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On fractional p-Laplacian type equations with general nonlinearities

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Abstract: In this paper, we study the existence and multiplicity of solutions for a class of quasi-linear elliptic problems driven by a nonlocal integro-differential operator with homogeneous Dirichlet boundary conditions. As a particular case, we study the following problem:

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian operator, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a generic Carathéodory function satisfying either a $p-$sublinear or a $p-$superlinear growth condition.

Key words: Fractional p-Laplacian problem, variational method, fractional Sobolev space, Palais–Smale condition

1. Introduction

In recent decades, fractional and nonlocal problems have attracted considerable attention. The basic operator involved in this kind of problems is the so-called fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$. This operator and its generalization appear in many areas of mathematics, such as harmonic analysis, probability theory, potential theory, quantum mechanics, statistical physics, and cosmology, as well as in many applications, see, for instance [3, 6, 19] and the references therein.

Lately, many works are devoted to the study of existence, nonexistence and regularity of solutions for nonlocal elliptic equations; we refer the interested reader to [10, 11, 13, 16, 20, 22]-[26] and the references therein.

In this paper we are concerned with the existence and multiplicity of solutions for the following problem:

$$\begin{cases} \mathcal{L}_K^p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\mathcal{L}_K^p$ is the non local operator defined as follows:

$$\mathcal{L}_K^p \varphi(x) = \lim_{\epsilon \to 0^+} 2 \int_{\mathbb{R}^N \setminus B_\epsilon(x)} |\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))K(x-y)dy, \ x \in \mathbb{R}^N,$$

along any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $B_\epsilon(x)$ denotes the ball in $\mathbb{R}^N$ of radius $\epsilon > 0$ at the center $x \in \mathbb{R}^N$ and

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$K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}+$ is a measurable function with the following property:

$$\begin{cases} mK \in L^1(\mathbb{R}^N), \ \textit{where}\ m = \min\{1, |x|^p\}, \\ \exists\ s \in (0,1), K_0 > 0 : K(x) \geq K_0|x|^{-(N+sp)}, \text{ a.e. in } \mathbb{R}^N \setminus \{0\}, \\ K(-x) = K(x), \text{ a.e. in } \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{1.2}$$

Throughout the paper, we assume that $s \in (0,1), N > sp,\ p \in (1,+\infty), r \in (p, p_s^\star)$ where $p_s^* = \frac{Np}{N-sp}$ and the condition (1.2) is fulfilled.

A typical example for K is given by the singular kernel $K(x) = |x|^{-(N+ps)}$. In this case, the operator $\mathcal{L}_K^p$ becomes the fractional p-Laplacian denoted by $(-\Delta)_p^s$ and problem (1.1) is equivalent to

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

When $p = 2$, the operator $(-\Delta)_p^s$ reduces to the usual linear fractional Laplacian operator $(-\Delta)^s$.

Recently, Servadei and Valdinoci [16] considered the problem (1.3). Under suitable conditions on the growth of the nonlinearity f, mainly the Ambrosetti–Rabinowitz growth condition, they got the existence of nontrivial weak solutions for problem (1.3). The same conclusion is obtained by Ambrosio in [2] by relaxing the Ambrosetti–Rabinowitz condition.

Later, Iannizzotto et al. [11] examined the following problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

By means of Morse theory and the spectral properties of the operator $(-\Delta)_p^s$, they proved the existence of a nonzero solution for (1.4) for all $\lambda \in \mathbb{R}$. They treated, respectively, the cases where f is p-superlinear, $p-$sublinear or asymptotically p-linear. Using the same tools, Ho et al. extended the above results in [12] to potentials of the form $f(x,u) = \lambda h(x)|u|^{p-2}u + k(x)|u|^{r-2}u + g(x,u)$ where h and k are two measurable functions belong to a class of singular weights (for more details see [12]).

Newly, in [20, 22]-[24], the authors studied the existence and multiplicity of solutions for fractional Kirchhoff problems involving different kinds of nonlinearities: logarithmic, superquadratic or critical. Their study is mainly based on the use of the Nehari manifold approach and variational arguments.

In the present paper, motivated by the above papers, we study the existence of weak solutions for the problem (1.1) involving more general nonlocal fractional operators with various classes of nonlinearities including those of the form $f(x,u) = \lambda V(x)|u|^{p-2}u + \mu g(x)|u|^{r-2}u + g(x,u)$ via critical point theory.

This paper is organized as follows: In Section 2, we present the variational framework of problem (1.1) and some preliminary results. In Section 3, we treat the case of $p-$sublinear nonlinearities. Section 4 is devoted to study the case of $p-$superlinear and subcritical nonlinearities.

2. Functional framework

Throughout the paper we assume that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $p \in (1, +\infty)$ and $s \in (0,1)$. We consider the space

$$X = \{v \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} |v(x) - v(y)|^p K(x-y) dx\, dy < \infty\},$$

endowed with the norm

$$\|v\|_X = |v|_{L^p(\mathbb{R}^N)} + [v]_{s,p,K},$$

where

$$[v]_{s,p,K} = \left(\iint_{\mathbb{R}^{2N}} |v(x) - v(y)|^p K(x-y) dx\, dy \right)^{\frac{1}{p}}.$$

Lemma 2.1 *([24])* $(X, \|.\|_X)$ *is a separable and reflexive Banach space.*

In the following, we denote $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$, where $\mathcal{O} = \mathcal{C}(\Omega) \times \mathcal{C}(\Omega) \subset \mathbb{R}^{2N}$, and $\mathcal{C}(\Omega) = \mathbb{R}^N \setminus \Omega$. We shall work in the closed linear subspace

$$X_0 = \big\{ u \in X : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \big\},$$

endowed with the norm

$$\|u\| := [u]_{s,p,K} = \left(\iint_Q \big|u(x) - u(y)\big|^p K(x-y)\, dx\, dy \right)^{1/p}.$$

Then $(X_0, \|.\|)$ is a separable and reflexive Banach space, (see [24]). Moreover, $C_0^\infty(\Omega)$ is dense in X_0, (see [9]).

Lemma 2.2 *([8]) Let* $K : R^N \setminus \{0\} \to \mathbb{R}+$ *satisfies* (1.2) *and let* (v_i) *be a bounded sequence in* X_0. *Then there exists* $v \in L^\nu(\mathbb{R}^N)$ *with* $v = 0$ *a.e. in* $\mathbb{R}^N \setminus \Omega$ *such that, up to subsequence,*

$$v_i \to v \quad \textit{strongly in } L^\nu(\mathbb{R}^N), \forall\, \nu \in [1, p_s^\star).$$

Furthermore, there is a constant $C_\nu > 0$ *such that*

$$|v|_{L^\nu(\Omega)} \leq C_\nu [v]_{s,p,K}, \forall v \in X_0.$$

Lemma 2.3 *Under the assumption* (1.2)*, we have*

(a) $\mathcal{L}_K^p : X_0 \to X_0'$ *is a continuous, bounded and strictly monotone operator.*

(b) $\mathcal{L}_K^p$ *is a mapping of* (S_+)*, i.e. if* $v_n \rightharpoonup v$ *and* $\mathcal{L}_K^p(v_n).(v_n - v) \to 0$*, then*

$$v_n \to v \quad \textit{strongly in } X_0.$$

Proof Let us prove part (a), by the Hölder inequality, for all $v, w \in X_0$ we have:

$$\begin{aligned} |\mathcal{L}_K^p(v).w| &\le \iint_Q |v(x)-v(y)|^{p-1}|w(x)-w(y)||K(x-y)|^{\frac{p-1}{p}}|K(x-y)|^{\frac{p-1}{p}}dx\ dy,\\ &\le \left(\iint_Q |v(x)-v(y)|^p K(x-y)dx\ dy\right)^{\frac{p-1}{p}}\left(\iint_Q |w(x)-w(y)|^p K(x-y)dx\ dy\right)^{\frac{1}{p}},\\ &\le [v]_{s,p,K}^{p-1}[w]_{s,p,K}. \end{aligned}$$

Hence $\|\mathcal{L}_K^p(v)\|_\star \le [v]_{s,p,K}^{p-1}$.
Let us now recall the well-known Simon inequality (see [24]): For all $\xi, \varsigma \in \mathbb{R}$, there exists $\kappa_p > 0$ such that

$$\kappa_p(|\xi|^{p-2}\xi - |\varsigma|^{p-2}\varsigma)(\xi-\varsigma) \ge \begin{cases} |\xi-\varsigma|^p & \text{if } \ p\ge 2\\ |\xi-\varsigma|^2(|\xi|+|\varsigma|)^{p-2} & \text{if } \ 1<p<2. \end{cases} \tag{2.1}$$

By inequality (2.1), it is easy to see that the operator $\mathcal{L}_K^p$ is strictly monotone.
For part (b), we know that if $v_n \rightharpoonup v$ then

$$\lim_{n\to+\infty}(\mathcal{L}_K^p(v_n)-\mathcal{L}_K^p(v), v_n - v) = \lim_{n\to+\infty}(\mathcal{L}_K^p(v_n), v_n - v) - (\mathcal{L}_K^p(v), v_n - v) = 0.$$

Using (2.1), we obtain, for $p \ge 2$,

$$[v_n - v]_{s,p,K}^p \le \kappa_p(\mathcal{L}_K^p(v_n) - \mathcal{L}_K^p(v), v_n - v) = o(1).$$

For $1 < p < 2$, let $w_n(x,y) = v_n(x) - v_n(y), w(x,y) = v(x) - v(y)$, so, we get

$$\begin{aligned} [v_n - v]_{s,p,K}^p &= \iint_Q |w_n(x,y) - w(x,y)|^p K(x-y)\ dx\ dy,\\ &\le \kappa_p^{\frac{p}{2}} \iint_Q \left((|w_n|^{p-2}w_n - |w|^{p-2}w)(w_n - w)\right)^{\frac{p}{2}}\left(|w_n|+|w|^{\frac{(2-p)p}{2}}\right)K(x-y)dxdy,\\ &\le \kappa_p^{\frac{p}{2}}\left(\iint_Q (|w_n|^{p-2}w_n - |w|^{p-2}w)(w_n-w)K(x-y)\ dx\ dy\right)^{\frac{p}{2}},\\ &\times \left(\int_Q (|w_n|+|w|)^p K(x-y)\ dx\ dy\right)^{\frac{2}{2-p}},\\ &\le \kappa_p^{\frac{p}{2}}(\mathcal{L}_K^p(v_n) - \mathcal{L}_K^p(v), v_n - v)^{\frac{p}{2}} \times \left(\iint_Q (|w_n|+|w|)^p K(x-y)\ dx\ dy\right)^{\frac{2}{2-p}} = o(1). \end{aligned}$$

Then $v_n \to v$ in X_0 as $n \to +\infty$, i.e. $\mathcal{L}_K^p$ is of type (S_+). □

Since X_0 is separable, there exist $(e_n) \subset X_0$ and $(f_n) \subset X_0'$ such that (see [28], Section 17)

$$X_0 = \overline{\text{span}\{e_n\}_{n\ge1}},\ X_0' = \overline{\text{span}\{f_n\}_{n\ge1}},\ < f_i, e_j >= \begin{cases} 1 & \text{if } i = j,\\ 0 & \text{if } i \ne j. \end{cases}$$

Let

$$X_n = \text{span}\{e_n\},\ Y_n = \bigoplus_{k=1}^{n} X_k,\ Z_n = \overline{\bigoplus_{k=n}^{+\infty} X_k}. \tag{2.2}$$

Lemma 2.4 *Let $\delta > 0$ and for any $n \in \mathbb{N}$,*

$$\beta_n = \sup\{|u|_{L^q(\Omega)},\ u \in Z_n\ :\ \|u\| \leq \delta\}.$$

Then $\beta_n \to 0$ as $n \to +\infty$.

Proof By definition of Z_n, we have $Z_{n+1} \subset Z_n$, and consequently $0 < \beta_{n+1} \leq \beta_n$ for any $n \in \mathbb{N}$. Hence there exists $\beta \geq 0$ such that

$$\beta_n \to \beta,\ \text{as } n \longrightarrow +\infty.$$

Moreover, by the definition of β_n, for any $n \in \mathbb{N}$ there exists $u_n \in Z_n$ such that

$$\|u_n\| \leq \delta \textit{ and } |u_n|_{L^q(\Omega)} > \frac{\beta_n}{2}. \tag{2.3}$$

Since X_0 is a reflexive space, there exist $u_\infty \in X_0$ and a subsequence of (u_n) (still denoted by u_n) such that $u_n \rightharpoonup u_\infty$. Since each of Z_n is convex and closed, hence it is closed for the weak topology. Consequently,

$$u_\infty \in \bigcap_{n=1}^{+\infty} Z_n = \{0\}.$$

By the Sobolev embedding theorem, we get

$$u_n \to 0 \textit{ in } L^q(\Omega). \tag{2.4}$$

Since β is nonnegative, from (2.3) and (2.4) we get that $\beta_n \to 0$ as $n \to +\infty$ which completes the proof of Lemma 2.4. □

Next, we need the following results in critical point theory.

Theorem 2.5 *([7]) Let $(X, \|.\|)$ be a reflexive Banach space and $\mathcal{J} : X \longrightarrow \mathbb{R}$ be a functional which is weakly lower semicontinuous and coercive, namely*

$$\mathcal{J}(u) \to +\infty \textit{ as } \|u\| \to +\infty.$$

Then $\mathcal{J}$ is bounded from below and attains its minimum.

Theorem 2.6 *(Mountain Pass Theorem [1])*
Let $(X, \|.\|)$ be a real Banach space and $\mathcal{J} \in C^1(X, \mathbb{R})$. Suppose that $\mathcal{J}$ satisfies the Palais–Smale condition and

(a) $\mathcal{J}(0) = 0$.

(b) $\exists\ \rho, \gamma > 0 : \mathcal{J}(u) \geq \gamma, \forall u \in X$ *with* $\|u\| = \rho$.

(c) $\exists\ e \in X, R > 0 : \|e\| = R$ *and* $\mathcal{J}(e) < 0$.

Then $\mathcal{J}$ *possesses a critical value* $c > \gamma$ *which can be characterized as*

$$c = \inf_{g \in \Gamma} \max_{t \in [0,1]} \mathcal{J}(g(t)),$$

where

$$\Gamma = \{g \in ([0,1], X) : g(0) = 0, g(1) = e\}.$$

Theorem 2.7 *(Fountain Theorem [21])*
Let $(X, \|.\|)$ *be a reflexive and separable Banach space,* $\mathcal{J} \in C^1(X, \mathbb{R})$ *be an even functional and the subspaces* X_n, Y_n, Z_n *as defined in* (2.2).
If for each $n \in \mathbb{N}$, *there exists* $\rho_n > r_n > 0$ *such that*

(a) $\inf_{u \in Z_n, \|u\| = r_n} \mathcal{J}(u) \to +\infty$ *as* $n \to +\infty$,

(b) $\max_{u \in Y_n, \|u\| = \rho_n} \mathcal{J}(u) \leq 0$,

(c) $\mathcal{J}$ *satisfies the Palais–Smale condition at any level* $c > 0$.

Then $\mathcal{J}$ *has a sequence of critical values tending to* $+\infty$.

Let $(X, \|.\|)$ be a real Banach space and let $\Sigma(X)$ denote the class of closed subsets of $X \backslash \{0\}$ symmetric with respect to the origin. A nonempty set $A \in \Sigma$ is said to have genus k (denoted $\gamma(A) = k$) if k is the smallest integer with the property that there exists an odd continuous mapping $h : A \longrightarrow \mathbb{R}^k \setminus \{0\}$. If such an integer does not exist, $\gamma(A) = +\infty$. For properties and more details of the notion of genus we refer the reader to [1].

Theorem 2.8 *([1]) Let* $(X, \|.\|)$ *be a Banach space and* $\mathcal{J} \in C^1(X, \mathbb{R})$ *be an even functional satisfying the Palais–Smale condition and* $\mathcal{J}(0) = 0$. *Set, for all* $n \in \mathbb{N}$,

$$\begin{aligned} \Sigma_n(X) &= \{A \in \Sigma(X) : \gamma(A) \geq n\}, \\ K(c) &= \{u \in X : \mathcal{J}(u) = c \text{ and } \mathcal{J}'(u) = 0\}, \\ c_n &= \inf_{A \in \Sigma_n(X)} \sup_{u \in A} \mathcal{J}(u),\ n \in \mathbb{N}. \end{aligned}$$

If for all $n \in \mathbb{N}, -\infty < c_n < 0$ *then each* c_n *is a critical value of* $\mathcal{J}$, $c_n \leq c_{n+1} < 0$ *for all* $n \in \mathbb{N}$. *Moreover, if there exists* $k \in \mathbb{N}$ *such that* $c_n = c_{n+1} = ... = c_{n+k}$, *then* $\gamma(K(c_n)) \geq n + 1$.

3. p-sublinear case

In this section, we investigate the existence of solutions for the following problem

$$\begin{cases} \mathcal{L}^p_K u(x) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.1}$$

Suppose that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying:
(H_1) there exist $q\in(1,p), b>0, a\in L^{q'}(\Omega), a\geq 0$ *a.e. in* Ω such that

$$|f(x,t)|\leq a(x)+b|t|^{q-1}, \text{for all } t\in\mathbb{R}, \quad \text{and a.e.} \quad x\in\Omega,$$

where $q'\in(1,+\infty)$ such that $\frac{1}{q}+\frac{1}{q'}=1$.

In [15], H. Qiu and M. Xiang used Leray-Schauder's nonlinear alternative to prove the existence of a weak solution for (3.1) under the $p-$sublinear condition (H_1). The problem is also studied in [11] via Morse theory. Here, we treat the problem (3.1) using variational techniques and a minimisation argument. Moreover, we give a result of multiplicity in case of odd potential.

The corresponding energy functional of the problem (3.1) is

$$I(u)=\frac{1}{p}\|u\|^p-\int_\Omega F(x,u(x))\;dx,$$

where F denotes the primitive function of f with respect to the second variable,i.e., $F(x,t)=\int_0^t f(x,\xi)\;d\xi$. We notice that, under the condition (H_1), I is a C^1 functional and the critical points of I are weak solutions of the problem (3.1). Moreover,

Lemma 3.1 *Under the condition* (H_1), *the functional* I *satisfies the Palais–Smale condition.*

Proof Let the sequence $(u_n)\subset X_0$ be such that

$$(I(u_n)) \text{ is bounded}, I'(u_n)\to 0, \text{ as } n\longrightarrow+\infty.$$

By (H_1) we have

$$\begin{aligned} I(u_n) &= \frac{1}{p}\|u_n\|^p-\int_\Omega F(x,u_n(x))dx,\\ &\geq \frac{1}{p}\|u_n\|^p-\frac{b\;C_q^q}{q}\|u_n\|^q-C_q|a|_{L^{q'}(\Omega)}\|u_n\|. \end{aligned}$$

From $p>q>1$ and the last inequality we deduce that the sequence (u_n) is bounded in X_0. Since X_0 is a reflexive space, up to a subsequence, still denoted by (u_n), there exists $u\in X_0$ such that $u_n\rightharpoonup u$ weakly in X_0.
Firstly, we have

$$I'(u_n).(u_n-u)=\mathcal{L}_K^p(u_n).(u_n-u)-\int_\Omega f(x,u_n)(u_n-u)dx.$$

By Lemma 2.2, $u_n \to u$ strongly in $L^q(\Omega)$. Moreover,

$$\begin{aligned} |\int_\Omega f(x,u_n(x))(u_n(x)-u(x))dx| &\le \int_\Omega |f(x,u_n(x))||u_n(x)-u(x)|dx, \\ &\le \int_\Omega |a(x)||u_n(x)-u(x)|dx \\ &+ b\int_\Omega |u_n(x)|^{q-1}|u_n(x)-u(x)|dx, \\ &\le |a|_{L^{q'}(\Omega)}|u_n-u|_{L^q(\Omega)} + b|u_n|^{q-1}_{L^q(\Omega)}|u_n-u|_{L^q(\Omega)}. \end{aligned}$$

Then

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n(x))(u_n(x)-u(x))dx = 0. \tag{3.2}$$

Hence

$$\mathcal{L}_K^p(u_n).(u_n-u) \to 0,$$

and by Lemma 2.3 we have $u_n \to u$ strongly in X_0. □

Theorem 3.2 *Under the condition* (H_1) *and*
(H_2) *there exist* $\delta > 0, d > 0$*, and* $1 < \theta < p$ *such that*

$$f(x,t) \ge dt^{\theta-1} \text{ a.e. } x \in \Omega \text{ and } 0 < t \le \delta,$$

the problem (3.1) *has at least one weak nontrivial solution.*

The proof is based on the direct method of calculus of variation via Theorem 2.5

Proof By (H_1), I is weakly lower semicontinuous on X_0. Moreover, we have

$$I(u) \ge \frac{1}{p}\|u\|^p - \frac{b\,C_q^q}{q}\|u\|^q - C_q|a|_{L^{q'}(\Omega)}\|u\|. \tag{3.3}$$

Since $p > q > 1$, the latter gives the coercivity of I. Hence I attains its minimum on X_0 and provides a weak solution of (3.1) denoted by u_0.
It remains to prove that $u_0 \neq 0$. Let $u \in C_0^\infty(\Omega)$ such that $u \ge 0, u \ne 0$ and $|u|_{L^\infty(\Omega)} \le \delta$. Then, by (H_2), we have for all $t \in (0,1)$,

$$\begin{aligned} I(tu) &= \frac{1}{p}\|tu\|^p - \int_\Omega F(x,tu(x))dx, \\ &\le \frac{t^p}{p}\|u\|^p - \frac{d}{\theta}\,t^\theta|u|^\theta_{L^\theta(\Omega)}, \\ &\le t^p\left(\frac{\|u\|^p}{p} - \frac{d}{\theta}|u|^\theta_{L^\theta(\Omega)}t^{\theta-p}\right). \end{aligned}$$

As $1 < \theta < p$, we can find $t \in (0,1)$ small enough such that $I(tu) < 0$. Hence $\inf\limits_{v \in X_0} I(v) = I(u_0) < 0$ and accordingly $u_0 \neq 0$.

□

Now, we treat the question of multiplicity in case where the function f is odd.

Theorem 3.3 *Suppose that f satisfies the conditions $(H_1), (H_2)$ and*
(H_3) $f(x,-t) = -f(x,t)$, *for a.e.* $x \in \Omega$ *and for all* $t \in \mathbb{R}$.
Then problem (3.1) *has a sequence of weak solutions* $(u_n) \subset X_0$ *such that*

$$I(u_n) < 0 \text{ and } I(u_n) \to 0 \text{ as } n \to +\infty.$$

Proof

First of all, by Lemma 3.1, we know that I satisfies the Palais–Smale condition. Moreover, by (H_3), I is an even functional. Now, for any $n \in \mathbb{N}$, we can choose a $n-$dimentional linear subspace $X_n \subset X_0$ such that $X_n \subset C_0^\infty(\Omega)$. As the norms on X_n are equivalent, there exists $\rho_n \in (0,1)$ such that

$$\|u\| \leq \rho_n \Rightarrow |u|_{L^\infty(\Omega)} \leq \delta, u \in X_n.$$

Set

$$S_{\rho_n}^{(n)} = \{u \in X_n : \ \|u\| = \rho_n\}.$$

By (H_2), for $u \in S_{\rho_n}^{(n)}$ and $t \in (0,1)$, we have

$$\begin{aligned} I(tu) &= \frac{1}{p}\|tu\|^p - \int_\Omega F(x, tu(x))dx, \\ &\leq \frac{t^p}{p}\rho_n^p - \frac{d}{\theta}\, t^\theta |u|^\theta_{L^\theta(\Omega)}, \\ &\leq \frac{t^p}{p}\rho_n^p - \tilde{d}_n t^\theta, \\ &\leq \tilde{d}_n t^p \left(\frac{\rho_n^p}{p\tilde{d}_n} - t^{\theta-p}\right). \end{aligned}$$

As $1 < \theta < p$, we can find $t_n \in (0,1)$ and $\epsilon_n > 0$ such that

$$I(t_n u) \leq -\epsilon_n < 0, \forall u \in S_{\rho_n}^{(n)},$$

that is

$$I(u) \leq -\epsilon_n < 0, \forall u \in S_{t_n\rho_n}^{(n)}.$$

We know that $\gamma(S_{t_n\rho_n}^{(n)}) = n$ (see [1]), so $c_n \leq -\epsilon_n < 0$. By Theorem 2.8, each c_n is a critical value of I. Hence there is a sequence of solutions $(u_n) \subset X_0$ of (3.1) such that $I(u_n) = c_n < 0$.
It remains to prove that $c_n \to 0$ as $n \to +\infty$. By the coerciveness of I, there exists a constant $\gamma > 0$ such that $I(u) > 0$ when $\|u\| \geq \gamma$. Taking arbitrary $A \in \Sigma_n$, then $\gamma(A) \geq n$. Let Y_n, Z_n the subspaces of X_0 as mentioned in (2.2), according to the properties of genus (see [1]), we know that $A \cap Z_n \neq \varnothing$. Let

$$\beta_n = \sup\{|u|_{L^q(\Omega)},\ u \in Z_n \ : \ \|u\| \leq \delta\}.$$

By Lemma 2.4 we have $\beta_n \to 0$ as $n \to +\infty$. When $u \in Z_n$ and $\|u\| \leq \delta$, we have

$$\begin{aligned} I(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(x,u)dx \quad &\geq \quad -\int_\Omega F(x,u(x))dx, \\ &\geq \quad -|a|_{L^{q'}(\Omega)}|u|_{L^q(\Omega)} - \frac{b}{q}|u|^q_{L^q(\Omega)}, \\ &\geq \quad -|a|_{L^{q'}(\Omega)}\beta_n - \frac{b}{q}\beta_n^q, \end{aligned}$$

and then $c_n \geq -|a|_{L^{q'}(\Omega)}\beta_n - \frac{b}{q}\beta_n^q$, which implies that $c_n \to 0$ as $n \to +\infty$. □

Remark 3.4 *To the best knowledge of the authors, there are only few works treating problem* (3.1) *with* p-*sublinear nonlinearity (see [10, 11, 15]). In [15], the authors established the existence of a weak solution under the condition* (H_1) *but they are unable to check its nontriviality due to the method used (Leray-Schauder's nonlinear alternative). In [11], the authors proved the existence and multiplicity of solutions for the problem* (3.1) *in the coercive case, including the case when* $f(x,.)$ *is* $p-$*sublinear at infinity. They used Morse theory and suppose, among others assumption,* $F(t,x) \geq 0$ *for a.e. in* Ω *and for all* $t \in \mathbb{R}$, *which is very restrictive compared to our assumptions. Earlier, in the paper [10], by means of critical point theory, the authors treated the case of fractional Laplacian equations for reaction term* f *with sublinear growth and oscillatory behavior. They established the existence and multiplicity of positive, negative and sign changing solutions. In Theorem 3.2 and Theorem 3.3, we generalize and complete the above works by involving more general operators or weakening the assumptions on the reaction term.*

4. p-superlinear case

In this section, we study the existence and multiplicity of solutions for the following problem

$$\begin{cases} \mathcal{L}_K^p u(x) = \lambda V(x)|u|^{p-2}u + \mu g(x)|u|^{r-2}u + f(x,u(x)) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.1}$$

where $V : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are two measurable functions satisfying:

(V) there exists $V_0 > 0$ such that $|V(x)| \leq V_0$ a.e. in Ω,

(G) there exists $\overline{g} > 0$ such that $0 < g(x) \leq \overline{g}$ a.e. in Ω.

In a recent paper the authors of [11] considered the fractional problem

$$\begin{cases} (-\Delta_p)^s u(x) = \lambda|u|^{p-2}u + g(x,u(x)) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.2}$$

where the primitive function of g with respect to the second variable satisfies the usual Ambrosetti–Rabinowitz $p-$superquadratic growth condition and does not change sign near the origin. They provide existence and multiplicity results as well as characterization of critical groups of the variational functional associated to problem (4.2).

More recently, the authors of [12], considered the following problem

$$\begin{cases}(-\Delta_p)^s u(x) = \lambda h(x)|u|^{p-2}u + k(x)|u|^{r-2}u + f(x,u(x)) & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,\end{cases} \tag{4.3}$$

with weights which are possibly singular on the boundary of the domain. Using Morse theory and some Hardy-type inequalities, the authors proved the existence and multiplicity of solutions for (4.3).

Here, motivated by the above works, we consider the more general problem (4.1) with a $p-$sublinear perturbation of a $p-$superlinear and subcritical nonlinearity with bounded weights.

For all $\lambda, \mu > 0$, define $\Phi_{\lambda,\mu} : X_0 \to \mathbb{R}$ by

$$\Phi_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega V(x)|u(x)|^p\ dx - \frac{\mu}{r}\int_\Omega g(x)|u(x)|^r\ dx - \int_\Omega F(x,u)dx.$$

Under the conditions $(H_1), (V)$ and (G), $\Phi_{\lambda,\mu} \in C^1(X_0, \mathbb{R})$ and for all $u, v \in X_0$ we have

$$\Phi'_{\lambda,\mu}(u).v = \mathcal{L}^p_K u.v - \lambda\int_\Omega V(x)|u|^{p-2}u(x)v(x)\ dx - \mu\int_\Omega g(x)|u|^{r-2}u(x)v(x)\ dx - \int_\Omega f(x,u(x))v(x)dx.$$

Denote by

$$\lambda_1 = \inf_{u\in X_0}\frac{\|u\|^p}{|u|^p_{L^p(\Omega)}} \geq \frac{1}{C_p^p}.$$

Lemma 4.1 *Under the conditions $(H_1), (V)$ and (G), for all $\lambda \in (0, \frac{\lambda_1}{V_0})$, $\Phi_{\lambda,\mu}$ satisfies the Palais–Smale condition.*

Proof Let $(u_n) \subset X_0$ such that $(\Phi_{\lambda,\mu}(u_n))$ is bounded and *and* $\Phi'_\lambda(u_n) \to 0$. By (H_1) and (V), we get

$$\begin{aligned} r\Phi_{\lambda,\mu}(u_n) - \Phi'_{\lambda,\mu}(u_n).u_n \quad &= \quad (\frac{r}{p}-1)\|u_n\|^p - (\frac{r}{p}-1)\lambda\int_\Omega V(x)|u_n(x)|^p dx \\ &+ \quad \int_\Omega (f(x,u_n)u_n - rF(x,u_n(x))\ dx, \\ &\geq \quad (\frac{r}{p}-1)\left(\|u_n\|^p - V_0\lambda|u_n|^p_{L^p(\Omega)}\right) - (r+1)\left(|a|_{L^{q'}(\Omega)}C_q\|u_n\|\right) \\ &- \quad (r+1)bC_q^q\|u_n\|^q, \\ &\geq \quad (\frac{r}{p}-1)\left(1 - \frac{V_0\lambda}{\lambda_1}\right)\|u_n\|^p - (r+1)\left(|a|_{L^{q'}(\Omega)}C_q\|u_n\|\right) \\ &- \quad (r+1)bC_q^q\|u_n\|^q. \end{aligned}$$

Now $\lambda < \frac{\lambda_1}{V_0}$ and $r > p > q > 1$ give that (u_n) is bounded in X_0. Since X_0 is reflexive space, up to a subsequence, still denoted by (u_n), there exists $u \in X_0$ such that $u_n \rightharpoonup u$ weakly in X_0. By Lemma 2.2,

$u_n \to u$ strongly in $L^\alpha(\Omega)$ for all $\alpha \in [1, p_s^\star)$. Moreover, one has

$$\Big| \int_\Omega g(x)|u_n(x)|^{r-2}u_n(x)(u_n(x)-u(x))dx \Big| \le \overline{g}|u_n|^{r-1}_{L^r(\Omega)}|u_n-u|_{L^r(\Omega)}. \tag{4.4}$$

$$\Big| \int_\Omega V(x)|u_n(x)|^{p-2}u_n(x)(u_n(x)-u(x))dx \Big| \le V_0|u_n|^{p-1}_{L^p(\Omega)}|u_n-u|_{L^p(\Omega)}. \tag{4.5}$$

Now, combining (3.2), (4.4) and (4.5) we get

$$\lim_{n\to+\infty} \mathcal{L}_K^p u_n.(u_n-u) = 0, \tag{4.6}$$

and by Lemma 2.3 we have $u_n \to u$ strongly in X_0. □

Lemma 4.2 *If $\lambda \in (0, \frac{\lambda_1}{V_0})$, then, there exists $\mu_\star > 0$ such that for any $\mu \in (0, \mu_\star)$, we can choose $\rho, \gamma > 0$ so that $\Phi_{\lambda,\mu}(u) > \gamma$ for all $u \in X_0$ with $\|u\| = \rho$.*

Proof By $(H_1), (V)$ and (G) and for all $u \in X_0$, we have

$$\begin{aligned}
\Phi_{\lambda,\mu}(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega V(x)|u(x)|^p\,dx - \frac{\mu}{r}\int_\Omega g(x)|u(x)|^r\,dx - \int_\Omega F(x,u(x))dx,\\
&\ge \frac{1}{p}\|u\|^p - \frac{\lambda}{p}V_0|u|^p_{L^p(\Omega)} - \frac{\mu}{r}\overline{g}|u|^r_{L^r(\Omega)} - b|u|^q_{L^q(\Omega)} - |a|_{L^{q'}(\Omega)}|u|_{L^q(\Omega)},\\
&\ge \frac{1}{p}\|u\|^p\left(1-\frac{V_0\lambda}{\lambda_1}\right) - \frac{\mu}{r}\overline{g}C_r^r\|u\|^r - C_q^q b\|u\|^q - |a|_{L^{q'}(\Omega)}C_q\|u\|,\\
&\ge \|u\|^p\left(\frac{1}{p}(1-\frac{V_0\lambda}{\lambda_1}) - bC_q^q\|u\|^{q-p} - C_q|a|_{L^{q'}(\Omega)}\|u\|^{1-p}\right) - \frac{\mu}{r}\overline{g}C_r^r\|u\|^r.
\end{aligned}$$

If $\lambda < \frac{\lambda_1}{V_0}$, then there exists $\rho > 0$ such that $bC_q^q\rho^{q-p} + C_q|a|_{L^{q'}(\Omega)}\rho^{1-p} < \frac{1}{2p}(1-\frac{V_0\lambda}{\lambda_1})$.
Now let $u \in X_0$ be such that $\|u\| = \rho$, then

$$\Phi_{\lambda,\mu}(u) \ge \frac{1}{2p}(1-\frac{V_0\lambda}{\lambda_1})\rho^p - \frac{\mu}{r}\overline{g}C_r^r\rho^r \to \frac{1}{2p}(1-\frac{V_0\lambda}{\lambda_1})\rho^p > 0 \ as\ \mu \to 0.$$

Then, there exists $\mu_\star > 0$ such that for all $\mu \in (0, \mu_\star)$, $\Phi_{\lambda,\mu}(u) = \gamma > 0$ for all $u \in X_0$ with $\|u\| = \rho$. □

Lemma 4.3 *There exist $e \in X_0, R > \rho$ such that $\|e\| = R$ and $\Phi_{\lambda,\mu}(e) < 0$.*

Proof By (H_1), for all $u \in X_0$ we have

$$\begin{aligned}
\Phi_{\lambda,\mu}(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega V(x)|u(x)|^p\,dx - \frac{\mu}{r}\int_\Omega g(x)|u(x)|^r\,dx - \int_\Omega F(x,u(x))dx,\\
&\le \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega V(x)|u(x)|^p\,dx - \frac{\mu}{r}\int_\Omega g(x)|u(x)|^r\,dx + \int_\Omega a(x)|u(x)|dx,\\
&+ \frac{b}{q}\int_\Omega |u(x)|^q\,dx.
\end{aligned}$$

Now let $t \geq 0$, $u_0 \in X_0$ such that $\|u_0\| = 1$, then

$$\begin{aligned}\Phi_{\lambda,\mu}(tu_0) &\leq \frac{t^p}{p}\|u_0\|^p - \frac{t^p\lambda}{p}\int_\Omega V(x)|u_0(x)|^p\ dx - \frac{t^r\mu}{r}\int_\Omega g(x)|u_0(x)|^r\ dx \\ &+ t\int_\Omega a(x)|u_0(x)|dx + \frac{bt^q}{q}\int_\Omega |u_0(x)|^q\ dx.\end{aligned}$$

Hence

$$\lim_{t\to+\infty}\Phi_{\lambda,\mu}(tu_0) = -\infty. \tag{4.7}$$

From (4.7), there exists $R > \rho$ such that $\Phi_{\lambda,\mu}(Ru_0) < 0$. Let $e = Ru_0$, then $\|e\| = R$ and $\Phi_{\lambda,\mu}(e) < 0$. □

Theorem 4.4 *Under the conditions* $(H_1), (V)$ *and* (G), *if* $\lambda \in (0, \frac{\lambda_1}{V_0})$, *then there exists* $\mu_\star > 0$ *such that for all* $\mu \in (0, \mu_\star)$, *problem* (4.1) *has a nontrivial weak solution which corresponds to a positive critical value.*

Proof According to Lemmas 4.1, 4.2 and 4.3, the functional $\Phi_{\lambda,\mu}$ satisfies all the assumptions of the Mountain Pass Theorem. Then, there exists $u \in X_0$ a nontrivial critical point of the functional $\Phi_{\lambda,\mu}$ with $\Phi_{\lambda,\mu}(u) > 0$ and thus a nontrivial weak solution of problem (4.1). □

Now, we show the existence of a sequence of weak solutions via Fountain Theorem.

Theorem 4.5 *Under the conditions* $(H_1), (H_3), (V), (G)$ *and* $\lambda \in (0, \frac{\lambda_1}{V_0})$, *problem* (4.1) *admits a sequence of weak solutions* $(u_n) \subset X_0$, *for all* $\mu > 0$, *such that*

$$\Phi_{\lambda,\mu}(u_n) \to +\infty \ \textit{as}\ n \to +\infty.$$

Proof By (H_3), $\Phi_{\lambda,\mu}$ is an even functional. Due to Lemma 4.1, we only need to verify conditions (a) and (b) of Theorem 2.7. Let

$$\begin{aligned}\beta_n^1 &= \sup\{|v|_{L^r(\Omega)}, u \in Z_n : \|u\| \leq 1\}, \\ \beta_n^2 &= \sup\{|v|_{L^p(\Omega)}, u \in Z_n : \|u\| \leq 1\}, \\ \beta_n^3 &= \sup\{|v|_{L^q(\Omega)}, u \in Z_n : \|u\| \leq 1\}, \\ \beta_n &= \max\{\beta_n^k, k = 1,2,3\}.\end{aligned}$$

According to Lemma 2.4, we have $\beta_n \to 0$, as $n \to +\infty$. From $(H_1), (V)$ and (G) we have

$$\begin{aligned}\Phi_{\lambda,\mu}(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega V(x)|u(x)|^p\ dx - \frac{\mu}{r}\int_\Omega g(x)|u(x)|^r\ dx - \int_\Omega F(x,u(x))dx, \\ &\geq \frac{1}{p}\|u\|^p - \frac{\lambda}{p}V_0|u|^p_{L^p(\Omega)} - \frac{\mu}{r}\overline{g}|u|^r_{L^r(\Omega)} - b|u|^q_{L^q(\Omega)} - |a|_{L^{q'}(\Omega)}|u|_{L^q(\Omega)}, \\ &\geq \frac{1}{p}\left(1 - \frac{V_0\lambda}{\lambda_1}\right)\|u\|^p - \frac{\mu}{r}\overline{g}\|u\|^r\Big|\frac{u}{\|u\|}\Big|^r_{L^r(\Omega)} - b\|u\|^q\Big|\frac{u}{\|u\|}\Big|^q_{L^q(\Omega)} \\ &- |a|_{L^{q'}(\Omega)}\|u\|\Big|\frac{u}{\|u\|}\Big|_{L^q(\Omega)}, \\ &\geq \frac{1}{p}\left(1 - \frac{V_0\lambda}{\lambda_1}\right)\|u\|^p - \frac{\mu}{r}\overline{g}\|u\|^r\beta_n^r - b\|u\|^q\beta_n^q - |a|_{L^{q'}(\Omega)}\|u\|\beta_n.\end{aligned}$$

For each $n \in \mathbb{N}$, let $r_n = \frac{1}{\beta_n}$, then $r_n \to +\infty$ as $n \to +\infty$.

Now if $\lambda \in (0, \frac{\lambda_1}{V_0})$, $u \in Z_n$ with $\|u\| = r_n$ we get

$$\Phi_{\lambda,\mu}(u) \geq \frac{1}{p}\left(1 - \frac{V_0\lambda}{\lambda_1}\right) r_n^p - (\frac{\mu}{r}\overline{g} + b + |a|_{L^{q'}(\Omega)}) \to +\infty \ as \ n \to +\infty.$$

Hence

$$\lim_{n\to+\infty} \inf_{u\in Z_n, \|u\|=r_n} \Phi_{\lambda,\mu}(u) = +\infty.$$

For (b), let $u \in Y_n$ we have

$$\begin{aligned}
\Phi_{\lambda,\mu}(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega V(x)|u(x)|^p \, dx - \frac{\mu}{r}\int_\Omega g(x)|u(x)|^r \, dx - \int_\Omega F(x, u(x))dx, \\
&\leq \frac{1}{p}\|u\|^p + \frac{\lambda}{p}V_0|u|^p_{L^p(\Omega)} - \frac{\mu}{r}|g^{\frac{1}{r}}u|^r_{L^r(\Omega)} + |a|_{L^{q'}(\Omega)}|u|_{L^q(\Omega)} + b|u|^q_{L^q(\Omega)}.
\end{aligned}$$

Since $\dim(Y_n) < \infty$, the norms $\|.\|, |.|_{L^p(\Omega)}, |g^{\frac{1}{r}}.|_{L^r(\Omega)}, |.|_{L^q(\Omega)}$ are equivalent. As $q < p < r$, the last estimate yields $\Phi_{\lambda,\mu}(u) \leq 0$ for all $u \in Y_n$ with $\|u\|$ large enough. This completes the proof of Theorem 4.5. $\square$

Remark 4.6 *Compared to the results in [11, 12], a part the general character of the operator involved in Theorem 4.4 and Theorem 4.5, note that the weight functions satisfy weaker conditions than those in [11, 12]. Especially the following assumptions in problem* (4.3)*,*

$$|\{x \in \Omega;\ h(x) > 0\}| > 0 \quad and \quad \inf_{x\in\Omega} k(x) > 0,$$

will be omitted. Hence, from this point of view, Theorems 4.4 and 4.5 complete the results in [11, 12] in case of bounded weights.

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