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Weak c-ideals of a Lie algebra

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Abstract: A subalgebra B of a Lie algebra L is called a weak c-ideal of L if there is a subideal C of L such that $L = B+C$ and $B\cap C \leq B_L$ where B_L is the largest ideal of L contained in B. This is analogous to the concept of weakly c-normal subgroups, which has been studied by a number of authors. We obtain some properties of weak c-ideals and use them to give some characterisations of solvable and supersolvable Lie algebras. We also note that one-dimensional weak c-ideals are c-ideals.

Key words: Weak c-ideal, Frattini ideal, Lie algebras, nilpotent, solvable, supersolvable

1. Introduction

Throughout L, a finite-dimensional Lie algebra over a field F will be denoted. If B is a subalgebra of L we define B_L, the *core* (with respect to L) of B to be the largest ideal of L contained in B. We say that a subalgebra B of L is a *weak c-ideal* of L if there is a subideal C of L such that $L = B + C$ and $B \cap C \leq B_L$. This is a generalisation of the concept of a c-ideal, which was studied in [9]. It is analogous to the concept of weakly c-normal subgroup as introduced by Zhu, Guo and Shum in [15]; this concept has since been further studied by a number of authors, including Zhong and Yang ([14]), Zhong, Yang, Ma and Lin ([13]), Tashtoush ([7]) and Jehad ([4]) who called them c-subnormal subgroups.

The maximal subalgebras of a Lie algebra L and their relationship to the structure of L have been studied extensively. It is well known that L is nilpotent if and only if every maximal subalgebra of L is an ideal of L (see [1]). A further result is that if L is solvable then every maximal subalgebra of L has codimension one in L if and only if L is supersolvable (see [2]). In [9], similar characterisations of solvable and supersolvable Lie algebras were obtained in terms of c-ideals. The purpose here is to generalise these results to ones relating to weak c-ideals.

In Section 2, we give some basic properties of weak c-ideals; in particular, it is shown that weak c-ideals inside the Frattini subalgebra of a Lie algebra L are necessarily ideals of L. In section three we first show that all maximal subalgebras of L are weak c-ideals of L if and only if L is solvable and that L has a solvable maximal subalgebra that is a weak c-ideal if and only if L is solvable. Unlike the corresponding results for c-ideals, it is necessary to restrict the underlying field to characteristic zero, as is shown by an example. Finally, we have that if all maximal nilpotent subalgebras of L are weak c-ideals, or if all Cartan subalgebras of L are weak c-ideals and F has characteristic zero, then L is solvable.

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In Section 4, we show that if L is a solvable Lie algebra over a general field and every maximal subalgebra of each maximal nilpotent subalgebra of L is a weak c-ideal of L, then L is supersolvable. If each of the maximal nilpotent subalgebras of L has dimension at least two, then the assumption of solvability can be removed. Similarly, if the field has characteristic zero and L is not three-dimensional simple, then this restriction can be removed. In the final section, we see that every one-dimensional subalgebra is a weak c-ideal if and only if it is a c-ideal.

If A and B are subalgebras of L for which $L = A + B$ and $A \cap B = 0$, we will write $L = A \oplus B$. The ideals $L^{(k)}$ and L^k are defined inductively by $L^{(1)} = L^1 = L$, $L^{(k+1)} = [L^{(k)}, L^{(k)}]$, $L^{k+1} = [L, L^k]$ for $k \geq 1$. If A is a subalgebra of L, the *centralizer* of A in L is $C_L(A) = \{x \in L : [x, A] = 0\}$.

2. Preliminary results

Definition 2.1 Let I be a subalgebra of L. We call I a subideal of L if there is a chain of subalgebras

$$I = I_0 < I_1 < \ ... < I_n = L,$$

where I_j is an ideal of I_{j+1} for each $0 \leq j \leq n-1$.

Definition 2.2 A subalgebra B of a Lie algebra L is a weak *c-ideal* of L if there exists a subideal C of L such that

$$L = B + C \text{ and } B \cap C \leq B_L,$$

where B_L, the core of B, is the largest ideal of L contained in B.

Definition 2.3 A Lie algebra L is called weak c-simple if L does not contain any weak c-ideals except the trivial subalgebra and L itself.

Lemma 2.4 Let L be a Lie algebra. Then, the following statements hold:

(1) Let B be a subalgebra of L. If B is a *c-ideal* of L, then B is a weak c-ideal of L.

(2) L is *weak c-simple* if and only if L is simple.

(3) If B is a *weak c-ideal* of L and K is a subalgebra with $B \leq K \leq L$, then B is a *weak c-ideal* of K.

(4) If I is an ideal of L and $I \leq B$, then B is a *weak c-ideal* of L if and only if B/I is a *weak c-ideal* of L/I.

Proof (1) By the definition every ideal is a c-ideal and every c-ideal is a weak c-ideal so the proof is obvious.

(2) Suppose first that L is simple and let B be a weak c-ideal with $B \neq L$. Then

$$L = B + C \text{ and } B \cap C \leq B_L$$

where C is a subideal of L. But, since L is simple, B_L must be 0. Moreover, $C \neq 0$ so $C = L$. Hence, $B = 0$ and L is weak c-simple.

Conversely, suppose L is weak c-simple. Then, since every ideal of L is a weak c-ideal, L must be simple.

(3) If B is a weak c-ideal of L, then there exists a subideal C of L such that

$$L = B + C \text{ and } B \cap C \leq B_L$$

Then $K = K \cap L = K \cap (B + C) = B + (K \cap C)$. Since C is a subideal of L, there exists a chain of subalgebras

$$C = C_0 < C_1 < \ ... < C_n = L$$

where C_j is an ideal of C_{j+1} for each $0 \leq j \leq n-1$. If we intersect this chain with K, we get

$$C \cap K = C_0 \cap K < C_1 \cap K < \ ... < C_n \cap K = L \cap K = K$$

and obviously $C_j \cap K$ is an ideal of $C_{j+1} \cap K$ for each $0 \leq j \leq n-1$. Hence, $C \cap K$ is a subideal of K. Also,

$$B \cap (C \cap K) \leq B_K$$

so that B is a weak c-ideal of L.

(4) Suppose first that B/I is a weak c-ideal of L/I. Then there exists a subideal C/I of L/I such that

$$L/I = B/I + C/I \text{ and } B/I \cap C/I \leq (B/I)_{L/I} = B_L/I$$

It follows that $L = B + C$ and $B \cap C \leq B_L$ where C is a subideal of L.

Suppose conversely that I is an ideal of L with $I \leq B$ and B is a weak c-ideal of L. Then there exists a C subideal of L such that

$$L = B + C \text{ and } B \cap C \leq B_L.$$

Since I is an ideal and $I \leq B$ the factor algebra

$$L/I = (B + C)/I = B/I + (C + I)/I$$

where $(C + I)/I$ is a subideal of L/I and

$$(B/I) \cap (C + I)/I = (B \cap (C + I))/I = (I + B \cap C)/I \leq B_L/I = (B/I)_{L/I}$$

so B/I is a *weak c-ideal* of L/I. □

The *Frattini subalgebra* of L, $F(L)$, is the intersection of all of the maximal subalgebras of L. The *Frattini ideal*, $\varphi(L)$, of L is $F(L)_L$. The next result is a generalisation of [9, Proposition 2.2]. The same proof works, but we will include it for completeness.

Proposition 2.5 Let B, C be subalgebras of L with $B \leq F(C)$. If B is a *weak c-ideal* of L, then B is an ideal of L and $B \leq \varphi(L)$.

Proof Suppose that $L = B + K$ where K is a subideal of L and $B \cap K \leq B_L$. Then $C = C \cap L = C \cap (B + K) = B + C \cap K = C \cap K$ since $B \leq F(C)$. Hence, $B \leq C \leq K$, giving $B = B \cap K \leq B_L$ and B is an ideal of L. It then follows from [8, Lemma 4.1] that $B \leq \varphi(L)$. □

An ideal A is *complemented* in L if there is a subalgebra U of L such that $L = A + U$ and $A \cap U = 0$. We adapt this to define a complemented weak c-ideal as follows.

Definition 2.6 Let L be a Lie algebra and B is a *weak c-ideal* of L. A *weak c-ideal* B is complemented in L if there is a subideal C of L such that $L = B + C$ and $B \cap C = 0$.

Then we can give the following lemma:

Lemma 2.7 If B is a *weak c-ideal* of a Lie algebra L, then B/B_L has a subideal complement in L/B_L, *i.e.* there exists a subideal subalgebra C/B_L of L/B_L such that L/B_L is semidirect sum of C/B_L and B/B_L. Conversely, if B is a subalgebra of L such that B/B_L has a subideal complement in L/B_L, then B is a *weak c-ideal* of L.

Proof Let B be a weak c-ideal of L. Then there exists a subideal C of L such that $B+C=L$ and $B\cap C\leq B_L$. If $B_L=0$, then $B\cap C=0$ and so that C is a subideal complement of B in L. Assume that $B_L\neq 0$, then we can construct the factor algebras B/B_L and $(C+B_L)/B_L$. If we intersect these two factor algebras, we have

$$\begin{aligned}\frac{B}{B_L}\cap\frac{C+B_L}{B_L} &= \frac{B\cap(C+B_L)}{B_L}\\ &= \frac{B_L+(B\cap C)}{B_L}\\ &= \frac{B_L}{B_L}=0\end{aligned}$$

Hence, $(C+B_L)/B_L$ is a subideal complement of B/B_L in L/B_L. Conversely, if K is a subideal of L such that K/B_L is a subideal complement of B/B_L in L/B_L, then we have that

$$L/B_L=(B/B_L)+(K/B_L)\ \text{and}\ (B/B_L)\cap(K/B_L)=0$$

Then $L=B+K$ and $B\cap K\leq B_L$. Therefore, B is a weak c-ideal of L. □

3. Some characterisations of solvable algebras

We will use the following Lemma, which is due to Stewart [6, Lemma 4.2.5]

Lemma 3.1 Let L be a Lie algebra over any field having two subideals H and K such that K is simple and not abelian. Suppose that $H\cap K=0$. Then $[H,K]=0$

Theorem 3.2 Let L be a Lie-algebra over a field F of characteristic zero and let B be an ideal of L. Then B is solvable if and only if every maximal subalgebra of L not containing B is a weak c-ideal of L.

Proof Suppose every maximal subalgebra of L not containing B is a weak c-ideal of L. Then we need to show B is solvable. Assume that this is false and let L be a minimal counter-example. Let A be a minimal ideal of L and assume that M/A is a maximal subalgebra of L/A such that $(B+A)/A\nsubseteq M/A$. Then M is a maximal subalgebra of L with $B\nsubseteq M$, so M is a weak c-ideal of L. It follows that M/A is a weak c-ideal of L/A, and, hence, that $(B+A)/A$ is solvable. If $B\cap A=0$, then $B\cong B/B\cap A\cong(B+A)/A$ is solvable. So, we can assume that every minimal ideal of L is contained in B. Moreover, B/A is solvable for each such minimal ideal. If L has two distinct minimal ideals A_1 and A_2, then $B\cong B/A_1\cap A_2$ is solvable, so L is monolithic with monolith A, say.

If A is abelian, then B is solvable, so we must have that A is simple. Clearly, $B\nsubseteq\varphi(L)$, since $\varphi(L)$ is nilpotent, so there is a maximal subalgebra M of L such that $B\nsubseteq M$. Then M must be a weak c-ideal of L, so there is a subideal C of L such that $L=M+C$ and $M\cap C\subseteq M_L$. Since $B\nsubseteq M_L$ we have that $M_L=0$.

It follows that L is primitive of type 2 and, hence, that $C_L(A) = 0$, by [10, Theorem 1.1]. But $[C, A] = 0$ by Lemma 3.1, so $C = 0$, a contradiction. Hence, B is solvable. So, suppose now that B is solvable and let M be a maximal ideal of L not containing B. Then there exists $k \in \mathbb{N}$ such that $B^{(k+1)} \subseteq M$, but $B^{(k)} \nsubseteq M$. Clearly $L = M + B^{(k)}$ and $B^{(k)} \cap M$ is an ideal of L, so $B^{(k)} \cap M \subseteq M_L$. It follows that M is a c-ideal and, hence, a weak c-ideal of L. □

Corollary 3.3 Let L be a Lie algebra over a field F of characteristic zero. Then L is solvable if and only if every maximal subalgebra of L is a weak c-ideal of L.

Unlike the corresponding results for c-ideals, the above two results do not hold in characteristic $p > 0$, as the following example shows.

Example 3.4 Let $L = sl(2) \otimes \mathcal{O}_1 + 1 \otimes F(\frac{\partial}{\partial x} + x\frac{\partial}{\partial x})$, where $\mathcal{O}_1 = F[x]$ with $x^p = 0$ is the truncated polynomial algebra in 1 indeterminate and the ground field, F, is algebraically closed of characteristic $p > 2$. Then $A = sl(2) \otimes \mathcal{O}_1$ is the unique minimal ideal of L. Put $S = sl(2) = Fu_{-1} + Fu_0 + Fu_1$ with $[u_{-1}, u_0] = u_{-1}$, $[u_{-1}, u_1] = u_0$, $[u_0, u_1] = u_1$ and let $M = (Fu_0 + Fu_1) \otimes \mathcal{O}_1 + 1 \otimes F(\frac{\partial}{\partial x} + x\frac{\partial}{\partial x})$. This is a maximal subalgebra of L, which doesn't contain A. Suppose that it is a weak c-ideal of L. Then there is a subideal C of L such that $L = C + M$ and $C \cap M \subseteq M_L = 0$.

Let

$$C = C_0 < C_1 < \dots < C_n = L$$

where C_j is an ideal of C_{j+1} for each $0 \leq j \leq n-1$. Then $A \subseteq C_{n-1}$, so $A = C_{n-1}$ or $C_{n-1} = A + 1 \otimes F\frac{\partial}{\partial x}$. In the latter case it is straightforward to check that $C_{n-2} \subseteq A$. In either case, C must be inside a proper ideal of $A$and, hence, inside $S \oplus O_1^+$, where O_1^+ is spanned by $x, x^2, \dots, x^{p-1}$. But now $u_{-1} \otimes 1 \notin C + M$. Hence, M is not a weak c-ideal of L.

Lemma 3.5 Let $L = U + C$ be a Lie algebra, where U is a solvable subalgebra of L and C is a subideal of L. Then there exists $n_0 \in \mathbb{N}$ such that $L^{(n_0)} \subseteq C$.

Proof Let $C = C_0 < C_1 < \dots < C_k = L$ where C_i is an ideal of C_{i+1} for $0 \leq i \leq k-1$. Then L/C_{k-1} is solvable, and so there exists n_{k-1} such that $L^{(n_{k-1})} \subseteq C_{k-1}$. Suppose that $L^{(n_i)} \subseteq C_i$ for some $0 \leq i \leq k-1$. Now C_i/C_{i-1} is solvable, and so there is r_i such that $C_i^{(r_i)} \subseteq C_{i-1}$. Hence, $L^{(n_i+r_i)} = (L^{(n_i)})^{(r_i)} \subseteq C_{i-1}$. Put $n_{i-1} = n_i + r_i$. The result now follows by induction. □

Theorem 3.6 Let L be a Lie algebra over a field F of characteristic zero. Then L has a solvable maximal subalgebra that is a weak c-ideal of L if and only if L is solvable.

Proof Suppose first that L has a solvable maximal subalgebra M that is a weak c-ideal of L. We show that L is solvable. Let L be a minimal counter-example. Then there is a subideal K of L such that $L = M + K$ and $M \cap K \leq M_L$. If $M_L \neq 0$, then L/M_L is solvable, by the minimality assumption, and M_L is solvable, whence L is solvable, a contradiction. It follows that $M_L = 0$ and $L = M \dot{+} K$. If R is the solvable radical of L then $R \leq M_L = 0$, so L is semisimple. But now, for all $n \geq 1$, $L = L^{(n)} \leq K \neq L$, by Lemma 3.5, a contradiction. The result follows. The converse follows from Corollary 3.3. □

Theorem 3.7 Let L be a Lie algebra over a field of characteristic zero such that all maximal nilpotent subalgebras are weak c-ideals of L. Then L is solvable.

Proof Suppose that L is not solvable but that all maximal nilpotent subalgebras of L are weak c-ideals of L. Let $L = R \oplus S$ be the Levi decomposition of L, where $S \neq 0$. Let B be a maximal nilpotent subalgebra of S and U be a maximal nilpotent subalgebra of L containing it. Then there is a subideal C of L such that $L = U + C$ and $U \cap C \subseteq U_L$. It follows from Lemma 3.5 that $S = S^{(n_0)} \subseteq L^{(n_0)} \subseteq C$, and so $B \subseteq U \cap C \subseteq U_L$, whence $S \cap U_L \neq 0$. But $S \cap U_L$ is an ideal of S and so is semisimple. Since U is nilpotent, this is a contradiction. □

Theorem 3.8 Let L be a Lie algebra, over a field F of characteristic zero in which every Cartan subalgebra of L is a weak c-ideal of L. Then L is solvable.

Proof Suppose that every Cartan subalgebra of L is a weak c-ideal of L and that L has a non-zero Levi factor S. Let H be a Cartan subalgebra of S and let B be a Cartan subalgebra of its centralizer in the solvable radical of L. Then $C = H + B$ is a Cartan subalgebra of L (see [3]), and there is a subideal K of L such that $L = C + K$ and $C \cap K \leq C_L$. Now there is an $r \geq 2$ such that $L^{(r)} \leq K$, by Lemma 3.5. But $S \leq L^{(r)} \leq K$, so $C \cap S \leq C \cap K \leq C_L$ giving $C \cap S \leq C_L \cap S = 0$, a contradiction. It follows that $S = 0$ and, hence, that L is solvable. □

4. Some characterisations of supersolvable algebras

The following is proved in [9, Lemma 4.1]

Lemma 4.1 Let L be a Lie algebra over any field F, let A be an ideal of L and let U/A be a maximal nilpotent subalgebra of L/A. Then $U = C + A$, where C is a maximal nilpotent subalgebra of L.

We will also need the following result.

Lemma 4.2 Let L be a Lie algebra over any field F and suppose that $L = B + K$, where B is a nilpotent subalgebra and K is a subideal of L. Then there exists $s \in \mathbb{N}$ such that $L^s \subseteq K$. Moreover, if A is a minimal ideal of L then either $A \subseteq K$ or $[L, A] = 0$.

Proof Since K is a subideal of L, there exists $r \in \mathbb{N}$ such that $L\,(\text{ad } K)^r \subseteq K$. As B is nilpotent, there exists $s \in \mathbb{N}$ such that $L^s = (B + K)^s \subseteq K$. Now $[L, A] = A$ or $[L, A] = 0$, and the former implies that $A \subseteq L^s \subseteq K$. □

Lemma 4.3 Let L be a Lie algebra over any field F in which every maximal subalgebra of each maximal nilpotent subalgebra of L is a weak c-ideal of L and let A be a minimal abelian ideal of L. Then every maximal subalgebra of each maximal nilpotent subalgebra of L/A is a weak c-ideal of L/A.

Proof Suppose that U/A is a maximal nilpotent subalgebra of L/A. Then $U = C + A$ where C is a maximal nilpotent subalgebra of L by Lemma 4.1. Let B/A be a maximal subalgebra of U/A. Then $B = B \cap (C + A) = B \cap C + A = D + A$ where D is a maximal subalgebra of C with $B \cap C \leq D$. Now D is a weak c-ideal of L so there is a subideal K of L with $L = D + K$ and $D \cap K \leq D_L$.

If $A \leq K$ we have

$$\frac{L}{A} = \frac{D+K}{A} = \frac{D+A}{A} + \frac{K}{A} = \frac{B}{A} + \frac{K}{A},$$

and

$$\frac{B}{A} \cap \frac{K}{A} = \frac{B \cap K}{A} = \frac{(D+A) \cap K}{A} = \frac{D \cap K + A}{A} \leq \frac{D_L + A}{A} \leq \left(\frac{B}{A}\right)_{L/A}.$$

So suppose that $A \nleq K$. Then Lemma 4.2 shows that $[L, A] = 0$. It follows that $A \leq C$ and $B = D$. We have $L = B + K$ and $B \cap K \leq B_L$, so

$$\frac{L}{A} = \frac{B}{A} + \frac{K+A}{A}$$

and

$$\frac{B}{A} \cap \frac{K+A}{A} = \frac{B \cap (K+A)}{A} = \frac{B \cap K + A}{A} \leq \frac{B_L + A}{A} \leq \left(\frac{B}{A}\right)_{L/A}.$$

□

Lemma 4.4 Let L be a Lie algebra over any field F, in which every maximal nilpotent subalgebra of L is a weak c-ideal of L, and suppose that A is a minimal abelian ideal of L and M is a core-free maximal subalgebra of L. Then A is one dimensional.

Proof We have that $L = A \dot{+} M$ and A is the unique minimal ideal of L, by [10, Theorem 1.1]. Let C be a maximal nilpotent subalgebra of L with $A \leq C$. If $A = C$, choose B to be a maximal subalgebra of A, so that $A = B + Fa$ and $B_L = 0$. Then B is a weak c-ideal of L. So there is a subideal of K of L with $L = B + K$ and $B \cap K \leq B_L = 0$. Now $L = B + K = B + K^L = K^L$, since $B \leq A \leq K^L$. It follows that $K = L$, whence $B = 0$ and $A = Fa$ is one dimensional.

So suppose that $C \neq A$. Then $C = A + M \cap C$. Let B be a maximal subalgebra of C containing $M \cap C$. Then B is a weak c-ideal of L, so there is a subideal K of L with $L = B + K$ and $B \cap K \leq B_L$. If $A \leq B_L \leq B$, we have $C = A + M \cap C \leq B$, a contradiction. Hence, $B_L = 0$ and $L = B \dot{+} K$. Now, $C = B + C \cap K$ and $B \cap C \cap K = B \cap K = 0$. As C is nilpotent, this means that $\dim(C \cap K) = 1$. If $A \subseteq K$, we have that $A \leq C \cap K$, so $\dim A = 1$, as required. Otherwise, $[L, A] = 0$, by Lemma 4.2 and again $\dim A = 1$. □

We can now prove our main result.

Theorem 4.5 Let L be a solvable Lie algebra over any field F in which every maximal subalgebra of each maximal nilpotent subalgebra of L is a weak c-ideal of L. Then L is supersolvable.

Proof Let L be a minimal counter-example and let A be a minimal abelian ideal of L. Then L/A satisfies the same hypothesis by Lemma 4.3 We, thus, have that L/A is supersolvable, and it remains to show that $\dim A = 1$.

If there is another minimal ideal I of L, then

$$A \cong (A + I)/I \leq L/I$$

which is supersolvable and so $\dim A = 1$. So, we can assume that A is the unique minimal ideal of L. Also, if $A \leq \varphi(L)$, we have that $L/\varphi(L)$ is supersolvable, whence L is supersolvable by [2, Theorem 7]. We, therefore, further assume that $A \nleq \varphi(L)$. It follows that $L = A \dot{+} M$, where M is a core-free maximal subalgebra of L. The result now follows from Lemma 4.4. □

If L has no one-dimensional maximal nilpotent subalgebras, we can remove the solvability assumption from the above result provided that F has characteristic zero.

Corollary 4.6 Let L be a Lie algebra over a field F of characteristic zero in which every maximal nilpotent subalgebra has dimension at least two. If every maximal subalgebra of each maximal nilpotent subalgebra of L is a weak c-ideal of L, then L is supersolvable.

Proof Let N be the nilradical of L, and let $x \notin N$. Then $x \in C$ for some maximal nilpotent subalgebra C of L. Since $\dim C > 1$, there is a maximal subalgebra B of C with $x \in B$. Then there is a subideal K of L such that $L = B + K$ and $B \cap K \subseteq B_L \leq C_L \leq N$. Clearly, $x \notin K$, since otherwise $x \in B \cap K \leq N$. Moreover, $L^r \subseteq K$ for some $r \in \mathbb{N}$, by Lemma 4.2. We have shown that if $x \notin N$ there is a subideal K of L with $x \notin K$ and $L^r \subseteq K$.

Suppose that L is not solvable. Then there is a semisimple Levi factor S of L. Choose $x \in S$. Then $x \in S = S^r \subseteq K$, a contradiction. Thus, L is solvable, and the result follows from Theorem 4.5. □

If L has a one-dimensional maximal nilpotent subalgebra, then we can also remove the solvability assumption from Theorem 4.4, provided that underlying field F has again characteristic zero and L is not three-dimensional simple.

Corollary 4.7 Let L be a Lie algebra over a field F of characteristic zero. If every maximal subalgebra of each maximal nilpotent subalgebra of L is a weak c-ideal of L, then L is supersolvable or three dimensional simple.

Proof If every maximal nilpotent subalgebra of L has dimension at least two, then L is supersolvable by Corollary 4.6. So we need only to consider the case where L has a one-dimensional maximal nilpotent subalgebra, say Fx. Suppose first that L is semisimple, so $L = S_1 \oplus \dots \oplus S_n$, where S_i is a simple ideal of L for $1 \leq i \leq n$. Let $n > 1$. If $x \in S_i$, then choosing $s \in S_j$ with $j \neq i$, we have that $Fx + Fs$ is a two dimensional abelian subalgebra, which contradicts the maximality of Fx. If $x \notin S_i$, for every $1 \leq i \leq n$, then x has nonzero projections in at least two of the S_k's, say $s_i \in S_i$ and $s_j \in S_j$. But then $Fx + Fs_i$ is a two-dimensional abelian subalgebra, a contradiction again. It follows that L is simple. But then Fx is a Cartan subalgebra of L, which yields that L has rank one and, thus, is three dimensional.

So now let L be a minimal-counter example. We have seen that L is not semisimple, so it has a minimal abelian ideal A. By Lemma 4.3, L/A is supersolvable or three-dimensional simple. In the former case, L is solvable and so is supersolvable by Theorem 4.5.

In the latter case, $L = A \oplus S$ where S is three-dimensional simple, and so a core-free maximal subalgebra of L. It follows from Lemma 4.4 that $\dim A = 1$. But now $C_L(A) = A$ or L. In the former case, $S \cong L/A = L/C_L(A) \cong Inn(A)$, a subalgebra of $Der(A)$, which is impossible. Hence, $L = A \oplus S$, where A and S are both ideals of L and again L has no one-dimensional maximal nilpotent subalgebras. □

5. One dimensional weak c-ideals

Lemma 5.1 Let L be a Lie algebra over any field F. Then the one-dimensional subalgebra Fx of L is a weak c-ideal of L if and only if it is a c-ideal of L.

Proof Let Fx be a weak c-ideal of L. Then there is a subideal K of L such that $L = Fx + K$ and $Fx \cap K \leq (Fx)_L$. Since either $K = L$ or K has codimension one in L, it is an ideal of L and Fx is a c-ideal of L. □

We say that L is *almost abelian* if $L = L^2 \oplus Fx$, where L^2 is abelian and $[x, y] = y$ for all $y \in L^2$. Then the following result follows from Lemma 5.1 and [9, Theorem 5.2].

Theorem 5.2 Let L be a Lie algebra over any field F. Then all one-dimensional subalgebras of L are weak c-ideals of L if and only if:

(i) $L^3 = 0$; or

(ii) $L = A \oplus B$, where A is an abelian ideal of L and B is an almost abelian ideal of L.

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