is a 2-absorbing δ -primary ideal of A . Suppose not. Then there exist $a, b, c \in A$ such that $abc \in \varphi(Q)$, $ab \notin Q$ and $ac, bc \notin \delta(Q)$. Then (a, b, c) is a φ - δ -triple zero of Q . By Theorem 2.18, we have $Q^3 \subseteq \varphi(Q)$, which is a contradiction.

(ii) : Suppose that Q is a 2-absorbing φ - δ -primary ideal of A that is not a 2-absorbing δ -primary ideal of A . Then there exists a φ - δ -triple zero (a, b, c) of Q . Then by Theorem 2.18, we have $Q^3 \subseteq \varphi(Q) \subseteq Q$. This implies that $\sqrt{Q} = \sqrt{\varphi(Q)}$. \square

Let S be a nonempty subset of R . Then S is said to be a multiplicatively closed set if whenever (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $st \in S$ for every $s, t \in S$. Let Q be a proper ideal of A . Then the set $\{x \in A : xy \in Q \text{ for some } y \in A - Q\}$ is denoted by $Z_Q(A)$.

Proposition 2.21 *Let Q be a 2-absorbing φ - δ -primary ideal of A and $S \subseteq A$ a multiplicatively closed set. Suppose that $\delta_S : \mathcal{I}(A_S) \rightarrow \mathcal{I}(A_S)$ is an expansion function such that $\delta_S(I_S) = (\delta(I))_S$ for every ideal I of A and $\varphi_S : \mathcal{I}(A_S) \rightarrow \mathcal{I}(A_S)$ is a reduction function such that $\varphi_S(I_S) = (\varphi(I))_S$ for every ideal I of A . Furthermore, assume that $S \cap Z_{\varphi(Q)}(A) = S \cap Q = S \cap Z_Q(A) = S \cap Z_{\delta(Q)}(A) = \emptyset$. The following statements are equivalent.*

- (i) Q is a 2-absorbing φ - δ -primary ideal of A .
- (ii) Q_S is a 2-absorbing φ_S - δ_S -primary ideal of A_S .

Proof (i) \Rightarrow (ii) : Suppose that Q is a 2-absorbing φ - δ -primary ideal of A . Let $\frac{a}{s} \frac{b}{t} \frac{c}{u} \in Q_S - \varphi_S(Q_S)$ for some $\frac{a}{s}, \frac{b}{t}, \frac{c}{u} \in A_S$. Then there exists $w \in S$ such that $wabc \in Q - \varphi(Q)$. As Q is a 2-absorbing φ - δ -primary ideal of A , we conclude that $wab \in Q$ or $wac \in \delta(Q)$ or $bc \in \delta(Q)$. Since $\delta(Q)_S = \delta_S(Q_S)$, we get $\frac{a}{s} \frac{b}{t} \in Q_S$ or $\frac{a}{s} \frac{c}{u} \in \delta_S(Q_S)$ or $\frac{b}{t} \frac{c}{u} \in \delta_S(Q_S)$. Therefore, Q_S is a 2-absorbing φ_S - δ_S -primary ideal of A_S .

(ii) \Rightarrow (i) : Suppose that Q_S is a 2-absorbing φ_S - δ_S -primary ideal of A_S and let $abc \in Q - \varphi(Q)$. Then we have $\frac{a}{1} \frac{b}{1} \frac{c}{1} \in Q_S$. Assume that $\frac{a}{1} \frac{b}{1} \frac{c}{1} \in \varphi_S(Q_S) = (\varphi(Q))_S$, there exists $t \in S$ such that $tabc \in \varphi(Q)$. As $abc \notin \varphi(Q)$, we conclude $t \in S \cap Z_{\varphi(Q)}(A)$, which is a contradiction. So we have $\frac{a}{1} \frac{b}{1} \frac{c}{1} \in Q_S - \varphi_S(Q_S)$. Since Q_S is a 2-absorbing φ_S - δ_S -primary ideal of A_S , we have $\frac{a}{1} \frac{b}{1} \in Q_S$ or $\frac{a}{1} \frac{c}{1} \in \delta_S(Q_S)$ or $\frac{b}{1} \frac{c}{1} \in \delta_S(Q_S)$. Then there exists $s \in S$ such that $sab \in Q$ or $sac \in \delta(Q)$ or $sbc \in \delta(Q)$. **Case 1:** Let $sab \in Q$. If $ab \notin Q$, then $s \in S \cap Z_Q(A)$ which is a contradiction. Thus, $ab \in Q$. **Case 2:** Let $sac \in \delta(Q)$. If $ac \notin \delta(Q)$, then we have $s \in S \cap Z_{\delta(Q)}(A)$ which is again a contradiction. Hence, we have $ac \in \delta(Q)$. Moreover, note that if $sbc \in \delta(Q)$, then $bc \in \delta(Q)$. □

Let Q be a weakly 2-absorbing δ -primary ideal of A and $0 \neq I_1 I_2 I_3 \subseteq Q$ for some ideals I_1, I_2, I_3 of A . Recall from [6] that Q is said to be a δ -free triple zero with respect to $I_1 I_2 I_3$ if (a, b, c) is not a δ -triple zero of Q for every $a \in I_1, b \in I_2$ and $c \in I_3$. In addition, Q is said to be a δ -free triple zero if whenever $0 \neq I_1 I_2 I_3 \subseteq Q$ for some ideals I_1, I_2, I_3 of A , then I is a δ -free triple zero with respect to $I_1 I_2 I_3$.

Definition 2.22 *Suppose that Q is a 2-absorbing φ - δ -primary ideal of A and $I_1 I_2 I_3 \subseteq Q$ with $I_1 I_2 I_3 \not\subseteq \varphi(Q)$ for some ideals I_1, I_2, I_3 of A . Then Q is said to be a φ - δ -free triple zero with respect to $I_1 I_2 I_3$ if (a, b, c) is not a φ - δ -triple zero of Q for every $a \in I_1, b \in I_2$ and $c \in I_3$. In particular, Q is said to be a φ - δ -free triple zero if whenever $I_1 I_2 I_3 \subseteq Q$ with $I_1 I_2 I_3 \not\subseteq \varphi(Q)$ for some ideals I_1, I_2, I_3 of A , then I is a φ - δ -free triple zero with respect to $I_1 I_2 I_3$.*

Theorem 2.23 *Let Q be a 2-absorbing φ - δ -primary ideal of A and $I_1 I_2 I_3 \subseteq Q$ with $I_1 I_2 I_3 \not\subseteq \varphi(Q)$ for some ideals I_1, I_2, I_3 of A . Suppose that Q is a φ - δ -free triple zero with respect to $I_1 I_2 I_3$. Then $I_1 I_2 \subseteq Q$ or $I_1 I_3 \subseteq \delta(Q)$ or $I_2 I_3 \subseteq \delta(Q)$.*

Proof Suppose that Q is a 2-absorbing φ - δ -primary ideal of A and $I_1 I_2 I_3 \subseteq Q$ with $I_1 I_2 I_3 \not\subseteq \varphi(Q)$ for some ideals I_1, I_2, I_3 of A . Further assume that Q is a φ - δ -free triple zero with respect to $I_1 I_2 I_3$. Then note

that

$$((I_1 + \varphi(Q))/\varphi(Q))((I_2 + \varphi(Q))/\varphi(Q))((I_3 + \varphi(Q))/\varphi(Q)) \subseteq Q/\varphi(Q)$$

and also

$$((I_1 + \varphi(Q))/\varphi(Q))((I_2 + \varphi(Q))/\varphi(Q))((I_3 + \varphi(Q))/\varphi(Q)) \neq 0_{A/\varphi(Q)}.$$

Now, we will show that $Q/\varphi(Q)$ is a δ_q -free triple zero with respect to

$((I_1 + \varphi(Q))/\varphi(Q))((I_2 + \varphi(Q))/\varphi(Q))((I_3 + \varphi(Q))/\varphi(Q))$. Suppose to the contrary. Then for each $i \in \{1, 2, 3\}$, $x_i + \varphi(Q) \in ((I_i + \varphi(Q))/\varphi(Q))$ such that $(x_1 + \varphi(Q))(x_2 + \varphi(Q))(x_3 + \varphi(Q)) = \varphi(Q)$, $(x_1 + \varphi(Q))(x_2 + \varphi(Q)) \notin Q/\varphi(Q)$ and $(x_1 + \varphi(Q))(x_3 + \varphi(Q))$, $(x_2 + \varphi(Q))(x_3 + \varphi(Q))$ are not in $\delta_q(Q/\varphi(Q))$. Since $x_i \in I_i + \varphi(Q)$ for each i , we can write $x_i = a_i + b_i$ for some $a_i \in I_i$ and $b_i \in \varphi(Q)$. Then note that $x_1 x_2 x_3 = (a_1 + b_1)(a_2 + b_2)(a_3 + b_3) \in \varphi(Q)$. As $b_1, b_2, b_3 \in \varphi(Q)$, we conclude that $a_1 a_2 a_3 \in \varphi(Q)$. On the other hand, since $x_1 x_2 = (a_1 + b_1)(a_2 + b_2) \notin Q$ and $b_1, b_2 \in \varphi(Q) \subseteq Q$, we get $a_1 a_2 \notin Q$. Similarly, we have $a_1 a_3, a_2 a_3 \notin \delta(Q)$. Thus, (a_1, a_2, a_3) is a φ - δ -free triple zero of Q and $a_i \in I_i$ which is a contradiction. Thus, we have $Q/\varphi(Q)$ is a δ_q -free triple zero with respect to $((I_1 + \varphi(Q))/\varphi(Q))((I_2 + \varphi(Q))/\varphi(Q))((I_3 + \varphi(Q))/\varphi(Q))$. Then by [6, Theorem 3.5], we have $((I_1 + \varphi(Q))/\varphi(Q))((I_2 + \varphi(Q))/\varphi(Q)) \subseteq Q/\varphi(Q)$ or $((I_1 + \varphi(Q))/\varphi(Q))((I_3 + \varphi(Q))/\varphi(Q)) \subseteq \delta_q(Q/\varphi(Q)) = \delta(Q)/\varphi(Q)$ or $((I_2 + \varphi(Q))/\varphi(Q))((I_3 + \varphi(Q))/\varphi(Q)) \subseteq \delta(Q)/\varphi(Q)$, which implies that $I_1 I_2 \subseteq Q$ or $I_1 I_3 \subseteq \delta(Q)$ or $I_2 I_3 \subseteq \delta(Q)$. \square

Let A_i be commutative rings with identity for each $i = 1, 2$ and $A = A_1 \times A_2$ denote the direct product of rings A_1, A_2 . Suppose that $\varphi_i : \mathcal{I}(A_i) \rightarrow \mathcal{I}(A_i) \cup \{\emptyset\}$ is a reduction function and $\delta_i : \mathcal{I}(A_i) \rightarrow \mathcal{I}(A_i)$ is an expansion function for each $i = 1, 2$. Now, define the following two functions:

$$\begin{aligned} \delta_{\times}(I_1 \times I_2) &= \delta_1(I_1) \times \delta_2(I_2), \\ \varphi_{\times}(I_1 \times I_2) &= \varphi_1(I_1) \times \varphi_2(I_2). \end{aligned}$$

Then it is easy to see that δ_{\times} and φ_{\times} are expansion and reduction functions of $\mathcal{I}(A)$, respectively.

Theorem 2.24 *Let A_1, A_2 be two commutative rings, Q_1 a proper ideal of A_1 , and $A = A_1 \times A_2$. Suppose that $\delta_i : \mathcal{I}(A_i) \rightarrow \mathcal{I}(A_i) \cup \{\emptyset\}$ is an expansion function and $\varphi_i : \mathcal{I}(A_i) \rightarrow \mathcal{I}(A_i) \cup \{\emptyset\}$ is a reduction function for each $i = 1, 2$ such that $\varphi_2(A_2) \neq A_2$. Then the following statements are equivalent.*

- (i) $Q_1 \times A_2$ is a 2-absorbing φ_{\times} - δ_{\times} -primary ideal of A .
- (ii) $Q_1 \times A_2$ is a 2-absorbing δ_{\times} -primary ideal of A .
- (iii) Q_1 is a 2-absorbing δ_1 -primary ideal of A_1 .

Proof (i) \Rightarrow (ii) : Suppose that $Q_1 \times A_2$ is a 2-absorbing φ_{\times} - δ_{\times} -primary ideal of A . Let $abc \in Q_1$ for some $a, b, c \in A_1$. Then we have $(a, 1)(b, 1)(c, 1) \in Q_1 \times A_2 - \varphi_{\times}(Q_1 \times A_2)$. This implies that $(a, 1)(b, 1) \in Q_1 \times A_2$ or $(a, 1)(c, 1) \in \delta_{\times}(Q_1 \times A_2)$ or $(b, 1)(c, 1) \in \delta_{\times}(Q_1 \times A_2)$. Then we conclude that $ab \in Q_1$ or $ac \in \delta_1(Q_1)$ or $bc \in \delta_1(Q_1)$. Thus, Q_1 is a 2-absorbing δ_1 -primary ideal of A_1 . If $Q_1 \times A_2$ is not a 2-absorbing δ_{\times} -primary ideal of A , then by Theorem 2.18, we have $(Q_1 \times A_2)^3 \subseteq \varphi_{\times}(Q_1 \times A_2)$ which implies that $A_2 = \varphi_2(A_2)$, a contradiction. Thus, $Q_1 \times A_2$ is a 2-absorbing δ_{\times} -primary ideal of A .

(ii) \Rightarrow (iii) : It is clear.

(iii) \Rightarrow (i) : Follows from [6, Theorem 4.1]. \square

Theorem 2.25 Let A_1, A_2 be two commutative rings, Q_1 a proper ideal of A_1 and $A = A_1 \times A_2$. Suppose that $\delta_i : \mathcal{I}(A_i) \rightarrow \mathcal{I}(A_i)$ is an expansion function and $\varphi_i : \mathcal{I}(A_i) \rightarrow \mathcal{I}(A_i) \cup \{\emptyset\}$ is a reduction function for each $i = 1, 2$. Then the following statements are equivalent.

- (i) $Q_1 \times A_2$ is a 2-absorbing φ_\times - δ_\times -primary ideal of A that is not a 2-absorbing δ_\times -primary ideal of A .
- (ii) $\varphi_\times(Q_1 \times A_2) \neq \emptyset$, $\varphi_2(A_2) = A_2$ and Q_1 is a 2-absorbing φ_1 - δ_1 -primary that is not 2-absorbing δ_1 -primary ideal.

Proof (i) \Rightarrow (ii) : Suppose that $Q_1 \times A_2$ is a 2-absorbing φ_\times - δ_\times -primary ideal of A that is not a 2-absorbing δ_\times -primary ideal of A . Then it is clear that $\varphi_\times(Q_1 \times A_2) \neq \emptyset$. If $\varphi_2(A_2) \neq A_2$, then by Theorem 2.24, $Q_1 \times A_2$ is a 2-absorbing δ_\times -primary ideal of A which is a contradiction. Thus, $\varphi_2(A_2) = A_2$. Moreover, it is easy to see that Q_1 is a 2-absorbing φ_1 - δ_1 -primary ideal that is not a 2-absorbing δ_1 -primary.

(ii) \Rightarrow (i) : Let $(x_1, y_1)(x_2, y_2)(x_3, y_3) \in Q_1 \times A_2 - \varphi_\times(Q_1 \times A_2)$. Then we have $x_1x_2x_3 \in Q_1 - \varphi_1(Q_1)$ since $\varphi_2(A_2) = A_2$. This implies that $x_1x_2 \in Q_1$ or $x_1x_3 \in \delta_1(Q_1)$ or $x_2x_3 \in \delta_1(Q_1)$. Then we conclude that $(x_1, y_1)(x_2, y_2) \in Q_1 \times A_2$ or $(x_1, y_1)(x_3, y_3) \in \delta_\times(Q_1 \times A_2)$ or $(x_2, y_2)(x_3, y_3) \in \delta_\times(Q_1 \times A_2)$. Thus, $Q_1 \times A_2$ is a 2-absorbing φ_\times - δ_\times -primary ideal of A . If $Q_1 \times A_2$ is a 2-absorbing δ_\times -primary, then one can easily see that Q_1 is a 2-absorbing δ_1 -primary ideal which is a contradiction. Hence, $Q_1 \times A_2$ is not a 2-absorbing δ_\times -primary ideal of A . □

Recall from [20] that an ideal expansion δ of $\mathcal{I}(A)$ is said to satisfy (*)-condition if $\delta(I) = A$ implies $I = A$, or equivalently, $I \neq A$ implies $\delta(I) \neq A$. Note that the radical operation is an example of expansion function satisfying the (*)-condition.

Theorem 2.26 Let A_i be a commutative ring, Q_i an ideal of A_i for each $i = 1, 2$, and $A = A_1 \times A_2$. Suppose that $\delta_i : \mathcal{I}(A_i) \rightarrow \mathcal{I}(A_i)$ is an expansion function satisfying the (*)-condition and $\varphi_i : \mathcal{I}(A_i) \rightarrow \mathcal{I}(A_i) \cup \{\emptyset\}$ is a reduction function for each $i = 1, 2$. Let $Q = Q_1 \times Q_2$ be a proper ideal of A such that $\varphi_i(Q_i) \neq Q_i$. Then the following statements are equivalent.

- (i) Q is a 2-absorbing φ_\times - δ_\times -primary ideal of A .
- (ii) $Q_1 = A_1$ and Q_2 is a 2-absorbing δ_2 -primary ideal of A_2 or $Q_2 = A_2$ and Q_1 is a 2-absorbing δ_1 -primary ideal of A_1 or Q_i is a δ_i -primary ideal of A_i for each $i = 1, 2$.
- (iii) Q is a 2-absorbing δ_\times -primary ideal of A .

Proof (i) \Rightarrow (ii) : Suppose that Q is a 2-absorbing φ_\times - δ_\times -primary ideal of A . If $Q_2 = A_2$, then by Theorem 2.24, Q_1 is a 2-absorbing δ_1 -primary ideal of A_1 . If $Q_1 = A_1$, similarly, Q_2 is a 2-absorbing δ_2 -primary ideal of A_2 . Thus, assume that Q_1, Q_2 are proper. Now, we will show that Q_i is a δ_i -primary ideal of A_i . Let $ab \in Q_1$ for some $a, b \in A_1$. Then choose $x \in Q_2 - \varphi_2(Q_2)$. Then note that $(1, x)(a, 1)(b, 1) = (ab, x) \in Q - \varphi_\times(Q)$. Since Q is a 2-absorbing φ_\times - δ_\times -primary ideal of A , we conclude that $(1, x)(a, 1) \in Q$ or $(1, x)(b, 1) \in \delta_\times(Q)$ or $(a, 1)(b, 1) \in \delta_\times(Q)$. If $(a, 1)(b, 1) \in \delta_\times(Q)$, then $1 \in \delta_2(Q_2)$ which implies that $Q_2 = A_2$ since δ_2 satisfies the (*)-condition. Hence, we have $(1, x)(a, 1) \in Q$ or $(1, x)(b, 1) \in \delta_\times(Q)$. This implies that $a \in Q_1$ or $b \in \delta_1(Q_1)$. Thus, Q_1 is a δ_1 -primary ideal of A_1 . Likewise, Q_2 is a δ_2 -primary ideal of A_2 .

(ii) \Rightarrow (iii) : Follows from [6, Theorem 4.2].

(iii) \Rightarrow (i) : Follows from Proposition 2.2. □

3. Conclusion

In this paper, the theoretical point of view of 2-absorbing φ - δ -primary ideals which is the generalization of 2-absorbing ideal and 2-absorbing primary ideals are examined. In order to extend this study, one could study other algebraic structures and do some further study on their properties.

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