

Count of genus zero J -holomorphic curves in dimensions four and six

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Abstract: An application of Gromov–Witten invariants is that they distinguish the deformation types of symplectic structures on a smooth manifold. In this manuscript, it is proven that the use of Gromov–Witten invariants in the class of embedded J -holomorphic spheres is restricted. This restriction is in the sense that they cannot distinguish the deformation types of symplectic structures on $X_1 \times S^2$ and $X_2 \times S^2$ for two minimal, simply connected, symplectic 4-manifolds X_1 and X_2 with $b_2^+(X_1) > 1$ and $b_2^+(X_2) > 1$. The result employs the adjunction inequality for symplectic 4-manifolds which is derived from Seiberg–Witten theory.

Key words: Symplectic manifolds, J -holomorphic curves, symplectic deformation equivalence

1. Introduction

The count of J -holomorphic curves in a symplectic manifold carries information about the properties of the symplectic structure on the manifold. This kind of study was first established by M. Gromov in 1985 [5]. Later, it was improved and applied in many ways [7, 13, 15, 19].

The study of symplectic structures on six dimensional manifolds is so incomplete that even for manifolds with a simple topology like the homotopy projective spaces, it is not clear if they admit any symplectic structures except $\mathbb{C}P^3$ itself. Nevertheless, it is conjectured that given a topological 4-manifold X which admits symplectic structures, the classification of smooth structures on X is equivalent to the classification of deformation types of symplectic structures on $X \times S^2$ ([10] page 437, [14]). It is shown that this conjecture holds for elliptic surfaces by Ruan and Tian [16]. Moreover, it was proven that if $X_1 \times S^2$ and $X_2 \times S^2$ are deformation equivalent, then some branched covers of X_1 and X_2 are diffeomorphic [14].

The main result of this paper exploits the adjunction inequality for symplectic 4-manifolds which is derived from Seiberg–Witten theory. In Remark 3.6, the relation of Gromov–Witten invariants of a 4-manifold X to its Seiberg–Witten invariants is discussed in a nutshell. This suggests that some of Gromov–Witten invariants of $X \times S^2$ can be given in terms of the Seiberg–Witten invariants of X . However, our results imply that in genus zero case, one cannot get information except for the class of an exceptional sphere. In particular, for a minimal symplectic 4-manifold, Seiberg–Witten invariants do not contribute to genus zero Gromov–Witten theory. In the last section, we prove Theorem 4.5, which lays out that the efficiency of genus zero Gromov–Witten invariants to distinguish the symplectic deformation types on a smooth 6-manifold is restricted in the sense that they cannot distinguish the symplectic structures on $X_1 \times S^2$ and $X_2 \times S^2$ for two minimal, simply

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connected, symplectic 4-manifolds X_1 and X_2 (with $b_2^+(X_1) > 1$ and $b_2^+(X_2) > 1$) in the pushforwards of second homology classes with minimal genus zero. Another consequence of Theorem 4.5 is Corollary 4.8 which states that if X_1 and X_2 are homeomorphic symplectic 4-manifolds with $b_2^+(X_1) > 1$ and $b_2^+(X_2) > 1$ and if X_1 is not minimal and X_2 is minimal, then $X_1 \times S^2$ and $X_2 \times S^2$ are not symplectic deformation equivalent. These two 6-manifolds are known to be diffeomorphic.

The outline of the paper is as follows. In Section 2, genus zero Gromov–Witten invariants of a symplectic manifold are defined tracking the definitions in [11]. In Section 3, we explain how these invariants are applied in dimension four with some examples. Conditions to have nonzero invariants are discussed in the same section and the first main result, Theorem 3.5, is proved. In the last section, the second main result, Theorem 4.5, is proved and its consequences are discussed. This final section is mostly about applications and limitations of the Gromov–Witten invariants for distinguishing symplectic 6-manifolds.

In the following discussion, as a convention, $g(A)$ will be the genus of a surface and $g([A])$ will be the minimal genus of embedded representatives of $[A] \in H^2(X; \mathbb{Z})$ in a 4-manifold X . The self intersection of A and $[A]$ in X is denoted by A^2 and $[A]^2$, respectively.

2. Genus zero Gromov–Witten invariants

In this section the definition of Gromov–Witten invariants in genus zero is reviewed as given by McDuff and Salamon in [11]. A Gromov–Witten type invariant is an invariant of the deformation type of symplectic structures on a manifold. Gromov–Witten invariants of symplectic manifolds count the number of connected J -holomorphic curves in a particular homology class which pass through a number of points. In practice, for genus zero invariants, for a given homology class this count is done by tracing the oriented intersection points of a moduli space and a number of cohomology elements. The invariants are defined using regular curves.

A compact symplectic manifold (M, ω) of dimension $2n$ is called semipositive if there are no spherical homology classes $[A] \in H_2(M; \mathbb{Z})$ such that $\omega([A]) > 0$ and $2 - n < c_1(M)[A] < 0$. In particular, if the dimension of the manifold in this definition is less than or equal to six, then the manifold is semipositive.

2.1. Gromov–Witten invariants

For a $2n$ dimensional symplectic manifold (M, ω) (we write M when there is no ambiguity), let J be a generic compatible almost complex structure. A J -holomorphic curve in M is a smooth map u from a genus g complex curve into M such that $J \circ du = du \circ i$ where i is the complex structure on the curve.

Given a nonzero homology class $[A] \in H_2(M; \mathbb{Z})$ and a positive integer k , consider the moduli space $\mathcal{M}_{[A],g,k}^M$ of all simple genus g maps into M with k distinct marked points where the homology class of the image is $[A]$, up to reparametrization. $\mathcal{M}_{[A],g,k}^M$ consists of the equivalence classes $[u, x_1, \dots, x_k]$. Since we deal with genus zero Gromov–Witten invariants, i.e. $g = 0$, we drop the subscript g from the notation.

Let M^k denote the Cartesian product of k copies of M for $k > 0$. For $1 < j < k$, let π_j denote the projection map $M^k \rightarrow M$ onto the j th factor. The evaluation map $ev : \mathcal{M}_{[A],k}^M \hookrightarrow M^k$ is defined by $ev([u, x_1, \dots, x_k]) = (u(x_1), \dots, u(x_k))$.

Theorem 6.6.1 of [11] states the following lemma. For the definition of a pseudocycle see Definition 6.5.1 of [11].

Lemma 2.1 *Let M be semipositive and if $[A] \in H_2(M; \mathbb{Z})$ such that $[A] = m[B]$ and $c_1(M)[B] = 0 \Rightarrow m = 1$ for every positive integer $m > 0$ and for all spherical homology classes $[B] \in H_2(M; \mathbb{Z})$. Then the evaluation map $ev : \mathcal{M}_{[A],k}^M \hookrightarrow M^k$ is a pseudocycle of real dimension $2n - 6 + 2c_1(M)[A] + 2k$.*

The condition in the statement of Lemma 2.1 on $[A]$ that $[A] = m[B]$ and $c_1(M)[B] = 0$ implies $m = 1$ for every integer $m > 0$ and for all spherical homology class $[B] \in H_2(M; \mathbb{Z})$ is not restrictive in the context of this paper because such classes cannot be represented by embedded J -holomorphic spheres in a symplectic 4-manifold with $b_2^+ > 1$. If there was any such class $[B]$, it should have self intersection -2 by the adjunction formula and this is not the case as we are going to see.

According to Theorem 7.1.1 of [11], the Gromov–Witten invariants are calculated as in Lemma 2.2 below. See also Lemma 6.5.5(iii) of [11].

Lemma 2.2 *The k -pointed genus zero Gromov–Witten invariant of M in the class $[A]$ is defined as*

$$GW_{[A],k}^M(\alpha_1, \dots, \alpha_k) = ev \cdot f$$

where $\alpha_1, \dots, \alpha_k$ are cohomology classes of M and f is a pseudocycle which is Poincaré dual to $\pi_1^*(\alpha_1) \cup \dots \cup \pi_k^*(\alpha_k)$.

To get nonzero invariants, the sum of the degrees of the cohomology elements must be equal to the dimension of $\mathcal{M}_{[A],k}^M$, which is $2n - 6 + 2c_1(M)[A] + 2k$. This is called the dimension condition.

Gromov–Witten invariants can be consistently extended to the case where k is zero. If $[A]$ is zero, $GW_{0,0}^M$ is set as zero. When k is zero, M^k is a point and any pseudocycle is trivial. If $[A]$ is nonzero, for the dimension condition to be satisfied $2n - 6 + 2c_1(M)[A]$ should be zero. When the dimension $2n$ of M is four, under the assumption that $[A]$ has an embedded sphere representative, this implies that $[A]^2$ should be -1 and $GW_{[A],0}^M$ counts the exceptional spheres in the class $[A]$. This is either zero or one (not -1 as a convention) as in Example 3.1. If $2n$ is six, then for $GW_{[A],0}^M$ to be nonzero, $c_1(M)[A]$ should be zero.

Two facts about Gromov–Witten invariants which are used in the subsequent sections are the following two lemmas which are known as the fundamental class axiom and the divisor axiom for Gromov–Witten invariants (Proposition 7.5.6 of [11]).

Lemma 2.3 *Let M be a semipositive symplectic manifold, $[A]$ be a nonzero second homology class and $k \geq 1$. Then $GW_{[A],k}^M(\alpha_1, \dots, \alpha_{k-1}, PD([M]))$ is zero. In other words, there cannot be a degree zero cohomology class among α_i 's.*

Lemma 2.4 *Let M be a semipositive symplectic manifold, $[A]$ be a nonzero second homology class and $k \geq 1$. If the degree of α_k is two, then*

$$GW_{[A],k}^M(\alpha_1, \dots, \alpha_{k-1}, \alpha_k) = (\alpha_k \cdot [A]) GW_{[A],k-1}^M(\alpha_1, \dots, \alpha_{k-1})$$

3. Dimension four

In the proofs, we are going to apply different results which appeal to a generic set of compatible almost complex structures. These are Baire sets as well as their intersections; thus, one can find a compatible almost complex structure J which is in all of these sets.

3.1. Exceptional spheres

We start this subsection with an example on calculations of Gromov–Witten invariants.

Example 3.1 *Let Y be a simply connected symplectic 4-manifold and X be its blowup. Topologically X is diffeomorphic to $Y \# \overline{\mathbb{C}P^2}$. In the blowup of a 4-manifold, there is an exceptional sphere which is a smooth sphere with self intersection -1 . Let us find the Gromov–Witten invariant of X for the homology class $[E]$ of the exceptional sphere in $H_2(X; \mathbb{Z})$ with no other constraints. No constraint means that the number of points through which the exceptional sphere must pass is zero, that is k is zero. By the adjunction formula for symplectic 4-manifolds, $c_1(X)[E]$ is calculated as one. The expected dimension of the moduli space is $2n - 6 + 2c_1(X)[E]$ which is equal to zero. The moduli space $\mathcal{M}_{[E],0}^X$ is a finite set of points with orientation. By the positivity of intersections of J -holomorphic curves in an almost complex manifold, there is only one J -holomorphic sphere E which represents $[E]$ in X , so $GW_{[E],0}^X$ is 1.*

Similar examples are studied in [14] (pages 140 and 141) where k is taken to be three instead of zero. The next theorem is an extension of Example 3.1 and examples in [14] to all positive values of k .

Theorem 3.2 *Let Y be a symplectic 4-manifold and X be $Y \# \overline{\mathbb{C}P^2}$. If $[E]$ is the homology class of the exceptional sphere, then $GW_{[E],k}^X(PD[E], \dots, PD[E])$ is equal to $(-1)^k$.*

Proof Let J be a compatible almost complex structure on X . By positivity of intersections of J -holomorphic curves in an almost complex manifold, there is only one J -holomorphic sphere which represents $[E]$ in X , which will be denoted by E . $GW_{[E],0}^X$ is one. $PD([E]) \cdot [E]$ is -1 . Applying the divisor axiom (Lemma 2.4) inductively, we find $GW_{[E],k}^X(PD[E], \dots, PD[E]) = (-1)^k$. □

Theorem 3.3 *Let $k > 0$. If Y is a simply connected, symplectic 4-manifold and X is $Y \# \overline{\mathbb{C}P^2}$, then the Gromov–Witten invariant $GW_{[E],k}^X(\alpha_1, \dots, \alpha_k)$ is zero unless $\alpha_i \in H^2(X; \mathbb{Z})$. If $\alpha_i \in H^2(X; \mathbb{Z})$ for all $i \in \mathbb{Z}$ such that $1 \leq i \leq k$ ($k > 0$), then*

$$GW_{[E],k}^X(\alpha_1, \dots, \alpha_k) = (\alpha_1 \cdot [E]) \cdots (\alpha_k \cdot [E])$$

Proof By the dimension condition, the sum of degrees of α_i should be equal to the dimension of the moduli space $\mathcal{M}_{[E],k}^X$, which is $2n - 6 + 2c_1(X)[E] + 2k = 2k$. There is no odd degree cohomology because X is simply connected. By the fundamental class axiom (Lemma 2.3), in order to get a nonzero invariant, all classes must be of degree two. By the divisor axiom, the result follows. □

3.2. Nonzero invariants

This subsection is on the conditions for the invariants to be nonzero. Since a homologically essential embedded 2-sphere in a symplectic 4-manifold with $b^+ > 1$ always has negative self intersection (otherwise Seiberg–Witten invariants would vanish which is not the case for a symplectic manifold, [9]), we necessarily have $A^2 \leq -1$. A symplectic 4-manifold is of Seiberg–Witten simple type. The following lemma, which is a version of the adjunction inequality of Seiberg–Witten theory (Theorem 2.4.8 in [4]), is critical in the proofs of the main theorems.

Lemma 3.4 *Let X be a symplectic 4-manifold with $b_2^+(X) > 1$ and J be a generic almost complex structure on X which is compatible with the symplectic structure. Let $A \subset X$ be a connected J -holomorphic sphere in the class of $[A] \in H_2(X; \mathbb{Z})$ such that $[A]$ is nonzero and $[A]$ can be represented by an embedded, connected J -holomorphic sphere. Then A^2 is less than or equal to -1 .*

The next theorem is one of the main results.

Theorem 3.5 *Let X be a symplectic 4-manifold with $b_2^+(X) > 1$ and J be a generic almost complex structure on X which is compatible with the symplectic structure. Let $[A]$ be a nonzero second homology class of X which can be represented by an embedded, connected J -holomorphic sphere. Let $\alpha_1, \dots, \alpha_k$ be cohomology classes of X . If $GW_{[A],k}^X(\alpha_1, \dots, \alpha_k)$ is nonzero, then $[A]$ is the class of an exceptional sphere in X .*

Proof If the Gromov–Witten invariant is nonzero then $[A]$ must have a connected J -holomorphic sphere representative, say A . A can be chosen to be embedded. Lemma 3.4 implies that $A^2 \leq -1$. By the regularity of J and Corollary 3.3.4 of [11] $A^2 \geq -1$, so A^2 is -1 . Therefore, $[A]$ is the class of an exceptional sphere. \square

The case where $[A]$ can be the zero class is excluded. We refer the reader to a more general source [11] for a discussion of the zero class.

Remark 3.6 *These results are compatible with the results of Taubes [17]. If X is a symplectic manifold with $b_2^+(X) > 1$ and $[A]$ is a second homology class of X such that all of its representatives are connected and $g([A]) \neq 1$, then the relation between the Gromov–Witten invariants and the Gromov invariants of Taubes is*

$$Gr^X([A]) = GW_{[A],k_{[A]}}^X(PD([point]), \dots, PD([point]))$$

where $k_{[A]} = [A]^2 + 1 - g([A])$ and $PD([point])$ is repeated $k_{[A]}$ times.

The number of points in X for $[E]$ is determined by Taubes as $k_{[E]} = [E]^2 + 1 - g([E]) = 0$. Keeping in mind that the representative for $[E]$ is connected, according to the formula which gives the relation between the Gromov invariants of Taubes and the Seiberg–Witten invariants ([2]), $GW_{[E],0}^X = Gr^X([E]) = SW^X(2[E] + c_1(X)) = \pm 1$. The last equality is justified by the blowup formula for the Seiberg–Witten invariants, i.e. $SW^X(2[E] + c_1(X)) = SW^X(2[E] + c_1(Y) - [E]) = \pm SW^Y(c_1(Y)) = \pm 1$.

The similarity of the calculations in Example 3.1 and to the calculations in Lemma 4.2 inspires that there may be relation between the invariants of the symplectic structure on a 4-manifold X and the Gromov–Witten invariants of $X \times S^2$. Unfortunately, we see that no interesting relation may occur in the genus zero case.

4. Exotic symplectic manifolds in dimension six

Example 4.1 is brought first by Ruan in [12] in the context of exotic symplectic structures on smooth 6-manifolds. See [11] (page 335) for another explanation of this example. The manifolds in these examples have $b_2^+ = 1$.

Example 4.1 Let X_1 be $\mathbb{C}P^2 \#_8 \overline{\mathbb{C}P^2}$ and $X_2 = B_8$ be the Barlow surface. X_1 and X_2 are homeomorphic but they are not diffeomorphic [1, 3, 8]. The Barlow surface is minimal and $\mathbb{C}P^2 \#_8 \overline{\mathbb{C}P^2}$ is not minimal. $X_1 \times S^2$ and $X_2 \times S^2$ are not symplectic deformation equivalent [12].

The next lemma is based on this example.

Lemma 4.2 Let Y be a symplectic 4-manifold and X be its blowup, i.e $X = Y \# \overline{\mathbb{C}P^2}$. Let $[E]$ be the class of the exceptional sphere and let $[\overline{E}]$ denote the pushforward of the homology class $[E]$ under the inclusion map in $H_2(X \times S^2)$. Then $GW_{[\overline{E}],1}^{X \times S^2}(PD([\overline{E}]))$ is equal to -1 .

Proof Let J be a compatible almost complex structure on $X \times S^2$. The dimension of $\mathcal{M}_{[\overline{E}],1}^{X \times S^2}$ is $2c_1(X \times S^2)[\overline{E}] + 2 = 2c_1(X)[E] + 2$ which is equal to four, and the dimension condition is satisfied. By positivity of intersections of J -holomorphic curves in an almost complex manifold, there is only one curve which represents $[E]$ in X , which will be denoted by E . For each point of S^2 factor, we have the curve \overline{E} in $X \times \cdot$, where \overline{E} is the image of E in $X \times S^2$. If we put the condition of passing through a marked point, this adds two real dimensions to the moduli space for the freedom of choosing a point on the sphere E . The moduli space is diffeomorphic to $\overline{E} \times S^2$ in $X \times S^2$ which is compact and smooth and the evaluation map is the diffeomorphism. In this case, the intersection of the pseudocycles in the definition of Gromov–Witten invariants are in fact an intersection of cycles in M . Therefore, $GW_{[\overline{E}],1}^{X \times S^2}(PD([\overline{E}]))$ is equal to $[\overline{E} \times S^2] \cdot [\overline{E}]$ which is -1 . □

When k is zero, as discussed at the end of Section 2, the dimension condition does not hold since $c_1(X \times S^2)[\overline{E}]$ is nonzero. Thus, $GW_{[\overline{E}],0}^{X \times S^2}$ is zero.

The following theorem extends Lemma 4.2 to the cases where k is greater than one. See also [14].

Theorem 4.3 Let Y be a symplectic 4-manifold, X be $Y \# \overline{\mathbb{C}P^2}$ and $k \geq 1$. Then

$$GW_{[\overline{E}],k}^{X \times S^2}(PD([\overline{E}]), PD([\overline{E} \times S^2]), \dots, PD([\overline{E} \times S^2])) = (-1)^k$$

where $PD([\overline{E} \times S^2])$ is repeated $k - 1$ times.

Proof The dimension of $\mathcal{M}_{[\overline{E}],k}^{X \times S^2}$ is $2c_1(X \times S^2)[\overline{E}] + 2k = 2 + 2k$. If k is one, then by Lemma 4.2 $GW_{[\overline{E}],1}^{X \times S^2}(PD([\overline{E}]))$ is -1 . $PD([\overline{E} \times S^2])$ is a degree two cohomology class of $X \times S^2$. Thus, we can apply the divisor axiom (Lemma 2.4). Applying the divisor axiom inductively, we conclude that $GW_{[\overline{E}],k}^{X \times S^2}(PD([\overline{E}]), PD([\overline{E} \times S^2]), \dots, PD([\overline{E} \times S^2]))$ is $(-1)^k$. □

The second homology classes of $X \times S^2$ are pushforwards of the second homology classes of X or $[\cdot \times S^2]$. The next theorem is on the former classes.

Theorem 4.4 *Let X be a symplectic 4-manifold with $b_2^+(X) > 1$ and J be a generic almost complex structure on X which is compatible with the symplectic structure. Let $[A]$ be a nonzero second homology class of X which can be represented by an embedded, connected J -holomorphic sphere. Let $\overline{[A]}$ be the pushforward of $[A]$ in $H_2(X \times S^2; \mathbb{Z})$ and $\alpha_1, \dots, \alpha_k$ be cohomology classes of $X \times S^2$. If $GW_{\overline{[A]},k}^{X \times S^2}(\alpha_1, \dots, \alpha_k)$ is nonzero, then $[A]$ is the homology class of an exceptional sphere in X .*

Proof This result is a straightforward consequence of Theorem 3.5. □

The next theorem says that Ruan’s example (Example 4.1) is the only meaningful application of genus zero Gromov–Witten invariants in the case of 6-manifolds that appear as the Cartesian products of 4-manifolds with S^2 .

Theorem 4.5 *Let X be a simply connected, symplectic 4-manifold with $b_2^+(X) > 1$ and J be a generic almost complex structure on X which is compatible with the symplectic structure. Let $[A]$ be a nonzero second homology class of X which can be represented by an embedded, connected J -holomorphic sphere. Let $\overline{[A]}$ be the pushforward of $[A]$ in $H_2(X \times S^2; \mathbb{Z})$ and $\alpha_1, \dots, \alpha_k$ be cohomology classes of $X \times S^2$. If $GW_{\overline{[A]},k}^{X \times S^2}(\alpha_1, \dots, \alpha_k)$ is nonzero, then the following conditions are satisfied.*

1. $[A]$ is the homology class $[E]$ of E for an exceptional sphere E in X .
2. For a unique j , α_j is a fourth cohomology class of $X \times S^2$ which evaluates nonzero on $[\overline{E} \times S^2] \in X \times S^2$ where \overline{E} is the image of E in $X \times S^2$ under the inclusion map,
3. For all i which are not equal to j , α_i is a second cohomology class of $X \times S^2$ which evaluates nonzero on $[\overline{E}] \in X \times S^2$.

Proof Assume that $GW_{\overline{[A]},k}^{X \times S^2}(\alpha_1, \dots, \alpha_k)$ is not zero. Theorem 4.4 imposes that $[A]$ should be the homology class of an exceptional sphere E in X , i.e. $[A]$ is identical with $[E]$ in $H_2(X; \mathbb{Z})$. By the dimension condition, the sum of degrees of α_i should be equal to the dimension of the moduli space $\mathcal{M}_{\overline{[A]},k}^{X \times S^2}$, which is $2n - 6 + 2c_1(X \times S^2)\overline{[A]} + 2k = 2 + 2k$. $X \times S^2$ is simply connected; thus, its odd cohomology groups are trivial, and all α_i ’s ($1 \leq i \leq k$) have even degrees. By the fundamental class axiom (Lemma 2.3), in order to get a nonzero invariant, there must be one class of degree four, and the remaining ones are of degree two.

Without loss of generality, since there is no odd degree cohomology class, assume that α_1 is the fourth degree class.

If $k = 1$, then the moduli space $\mathcal{M}_{\overline{[E]},1}^{X \times S^2}$ is diffeomorphic to $\overline{E} \times S^2$ in $X \times S^2$ which is compact and smooth and the evaluation map is the diffeomorphism. In this case, the intersection of the pseudocycles in the definition of Gromov–Witten invariants are in fact an intersection of cycles in M . Therefore, $GW_{\overline{[E]},1}^{X \times S^2}(\alpha_1)$ is equal to $\alpha_1 \cdot [\overline{E} \times S^2]$, the evaluation of α_1 on $[\overline{E} \times S^2]$, and is nonzero only if the latter is nonzero.

If $k > 1$, let us turn to the cohomology classes α_i , $2 \leq i \leq k$. Each α_i is of degree two, so the divisor axiom is applicable and the result follows. □

Corollary 4.6 *Let k be a positive integer, $\alpha_1 \in H^4(X; \mathbb{Z})$ and $\alpha_i \in H^2(X; \mathbb{Z})$ for all $i \in \mathbb{Z}$ such that $2 \leq i \leq k$. If X is a simply connected, symplectic 4-manifold, then*

$$GW_{[A],k}^{X \times S^2}(\alpha_1, \dots, \alpha_k) = (\alpha_1 \cdot [\overline{E} \times S^2]) \cdot (\alpha_k \cdot [\overline{E}]) \cdots (\alpha_k \cdot [\overline{E}])$$

Remark 4.7 *Let X be a 4-manifold as in Theorem 4.5. If $GW_{[A],k}^{X \times S^2}(\alpha_1, \dots, \alpha_k)$ is nonzero for a nonzero second homology class $[A]$ of X which can be represented by an embedded, connected J -holomorphic sphere and for some cohomology classes $\alpha_1, \dots, \alpha_k$ of $X \times S^2$, then X is a blowup of a 4-manifold Y . $H_2(X; \mathbb{Z})$ is isomorphic to the direct sum $H_2(Y; \mathbb{Z}) \oplus H_2(\overline{\mathbb{C}P^2}; \mathbb{Z})$. By a slight abuse of notation, suppose that $H_2(\overline{\mathbb{C}P^2}; \mathbb{Z})$ is generated by $[E]$. Then $H_2(X \times S^2; \mathbb{Z})$ is isomorphic to the direct sum $H_2(Y; \mathbb{Z}) \oplus H_2(\overline{\mathbb{C}P^2}; \mathbb{Z}) \oplus H_2(S^2; \mathbb{Z})$. A generator of $H_2(X \times S^2; \mathbb{Z})$ is either the pushforward of a generator $[B]$ of $H_2(Y; \mathbb{Z})$ under the inclusion map into $X \times S^2$, $[\overline{E}]$ or $[\cdot \times S^2]$. If the Poincaré dual of the degree four cohomology class α_1 in the proof of the theorem is written as a linear combination of these generators, then the coefficient of $[\overline{E}]$ cannot be zero. That is α_1 has a $PD([\overline{E}])$ term. A similar argument applies to α_i ($2 \leq i \leq k$) and $PD([\overline{E} \times S^2])$.*

Another corollary to Theorem 4.5 is the existence of exotic symplectic deformation types on a fixed smooth 6-manifold.

Corollary 4.8 *Let X_1 and X_2 be homeomorphic symplectic 4-manifolds with $b_2^+(X_1) > 1$ and $b_2^+(X_2) > 1$. If X_1 is not minimal and X_2 is minimal, then $X_1 \times S^2$ and $X_2 \times S^2$ are diffeomorphic but they are not symplectic deformation equivalent.*

Proof In Lemma 4.2, one of the invariants of $X_1 \times S^2$, $GW_{[E],1}^{X \times S^2}(PD([\overline{E}]))$, is found to be nonzero.

$X_1 \times S^2$ and $X_2 \times S^2$ are diffeomorphic [6, 18]. Let $h : X_1 \times S^2 \rightarrow X_2 \times S^2$ be a diffeomorphism. The diffeomorphism h induces an isomorphism of the homology, the cohomology and the triple intersection forms. Under this isomorphism $c_1(X_1 \times S^2)$ is taken to $c_1(X_2 \times S^2)$, and by Theorem 9 of [18] homotopy class of the compatible almost complex structures is preserved. Theorem 4.5 implies that for a generic almost complex structure $X_2 \times S^2$ has all its corresponding genus zero Gromov–Witten invariants zero since X_2 is minimal. Therefore, the symplectic structures on $X_1 \times S^2$ and $X_2 \times S^2$ are not symplectic deformation equivalent. □

As another outcome of Theorem 4.5, we see the following corollary.

Corollary 4.9 *Given a minimal, simply connected, symplectic 4-manifold X such that $b_2^+(X) > 1$, the genus zero Gromov–Witten invariant $GW_{[A],k}^{X \times S^2}$ vanishes for all $k \geq 0$ and for any nonzero second homology class $[A]$ of $X \times S^2$.*

Corollary 4.9 implies that for two minimal, simply connected, symplectic 4-manifolds X_1 and X_2 such that $b_2^+(X_1) > 1$ and $b_2^+(X_2) > 1$, genus zero Gromov–Witten invariants cannot distinguish the symplectic structures on $X_1 \times S^2$ and $X_2 \times S^2$, regardless of whether or not the two 6-manifolds are diffeomorphic.

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References

- [1] Donaldson SK. An application of gauge theory to four-dimensional topology. *Journal of Differential Geometry* 1983; 18 (2): 279-315. doi: 10.4310/jdg/1214437665
- [2] Fintushel R. Knot surgery revisited. In: Elwood DA, Ozsvath PS, Stipsicz SI, Szabo Z (editors). *Floer Homology, Gauge Theory, and Low-Dimensional Topology*, Clay Mathematics Proceedings, 5. Providence, RI, USA: American Mathematical Society, 2006, pp.195-224.
- [3] Freedman MH. The topology of four-dimensional manifolds. *Journal of Differential Geometry* 1982; 17 (3): 357-453. doi: 10.4310/jdg/1214437136
- [4] Gompf RE, Stipsicz AI. *4-Manifolds and Kirby Calculus*. Providence, RI, USA: American Mathematical Society, 1999.
- [5] Gromov M. Pseudoholomorphic curves in symplectic manifolds. *Inventiones Mathematicae* 1985; 82 (2): 307-347. doi: 10.1007/BF01388806
- [6] Jupp PE. Classification of certain 6-manifolds. *Mathematical Proceedings of the Cambridge Philosophical Society* 1973; 73 (2): 293-300. doi: 10.1017/S0305004100076854
- [7] Kontsevich M, Manin Y. Gromov–Witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics* 1994; 164 (3): 525-562. doi: 10.1007/BF02101490
- [8] Kotschick D. On manifolds homeomorphic to $\mathbf{C}P^2 \# 8\overline{\mathbf{C}P^2}$. *Inventiones Mathematicae* 1989; 95 (3): 591-600. doi: 10.1007/BF01393892
- [9] McDuff D. The structure of rational and ruled symplectic 4-manifolds. *Journal of the American Mathematical Society* 1990; 3 (3): 679-712. doi: 10.2307/1990934
- [10] McDuff D, Salamon D. *Introduction to Symplectic Topology*. 2nd ed. New York, NY, USA: Oxford University Press, 1998.
- [11] McDuff D, Salamon D. *J-Holomorphic Curves and Symplectic Topology*. Providence, RI, USA: American Mathematical Society, 2004.
- [12] Ruan Y. Symplectic topology on algebraic 3-folds. *Journal of Differential Geometry* 1994; 39 (1): 215-227. doi: 10.4310/jdg/1214454682
- [13] Ruan Y. Virtual neighborhoods and pseudo-holomorphic curves. In: *Turkish Journal of Mathematics - Proceedings of 6th Gökova Geometry-Topology Conference*; Gökova, Turkey; 1999. pp. 161-231.
- [14] Ruan Y. Gromov–Witten invariants and quantum cohomology. In: Bouwknegt P, Wu S (editors). *Geometric analysis and applications to quantum field theory*. Boston, MA, USA: Birkhäuser, 2002, pp. 137-156.
- [15] Ruan Y, Tian G. A mathematical theory of quantum cohomology. *Journal of Differential Geometry* 1995; 42 (2): 259-367. doi: 10.4310/jdg/1214457234
- [16] Ruan Y, Tian G. Higher genus symplectic invariants and sigma models coupled with gravity. *Inventiones Mathematicae* 1997; 130 (3): 455-516. doi: 10.1007/s002220050192

- [17] Taubes CH. Seiberg Witten and Gromov Invariants for Symplectic 4-Manifolds. Somerville, MA, USA: International Press, 2000.
- [18] Wall CTC. Classification Problems in Differential Topology. V. On Certain 6-Manifolds. *Inventiones Mathematicae* 1966; 1 (4): 355-374. doi: 10.1007/BF01389738
- [19] Zinger A. Standard versus reduced genus-one Gromov–Witten invariants. *Geometry and Topology* 2008; 12 (2): 1203-1241. doi: 10.2140/gt.2008.12.1203