Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2366 – 2381
© TÜBİTAK
doi:10.3906/mat-2109-60

Axes in non-associative algebras

Louis ROWEN[1,*], **Yoav SEGEV**[2]
[1]Department of Mathematics, Bar-Ilan University, Ramat Gan, Israel
[2]Department of Mathematics, Ben-Gurion University, Beer-Sheva 84105, Israel

Received: 13.09.2021 • **Accepted/Published Online:** 18.11.2021 • **Final Version:** 29.11.2021

Abstract: "Fusion rules" are laws of multiplication among eigenspaces of an idempotent. This terminology is relatively new and is closely related to axial algebras, introduced recently by Hall, Rehren and Shpectorov. Axial algebras, in turn, are closely related to 3-transposition groups and Vertex operator algebras.

In this paper we consider fusion rules for semisimple idempotents, following Albert in the power-associative case. We examine the notion of an axis in the non-commutative setting and show that the dimension d of any algebra A generated by a pair a, b of (not necessarily Jordan) axes of respective types (λ, δ) and (λ', δ') must be at most 5; d cannot be 4. If $d \leq 3$ we list all the possibilities for A up to isomorphism.

We prove a variety of additional results and mention some research questions at the end.

Key words: Axial algebra, axis, flexible algebra, power-associative, fusion rule, idempotent, Jordan type

1. Introduction

Our goal in this paper is to further the study of axes, i.e., semi-simple idempotents, in an arbitrary algebra A. Throughout this paper A is an algebra (not necessarily associative or commutative, not necessarily with a multiplicative unit element) over a field F.

Let $a \in A$ be an idempotent, and $\lambda, \delta \in F$.

1. We define the *left and right* multiplication maps $L_a(b) := a \cdot b$ and $R_a(b) := b \cdot a$.

2. We write $A_\lambda(X_a)$ for the eigenspace of λ with respect to the transformation X_a, $X \in \{L, R\}$, i.e., $A_\lambda(L_a) = \{v \in A : a \cdot v = \lambda v\}$, and similarly for $A_\lambda(R_a)$. Often we just write A_λ for $A_\lambda(L_a)$, when a is understood. We note that it may happen that $A_\lambda(X_a) = 0$.

3. We write $A_{\lambda,\delta}(a) := A_\lambda(L_a) \cap A_\delta(R_a)$. We just write $A_{\lambda,\delta}$ when the idempotent a is understood. An element in $A_{\lambda,\delta}(a)$ will be called a (λ, δ)-*eigenvector* of a, and (λ, δ) will be called its *eigenvalue*.

An idempotent $a \in A$ is a *left axis* (resp. *right* axis) if L_a (resp. R_a) is a semi-simple operator. An *axis* is an idempotent a which is both a left and a right axis, and satisfies $L_a R_a = R_a L_a$. If A is associative with an identity element, then $A = A_{1,1}(a) + A_{0,0}(a) + A_{1,0}(a) + A_{0,1}(a)$, for any idempotent $a \in A$.

*Correspondence: rowen@math.biu.ac.il
** The first author was supported by the Israel Science Foundation grant 1623/16
2010 *AMS Mathematics Subject Classification:* Primary: 17A15, 17A20, 17D99; Secondary: 17A01, 17A36, 17C27

Axes a in a power-associative algebra A (although not under that name) were already studied by Albert [1]. He showed that when A is commutative, $A = A_{1,1}(a) + A_{0,0}(a) + A_{1/2,\,1/2}(a)$, for any idempotent $a \in A$. In particular this is the case when A is a Jordan algebra.

In [3], Hall, Rehren and Shpectorov introduced axial algebras. These are commutative algebras, which are not necessaily power associative, and which are generated by axes. Axial algebras, and, in particular, “primitive axial algebras of Jordan type” are of interest because of their connection with group theory and with Vertex operator algebras. We refer the reader to the introduction of [3] for further information.

Axial (composition) algebras are (non-associative) algebras generated by axes. These algebras are not necessarily power associative and not necessarily commutative. However, they are generated by axes satisfying certain *fusion rules* (see [2] for the most general notion of fusion rules and for the notion of decomposition algebra).

Our main interest in this paper, continuing our work in [5], is to place (commutative) axial algebras in a general non-commutative framework. We hope that in addition to being interesting in its own right, this information might be used to understand (and prove) the finite dimensionality of various finitely generated primitive axial algebras.

Definitions 1.1

1. *The algebra A is flexible if it satisfies the identity $(xy)x = x(yx)$. In a flexible algebra we write xyx without parentheses since there is no ambiguity.*

2. *A is power-associative if $F[x]$ is associative (and therefore commutative) for each $x \in A$.*

Commutative algebras are flexible since

$$(xy)x = (yx)x = x(yx). \tag{1.1}$$

We first study eigenvalues of idempotents $a \in A$ in the spirit of [1]. We often assume flexibility. In subsection 2.1 we also often assume that A is power-associative, relying heavily on Albert [1].

Definitions 1.2 *Let $a \in A$ be an idempotent, and $\lambda, \delta \notin \{0,1\}$ in F.*

1. *a is a left axis if a is a left semisimple idempotent, i.e., L_a satisfies a polynomial $p[t] = t(t-1)\prod_{i=1}^m (t-\lambda_i)$ with only simple roots $0, 1, \lambda_1, \dots, \lambda_m$. Note that we do not require $p[t]$ to be the minimal polynomial of L_a. We call $\lambda_1, \dots, \lambda_m$ the type of a. The left axis a is primitive if $A_1 = Fa$.*

   For a primitive left axis a of type $\lambda_1, \dots, \lambda_m$, and $x \in A$, we write

   $$x = \alpha_x a + x_0 + \textstyle\sum_{i=1}^m x_{\lambda_i}, \qquad x_{\lambda_i} \in A_{\lambda_i}(L_a).$$

   and we call it the left decomposition of x with respect to a.

2. *Given a left axis a, let $B := A_1 + A_0$. We say that a satisfies the left basic fusion rule if*

   (i) B is a subalgebra of A.

   (ii) $BA_\lambda = A_\lambda B \subseteq A_\lambda$ for each eigenvalue $\lambda \neq 0, 1$.

(iii) For each λ *there is* λ' *such that* $A_\lambda A_{\lambda'} \subseteq B$.

The left involutory fusion rules are the basic fusion rules together with $A_\lambda^2 \subseteq B$, *for each* λ.

3. *Thus* a *is a primitive left axis of type* λ *with the left involutory fusion rule if*

   (i) $(L_a - \lambda)(L_a - 1)L_a = 0$. *(In particular the only left eigenvalues are contained in* $\{0, 1, \lambda\}$*.)*

   (ii) There is a direct sum decomposition of A *which is a* $\mathbb{Z}_2$*-grading:*

$$A = \overbrace{Fa \oplus A_0}^{+} \oplus \overbrace{A_\lambda}^{-},$$

   recalling that A_λ *means* $A_\lambda(L_a)$.

4. *A right axis of type* $\delta_1, \dots, \delta_n$ *is defined similarly. Also the right fusion rules are defined as in (2). As with a left axis, for a right axis* a *and* $x \in A$, *we write*

$$x = \beta_x a + {}_0x + \sum_{i=1}^m {}_{\lambda_i}x, \qquad {}_{\lambda_i}x \in A_{\lambda_i}(R_a).$$

   and we call it the right decomposition of x *with respect to* a.

5. a *is a (2-sided) primitive axis of type* $(\lambda_1, \dots, \lambda_m; \delta_1, \dots, \delta_n)$ *if* a *is a primitive left axis of type* $\lambda_1, \dots, \lambda_m$ *and a primitive right axis of type* $\delta_1, \dots, \delta_n$, *satisfying the left and right involutory fusion rules, and, in addition,* $L_a R_a = R_a L_a$. *In particular,* a *is a primitive axis of type* (λ, δ), *when* $m = n = 1$.

   Note that for any primitive axis a, $A_{1,1} = Fa = A_1(L_a) = A_1(R_a)$*; in particular* $A_{1,\mu} = A_{\nu,1} = 0$, *for any* ν, μ *distinct from* 1. *So if* a *is a primitive axis, and* $x \in A$, *we write*

$$x = \alpha_x a + x_{0,0} + \sum_{i=1}^m x_{\lambda_i,0} + \sum_{j=1}^n x_{0,\delta_i} + \sum_{i,j} x_{\lambda_i,\delta_j}.$$

   and we call it the decomposition of x *with respect to* a.

6. *([5]) An axis* a *is of Jordan type* (λ, δ) *if* $A_{\lambda,0}(a) = 0 = A_{0,\delta}(a)$. *Note that in this case*

$$x = \alpha_x a + x_{0,0} + x_{\lambda,\delta}, \qquad \forall x \in A.$$

   In this case we sometimes call a *an axis of Jordan type.*

◇

Hence, if a is an axis of type (λ, δ), then

$$A = \overbrace{A_{1,1} \oplus A_{0,0}}^{++} \oplus \overbrace{A_{0,\delta}}^{+-} \oplus \overbrace{A_{\lambda,0}}^{-+} \oplus \overbrace{A_{\lambda,\delta}}^{--},$$

is $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading of A (multiplication $(\epsilon, \epsilon')(\rho, \rho')$ is defined in the obvious way for $\epsilon, \epsilon', \rho, \rho' \in \{+, -\}$).

If a is an axis of Jordan type (λ, δ), then

$$A = A_1 \oplus A_0 \oplus A_{\lambda,\delta}.$$

Jordan type is close to commutative, but there are natural non-commutative examples given in [5, Examples 2.6]. Also see Examples 2.2 for axes not of Jordan type.

Note that besides being non-commutative, Definition 1.2(2) **does not** require the condition that A_0 is a subalgebra, contrary to the usual hypothesis in the commutative theory of primitive axial algebras. But this condition does not seem to pertain to any of the proofs.

If a is a primitive axis of type (λ, δ), then by Definition 1.2(5), we can write $x \in A$ as

$$x = \alpha_x a + x_{0,0} + x_{0,\delta} + x_{\lambda,0} + x_{\lambda,\delta}, \qquad x_{\mu,\nu} \in A_{\mu,\nu}(a), \tag{1.2}$$

which we call the (2*-sided*) *decomposition of* x with respect to a.

When a is of Jordan type, we have the much simpler decomposition

$$x = \alpha_x a + x_0 + x_{\lambda,\delta}$$

with $x_0 \in A_{0,0}$, and $x_{\lambda,\delta} \in A_{\lambda,\delta}$, which is both the left decomposition and right decomposition.

Next, for $\lambda \in F$, write

$$\mathring{A}_\lambda(a) = \{x \in A \mid ax + xa = 2\lambda x\}. \tag{1.3}$$

(We write $\mathring{A}_\lambda$, when a is understood.) Albert proved for A power-associative with $\mathrm{char}(F) \neq 2, 3$, that

$$A = A_{1,1} \oplus A_{0,0} \oplus \mathring{A}_{1/2}.$$

This is reproved directly as Theorem 2.8(1). The following theorem of Albert then describes the appropriate fusion laws in a power-associative algebra A :

Albert's Fusion Theorem ([1, Theorem 3, p. 560, Theorem 5, p. 562]) *Suppose that* A *is power-associative with* $\mathrm{char}(F) \neq 2, 3$. *Let* $a \in A$ *be an idempotent (not necessarily primitive).*

1. $A = A_{1,1} \oplus A_{0,0} \oplus \mathring{A}_{1/2}$.
2. $A_{0,0}, A_{1,1}$ *are subalgebras.*

   Furthermore for A *flexible,*
3. $A_{\lambda,\lambda}\mathring{A}_{1/2} \subseteq A_{1-\lambda,1-\lambda} + \mathring{A}_{1/2}, \quad \mathring{A}_{1/2}A_{\lambda,\lambda} \subseteq A_{1-\lambda,1-\lambda} + \mathring{A}_{1/2}, \quad$ *for* $\lambda = 0, 1$.
4. $a\mathring{A}_{1/2}, \mathring{A}_{1/2}a \subseteq \mathring{A}_{1/2}$. ◇

We prove

Theorem A (Theorem 4.3). Assume that A is generated by two axes a, b of type (λ, δ) and (λ', δ'). Then the dimension of A is at most 5.

In [5, Theorem 2.5], we classified all algebras as in Theorem A, of dimension 2. We prove

Theorem B (Theorem 4.10). Assume that A is generated by two axes a, b of type (λ, δ) and (λ', δ'). If $\dim(A) = 3$, then a and b are of Jordan type, and hence the possibilities for A are given in [5, Theorem B].

Proposition C. Assume that A is generated by two axes a, b of type (λ, δ) and (λ', δ'). Then

1. (Lemma 4.9) If $ab \in Fa + Fb$, then $\dim(A) = 2$.

2. (Lemma 4.8) If $ab = 0$, then $ba = 0$.

We prove a number of additional results and we mention some questions in §5.

Remark 1.3 *Along the period that this paper was refereed we proved that there are no algebras of dimension* 4 *generated by two primitive axes of type* (λ, δ) *and* (λ', δ')*. This result will appear somewhere else. The referee has pointed out that [7] contains some similar results for alternative algebras.*

2. General results about axes

The next proposition provides a useful decomposition result into eigenspaces.

Proposition 2.1 *Suppose* a *is a left axis, where* L_a *satisfies the polynomial* $\prod_{i=1}^{m}(x - \mu_i)$*, with only simple roots* $\mu_1, \dots, \mu_m$*. For* $y \in A$*, write* $y = \sum_{j=1}^{m} y_j$*, where* y_j *is the left* μ_j*-eigenvector in the* L_a*-eigenvector decomposition of* y*, and define the vector space* $V_a(y) = \sum_{i=0}^{m-1} F L_a^i y$*. Then*

$$V_a(y) = \oplus_{j=1}^{m} F y_j.$$

Proof By induction, $L_a^k(y) = \sum \mu_j^k y_j$ for $0 \le k \le m-1$. Thus

$$V_a(y) \subseteq \oplus_{j=1}^{m} F y_j.$$

But on the other hand, we have a system of m equations in the y_j with coefficients (μ_j^k), the Vandermonde matrix, which is non-singular, so there is a unique solution for the y_j in $V_a(y)$. □

This space $V_a(b)$ plays a key role in investigating a second axis b (see Proposition 4.7).

2.1. Basic properties of flexible idempotents

An idempotent $a \in A$ is called *flexible* if $(ax)a = a(xa)$ and $(xa)x = x(ax)$ for all $x \in A$. We present some examples of axes that are not flexible, to justify full generality.

Examples 2.2

1. *Let* $A = Fa \oplus Fb \oplus Fc$ *with multiplication given by*

$$a^2 = a, \qquad x^2 = a = y^2,$$
$$ax = \lambda x, \qquad xa = 0,$$
$$ay = 0, \qquad ya = \delta y, \qquad xy = yx = 0.$$

Then $a(xa) = 0 = (ax)a$*, and* $a(ya) = 0 = (ay)a$*, so* $L_a R_a = R_a L_a$*.*

$$(L_a - \lambda)x = \lambda(x - x) = 0 = L_a y,$$

implying $(L_a - \lambda)(L_a - 1)L_a = 0$*. Likewise* $(R_a - \delta)(R_a - 1)R_a = 0$*.*

Hence a *is a primitive axis of type* (λ, δ) *which is not of Jordan type, with* $Fx = A_{\lambda,0}(a)$ *and* $Fy = A_{0,\delta}(a)$*;* a *is the only axis in* A*.*

2. *Let* $0 \neq \lambda, \delta \in F$*, with* $\lambda + \delta = 1$*. Let*

$$A = Fa \oplus Fy \oplus Fx \oplus Fy \oplus Fz,$$

with multiplication defined by:

$$a^2 = a, \quad ac = ca = 0, \quad ax = \lambda x, xa = 0, \quad ay = 0, ya = \delta y,$$
$$az = \lambda z, \quad za = \delta z.$$

$$c^2 = cx = xc = cy = yc = cz = zc = 0.$$
$$x^2 = 0, \quad xy = 0 = yx, \quad xz = \lambda y, \quad zx = 0.$$
$$y^2 = c, \quad yz = \delta x, \quad zy = 0, \quad z^2 = 0.$$

So

$$A_{1,1} = Fa, \quad A_{0,0} = Fy, \quad A_{\lambda,0} = Fx, \quad A_{0,\delta} = Fy, \quad A_{\lambda,\delta} = Fz.$$

It is easy to check that $(av)a = a(va)$*, for* $v \in \{a, c, x, y, z\}$*, and* a *is a primitive axis of type* (λ, δ)*, not of Jordan type. Also,* a *is not a flexible axis since* $(xa)x = 0$*, while* $x(ax) = \lambda x^2 = -\lambda c$*.*

Let $b = a + c + x + y + z$*. Then*

$$\begin{aligned} b^2 =& a^2 + x^2 + (y)^2 + ax + ya + az + za + xz + yz \\ &= a - c + c + (\lambda + \delta)x + (\lambda + \delta)y + (\lambda + \delta)z = b. \end{aligned}$$

Note that A *has dimension 5. Since* $a, b, ab, ba,$ *and* aba *are independent, they span* A*.*

Flexible idempotents to axis are rather special.

Lemma 2.3 *If* a *is an idempotents to axis satisfying* $(xa)x = x(ax)$*, for all* $x \in A$*, then* $L_a^2 - L_a = R_a^2 - R_a$*.*

Proof We have:

$$a + ax + (xa)a + xax = (a + xa)(a + x) = ((a + x)a)(a + x) =$$
$$(a + x)(a(a + x)) = (a + x)(a + ax) = a + a(ax) + xa + xax,$$

so $ax + (xa)a = a(ax) + xa$. □

Lemma 2.4 *Suppose* $a \in A$ *is an idempotent satisfying* $L_a^2 - L_a = R_a^2 - R_a$*. (In particular this holds when* a *is an idempotent satisfying* $(xa)x = x(ax)$*, for all* $x \in A$*, by Lemma 2.3.)*

1. *If* $A_{\mu,\nu}(a) \neq 0$*, then either* $\nu = \mu$ *or* $\nu = 1 - \mu$*.*
2. A *decomposes as* $\oplus_\lambda A_{\lambda,\lambda} \oplus A_{\lambda,1-\lambda}$*, summed over all left eigenvalues* λ *of* a*.*

Proof (1) Let $z \neq 0$ is a (μ, ν)-eigenvector, then

$$(\mu^2 - \mu)z = a(az) - az = (za)a - za = (\nu^2 - \nu)z,$$

implying $\mu^2 - \mu = \nu^2 - \nu$, or $(\mu - \nu)(\mu + \nu - 1) = 0$.
(2) These are the only possibilities, in view of (1). □

Lemma 2.5 *Suppose $a \in A$ is a flexible idempotent. Then*

$$R_a(R_a + L_a - 1) = L_a(R_a + L_a - 1).$$

Proof By Lemma 2.3, and since $L_aR_a = R_aL_a$,

$$\begin{aligned} R_a(R_a + L_a - 1) &= R_aL_a + R_a^2 - R_a = R_aL_a + L_a^2 - L_a \\ &= L_a(R_a + L_a - 1). \end{aligned}$$

□

2.2. Albert's power-associative results

As an application, we reprove Albert's theorem on power-associative algebras, but first we need:

Lemma 2.6 *If A is flexible and power-associative over a field of characteristic $\neq 2, 3$, then*

$$(X_a - 1)Y_a(L_a + R_a - 1) = 0, \text{ for } X, Y \in \{R, L\}.$$

Proof We show that $(R_a - 1)R_a(L_a + R_a - 1) = 0$, the rest follows from Lemma 2.3 and Lemma 2.5.

By [1, Equation 12, p. 556], and applying Lemma 2.3,

$$\begin{aligned} 0 &= R_a^2 + L_aR_a + L_a^2 - R_a^3 - L_aR_a^2 - L_a \\ &= R_a^2 + L_aR_a + R_a^2 - R_a^3 - L_aR_a^2 - R_a \\ &= 2R_a^2 - R_a^3 + L_a(R_a - R_a^2) - R_a, \end{aligned} \tag{2.1}$$

so

$$(R_a - 1)^2R_a = R_a^3 - 2R_a^2 + R_a = L_a(R_a - R_a^2) = -L_aR_a(R_a - 1),$$

so

$$(R_a - 1)R_a(L_a + R_a - 1) = 0.$$

□

Remark 2.7 *In an easy argument in the first three lines of the proof of [1, Theorem 3, p. 560] Albert proves that $\mathring{A}_\lambda = A_{\lambda,\lambda}$ for $\lambda \in \{0, 1\}$ (see Equation (1.3) for notation). This is used in the following result of Albert on power-associative algebras ([1, (22), p. 559] and [1, Thm. 3, p. 562]).*

Theorem 2.8 ([1]) *Suppose that A is power-associative, and that $\mathrm{char}(F) \neq 2, 3$. Let $a \in A$ be an idempotent. Then*

1. $A = A_{1,1} \oplus A_{0,0} \oplus \mathring{A}_{1/2}$.
2. *We have*

$$A_1 = X_a(L_a + R_a - 1)A; \qquad A_0 = (X_a - 1)(L_a + R_a - 1)A;$$

$$\mathring{A}_{1/2} = (R_a + L_a)(R_a + L_a - 2)A, \quad X \in \{R, L\}.$$

Here is an alternate proof that is rather conceptual and reasonably quick, working directly in A.

Proof (1) Assume first that A is commutative. Then A is flexible (see Equation (1.1)), so we may use Lemma 2.5 and Lemma 2.6. Let $x \in A$.

Let $x_1 = R_a x$ and $x_2 = (1 - R_a)x$, so that $x = x_1 + x_2$. By Lemma 2.6 we have

$$(R_a - 1)(R_a + L_a - 1)(x_1) = 0 = (L_a - 1)(R_a + L_a - 1)(x_1). \tag{2.2}$$

and

$$R_a(R_a + L_a - 1)(x_2) = 0 = L_a(R_a + L_a - 1)(x_2). \tag{2.3}$$

Let

$$x_1' = (R_a + L_a - 1)(x_1), \qquad x_1'' = (R_a + L_a - 2)(x_1),$$
$$x_2' = (R_a + L_a - 1)(x_2), \qquad x_2'' = (R_a + L_a)(x_2),$$

Of course

$$x = x_1' - x_1'' + x_2'' - x_2'.$$

By Equations (2.2) and (2.3),

$$R_a(x_1') = x_1' = L_a(x_1'), \qquad R_a(x_2') = 0 = L_a(x_2').$$

So

$$x_1' \in A_1, \qquad \text{and} \qquad x_2' \in A_0.$$

Also by Equation (2.2), $(R_a + L_a - 1)(x_1'') = 0$, that is $x_1'' \in \mathring{A}_{1/2}$, and by Equation (2.3), $(R_a + L_a - 1)(x_2'') = 0$, that is $x_2'' \in \mathring{A}_{1/2}$.

For the general case where A is not commutative, $\mathring{A}$ is commutative and power-associative; hence $A = \mathring{A}_1 \oplus \mathring{A}_0 \oplus \mathring{A}_{1/2}$. However, $\mathring{A}_\lambda = A_{\lambda,\lambda}$, for $\lambda \in \{0, 1\}$, by Remark 2.7.

(2) follows from (1). Indeed

$$X_a(L_a + R_a - 1)A_0 = 0 = X_a(L_a + R_a - 1)\mathring{A}_{1/2},$$

and $X_a(L_a + R_a - 1)(x) = x$, for $x \in A_1$, and similarly for $(X_a - 1)(L_a + R_a - 1)$ and $(X_a + L_a)(R_a + L_a - 2)$. □

Note that if A is commutative then $\mathring{A}_{1/2} = A_{1/2}$, so Theorem 2.8 generalizes the commutative description of axes.

2.3. Properties of primitive axes of type (λ, δ)

Here are some computations.

Lemma 2.9 *Suppose a is a primitive axis of type (λ, δ), and let $y \in A$.*

1. $a(Fa + A_0) = Fa$.
2. $ay = \alpha_y a + \lambda y_\lambda$, *so* $y_\lambda = \frac{1}{\lambda}(ay - \alpha_y a)$.

3. $a(ay) = \alpha_y(1-\lambda)a + \lambda ay$.

4. $y_0 = y - \frac{1}{\lambda}(ay - (\lambda+1)\alpha_y a)$.

5. $(ay)a = \alpha_y a + \lambda\delta y_{\lambda,\delta}$, *implying* $y_{\lambda,\delta} = \frac{1}{\lambda\delta}((ay)a - \alpha_y a)$.

Proof (1) This is obvious.

(2) Apply L_a and solve.

(3) $a(ay) = \alpha_y a + \lambda^2 y_\lambda = \alpha_y a + \lambda(ay - \alpha_y a)$, yielding (3).

(4) $y_0 = y - y_\lambda - \alpha_y a = y - \frac{1}{\lambda}(ay - (\lambda+1)\alpha_y a)$.

(5) Apply R_a to (2). □

Lemma 2.10

1. $ab \notin Fa$ *or* $ab = 0$*, for any idempotent* a *and primitive right axis* b *satisfying the involutory fusion rules.*

2. *For any primitive left axis* a *of type* λ *and any primitive right axis* b *satisfying the involutory fusion rules, either* $ab = 0$ *or* $b_\lambda \neq 0$*.*

3. *If* a *is a primitive axis of Jordan type* (λ, δ) *with* $\lambda \neq \delta$*, and* b *is an axis, then* $b_{\lambda,\delta}^2 = 0$.

Proof (1) If $ab = \gamma a$, then a is an eigenvector of R_b, with eigenvalue γ. If $\gamma = 1$, then b is not primitive, a contradiction. Otherwise, by the fusion rules for b, $a = a^2 \in Fb + A_0(R_b)$. But then $ab \in Fb$. This together with $ab \in Fa$ forces $ab = 0$.

(2) If $b_\lambda = 0$, then $ab = \alpha_b a$, so we are done by (1).

(3) The proof is exactly as in [5, Lemma 2.4(iv)]. □

Corollary 2.11

1. *A primitive axis* a *of type* (λ, δ) *is in the center of* A*, iff it is of Jordan type with* $A_{\lambda,\delta} = 0$.

2. *Let* $a \neq b$ *be two primitive axes with* $ab = ba$ *and* a *of type* (λ, δ)*. Then either* $ab = 0$*, or* $\lambda = \delta$*. In particular, if* a *is of Jordan type, then either* a *commutes with all elements of* A *or* $ab = 0$*.*

Proof (1) ($\Rightarrow$) For any $x \in A$,

$$\alpha_x a + \lambda^2 x_\lambda = a(ax) = ax = \alpha_x a + \lambda x_\lambda,$$

implying $x_\lambda = 0$ since $\lambda \neq 0, 1$. Hence $x_{\lambda,0} = 0 = x_{\lambda,\delta}$, and by symmetry $x_{0,\delta} = 0$.

($\Leftarrow$) $x = \alpha_x a + x_{0,0}$ for any x in A, implying $ax = xa = \alpha_x a$ and also $xy = x_{0,0}y_{0,0} + \alpha_x\alpha_y a$, and thus $a(xy) = (ax)y = \alpha_x ay = \alpha_x\alpha_y a$ for all x, y.

(2) Let $b = \alpha_b a + b_{0,0} + b_{\lambda,0} + b_{0,\delta} + b_{\lambda,\delta}$ be the decomposition of b with respect to a. Then $\alpha_b a + \lambda b_{\lambda,0} + \lambda b_{\lambda,\delta} = ab = ba = \alpha_b a + \delta b_{0,\delta} + \delta b_{\lambda,\delta}$ implying $b_{\lambda,0} = b_{0,\delta} = 0$. If $b_{\lambda,\delta} \neq 0$, then $\lambda = \delta$, and we are done. Hence $b_{\lambda,\delta} = 0$, in which case $ab \in Fa$, so by Lemma 2.10(1), $ab = 0$. □

We generalize Seress' Lemma from [5, Lemma 2.7]:

Lemma 2.12 (General Seress' Lemma) *If a is a primitive left axis satisfying the left basic fusion rules then $a(xy) = (ax)y + a(x_0y)$, for any $x \in A$ and $y \in Fa + A_0(L_a)$. In particular, $a(xy) \in (ax)y + Fa$. Here*

$$x = \alpha_x a + x_0 + \sum_{i=1}^{m} x_{\lambda_i},$$

is the left decomposition of x with respect to a.

The symmetric statement holds: If a is a primitive right axis satisfying the right basic fusion rules then $(yx)a = y(xa) + a(y \cdot_0 x)$, for any $x \in A$ and $y \in Fa + A_0(R_a)$. In particular, $(yx)a \in y(xa) + Fa$.

Proof By linearity we need only check for x an eigenvector of L_a. If $x = x_0 \in A_0(L_a)$, then $a(x_0y) = 0 + a(x_0y) = (ax_0)y + a(x_0y)$. By the basic fusion rules $x_0y \in Fa + A_0(L_a)$, so $a(x_0y) \in Fa$.

If $x = x_\mu \in A_\mu(L_a)$ for $\mu \neq \{0,1\}$ then $x_\mu y \in A_\mu(L_a)$, by the basic fusion rules, hence $(ax_\mu)y = a(x_\mu y) = \mu x_\mu y$. □

3. Miyamoto involutions

It is easy to check that any $\mathbb{Z}_2$-grading of A induces an *involution*, i.e. an automorphism of order 2 of A. Indeed, if $A = A^+ \oplus A^-$, then $y \mapsto y^+ - y^-$ is such an automorphism, where $y \in A$ and $y = y^+ + y^-$ for $y^+ \in A^+, y^- \in A^-$.

So if $a \in A$ is a primitive axis of type (λ, δ), then we have three such automorphisms of order 2, which, conforming with the literature, we call the **Miyamoto involutions** *associated with a*:

(i) $\tau_{\lambda,a} = \tau_\lambda : y \mapsto y - 2y_\lambda = \alpha_y a + y_{0,0} + y_{0,\delta} - y_{\lambda,0} - y_{\lambda,\delta}$.

(ii) $\tau_{\delta,a} = \tau_\delta : y \mapsto y - 2\,{}_\delta y = \alpha_y a + y_{0,0} - y_{0,\delta} + y_{\lambda,0} - y_{\lambda,\delta}$.

(iii) $\tau_{\mathrm{diag},a} = \tau_{\mathrm{diag}} : y \mapsto \alpha_y a + y_{0,0} - y_{0,\delta} - y_{\lambda,0} + y_{\lambda,\delta}$.

In case a is of Jordan type, the first two Miyamoto involutions are the same, and the third is the identity, so we are left with a single non-trivial Miyamoto involution associated to a, which we write as τ_a.

Notation 3.1 *For any Miyamoto involution τ we write y^τ for $\tau(y)$. Let X be a set of axes, where each $x \in X$ is primitive of type (λ_x, δ_x). We denote by $G(X)$ the subgroup of $\mathrm{Aut}(A)$ generated by all the Miyamoto involutions associated with all the axes in X. We denote $\overline{X} := X^{G(X)} = \{x^g \mid x \in X,\ g \in G(X)\}$.*

Remark 3.2 *Let a be a primitive axis of type (λ, δ). Of course we can recover:*

(i) $y_{\lambda,\delta} + y_{\lambda,0} = \frac{1}{2}(y - y^{\tau_\lambda})$.

(ii) $y_{\lambda,\delta} + y_{0,\delta} = \frac{1}{2}(y - y^{\tau_\delta})$.

(iii) $y_{0,\delta} + y_{\lambda,0} = \frac{1}{2}(y - y^{\tau_{\mathrm{diag}}})$,

so each of $y_{\lambda,\delta}, y_{0,\delta}$, and $y_{\lambda,0}$ are linear combinations of y, y^{τ_λ}, y^{τ_δ}, and $y^{\tau_{\mathrm{diag}}}$, as is $+\alpha_y a = y - (y_{\lambda,\delta} + y_{0,\delta} + y_{\lambda,0})$. ◇

Theorem 3.3 *Suppose A is generated by a set of axes X, where each $x \in X$ is primitive of type (λ_x, δ_x).*

1. *(Generalizing [3, Corollary 1.2, p. 81].) A is spanned by the set $\overline{X}$.*
2. *Let $V \subseteq A$ be a subspace containing X such that $xV \subseteq V$ and $Vx \subseteq V$, for all $x \in X$. Then $V = A$.*

Proof (1) By induction on the length of a monomial in the axes $\overline{X}$, it suffices to show that ab is in the span of $\overline{X}$, for $a, b \in \overline{X}$. Write

$$b = \alpha_b a + b_{0,0} + b_{\lambda,0} + b_{0,\lambda} + b_{\lambda,\delta}.$$

Then $ab = \alpha_b a + \lambda b_{\lambda,0} + \lambda b_{\lambda,\delta}$. But by Remark 3.2, $b_{\lambda,0}$ and $b_{\lambda,\delta}$ are linear combinations of $b, b^{\tau_\lambda}, b^{\tau_\delta}, b^{\tau_{\mathrm{diag}}}$.

(2) Let $a \in X$, so that $aV \subseteq V$ and $Va \subseteq V$. We claim that if $v \in V$, then $v_{\mu,\nu} \in V$ for each $\mu \in \{\lambda, 0\}, \nu \in \{\delta, 0\}$. First, $v_{\lambda,\delta} \in V$ since $\alpha_v a + \lambda\delta v_{\lambda,\delta} = ava = a(va) \in V$. Hence $v_{\lambda,0} \in V$ since $av - ava \in V$, and $v_{0,\delta} \in V$ since $va - ava \in V$. Finally $v_{0,0} = v - \alpha_v a - v_{0,\delta} - v_{\lambda,0} - v_{\lambda,\delta} \in V$.

Next we claim that $V^\tau = V$, for all $\tau \in \{\tau_{\lambda,a}, \tau_{\delta,a}, \tau_{\mathrm{diag},a}\}$, and all $a \in X$. Indeed, for $v \in V$, $v^{\tau_\lambda} = v - 2(v_{\lambda,0} + v_{\lambda,\delta})$, $v^{\tau_\delta} = v - 2(v_{0,\lambda} + v_{\lambda,\delta})$, and $v^{\tau_{\mathrm{diag}}} = v - 2(v_{\lambda,0} + v_{0,\delta})$, Since all $v_{\mu,\nu} \in V$, we see that our claim holds.

Hence, in the notation of (1), V contains $\overline{X}$, whose span is A. □

4. Algebras generated by 2 primitive axes of type (λ, δ) and (λ', δ')

In this section, A is an algebra generated by 2 primitive axes a of type (λ, δ) and b of type (λ', δ'). When a and b have Jordan type, we showed in [5] that $\dim(A) \leq 3$, and, furthermore, we classified the possible algebras A (see [5, Theorem B]). In [5, Theorem A] we classified all algebras A with $\dim(A) = 2$. Here we continue the classification of the possible algebras A. Throughout this section we let

$$\sigma := ab - \delta' a - \lambda b \qquad \text{and} \qquad \sigma' := ba - \lambda' a - \delta b.$$

We write

$$b = \alpha_b a + b_{0,0} + b_{0,\delta} + b_{\lambda,0} + b_{\lambda,\delta} \quad \text{and} \quad a = \alpha_a b + a_{0,0} + a_{0,\delta'} + a_{\lambda',0} + a_{\lambda',\delta'}$$

For the decomposition of b (respectively a) with respect to a (resp. b), as in Equation (1.2).

Lemma 4.1 1. *We have*

$$\sigma = \gamma a - \lambda(b_{0,\delta} + b_{0,0}) = \gamma' b - \delta'(a_{0,0} + a_{\lambda',0}).$$

where $\gamma = \alpha_b(1-\lambda) - \delta'$, and $\gamma' = \alpha_a(1-\delta') - \lambda$. So

$$\sigma \in (Fa + A_0(L_a)) \cap (Fb + A_0(R_b)),$$

2. *$a\sigma = \gamma a$ and $\sigma b = \gamma' b$.*
3. *$\sigma' a \in Fa$ and $b\sigma' \in Fb$.*

Proof (1) We compute that

$$\begin{aligned}\sigma =& (\alpha_b a + \lambda(b_{\lambda,0} + b_{\lambda,\delta})) - \delta' a - \lambda\alpha_b a - \lambda(b_{0,0} + b_{0,\delta}) - \lambda(b_{\lambda,0} + b_{\lambda,\delta}) \\ &= (\alpha_b - \lambda\alpha_b - \delta')a - \lambda(b_{0,0} + b_{0,\delta}).\end{aligned}$$

and

$$\sigma = \alpha_a b + \delta'(a_{0,\delta'} + a_{\lambda',\delta'}) - \delta'\big(\alpha_a b + (a_{0,0} + a_{\lambda',0}) + (a_{0,\delta'} + a_{\lambda',\delta'})\big) - \lambda b$$
$$= (\alpha_a - \delta'\alpha_a - \lambda)b - \delta'(a_{0,0} + a_{\lambda',0}).$$

(2) This is obvious.

(3) Reverse a and b in (2). □

Proposition 4.2

1. *$a(ax) \in Fa + Fax$, for all $x \in A$.*
2. *$(xb)b \in Fb + Fxb$, for all $x \in A$.*
3. *$(ab)(ab) - a(bab) \in V := Fa + Fb + Fab + Fba$.*
4. *$a(bab), (bab)a, a(bab)a \in V' := Fa + Fb + Fab + Fba + Faba$.*
5. *Let V' be as in (4). Then $aV' \subseteq V' \supseteq V'a$.*
6. *$bab \in V'$, where V' is as in (4).*

Proof (1)&(2) Are special cases of Lemma 2.9.

(3) Write $x \cong y$ if $x - y \in V$. We claim that

$$a(b\sigma) \cong a(bab) - \delta' aba \tag{4.1}$$
$$(ab)\sigma \cong (ab)(ab) - \delta' aba \tag{4.2}$$

For (4.1), we have $b\sigma = b(ab - \delta' a - \lambda b)$, so

$$a(b\sigma) = a(bab) - \delta' aba - \lambda ab \cong a(bab) - \delta' aba.$$

For (4.2) we have

$$(ab)\sigma = (ab)(ab - \delta' a - \lambda b) = (ab)(ab) - \delta' aba - \lambda(ab)b \cong (ab)(ab) - \delta' aba.$$

Now $\sigma \in Fa + A_0(L_a)$, so

$$\begin{aligned} a(bab) &\cong a(b\sigma) + \delta' aba \quad &&\text{(by Equation (4.1))} \\ &\cong (ab)\sigma + \delta' aba &&\text{(by Seress' Lemma)} \\ &\cong (ab)(ab) &&\text{(by Equation (4.2))}. \end{aligned}$$

(4) By (3) and (1),

$$a\big((ab)(ab) - a(bab)\big) \in V'.$$

By Lemma 2.9(3), $a(a(bab)) - \lambda a(bab) \in V'$ implying

$$a((ab)(ab)) - \lambda a(bab) \in V'.$$

Now $aba = \alpha_b a + \lambda\delta b_{\lambda,\delta}$, so $b_{\lambda,\delta} \in V'$. Hence $b_{\lambda,0} \in V'$, because $ab = \alpha_b a + \lambda b_{\lambda,0} + \lambda b_{\lambda,\delta} \in V'$. By the fusion rules, $a((ab)(ab))$ is an F-linear combination of $a, b_{\lambda,0}$ and $b_{\lambda,\delta}$. Hence $a((ab)(ab)) \in V'$, so $a(bab) \in V'$.

By symmetry, $(bab)a \in V'$, and then clearly $a(bab)a \in V'$.

(5) This follows easily from (1), (2) and (4).

(6) By (4) and Lemma 2.9(5), $(bab)_{\lambda,\delta} \in V'$. By (4) and Lemma 2.9(2) and its right counterpart,

$$(bab)_{\lambda,0}, (bab)_{0,\delta} \in V'.$$

We have

$$\begin{aligned} bab &= (\alpha_b a + b_0 + b_\lambda)(\alpha_b a + \lambda b_\lambda) \\ &= \alpha_b^2 a + \alpha_b \lambda^2 b_\lambda + \alpha_b b_0 a + \lambda b_0 b_\lambda + \alpha_b b_\lambda a + \lambda b_\lambda^2 \\ &= \overbrace{\alpha_b^2 a + \lambda b_\lambda^2} + \overbrace{\alpha_b \lambda^2 b_\lambda + \alpha_b b_0 a + \lambda b_0 b_\lambda + \alpha_b b_\lambda a}^{\in A_{0,\delta} + A_{\lambda,0} + A_{\lambda,\delta}}. \end{aligned}$$

Hence $(bab)_{0,0} = \lambda(b_\lambda^2)_{0,0}$. Now By (3) and (4), $(ab)^2 \in V'$, so $b_\lambda^2 \in V'$, by Lemma 2.9(2) (with b in place of y). Denoting $x := b_\lambda^2$, we can write $x = \alpha_x a + x_{0,0} + x_{0,\delta}$. Since $V'a \subseteq V'$, we get $xa \in V'$, so $\alpha_x a + \delta x_{0,\delta} \in V'$. It follows that $x_{0,\delta} \in V'$, and consequently $(bab)_{0,0} = \lambda x_{0,0} \in V'$. Hence we see that $(bab)_{\mu,\nu} \in V'$, for all $\mu, \nu \in \{0, \lambda, \delta\}$, so $bab \in V'$. □

Theorem 4.3 *$A = Fa + Fb + Fab + Fba + Faba$, and thus has dimension ≤ 5.*

Proof Let

$$V' := Fa + Fb + Fab + Fba + Faba.$$

By Proposition 4.2, $V' = Fa + Fb + Fab + Fba + Faba + Fbab$. By Theorem 3.3 it suffices to show that $aV', bV', V'a, V'b \subseteq V'$. By symmetry (with respect both to a, b and to working on the left or on the right), it is enough to show that $aV' \subseteq V'$, but this is immediate from Proposition 4.2(1). □

Lemma 4.4 *Let $V = Fa + Fb + Fab$. If $ba \in V$, then $V = A$.*

Proof By Proposition 4.2(1), V is closed under L_a. In particular $aba \in V$, so V is closed under R_a. Similarly V is closed under R_b by Proposition 4.2(2), so $bab \in V$, and then V is closed under L_b. The Lemma follows from Theorem 3.3(2). □

Next we note that

Remark 4.5 *For any $x \in A$, Proposition 4.2((1)&(2)) shows that $L_a^2(x) \in FL_a(x) + Fa$, and $R_a^2(x) \in FR_a(x) + Fa$.*

We recall the 2-sided decomposition $x = \alpha_x a + x_{0,0} + x_{\lambda,0} + x_{0,\delta} + x_{\lambda,\delta}$ of equation (1.2).

Proposition 4.6 *For $x \in A$, define the vector space*

$$V_a(x) = Fa \oplus Fx \oplus Fax \oplus Fxa \oplus Faxa.$$

Then

$$V_a(x) = Fa \oplus Fx_{0,0} \oplus Fx_{\lambda,0} \oplus Fx_{0,\delta} \oplus Fx_{\lambda,\delta}.$$

Proof Applying a special case of Proposition 2.1 both on the left and right,

$$\sum_{i,j=0}^{2} FL_a^i R_a^j x = Fa \oplus Fx_{0,0} \oplus Fx_{\lambda,0} \oplus Fx_{0,\delta} \oplus Fx_{\lambda,\delta}.$$

We conclude with Remark 4.5, which lets us replace $L_a^2(y)$ and $R_a^2(y)$ by $L_a(y)$, $R_a(y)$, and Fa, for any y. □

As a useful corollary we get

Proposition 4.7 $A = Fa + Fb_{0,0} + Fb_{\lambda,0} + Fb_{0,\delta} + Fb_{\lambda,\delta}$.

Proof By Theorem 4.3,

$$A = Fa + Fb + Fab + Fba + Faba,$$

so the proposition follows from Proposition 4.6, because $A = V_a(b)$. □

Lemma 4.8 *If* $ab = 0$*, then* $ba = 0$*, and* $A = Fa + Fb$*.*

Proof Since $ab = 0$, we get that $\alpha_b a + \lambda b_{\lambda,0} + \lambda b_{\lambda,\delta} = 0$. It follows that $\alpha_b = b_{\lambda,0} = b_{\lambda,\delta} = 0$. Hence $b = b_{0,0} + b_{0,\delta}$. By Proposition 4.7, $\dim(A) \leq 3$.

If $b_{0,\delta} = 0$, then $ba = 0$, and we are done. So we may assume that $b_{0,\delta} \neq 0$. Now $0 = b(ab) = (ba)b = \delta b_{0,\delta} b$. Hence $b_{0,\delta} b = 0$. Thus $A_0(R_b) = Fa + Fb_{0,\delta}$ is 2-dimensional. Since $ab = 0$, we see that $a = a_{0,0} + a_{\lambda',0}$. If $a_{\lambda',0} = 0$, then $ba = 0$. If $a_{\lambda',0} \neq 0$, then $A_{\lambda',0}(b) \neq 0$, and we get that $\dim(A) \geq 4$, a contradiction. □

Lemma 4.9 *If* $ab \in Fa + Fb$*, then* $A = Fa + Fb$ *has dimension* 2*.*

Proof Write $ab = \alpha a + \beta b$. If $\beta = 0$, then, by Lemma 2.10, $ab = 0$, so we are done by Lemma 4.8. Hence $\beta \neq 0$. We then see that

$$\alpha_b a + \lambda(b_{\lambda,0} + b_{\lambda,\delta}) = ab = \alpha a + \beta(\alpha_a b + b_{0,0} + b_{0,\delta} + b_{\lambda,0} + b_{\lambda,\delta}),$$

so $b_{0,0} = 0 = b_{0,\delta}$. Thus $b = \alpha_b a + b_{\lambda,0} + b_{\lambda,\delta} = \alpha_b a + b_\lambda$. By Proposition 4.7, $\dim(A) = 3$, and A is spanned by $a, b_{\lambda,0}, b_{\lambda,\delta}$. In particular, $A_0(L_a) = 0$. Note that $b_\lambda^2 \in Fa + A_0(L_a)$, so $b_\lambda^2 \in Fa$. Now $ab = \alpha_b a + \lambda b_\lambda$, hence $b_\lambda \in Fa + Fb$. Write $b_\lambda = \alpha_1 a + \beta_1 b$, then, as above, using Lemma 2.10, $\alpha_1 \neq 0 \neq \beta_1$. But now we see that

$$Fa \ni b_\lambda^2 = \alpha_1^2 a + \beta_1^2 b + \alpha_1\beta_1 ab + \alpha_1\beta_1 ba.$$

This shows that $ba \in Fa + Fb$, so $A = Fa + Fb$. □

Theorem 4.10 *Assume* $\dim(A) = 3$*. Then* a *and* b *are of Jordan type.*

Proof By Lemma 4.9, a, b, and ab are independent; hence $\sigma \notin Fa + Fb$. But $\sigma = \gamma a - \lambda(b_{0,\delta} + b_{0,0})$, so at least one of $b_{0,0}, b_{0,\delta}$ is non-zero. Similarly, by Lemma 4.9, a, b, ba are independent, so $\sigma' \notin Fa + Fb$, and at least one of $b_{0,0}, b_{\lambda,0}$ is non-zero.

Assume first that $b_{0,0} = 0$. Then $b_{\lambda,0} \neq 0 \neq b_{0,\delta}$. Since $\dim(A) = 3$, Proposition 4.7 implies that $b_{\lambda,\delta} = 0$. Thus $b = \alpha_b a + b_{\lambda,0} + b_{0,\delta}$. Suppose $\alpha_b = 0$. Then

$$b = b^2 = b_{\lambda,0}^2 + b_{0,\delta}^2 + b_{\lambda,0}b_{0,\delta} + b_{0,\delta}b_{\lambda,0}.$$

Since $b_{\lambda,0}b_{0,\delta}+b_{0,\delta}b_{\lambda,0}\in A_{\lambda,\delta}=0$, and $b_{\lambda,0}^2+b_{0,\delta}^2\in Fa+A_{0,0}$, we see that $b\in Fa+A_{0,0}$, so $ab\in Fa$. By Lemma 2.10, $ab=0$, a contradiction.

Hence $\alpha_b\neq 0$. But then

$$(ba)b=(\alpha_b a+\delta b_{0,\delta})(\alpha_b a+b_{\lambda,0}+b_{0,\delta})=\alpha_b^2 a+\alpha_b\lambda b_{\lambda,0}+\alpha_b\delta^2 b_{0,\delta}+\delta b_{0,\delta}b_{\lambda,0}+\delta b_{0,\delta}^2$$

and

$$\begin{aligned} b(ab) &= (\alpha_b a+b_{\lambda,0}+b_{0,\delta})(\alpha_b a+\lambda b_{\lambda,0}) \\ &= \alpha_b^2 a+\alpha_b\delta b_{0,\delta}+\alpha_b\lambda^2 b_{\lambda,0}+\lambda b_{\lambda,0}^2+\lambda b_{0,\delta}b_{\lambda,0}. \end{aligned}$$

Since $(ba)b=b(ab)$, this implies $\alpha_b\lambda b_{\lambda,0}=\alpha_b\lambda^2 b_{\lambda,0}$, so $b_{\lambda,0}=0$, a contradiction.

Hence $b_{0,0}\neq 0$. Suppose $b_{\lambda,0}\neq 0$. Then $b=\alpha_b a+b_{0,0}+b_{\lambda,0}$. But then $ba=\alpha_b a$, and we saw that this implies $ba=0$, a contradiction. Thus $b_{\lambda,0}=0$. Symmetrically $b_{0,\delta}=0$, so a is of Jordan type. By symmetry so is b. □

5. A few observations and some questions.

In this section A is generated by a set of axes X, where each $x\in X$ is primitive of type (λ_x,δ_x). Recall the notation $\overline{X}$ from 3.1.

Lemma 5.1 *Let $a\in X$.*

1. *For any $z\in A$, there is a finite subset $Y\subseteq\overline{X}$ such that $z_{0,0}\in\sum_{x\in Y}Fx_{0,0}$. Here $z_{0,0},x_{0,0}\in A_{0,0}(a)$.*
2. *To determine whether $(A_{0,0}(a))^2\subseteq A_{0,0}(a)$, it is enough to check that $(B_{0,0}(a))^2\subseteq B_{0,0}(a)$, for any subalgebra B of A generated by a,x,x', with $x,x'\in\overline{X}$.*

Proof (1) This is obvious. Just write $z=\sum_{x\in Y}\gamma_x x$, for some finite subset $Y\subseteq X$. Then $z_{0,0}=\sum_{x\in Y}\gamma_x x_{0,0}$.
(2) Let $y',y\in A_{0,0}$. Using (1) write

$$y=\sum_{x\in Y}\gamma_x x_{0,0},\qquad y'=\sum_{x'\in Y'}\gamma'_{x'}x'_{0,0},$$

with Y,Y', finite subsets of X. Then

$$yy'\in\textstyle\sum_{x\in Y,x'\in Y'}Fx_{0,0}x'_{0,0}.$$

The assertion of (2) is now obvious. □

Lemma 5.2 *Suppose that whenever $B\subseteq A$ is generated by 2 primitive axes of type (λ,δ) and (λ',δ'), then* $\dim(B)\leq 3$. *Then all axes in A are of Jordan type.*

Proof This follows immediately from Theorem 4.10 and Theorem [5, Theorem A]. □

Some questions

1. Suppose A is generated by axes a and b of respective types (λ, δ) and (λ', δ'). Is it true that $\dim(A) \leq 3$? (see Lemma 5.2). We have recently shown that $\dim(A) \neq 4$, so the only remaining case is whether one can have $\dim(A) = 5$. Added in proof: The authors answered this question affirmatively.

2. Suppose that each axis $x \in X$ is of (Jordan) type (λ_x, δ_x). Is it true that $(A_{0,0}(x))^2 \subseteq A_{0,0}(x)$? (By Lemma 5.1 it is enough to consider triples of axes).

3. Suppose $|X| = 3$. Does it follow that $\dim(A)$ is finite?

References

[1] Albert AA, *Power-associative rings.* Transactions of the Amererican Mathematical Society 1948; 64: 552–593.

[2] De Medts T, Peacock SF, Shpectorov S, and Van Couwenberghe M, *Decomposition algebras and axial algebras.* Journal of Algebra 2020; 556: 287–314.

[3] Hall JI, Rehren F, Shpectorov S, *Primitive axial algebras of Jordan type,* Journal of Algebra 2015; 437: 79–11.

[4] Hall JI, Segev Y, Shpectorov S, *Miyamoto involutions in axial algebras of Jordan type half*, Israel Journal of Mathematics 2018; 223: 261–308.

[5] Rowen L, Segev Y, *Axes of Jordan type in non-commutative algebras*, submitted. J. Algebra Appl.

[6] Schafer RD, *non-commutative Jordan algebras of characteristic 0*, Proceedings of the American Mathematical Society 1955; 6: 472–475.

[7] Shestakov, I, Trushina, M, *Irreducible bimodules over alternative algebras and superalgebras.* Trans. Amer. Math. Soc. 2016; 368: 4657–4684.