Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2511 – 2533
© TÜBİTAK
doi:10.3906/mat-2107-20

Decompositions of semigroups

Rida-e ZENAB[*]
Department of Mathematics and Social Sciences Sukkur IBA University, Sukkur, Pakistan

Received: 08.07.2021 • **Accepted/Published Online:** 01.10.2021 • **Final Version:** 29.11.2021

Abstract: In this article we discuss the factorisations of semigroups and monoids in the context of direct, semidirect and Zappa–Szép products addressing the question of uniqueness. An equivalence between external and internal Zappa–Szép product of groups and monoids is known, but no such correspondence exists for semigroups in general. We prove the equivalence between external and internal Zappa–Szép product of semigroups subject to certain conditions in this article. We end with some illustrative examples of the Zappa–Szép product of the bisimple inverse monoids.

Key words: Zappa–Szép products, semigroups, bisimple inverse monoids

1. Introduction

Direct, semidirect and Zappa–Szép products, with each being a generalisation of its predecessor, are natural notions used to compose and decompose algebraic structures such as groups, monoids and semigroups. These constructions can be approached from two directions: internal and external- the two are closely related and often form the first step behind the development of structure theorems.

Over recent years, semidirect and Zappa–Szép products have become significant tools for groups, monoids and semigroups. Indeed, the Kaloujnine–Krasner Theorem [7] and the Krohn–Rhodes Theorem [8] have established semidirect products as key players. Zappa–Szép products, on the other hand, have become very useful in the construction of examples of C^*-algebras of semigroups and topological groupoids [2, 3].

The concept of Zappa–Szép product was first studied by Neumann for groups, who used the terminology general decompositions for these products [12]. Neumann pointed the equivalence between a general decomposition $G = HK$ and transitive actions of G with point stabilizer H and regular subgroup K. The first systematic study of such constructions was done by Zappa in 1940 [21]. Further developments in the theory for groups were made by Casadio [4], Redei [15] and Szép [16–19]. Redei introduced the term skew products for Zappa–Szép products [15]. Szép initiated the study of similar products in setting other than groups in [18] and [19]. Lavers who used the term general product, gave necessary and sufficient conditions for the Zappa–Szép product of two finitely presented monoids to be finitely presented [10].

There is a correspondence between the external and internal Zappa–Szép products (and also direct and semidirect products) of groups and monoids, but this is not true for semigroups in general. We discuss the internal and external direct and semidirect products of monoids and semigroups in Sections 2 and 3, respectively. We discuss some known results that describe the correspondence between external and internal direct and

*Correspondence: ridaezenab@iba-suk.edu.pk
1991 *AMS Mathematics Subject Classification:* 20 M 10

semidirect products of monoids and consider why such a correspondence for semigroups does not exist. We discuss the conditions under which we get an equivalence for direct and semidirect products of semigroups

In Section 4, we consider the conditions under which we get an equivalence between external and internal Zappa–Szép products of semigroups. We provide two proofs for this construction and will see that it generalises results on semidirect products.

A semigroup S is called inverse if every element $a \in S$ possesses a unique inverse in S, usually denoted a^{-1}, in the sense that

$$a = aa^{-1}a \text{ and } a^{-1} = a^{-1}aa^{-1}.$$

Canonical examples of inverse semigroups are groups (possessing exactly one idempotent) and semilattices (inverse semigroups all of whose elements are idempotent, which perforce possess the structure of a semilattice in the ordered sense).

Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are well known equivalence relations that characterise the elements of a semigroup in terms of principal ideals they generate. If S is an inverse semigroup, then for $a, b \in S$, $a \,\mathcal{R}\, b$ if and only if $aa^{-1} = bb^{-1}$. Dually, $a \;\mathcal{L}\; b$ if and only if $a^{-1}a = b^{-1}b$. The reader is referred to [6] for further details concerning inverse semigroups and Green's relations on such semigroups.

In Section 5, we provide examples of Zappa–Szép product of bisimple inverse monoids. A semigroup S is called bisimple if it has a single $\mathcal{D}$-class. In [5], Gould and Zenab studied Zappa–Szép products of semigroups with well behaved structures determined by idempotents in the broader sense away from canonical situation of inverse semigroups. They indicated the applications of their construction to bisimple inverse monoids. In this article we show that how the theory in [5] works explicitly in the inverse case to provide the reader new insights hidden in the broader sense.

We will denote the $\mathcal{R}$-class of the identity of a monoid by R_1, $\mathcal{L}$-class of the identity by L_1 and the $\mathcal{H}$-class of the identity by H_1.

2. Direct products of monoids and semigroups

The notion of the direct product is one of the well known techniques to construct a new algebra from two given ones. The structure theorem of finitely generated abelian groups demonstrates the importance of the direct product construction. In this section we discuss the equivalence between external and internal direct products of monoids and semigroups. We begin with the notion of the external direct product of two semigroups (monoids) taken from category theory [11].

Definition 2.1 *Let S, T and U be semigroups (monoids) and suppose*

$$U \xrightarrow{p_1} S \quad \textit{and} \quad U \xrightarrow{p_2} T$$

are morphisms. We say that U is the external direct product of semigroups (monoids) S and T if for any semigroup (monoid) U' and morphisms

$$U' \xrightarrow{f_1} S \quad \textit{and} \quad U' \xrightarrow{f_2} T$$

there exists a unique morphism $U' \xrightarrow{f} U$ *making the following diagram commute:*

$$
\begin{tikzcd}
 & U' \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\
S & U \arrow[l, "p_1"] \arrow[r, "p_2"'] & T
\end{tikzcd}
$$

From above definition we have $(u')f = ((u')f_1, (u')f_2)$, so that a point in U is determined by its S-coordinate and T-coordinate. This is to say that p_1 and p_2 are projection morphisms and U is in one to one correspondence with the Cartesian product $S \times T$. Thus the universal property of product not only determines $S \times T$ as a set but determines it up to isomorphism in the category of semigroups (monoids).

To give the definition of the internal direct product of semigroups, we first need to explain what we mean by a uniquely factorisable semigroup.

Definition 2.2 *Let U be a semigroup. We say that U is factorisable (with factors S and T) if there are subsemigroups S and T of U such that $U = ST$. If each element $u \in U$ can be uniquely written as a product of an element of S and an element of T, then we say that U is* uniquely factorisable*.*

It is worth mentioning here that this is also the definition of an internal Zappa–Szép product which we will discuss later in this paper.

If a semigroup U is uniquely factorisable as $U = ST$, then from [1, Lemma 3.4] there is an idempotent $e \in S \cap T$ which is a right identity for S and a left identity for T. We provide an alternative proof of this result in the following.

Lemma 2.3 *Let U be a semigroup and suppose U is uniquely factorisable as $U = ST$, where S and T are subsemigroups of U. Then there exists an idempotent $e \in S \cap T$ which is a right identity for S and a left identity for T.*

Proof Let $s \in S$. Then s can be written as $s = uv$ for $u \in S$ and $v \in T$. For $t \in T$, $st = uvt$ for all $t \in T$ so that $t = vt$ for all $t \in T$ by uniqueness of factorisation. Thus v is a left identity for T and in particular $v = v^2$.

Next as $v \in T$, so we can write v as $v = pq$ for $p \in S$, $q \in T$. For $w \in S$ $wv = wpq$ for all $w \in S$ so that $w = wp$ for all $w \in S$ which implies p is a right identity for S and in particular, $p = p^2$, so that

$$v = pq = ppq = pv.$$

Dually, there exists an element $l = l^2 \in T$ which is a left identity for T and we can write p as $p = pl$. Now

$$
\begin{array}{rrcll}
 & v & = & pv & \\
\Rightarrow & vl & = & pvl & \\
\Rightarrow & l & = & pl & \text{because } v \text{ is a left identity for } T \\
\Rightarrow & l & = & p & \text{because } p = pl.
\end{array}
$$

Hence $l = p = e \in S \cap T$ which is a right identity for S and a left identity for T. $\square$

We now explain what we mean by an internal direct product of semigroups and monoids.

Definition 2.4 *Let U be a semigroup (monoid) and S and T are subsemigroups (submonoids) of U. Then U is the internal direct product of S and T if:*

(1) U is uniquely factorisable with factors S and T;

(2) for all $s \in S$ and for all $t \in T$, $st = ts$.

There is an equivalence between internal and external direct product of monoids and we state this result without proof since the proof follows that for groups.

Proposition 2.5 *Let U be a monoid and S, T be submonoids of U. Suppose U is the internal direct product of S and T. Then U is isomorphic to the external direct product $S \times T$.*

Conversely, suppose that $U = S \times T$ is the external direct product of monoids S and T. Then there are submonoids S' and T' of U such that $S \cong S'$, $T \cong T'$ and $U = S'T'$ is the internal direct product of S' and T'.

There is not such a complete equivalence between internal and external direct products of semigroups as we have seen for monoids. One direction: from internal to external works as we now explain. When we consider the internal direct product $U = ST$ of semigroup U with subsemigroups S and T, then we notice that there is an element $e \in S \cap T$ which becomes the identity of S and T and hence of U. Thus U is forced to be a monoid with submonoids S and T as we see in the following lemma.

Lemma 2.6 *Let U be a semigroup and S and T be subsemigroups of U. Suppose U is the internal direct product of S and T. Then U is a monoid and S and T are submonoids of U.*

Proof As U is uniquely factorisable, so by Lemma 2.3, there exists an element $e \in S \cap T$ such that $se = s$ for all $s \in S$ and $et = t$ for all $t \in T$.

We note that for $s \in S$ $es = se = s$, because elements of S and T commute, so that e is identity of S. Thus for all $s \in S$ and for all $t \in T$ $est = st$, implies that e is left identity for U. Dually e is right identity for U and hence U is a monoid with submonoids S and T. Thus U is the internal direct product of S and T, which by Proposition 2.5 is isomorphic to the external direct product $S \times T$. □

From the above lemma, we observe that $S \cap T = \{e\}$ as if $w \in S \cap T$ is another element, then $w = ew = we$ gives $w = e$ because of unique factorisations. Hence $S \cap T = \{e\}$.

Thus an internal direct product of semigroups is just an internal direct product of monoids which correspond to an external direct product of semigroups. Naturally our next step is to see what happens if we consider an external direct product of semigroups S and T. If T (respectively S) contains an idempotent, then S (respectively T) is embedded in $S \times T$. In general it is not immediate to obtain an internal direct product from an external direct product of semigroups, because for semigroups there do not exist such mappings $s \mapsto (s, 1_T)$ and $t \mapsto (1_S, t)$ as they exists for monoids, but we can use Preston's techniques from [14] to characterise internal and external direct product of semigroups. Preston developed these in the more general context of semidirect products, we specialise here to direct products.

Lemma 2.7 *Suppose S and T are semigroups and $U = S \times T$ is the external direct product of S and T. Let $S^{(1)}$ and $T^{(1)}$ be the semigroups obtained from S and T, respectively by adjoining an identity and let $V = S^{(1)} \times T^{(1)}$. Then there are subsemigroups S' and T' of $V = S^{(1)} \times T^{(1)}$ with $S \cong S', T \cong T'$ such that $U = S'T'$ and every element of U is uniquely factorisable as $u = st$ for $s \in S'$ and $t \in T'$.*

Proof Let $S^{(1)}$ and $T^{(1)}$ be the semigroups obtained from S and T, respectively by adjoining an identity and suppose $V = S^{(1)} \times T^{(1)}$. Let

$$S'' = \{(s, 1); s \in S^{(1)}\} \text{ and } T'' = \{(1, t) : t \in T^{(1)}\}.$$

By Proposition 2.5, S'' and T'' are submonoids of $V = S^{(1)} \times T^{(1)}$ and $V = S''T''$ is the internal direct product of S'' and T'' where $\alpha : S^{(1)} \to S''$ and $\beta : T^{(1)} \to T''$ are isomorphisms. Next let

$$S' = \{(s, 1) : s \in S\} \text{ and } T' = \{(1, t) : t \in T\}.$$

Clearly S' and T' are subsemigroups of S'' and T'' and hence of $V = S^{(1)} \times T^{(1)}$. Then $\alpha|_S : S \to S'$ and $\beta|_T : T \to T'$ are isomorphisms. Also each element $(s, t) \in U$ can be written as $(s, t) = (s, 1)(1, t)$ where $(s, 1) \in S'$ and $(1, t) \in T'$. Moreover elements of S' and T' commute. Hence $U = S'T'$ is a subsemigroup of $V = S^{(1)} \times T^{(1)}$. □

The following example shows that there is not a natural correspondence between internal and external direct products of semigroups.

Example 2.8 *Consider the additive semigroup* $\mathbb{N} = \{1, 2, 3, \cdots\}$ *and let* $S = \mathbb{N} \times \mathbb{N}$. *Suppose* $S = UT$ *is the internal direct product of subsemigroups* U *and* T *where* $U \cong \mathbb{N}$ *and* $T \cong \mathbb{N}$. *Now*

$$U = \langle (p, q) \rangle = \{np, nq)\ n \in \mathbb{N}\} \text{ and } T = \langle (r, s) \rangle = \{(mr, ms) : m \in \mathbb{N}\},$$

for some (p, q), $(r, s) \in S$.

As $(1, 1) \in S = UT$, *so we may write* $(1, 1) = (np, nq) + (mr, ms)$, *but this is impossible because the element* 1 *is indecomposable in the additive semigroup* $\mathbb{N}$. *Thus our supposition is wrong and an external direct product of semigroups is not isomorphic to an internal direct product.*

3. Semidirect products of monoids and semigroups

In this section we discuss an equivalence between internal and external semidirect product of monoids/semigroups. The term semidirect product for semigroups was first used by Neumann [13] to construct wreath products of semigroups.

We first explain what we mean by an action of a monoid/semigroup S on a set X.

Definition 3.1 *Let* S *be a monoid and* X *be a set. Then* S *acts on* X *on the left if there is a map*

$$S \times X \to X,\ (s, x) \mapsto s \cdot x$$

such that for all $x \in X$ *and for all* $s, s' \in S$ *we have*

(S1) $ss' \cdot x = s \cdot (s' \cdot x)$.

(S2) $1_S \cdot x = x$.

Equivalently, an action of S is a monoid morphism from S to the transformation monoid $\mathcal{T}_X$. Diverting from usual practice, we write elements of $\mathcal{T}_X$ from right to left, that is, for $\alpha, \beta \in \mathcal{T}_X$, to compute $\alpha\beta$ we first do β then α and for each $s \in S$, we denote $(\alpha(s))x$ by $s \cdot x$.

Definition 3.2 *Let S and T be monoids and suppose T acts on S on the left satisfying (S1) and (S2). We say that T acts on S by morphisms if for all $t \in T$ and $s, s' \in S$ we have*

(S3) $t \cdot ss' = (t \cdot s)(t \cdot s')$.

This is equivalent to saying that there there is a homomorphism

$$\theta : T \to \operatorname{End} S$$

where $\operatorname{End} S$ *is monoid of endomorphisms of S. We say that $\theta : T \to \operatorname{End} S$ is a monoid morphism if $t \cdot 1 = 1$ for all $t \in T$.*

Define a binary operation on $S \times T$ by

$$(s,t)(s',t') = (s(t \cdot s'), tt').$$

With θ being a monoid morphism, it is easy to check that $S \times T$ is a monoid with identity $(1_S, 1_T)$ known as an external semidirect product of monoids S and T and denoted by $S \rtimes T$.

If S and T are semigroups and T acts on the left of S by endomorphisms satisfying (S1) and (S3), then $S \rtimes T$ is a semigroup known as the external semidirect product of semigroups S and T.

Just as in the case of direct products, it is not so easy to get hold of the notion of internal semidirect product for monoids and semigroups. We now give the definition of an internal semidirect product from [14].

Definition 3.3 *Let U be a monoid and S be a subset of U. An element $u \in U$ is said to be left permutable with S if*

$$uS \subseteq Su.$$

If T is a subset of U, then T is said to be locally left permutable with S if each element of T is left permutable with S.

Definition 3.4 *Let U be a monoid and S, T be submonoids of U. Then U is the internal semidirect product of S and T if U is uniquely factorisable with factors S and T and T is locally left permutable with S.*

There is an equivalence between internal and external semidirect products of monoids as we see in the following theorem.

Theorem 3.5 *Let U be a monoid and let $U = ST$ be the internal semidirect product of submonoids S and T. Then there is an action of T on S such that $U \cong S \rtimes T$, the external semidirect product of S by T.*

Conversely, let $U = S \rtimes T$ be the external semidirect product of monoids S and T. Let

$$S' = \{(s, 1_T) : s \in S\} \text{ and } T' = \{(1_S, t) : t \in T\}.$$

Then S' and T' are submonoids of U such that T' is locally left permutable with S', and $U = S'T'$ is the internal semidirect product of S' and T'. Further $S \cong S'$ and $T \cong T'$.

Proof Suppose $U = ST$ is the internal semidirect product of submonoids S and T. As factorisations are unique, so for each $s \in S$ and $t \in T$, there exists a unique element $t \cdot s$ of S such that

$$ts = (t \cdot s)t.$$

By associativity we have that $(tt')s = t(t's)$ for any $t, t' \in T$ and $s \in S$. Now

$$(tt')s = ((tt') \cdot s)tt'$$

and

$$t(t's) = t((t' \cdot s)t') = (t \cdot (t' \cdot s))tt'.$$

By uniqueness we have $tt' \cdot s = t \cdot (t' \cdot s)$ and so (S1) holds.

Next for $t \in T$ and $s, s' \in S$

$$t(ss') = (t \cdot (ss'))t$$

and

$$(ts)s' = (t \cdot s)ts' = (t \cdot s)(t \cdot s')t.$$

By associativity and uniqueness, we have that

$$t \cdot (ss') = (t \cdot s)(t \cdot s'),$$

and thus (S3) holds.

Also as U is a monoid and $1_U = 1 \in S \cap T$, so for $s \in S$, we see that

$$s1 = 1s = (1 \cdot s)1$$

and by uniqueness $1 \cdot s = s$. Thus (S2) holds. Hence we can form an external semidirect product $U' = S \rtimes T$ of S and T. Now define a map

$$\varphi : U \to U'$$

by $(st)\varphi = (s, t)$. By uniqueness of factorisation, it is easy to see that φ is well defined and one-one. Clearly φ is onto. To show that φ is a morphism, let $st, s't' \in U$. Then

$$\begin{array}{rcl} (st)\varphi(s't')\varphi & = & (s,t)(s',t') \\ & = & (s(t \cdot s'), tt') \\ & = & (s(t \cdot s')tt')\varphi \\ & = & (s(ts')t')\varphi \\ & = & ((st)(s't'))\varphi. \end{array}$$

Also $(1)\varphi = (1, 1)$. Hence $U \cong U'$.

Conversely, let $U = S \rtimes T$ be the external semidirect product of monoids S and T. As $t \cdot 1 = 1$ for all $t \in T$, it is easy to see that U contains submonoids

$$S' = \{(s, 1_T) : s \in S\} \text{ and } T' = \{(1_S, t) : t \in T\}.$$

We note that every element of U has a unique expression

$$(s, t) = (s, 1_T)(1_S, t).$$

Let $s' = (s, 1_T) \in S'$ and $t' = (1_S, t) \in T'$. Then

$$\begin{array}{rcl} t's' & = & (1_S, t)(s, 1_T) \\ & = & (t \cdot s, t) \\ & = & (t \cdot s, 1_T)(1_S, t) \\ & = & (t' \cdot s')t'. \end{array}$$

Thus $t'S' \subseteq S't'$ and hence T' is locally left permutable with S'. Thus $S \rtimes T$ is the internal semidirect product of S' and T'. Now define maps $\alpha : S \to S'$ and $\beta : T \to T'$ by

$$s \mapsto (s, 1_T) \text{ and } t \mapsto (1_S, t),$$

respectively. Then it is easy to check that $S \cong S'$ and $T \cong T'$. □

We see in the following lemma that if U is the internal semidirect product of subsemigroups S and T, then T is forced to be a monoid.

Lemma 3.6 *Let U be a semigroup and suppose that $U = ST$ is the internal semidirect product of subsemigroups S and T. Then there exists an element $e \in S \cap T$ which is a right identity for S and a left identity for T such that*

$$e \cdot s = es,\ e = t \cdot e \text{ for all } s \in S \text{ and } t \in T.$$

Moreover T is a monoid with identity e.

Proof As factorisations are unique, so from Lemma 2.3 there exists an element $e \in S \cap T$ which is a right identity for S and a left identity for T. Now for $s \in S$

$$es = (e \cdot s)e.$$

But e is right identity for S, so that

$$es = e(se) = (es)e.$$

Thus $(es)e = (e \cdot s)e$ and by uniqueness $es = e \cdot s$.

Next for $t \in T$

$$te = (t \cdot e)t.$$

But e is a left identity for T, so that

$$te = (et)e = e(te).$$

Thus $e(te) = (t \cdot e)t$ and by uniqueness $e = t \cdot e$ and $te = t$. Hence

$$e \cdot s = es,\ e = t \cdot e \text{ for all } s \in S \text{ and } t \in T.$$

Also as $te = t$ for all $t \in T$, so e is a right identity for T and hence T is a monoid. □

As for direct products, there is not such a complete equivalence between internal and external semidirect products of semigroups. It is not difficult to get an external semidirect product of semigroups from an internal semidirect product of semigroups, but the converse is not obvious. However we can characterise those external semidirect products of semigroups which are also internal semidirect products.

Theorem 3.7 *Let U be a semigroup and let $U = ST$ be the internal semidirect product of subsemigroups S and T. Then there is an action of T on S such that $U' = S \rtimes T$ is the external semidirect product of S by T and $U \cong U'$. Moreover there exists an element $e \in S \cap T$ which is a right identity for S and an identity for T such that*

$$e \cdot s = es,\ e = t \cdot e \text{ for all } s \in S \text{ and } t \in T.$$

Conversely, suppose that $U = S \rtimes T$ *is the external semidirect product of semigroups* S *and* T *such that* S *has a right identity* e_S *and* T *is a monoid with identity* e_T *and*

$$e_T \cdot s = e_S s,\ e_S = t \cdot e_S \quad \textit{for all } s \in S \textit{ and } t \in T.$$

Let

$$S' = \{(s, e_T) : s \in S\} \textit{ and } T' = \{(e_S, t) : t \in T\}.$$

Then S' *and* T' *are subsemigroups of* U, $S \cong S'$, $T \cong T'$ *and* $U = S'T'$ *is the internal semidirect product of* S' *and* T'.

Proof Suppose $U = ST$ is the internal semidirect product of subsemigroups S and T. Then because T is locally left permutable with S and factorisations are unique, so associativity and uniqueness gives us (S1) and (S3). Therefore we can form an external semidirect product $U' = S \rtimes T$ as in Theorem 3.5, so that $U \cong U'$. Also from Lemma 3.6, there exits an element $e \in S \cap T$ which is a right identity for S and an identity for T such that

$$e \cdot s = es,\ e = t \cdot e \text{ for all } s \in S \text{ and } t \in T.$$

Conversely, let $U = S \rtimes T$ be the external semidirect product of semigroups S and T such that S has a right identity e_S and T has an identity e_T with

$$e_T \cdot s = e_S s,\ e_S = t \cdot e_S \quad \text{for all } s \in S \text{ and } t \in T.$$

Put

$$S' = \{(s, e_T) : s \in S\} \text{ and } T' = \{(e_S, t) : t \in T\}.$$

Define maps $\alpha : S \to S'$ and $\beta : T \to T'$ by

$$s\alpha = (s, e_T) \text{ and } t\beta = (e_S, t),$$

respectively. Then it is easy to see that $S \cong S'$ and $T \cong T'$. Also each element in $(s, t) \in U$ can be written as

$$(s, t) = (s, e_T)(e_S, t)$$

and this decomposition is unique. Next let $s' = (s, e_S) \in S'$ and $t' = (e_S, t) \in T'$. Then

$$\begin{aligned}
t's' &= (e_S, t)(s, e_T) \\
&= (e_S(t \cdot s), te_T) \\
&= ((t \cdot e_S)(t \cdot s), t) && \text{because } e_S = t \cdot e_S \text{ and } te_T = t \\
&= (t \cdot (e_S s), t) && \text{using (S3)} \\
&= (t \cdot (e_T \cdot s), t) && \text{because } e_T \cdot s = e_S s \\
&= (te_T \cdot s, t) && \text{using (S1)} \\
&= (t \cdot s, t) && \text{because } te_T = t \\
&= (t \cdot s, e_T)(e_S, t) \\
&\in S't'.
\end{aligned}$$

Thus $t'S' \subseteq S't'$ and so T' is locally left permutable with S'. Hence $U = S'T'$ is the internal semidirect product of S' and T'. □

Preston provided a characterisation of external semidirect products of semigroups as we mentioned in previous section where we used it to characterise external direct products of semigroups. Preston proved that it is always possible to consider a semidirect product of semigroups as a subsemigroup of semidirect product of monoids [14]. His construction works as follows:

Let S and T be semigroups and $U = S \rtimes T$ be an external semidirect product with $\theta : T \to \operatorname{End} S$. Let $S^{(1)}$ and $T^{(1)}$ be semigroups obtained from S and T, respectively by adjoining an extra element 1, that acts as an identity element. Let $t \cdot 1 = 1$ and define $\theta^{(1)} : T^{(1)} \to \operatorname{End} S^{(1)}$ to be the extension of θ where $(1)\theta = 1_{S^1}$. Set $W = S^{(1)} \rtimes T^{(1)}$. Then $\alpha : s \mapsto (s, 1)$ and $\beta : t \mapsto (1, t)$ embed S and T, respectively in W. Observe that $S\alpha \cap T\beta = \emptyset$ because if $x = s\alpha = t\beta$, then $(s, 1) = (1, t)$, which is impossible. Therefore by identifying S with $S\alpha$ and T with $T\beta$, S and T become disjoint subsemigroups of W. Thus we have the following result:

Theorem 3.8 [14, Theorem 8] *Let S and T be semigroups and $U = S \rtimes T$ be an external semidirect product with $\theta : T \to \operatorname{End} S$. Then setting $W = S^{(1)} \rtimes T^{(1)}$ and θ^1 as above, S and T are subsemigroups of W and $U' = ST$ is a subsemigroup of W. Moreover, $U \cong U'$.*

4. Zappa–Szép products of semigroups

For convenience of the reader we begin this section with the definition of the Zappa–Szép product of semigroups.

Definition 4.1 *Let S and T be semigroups and suppose that we have maps*

$$T \times S \to S,\ (t, s) \mapsto t \cdot s \text{ and } T \times S \to T,\ (t, s) \mapsto t^s$$

such that for all $s, s' \in S, t, t' \in T$:

$$\begin{array}{ll} \text{(ZS1) } tt' \cdot s = t \cdot (t' \cdot s); & \text{(ZS3) } (t^s)^{s'} = t^{ss'}; \\ \text{(ZS2) } t \cdot (ss') = (t \cdot s)(t^s \cdot s'); & \text{(ZS4) } (tt')^s = t^{t' \cdot s} t'^s. \end{array}$$

From (ZS1) and (ZS3), we see that in the usual sense T acts on S from the left and S acts on T from the right, respectively. We define a binary operation on $S \times T$ by

$$(s, t)(s', t') = (s(t \cdot s'), t^{s'} t').$$

It is easy to see that this binary operation is associative. Thus $S \times T$ becomes a semigroup known as (external) Zappa–Szép product of S and T and denoted by $S \bowtie T$.

Note that if one of the above actions is trivial (that is, one semigroup acts by the identity map), then the second action is by morphisms, and we obtain the semidirect product $S \rtimes T$ (if S acts trivially) or $S \ltimes T$ (if T acts trivially). If both actions are trivial, then we obtain the direct product $S \times T$.

If S and T are monoids then we insist that the following four axioms also hold:

$$\begin{array}{ll} \text{(ZS5) } t \cdot 1_S = 1_S; & \text{(ZS7) } 1_T \cdot s = s; \\ \text{(ZS6) } t^{1_S} = t; & \text{(ZS8) } 1_T^s = 1_T. \end{array}$$

From (ZS6) and (ZS7) we see that the actions are monoid actions in the usual sense and (ZS5) and (ZS8) are saying that the identities are fixed under the actions. It follows that $S \bowtie T$ becomes a monoid with identity $(1_S, 1_T)$.

If we replace monoids by groups in the above definition, then $S \bowtie T$ is a group [21].

We now give definition of an internal Zappa–Szép product and then we explain the correspondence between external and internal Zappa–Szép products of semigroups and monoids.

Definition 4.2 *Let U be a monoid and S, T be subsemigroups (submonoids) of U. Then U is the internal Zappa–Szép product of S and T if U is uniquely factorisable with factors S and T, that is, $U = ST$ and every element $u \in U$ has a unique expression as*

$$u = st \text{ where } s \in S \text{ and } t \in T.$$

It is well known that there is an equivalence between internal and external Zappa–Szép products of groups. A similar correspondence exists for monoids [9], as we now explain. If $Z = S \bowtie T$ is the external Zappa–Szép product of monoids S and T, then putting

$$S' = \{(s, 1_T) : s \in S\} \text{ and } T' = \{(1_S, t) : t \in T\}$$

we have that S' and T' are submonoids of Z, isomorphic to S and T respectively, such that Z is the internal Zappa–Szép product of S' and T'. Conversely if $Z = ST$ is the internal Zappa–Szép product of submonoids S and T then uniqueness of decompositions and associativity enable us to show that Z is isomorphic to an external Zappa–Szép product $S \bowtie T$.

In general, there is no such correspondence between internal and external Zappa–Szép product of semigroups. It is not difficult to get an external Zappa–Szép product from an internal Zappa–Szép product of semigroups but the converse is not obvious. However we can characterise those external Zappa–Szép products of semigroups which are also internal ones.

Wazzan [20] noticed that if an idempotent exists as in Lemma 2.3, which is a right identity for S and a left identity for T then for $s \in S$, $t \in T$ in any external Zappa–Szép product we have

$$es = e \cdot s,\ e = e^s \text{ and } t \cdot e = e,\ t^e = te.$$

We use above information to show that the internal and external Zappa–Szép products of semigroups correspond to each other subject to certain conditions.

Theorem 4.3 *Let U be a semigroup and suppose that $U = ST$ is the internal Zappa–Szép product of subsemigroups S and T. Then there is an action of T on the left of S and an action of S on the right of T such that (ZS1)–(ZS4) hold and $U \cong S \bowtie T$. Further, there exists an idempotent $e \in S \cap T$ such that e is right identity for S and left identity for T with*

$$es = e \cdot s,\ e = e^s \textit{ and } t \cdot e = e,\ t^e = te$$

for all $s \in S$ and $t \in T$.

Conversely suppose that $U = S \bowtie T$ is an external Zappa–Szép product of semigroups S and T such that S has a right identity e_S and T has a left identity e_T with

$$e_S s = e_T \cdot s,\ e_T = e_T^s \textit{ and } t \cdot e_S = e_S,\ t^{e_S} = te_T$$

for all $s \in S$ and $t \in T$. Let

$$S' = \{(s, e_T) : s \in S\} \textit{ and } T' = \{(e_S, t) : t \in T\}.$$

Then $S \cong S'$, $T \cong T'$ and $U = S'T'$ is the internal Zappa–Szép product of S' and T'.

Proof If $U = ST$ is the internal Zappa–Szép product of subsemigroups S and T, then from associativity and uniqueness it is easy to show that there is an action of T on the left of S and an action of S on the right of T such that (ZS1)–(ZS4) hold and $U \cong S \bowtie T$. Also as the factorisation is unique, therefore there exists an idempotent e which is a right identity for S and a left identity for T such that

$$es = e \cdot s,\ e = e^s \text{ and } t \cdot e = e,\ t^e = te$$

for all $s \in S$ and $t \in T$.

Conversely, let $U = S \bowtie T$ be an external Zappa–Szép product of S and T. Suppose there is a right identity e_S of S and a left identity e_T of T satisfying

$$e_S s = e_T \cdot s,\ e_T = e_T^s \text{ and } t \cdot e_S = e_S,\ t^{e_S} = te_T$$

for all $s \in S$ and $t \in T$. Put

$$S' = \{(s, e_T) : s \in S\} \text{ and } T' = \{(e_S, t) : t \in T\}.$$

Define a map $\alpha : S \to S'$ by

$$(s)\alpha = (s, e_T).$$

Clearly α is well defined, one-one and onto. To check that α is a homomorphism, let $s, s' \in S$. Then

$$\begin{aligned}
(ss')\alpha &= (ss', e_T) \\
&= (se_S s', e_T) && \text{because } e_S \text{ is right identity for } S \\
&= (s(e_T \cdot s'), e_T^{s'} e_T) && \text{because } e_T \cdot s' = e_S s' \text{ and } e_T^{s'} = e_T \text{ by our supposition} \\
&= (s, e_T)(s', e_T) \\
&= (s)\alpha(s')\alpha.
\end{aligned}$$

Hence $S \cong S'$. Similarly, defining a map $\beta : T \to T'$ by $(t)\beta = (e_S, t)$, it is easy to check that $T \cong T'$. Now for $(s, e_T) \in S'$ and $(e_S, t) \in T'$, we see that

$$\begin{aligned}
(s, e_T)(e_S, t) &= (s(e_T \cdot e_S), e_T^{e_S} t) \\
&= (se_S, e_T t) && \text{because } e_T \cdot e_S = e_S \text{ and } e_T^{e_S} = e_T \\
&= (s, t) && \text{because } e_S \text{ is a right identity of } S \text{ and} \\
& && e_T \text{ is a left identity of } T.
\end{aligned}$$

Thus each element $(s, t) \in U$ can be written as

$$(s, t) = (s, e_T)(e_S, t)$$

and this composition is evidently unique. Hence $U = S'T'$ is the internal Zappa–Szép product of S' and T'. □

We can also use Preston's technique [14] to get an internal Zappa–Szép product from an external Zappa–Szép product $S \bowtie T$ of semigroups S and T by adjoining identities to S and T and extending the actions to $S^{(1)}$ and $T^{(1)}$.

Theorem 4.4 *Let S and T be semigroups and let $U = S \bowtie T$ be the external Zappa–Szép product of S and T. Let $S^{(1)}$ and $T^{(1)}$ be the semigroups obtained from S and T, respectively by adjoining an identity. Then, setting $t \cdot 1 = 1$ and $1^s = 1$ and defining*

$$(1, s) \mapsto 1 \cdot s = s,\ (t, 1) \mapsto t^1 = t$$

to be the identity maps, the extended actions satisfy (ZS1)–(ZS8). Thus we can form the Zappa–Szép product $S^{(1)} \bowtie T^{(1)}$ of $S^{(1)}$ and $T^{(1)}$. Moreover $S \cong S' = \{(s,1) : s \in S\}$ under the map $s \mapsto (s,1)$ and $T \cong T' = \{(1,t) : t \in T\}$ under the map $t \mapsto (1,t)$ and $U = S'T'$ is a subsemigroup of $S^{(1)} \bowtie T^{(1)}$, and is uniquely factorisable with factors S' and T'.

Proof We first check that $S^{(1)} \bowtie T^{(1)}$ satisfies all axioms of a Zappa–Szép product.

(ZS1) Let $t, t' \in T^{(1)}$ and $s \in S^{(1)}$. Then to check that $tt' \cdot s = t \cdot (t' \cdot s)$, we have eight cases as follows:

(i) $t = t' = s = 1$;	(v) $t = 1,\ t' = 1$ and $s \neq 1$;
(ii) $t = 1,\ t' \neq 1$ and $s = 1$;	(vi) $t = 1,\ t' \neq 1$ and $s \neq 1$;
(iii) $t \neq 1,\ t' \neq 1$ and $s = 1$;	(vii) $t \neq 1,\ t' = 1$and $s \neq 1$;
(iv) $t \neq 1,\ t' = 1$ and $s = 1$;	(viii) $t \neq 1,\ t' \neq 1$ and $s \neq 1$.

(i)–(iv) are obvious because $s = 1$ in all four cases and $t \cdot 1 = 1$ for all $t \in T^{(1)}$. For (v) we see that

$$tt' \cdot s = 11 \cdot s = 1 \cdot s = s = 1 \cdot (1 \cdot s) = t \cdot (t' \cdot s).$$

For (vi)

$$tt' \cdot s = 1t' \cdot s = t' \cdot s = 1 \cdot (t' \cdot s) = t \cdot (t' \cdot s).$$

For (vii)

$$tt' \cdot s = t \cdot s = t \cdot (1 \cdot s) = t \cdot (t' \cdot s).$$

For (viii) because $t \neq 1,\ t' \neq 1$ and $s \neq 1$ and $S^{(1)}$ and $T^{(1)}$ are the monoids obtained from S and T, respectively by adjoining an identity, so $t, t' \in T$ and $s \in S$. Also as $S \bowtie T$ is the external Zappa–Szép product of S and T, we have

$$tt' \cdot s = t \cdot (t' \cdot s).$$

Hence $T^{(1)}$ acts on $S^{(1)}$ from the left and thus (ZS1) holds.

(ZS2) Let $t \in T^{(1)}$ and $s, s' \in S^{(1)}$. We need to check that $t \cdot ss' = (t \cdot s)(t^s \cdot s')$. Now again we have eight cases as follows:

(i) $t = s = s' = 1$;	(v) $t \neq 1,\ s = 1$ and $s' = 1$;
(ii) $t = 1,\ s = 1$ and $s' \neq 1$;	(vi) $t \neq 1,\ s = 1$ and $s' \neq 1$;
(iii) $t = 1,\ s \neq 1$ and $s' = 1$;	(vii) $t \neq 1,\ s \neq 1$ and $s' = 1$;
(iv) $t = 1,\ s \neq 1$ and $s' \neq 1$;	(viii) $t \neq 1,\ s \neq 1$ and $s' \neq 1$.

(i)-(iv) are clear because $t = 1$ in all four cases and $1 \cdot s = s$, $1^s = 1$ for all $s \in S$. For (v), we see that $t \cdot (ss') = t \cdot 11 = t \cdot 1 = 1 = (t \cdot 1)(t^1 \cdot 1) = (t \cdot s)(t^s \cdot s')$. For (vi)

$$t \cdot (ss') = t \cdot s' = (t \cdot 1)(t^1 \cdot s') = (t \cdot s)(t^s \cdot s').$$

For (vii)

$$t \cdot (ss') = t \cdot s = (t \cdot s)(t^s \cdot 1) = (t \cdot s)(t^s \cdot s').$$

For (viii) because $t \neq 1,\ s \neq 1$ and $s' \neq 1$ and $S^{(1)}$ and $T^{(1)}$ are the monoids obtained from S and T, respectively by adjoining an identity, so $t \in T$ and $s, s' \in S$. Also, as $S \bowtie T$ is the external Zappa–Szép product of S and T, we have

$$t \cdot (ss') = (t \cdot s)(t^s \cdot s').$$

Hence (ZS2) holds.

(ZS3) is dual of (ZS1) and (ZS4) is dual of (ZS2).

Also (ZS5)–(ZS8) hold because of the way in which we have extended our actions of S on T and T on S to actions of $S^{(1)}$ on $T^{(1)}$ and $T^{(1)}$ on $S^{(1)}$ respectively.

Hence we may form the Zappa–Szép product $V = S^{(1)} \bowtie T^{(1)}$ of the monoids $S^{(1)}$ and $T^{(1)}$.

Next let

$$S'' = \{(s,1) : s \in S^{(1)}\} \text{ and } T'' = \{(1,t) : t \in T^{(1)}\}.$$

It is easy to check that S'', T'' are submonoids of V and $V = S''T''$ is the internal Zappa–Szép product of S'' and T''. Moreover

$$\alpha : S^{(1)} \to S'' \text{ and } \beta : T^{(1)} \to T''$$

are isomorphisms. Now let

$$S' = \{(s,1) : s \in S\} \text{ and } T' = \{(1,t) : t \in T\}.$$

Then S' and T' are subsemigroups of S'' and T'', respectively and hence of V. Clearly

$$\alpha|_S : S \to S' \text{ and } \beta|_T : T \to T'$$

are isomorphisms.

Note that any element $(s,t) \in U$ can be written as

$$(s,t) = (s,1)(1,t) \text{ for } (s,1) \in S' \text{ and } (1,t) \in T'$$

and this decomposition is unique.

Thus $U = S'T'$ is uniquely factorisable with factors S' and T'. □

5. Zappa–Szép product of bisimple inverse monoids

This section is devoted to the study of Zappa–Szép products of bisimple inverse monoids. For convenience of the reader we begin this section with the definition of the Bruck–Reilly extension of a monoid.

Definition 5.1 *Let S be a monoid and let θ be an endomorphism on S. Define a binary operation on $\mathbb{N}^0 \times S \times \mathbb{N}^0$ by the rule*

$$(m,r,n)(p,s,q) = \big(m-n+t, (r\theta^{t-n})(s\theta^{t-p}), q-p+t\big),$$

where $t = \max(n,p)$ and θ^0 is the identity map on S. It is easy to check that this binary operation is associative and so $\mathbb{N}^0 \times S \times \mathbb{N}^0$ is a monoid called Bruck–Reilly extension of a monoid determined by θ and denoted by $BR(S,\theta)$.

Note that the outer coordinates in the multiplication combine exactly as in the bicyclic semigroup $B = \mathbb{N}^0 \times \mathbb{N}^0$ and so the bicyclic semigroup is just the Bruck–Reilly extension of the trivial group. The bicyclic monoid is an important example of an inverse semigroup that is not a group, having left (right) invertible elements that are not right (left) invertible.

The above construction plays an important role in the theory of inverse semigroups and was used to classify several classes of simple inverse semigroups.

It is known that the Bruck–Reilly extension of a monoid S can be realised as a Zappa–Szép product of $(\mathbb{N},+)$ and a semidirect product $\mathbb{N} \rtimes S$ [9]. The author and Gould generalised this result to a broader class of semigroups which they call special $\widetilde{\mathcal{D}}_E$-simple restriction monoids [5]. The structure of such semigroups is determined by idempotents and is a generalisation of bisimple inverse monoids. In a bisimple inverse monoid all idempotents are $\mathcal{D}$-equivalent. The Bruck–Reilly extension of a monoid and bicyclic semigroups are natural examples of bisimple inverse monoids. In this section we provide an insight to the results of [5] by considering bisimple inverse monoids. We see that a bisimple inverse monoid is the Zappa–Szép product of its $\mathcal{R}$-class of identity and $\mathcal{L}$-class of identity. We notice that when $\mathcal{H}$ is a congruence on S, then we obtain a semidirect product instead of a Zappa–Szép product. Kunze's result on Bruck–Reilly extension of a monoid, thus can be viewed as a bisimple inverse monoid containing a monoid transversal of $\mathcal{H}$-classes of L_1.

The following theorem shows that a bisimple inverse monoid can be decomposed as a Zappa–Szép product of submonoids.

Theorem 5.2 *Suppose S is a bisimple inverse monoid. Suppose there is a submonoid L such that $L \subseteq L_1$ and $|L \cap H| = 1$ for every $\mathcal{H}$-class $H \subseteq L_1$. Then $Z = L \bowtie R_1$ is a Zappa–Szép product of L and R_1 under the actions defined by:*

$$r \cdot l = d \text{ where } d \in L \text{ and } d \,\mathcal{R}\, rl$$

and

$$r^l = d^{-1}rl \text{ where } d \in L, \text{ and } d \,\mathcal{R}\, rl$$

for $l \in L$ and $r \in R_1$. Further $S \cong Z$.

Proof Let $r \in R_1$ and $l, d \in L$ with $d \,\mathcal{R}\, rl$. We note that as $d \,\mathcal{R}\, rl$ and $\mathcal{R}$ is a left congruence, so

$$d^{-1}d \,\mathcal{R}\, d^{-1}rl \Rightarrow 1 \,\mathcal{R}\, d^{-1}rl.$$

Now as $rl = dd^{-1}rl$, we have that

$$rl \,\mathcal{L}\, d^{-1}rl.$$

We now check the axioms of a Zappa–Szép product.

(ZS1) Let $l \in L$ and $r, s \in R_1$. Then

$$rs \cdot l = d \text{ where } d \in L \text{ and } d \,\mathcal{R}\, rsl \longrightarrow (1)$$

Also

$$r \cdot (s \cdot l) = r \cdot u = v \text{ where } u, v \in L \text{ and } u \,\mathcal{R}\, sl,\; v \,\mathcal{R}\, ru.$$

Now

$$v \,\mathcal{R}\, ru \,\mathcal{R}\, rsl \,\mathcal{R}\, d. \longrightarrow (2)$$

From (1) and (2) and the fact $|L \cap H_d| = |L \cap H_v| = 1$, we see that (ZS1) holds.

(ZS2) For $l, k \in L$ and $r \in R_1$,

$$r \cdot lk = d \text{ where } d \in L, \text{ and } d \,\mathcal{R}\, rlk$$

and

$$(r \cdot l)(r^l \cdot k) = u(u^{-1}rl \cdot k) = uv$$

where $u \in L$ with $u \,\mathcal{R}\, rl$ and $v \in L$ with $v \,\mathcal{R}\, u^{-1}rlk$.

We therefore have

$$\begin{array}{lll} & uv \,\mathcal{R}\, uu^{-1}rlk & \text{since } \mathcal{R} \text{ is left congruence} \\ \Rightarrow & uv \,\mathcal{R}\, rl(rl)^{-1}rlk & \text{since } uu^{-1} = rl(rl)^{-1} \text{ as } rl \,\mathcal{R}\, u \\ \Rightarrow & uv \,\mathcal{R}\, rlk & \\ \Rightarrow & uv \,\mathcal{R}\, d. & \end{array}$$

Thus $uv = d$ as u, v and $d \in L$. Hence (ZS2) holds.

(ZS3) For $l, k \in L$ and $r \in R_1$,

$$r^{lk} = d^{-1}rlk \text{ where } d \in L \text{ and } d \,\mathcal{R}\, rlk$$

and

$$\begin{array}{rcll} (r^l)^k & = & (u^{-1}rl)^k & \text{where } u \in L \text{ and } u \,\mathcal{R}\, rl \\ & = & v^{-1}u^{-1}rlk & \text{where } v \in L \text{ and } v \,\mathcal{R}\, u^{-1}rlk \\ & = & (uv)^{-1}rlk. & \end{array}$$

Now as $v \,\mathcal{R}\, u^{-1}rlk$ and $\mathcal{R}$ is a left congruence, so using the argument from (ZS2), we see that $uv \,\mathcal{R}\, rlk \,\mathcal{R}\, d$ and $uv = d$ and hence (ZS3) holds.

(ZS4) Let $l \in L$ and $r, s \in R_1$. Then

$$(rs)^l = d^{-1}rsl \text{ where } d \in L \text{ and } d \,\mathcal{R}\, rsl$$

and

$$\begin{array}{rcll} r^{s \cdot l}s^l & = & r^u u^{-1}sl & \text{where } u \in L \text{ and } u \,\mathcal{R}\, sl \\ & = & v^{-1}ruu^{-1}sl & \text{where } v \in L \text{ and } v \,\mathcal{R}\, ru \\ & = & v^{-1}rsl(sl)^{-1}sl & \text{because } u \,\mathcal{R}\, sl \\ & = & v^{-1}rsl. & \end{array}$$

From $u \,\mathcal{R}\, sl$ we have

$$v \,\mathcal{R}\, ru \,\mathcal{R}\, rsl \,\mathcal{R}\, d,$$

so that $v = d$ and thus (ZS4) holds and hence $L \bowtie R_1$ is a Zappa-Szép product of L and R_1.

Now we have to show that $Z \cong S$. For this define a map $\varphi : Z \longrightarrow S$ by $(l, r)\varphi = lr$. This map is obviously well defined. We check that it is one-one, onto and a homomorphism.

Let $(l, r), (l', r') \in Z$ be such that

$$(l, r)\varphi = (l', r')\varphi,$$

so that

$$lr = l'r'.$$

Now we see that

$$lr \,\mathcal{R}\, l \cdot 1 = l \text{ and } l'r' \,\mathcal{R}\, l' \cdot 1 = l'.$$

Thus using $lr = l'r'$, we get $l \,\mathcal{R}\, l'$ and so $l = l'$. Also

$$lr = l'r' = lr' \Rightarrow l^{-1}lr = l^{-1}lr' \Rightarrow r = r'$$

because $l^{-1}l = 1$. Thus $(l, r) = (l', r')$ and hence φ is one-one.

Now to show φ is onto, let $s \in S$. Then there exists $u \in L$ such that $s \mathcal{R} u$.

As $\mathcal{R}$ is a left congruence, $u^{-1}s \mathcal{R} u^{-1}u = 1$. Also $s \mathcal{R} u$, we have that $uu^{-1}s = s$, and so $u^{-1}s \mathcal{L} s$. Thus there exist $u \in L$ and $u^{-1}s \in R_1$ such that

$$(u, u^{-1}s)\varphi = uu^{-1}s = s.$$

Hence φ is onto.

Let $(l, r), (l', r') \in Z$, then

$$\begin{array}{rcll} \big((l,r)(l',r')\big)\varphi & = & (l(r \cdot l'), r^{l'}r')\varphi & \\ & = & (lu, u^{-1}rl'r')\varphi & \quad \text{where } u \in L \text{ and } u \mathcal{R} rl' \\ & = & luu^{-1}rl'r' & \\ & = & lrl'r' & \\ & = & (l,r)\varphi(l',r')\varphi. & \end{array}$$

Hence φ is a homomorphism and hence an isomorphism. □

Remark 5.3 *As S is monoid, it is easy to check that axioms (ZS5)–(ZS8) of Zappa–Szép product also hold.*

Corollary 5.4 *Suppose S is a combinatorial bisimple inverse monoid. Let $L = L_1$ be the $\mathcal{L}$-class of the identity and $R = R_1$ be the $\mathcal{R}$-class of the identity. Then $Z = L \bowtie R$ is a Zappa–Szép product of L and R under the actions defined by:*

$$r \cdot l = d \text{ where } d \mathcal{R} rl$$

and

$$r^l = u \text{ where } u \mathcal{L} rl$$

for $l \in L$ and $r \in R$. Further $Z \cong S$.

Proof As S is combinatorial, it is clear that $L = L_1$ has the property required for Theorem 5.2. It is also clear that action of R on L coincides with that in Theorem 5.2.

Now let $l \in L$ and $r \in R$, and suppose that $r \cdot l = d$, $d \in L$ and so that $d \mathcal{R} rl$. Then

$$1 = d^{-1}d \mathcal{R} d^{-1}rl$$

and as $rl = dd^{-1}rl$, we have $rl \mathcal{L} d^{-1}rl$.

It follows that action of L on R coincides with that in Theorem 5.2. □

We now show that if $\mathcal{H}$ is a congruence, then we get a semidirect product instead of a Zappa–Szép product.

Theorem 5.5 *Suppose S is a bisimple inverse monoid. Suppose there is a submonoid L such that $L \subseteq L_1$ and $|L \cap H| = 1$ for every $\mathcal{H}$-class $H \subseteq L_1$. Suppose in addition that $\mathcal{H}$ is congruence. Let $R = \{u^{-1} : u \in L\}$. Then R is a submonoid of R_1, $|R \cap H| = 1$ for all $H \subseteq R_1$ and $H_1 \ltimes R$ is a semidirect product isomorphic to R_1 under the action defined by*

$$r \cdot h = rhr^{-1} \text{ for } r \in R \text{ and } h \in H_1.$$

Proof This action is well defined, as we see that

$$rhr^{-1}\,\mathcal{H}\,r1r^{-1}=rr^{-1}=1$$

giving $rhr^{-1}\in H$.

It is easy to check that R is submonoid of R_1 and $|R\cap H|=1$ for all $H\subseteq R_1$.

Now we check the axioms of a semidirect product.

(1) For $1\in R$ and $h\in H_1$,

$$1\cdot h=1h1^{-1}=h.$$

(2) Let $r,s\in R$ and $h\in H_1$. Then

$$r\cdot(s\cdot h)=r\cdot(shs^{-1})=r(shs^{-1})r^{-1}=rsh(rs)^{-1}=rs\cdot h.$$

(3) For $r\in R$ and $h,k\in H_1$,

$$(r\cdot h)(r\cdot k)=rhr^{-1}rkr^{-1}.$$

As $rh\,\mathcal{H}\,r\,\mathcal{L}\,r^{-1}r$, so $rhr^{-1}r=rh$ and thus

$$(r\cdot h)(r\cdot k)=rhkr^{-1}=r\cdot hk.$$

Thus $H_1\ltimes R$ is semidirect product of H_1 and R.

Define $\alpha:H_1\ltimes R\longrightarrow R_1$ by $(h,r)\alpha=hr$. Clearly α is well defined as $hr\,\mathcal{H}\,r$. Now let (h,r), $(k,s)\in H_1\ltimes R$, then

$$\begin{aligned}(h,r)\alpha &= (k,s)\alpha\\ \Rightarrow hr &= ks.\end{aligned}$$

Now $hr\,\mathcal{H}\,r$ and $ks\,\mathcal{H}\,s$. So $hr=ks$ gives $r\,\mathcal{H}\,s$ and so $r=s$. Also

$$hr=ks\Rightarrow hr=kr\Rightarrow hrr^{-1}=krr^{-1}\Rightarrow h=k$$

because $rr^{-1}=1$. Thus α is one-one.

To show that α is onto, let $t\in R_1$. For $r\in R$ with $r\,\mathcal{H}\,t$, we see that

$$1\,\mathcal{R}\,r \text{ and } 1r=r,$$

so by Green's Lemma $\rho_r:\,H_1\to H_r$ is a bijection. Therefore

$$\begin{aligned}t &= u\rho_r \text{ for some } u\in H_1\\ &= ur\\ &= (u,r)\alpha.\end{aligned}$$

Thus α is onto.

Now let (h,r), $(k,s)\in H_1\ltimes R$. Then

$$\begin{aligned}\big((h,r)(k,s)\big)\alpha &= (hrkr^{-1},rs)\alpha\\ &= hrkr^{-1}rs\\ &= hrks \qquad \text{because } rk\,\mathcal{L}\,r^{-1}r\\ &= (h,r)\alpha(k,s)\alpha.\end{aligned}$$

Thus α is a homomorphism and hence $H_1 \ltimes R \cong R_1$.

$\square$

The left (right) dual of Theorem 5.5 also holds as we see in the following.

Corollary 5.6 *Let S be a bisimple inverse monoid. Suppose there is a submonoid R such that $R \subseteq R_1$ and $|R \cap H| = 1$ for every $\mathcal{H}$-class $H \subseteq R_1$. Suppose in addition that $\mathcal{H}$ is congruence. Let $L = \{u^{-1} : u \in R\}$. Then L is a submonoid of L_1, $|L \cap H| = 1$ for all $H \subseteq L_1$ and $L \rtimes H_1$ is semidirect product isomorphic to L_1 under the action defined by*

$$h^l = l^{-1}hl \text{ for } h \in H \text{ and } l \in L.$$

Deduction 5.1 *From Corollary 5.6 we deduce Kunze's result proved in [9]. Kunze showed that if S is a monoid, θ an endomorphism of S and $\mathbb{N}$ is the set of natural numbers under addition, then a semidirect product $\mathbb{N} \rtimes S$ can be formed under the multiplication,*

$$(k, s)(l, t) = (k + l, (s\theta^l)t).$$

Now we see that

$$L_1 = \{(l, s, 0) : l \in \mathbb{N}^0, s \in S\},$$

so that if we put

$$L = \{(l, e, 0) : l \in \mathbb{N}^0\} \cong \mathbb{N}^0,$$

then L is submonoid of L_1 such that $|L \cap H| = 1$ for all $H \subseteq L_1$. Further,

$$H_1 = \{(0, s, 0) : s \in H\}.$$

For $(l, e, 0) \in L$ and $(0, s, 0) \in H_1$,

$$\begin{aligned} (0, s, 0)^{(l,e,0)} &= (l, e, 0)^{-1}(0, s, 0)(l, e, 0) \\ &= (0, s\theta^l, 0) \in H_1. \end{aligned}$$

Thus $L \rtimes H_1$ is semidirect product under multiplication defined by

$$\begin{aligned} \big((k, e, 0), (0, s, 0)\big)\big((l, e, 0), (0, t, 0)\big) &= \big((k, e, 0)(l, e, 0), (0, s, 0)^{(l,e,0)}(0, t, 0)\big) \\ &= \big((k + l, e, 0), (0, s\theta^l t, 0)\big). \end{aligned}$$

We are now going to prove a result corresponding to Theorem 5.5 in the case where $\mathcal{H}$ is not a congruence and thus we show that the $\mathcal{R}$-class of identity is itself a Zappa–Szép product, rather than a semidirect product.

Theorem 5.7 *Let S be a bisimple inverse monoid. Suppose there is a submonoid L such that $L \subseteq L_1$ and $|L \cap H| = 1$ for every $\mathcal{H}$-class $H \subseteq L_1$. Let $R = \{u^{-1} : u \in L\}$. Then R is a submonoid of R_1, $|R \cap H| = 1$ for every $H \subseteq R_1$ and $H_1 \bowtie R$ is Zappa–Szép product isomorphic to R_1 under the action of R on H_1 defined by*

$$r \cdot h = rht^{-1} \text{ where } t\,\mathcal{L}\,rh \text{ and } t \in R.$$

and action of H_1 on R by

$$r^h = t \text{ where } t\,\mathcal{L}\,rh \text{ and } t \in R.$$

From here we see that if $\mathcal{H}$ is a congruence, then

$$rh\,\mathcal{H}\,r1 = r$$

and thus we get back to our semidirect product as in Theorem 5.5.

Proof From Theorem 5.5 we know that R is a submonoid of R_1 and $|R \cap H| = 1$ for every $H \subseteq R_1$. Now we have to prove that $H_1 \bowtie R$ is a Zappa–Szép product isomorphic to R_1.

First of all we see that action of R on H_1 is well defined as

$$rht^{-1}\,\mathcal{L}\,tt^{-1} = 1$$

and

$$rht^{-1}\,\mathcal{R}\,rht^{-1}t = rh\,\mathcal{R}\,r1 = r\,\mathcal{R}\,1,$$

so that $rht^{-1}\,\mathcal{H}\,1$, that is, $rht^{-1} \in H_1$.

We now check that the axioms of a Zappa–Szép product hold.

(ZS1) For $r, s \in R$ and $h \in H_1$,

$$rs \cdot h = rsht^{-1} \text{ where } t \in R \text{ and } t\,\mathcal{L}\,rsh$$

and

$$\begin{aligned} r \cdot (s \cdot h) &= r \cdot shu^{-1} && \text{where } u \in R \text{ and } u\,\mathcal{L}\,sh \\ &= rshu^{-1}v^{-1} && \text{where } v \in R \text{ and } v\,\mathcal{L}\,rshu^{-1} \\ &= rsh(vu)^{-1}. \end{aligned}$$

Now as $v\,\mathcal{L}\,rshu^{-1}$, we have $vu\,\mathcal{L}\,rshu^{-1}u$. But $sh\,\mathcal{L}\,u^{-1}u$, so that $vu\,\mathcal{L}\,rsh\,\mathcal{L}\,t$ by choice of t.

As $v, u, vu, t \in R$, we must have $vu = t$, so that (ZS1) holds.

(ZS2) Let $r \in R$ and $h, k \in H_1$. Then

$$r \cdot hk = rhku^{-1} \text{ where } u \in R \text{ and } u\,\mathcal{L}\,rhk$$

and

$$\begin{aligned} (r \cdot h)(r^h \cdot k) &= (rhv^{-1})(v \cdot k) && \text{where } v \in R \text{ and } v\,\mathcal{L}\,rh \\ &= rhv^{-1}vkx^{-1} && \text{where } x \in R \text{ and } x\,\mathcal{L}\,vk \\ &= rhkx^{-1} && \text{because } rh\,\mathcal{L}\,v^{-1}v. \end{aligned}$$

Now $x\,\mathcal{L}\,vk\,\mathcal{L}\,rhk\,\mathcal{L}\,u \Rightarrow x\,\mathcal{L}\,u$, so that $x = u$ and thus (ZS2) holds.

(ZS3) For $r \in R$ and $h, k \in H_1$,

$$r^{hk} = t \text{ where } t \in R \text{ and } t\,\mathcal{L}\,rhk$$

and

$$\begin{aligned} (r^h)^k &= u^k && \text{where } u \in R \text{ and } u\,\mathcal{L}\,rh \\ &= v && \text{where } v \in R \text{ and } v\,\mathcal{L}\,uk. \end{aligned}$$

So $v\,\mathcal{L}\,uk\,\mathcal{L}\,rhk\,\mathcal{L}\,t$ implies that $v = t$.

Thus (ZS3) also holds.

(ZS4) For $r, s \in R$ and $k \in H_1$,

$$(rs)^k = t \text{ where } t \in R \text{ and } t\,\mathcal{L}\,rsk$$

and

$$\begin{aligned} r^{s\cdot k}s^k &= r^{sku^{-1}}u \quad \text{where } u \in R \text{ and } u\,\mathcal{L}\,sk \\ &= vu \qquad\quad \text{where } v \in R \text{ and } v\,\mathcal{L}\,rsku^{-1}. \end{aligned}$$

So $v \ \mathcal{L} \ rsku^{-1}$ and as $\mathcal{L}$ is right congruence, so we have

$$vu\,\mathcal{L}\,rsku^{-1}u.$$

But $sk \ \mathcal{L} \ u^{-1}u$, so $vu \ \mathcal{L}$ rsk $\mathcal{L} \ t$ and thus $vu = t$.

Hence (ZS4) holds and thus $Z = H_1 \bowtie R$ is a Zappa–Szép product of H_1 and R under multiplication defined by:

$$(h, r)(k, s) = (hrkt^{-1}, ts) \text{ where } t\,\mathcal{L}\,rk.$$

Define a map $\gamma : Z \to R_1$ by $(h, r)\gamma = hr$. Clearly γ is well defined.

Now let (h, r), $(k, s) \in H_1 \rtimes R$, then

$$(h, r)\gamma = (k, s)\gamma \Rightarrow hr = ks$$

We have that

$$hr\,\mathcal{L}\,r \text{ and } hr\,\mathcal{R}\,h\,\mathcal{R}\,r.$$

Thus $hr\,\mathcal{H}\,r$ and similarly $ks\,\mathcal{H}\,s$. So $hr = ks$ gives $r\,\mathcal{H}\,s$ and so $r = s$. Also

$$hr = ks \Rightarrow hr = kr \Rightarrow hrr^{-1} = krr^{-1} \Rightarrow h = k$$

as $rr^{-1} = 1$. Thus γ is one-one.

It is easy to check that γ is onto and a homomorphism. Hence $H_1 \bowtie R \cong R_1$. □

Remark 5.8 *As in Theorem 5.7, Z is Zappa–Szép product of monoids, so it is easy to check that axioms for identities of Zappa–Szép product must also hold.*

We end this paper with two examples of bisimple inverse monoids, but first we mention following result which is folklore and is easy to prove.

Proposition 5.9 *Let S and T be inverse monoids. Then $S \times T$ is an inverse monoid with $(s, t)^{-1} = (s^{-1}, t^{-1})$ and for any (s, t), $(u, v) \in S \times T$*

$$(s, t)\;\mathcal{R}\;(u, v) \textit{ if and only if } s\,\mathcal{R}\,u \textit{ and } t\,\mathcal{R}\,v$$

$$(s, t)\,\mathcal{L}\,(u, v) \textit{ if and only if } s\,\mathcal{L}\,u \textit{ and } t\,\mathcal{L}\,v.$$

Further if S and T are bisimple, so is $S \times T$. If $\mathcal{H}$ is a congruence on both S and T, then $\mathcal{H}$ is a congruence on $S \times T$.

Example 5.10 *Let $\mathbb{R}^* = \{r \in \mathbb{R} : r \geq 0\}$. Also let*

$$\mathbb{R}^* \times \mathbb{R}^* = \{(r, s) : r, s \in \mathbb{R}^*\}.$$

Define a binary operation on $\mathbb{R}^* \times \mathbb{R}^*$ *as follows:*

$$(r,s)(p,q) = \big(r - s + max(s,p), q - p + max(s,p)\big).$$

The proof that $\mathbb{R}^* \times \mathbb{R}^*$ *is a bisimple inverse monoid with identity element* $(0,0)$ *is a folklore and a direct generalisation of the result for the bicyclic monoid. Also* $\mathcal{H}$ *is the identity relation which implies that* $\mathbb{R}^* \times \mathbb{R}^*$ *is combinatorial.*

Next let $I = (\mathbb{R}^* \times \mathbb{R}^*) \times G = \{\big((r,s),g\big) : (r,s) \in \mathbb{R}^* \times \mathbb{R}^*,\ g \in G\}$ *be the direct product of* $\mathbb{R}^* \times \mathbb{R}^*$ *and* G*, where* G *is a group.*

From Proposition 5.9 I *is a bisimple inverse monoid with identity* $\big((0,0),e\big)$*. Any element* $\big((r,s),g\big) \in I$ *has inverse* $\big((r,s)^{-1}, g^{-1}\big)$*.*

Now for any $\big((r,s),g\big)$*,* $\big((u,v),h\big) \in I$*, we have from Proposition 5.9*

$$\begin{array}{ll} & \big((r,s),g\big)\,\mathcal{R}\,\big((u,v),h\big) \\ \Leftrightarrow & (r,s)\,\mathcal{R}\,(u,v) \text{ and } g\,\mathcal{R}\,h \\ \Leftrightarrow & r = u \text{ and } g\,\mathcal{R}\,h \end{array}$$

and

$$\begin{array}{ll} & \big((r,s),g\big)\,\mathcal{L}\,\big((u,v),h\big) \\ \Leftrightarrow & (r,s)\,\mathcal{L}\,(u,v) \text{ and } g\,\mathcal{L}\,h \\ \Leftrightarrow & s = v \text{ and } g\,\mathcal{L}\,h. \end{array}$$

Next we see that

$$R_1 = \{\big((0,r),h\big) : r \in \mathbb{R}^*\}$$

is the $\mathcal{R}$*-class of the identity and*

$$H_1 = \{\big(0,0),h\big) : h \in G\}$$

is the $\mathcal{H}$*-class of the identity. Again from Proposition 5.9* $\mathcal{H}$ *is a congruence on* I*. Take*

$$R = \{\big((0,r),e\big) : r \in \mathbb{R}^*\}$$

From Theorem 5.5 it is easy to check that $H_1 \rtimes R$ *is a semidirect product under the action defined by*

$$\big((0,r),e\big) \cdot \big((0,0),h\big) = \big((0,r),e\big)\big((0,0),h\big)\big((0,r),e\big)^{-1} = \big((0,0),h\big)$$

for $r \in R$ *and* $h \in H_1$*.*

The following example shows that the Bicyclic semigroup B can be written as a Zappa–Szép product of $\mathcal{L}$-classes and $\mathcal{R}$-classes.

Example 5.11 *The bicyclic semigroup* B *is the Zappa-Szép product of* $L = L_1$ *and* $R = R_1$*, where*

$$L = \{(m,0) : m \in \mathbb{N}^0\} \cong \mathbb{N}^0$$

$$R = \{(0,n) : n \in \mathbb{N}^0\} \cong \mathbb{N}^0.$$

The above example is a consequence of Corollary 5.4 as R and L act on each other from the left and right respectively and $B \cong L \bowtie R$.

Acknowledgements

The author is grateful to the Sukkur IBA University for the constant support and encouragement. Some of the work in this article is taken from the PhD of the author [22], written under the guidance of Professor Victoria Gould. The author is grateful to her and the Department of Mathematics of the University of York for their support and guidance. The author is also thankful to the Schlumberger Foundation for funding her research during her stay at the University of York.

References

[1] Brin MG. On the Zappa-Szép product. Communications in Algebra. 2005; 33 (2): 393-424. doi: 10.1081/AGB-200047404

[2] Brownlowe N, Ramagge J, Robertson D, Whittaker MF. Zappa-Szép products of semigroups and their C^*-algebras. Journal of Functional Analysis. 2014; 266: 3937-3967. doi: 10.1016/j.jfa.2013.12.025

[3] Brownlowe N, Pask D, Ramagge J, Robertson D, Whittaker MF. Zappa-Szép product groupoids and C^*-blends. Semigroup Forum. 2017; 94: 500-519. doi: 10.1007/s00233-016-9775-z

[4] Casadio G. Construzione di gruppi come prodotto di sottogruppi permutabili. Rendiconti Di Matematica e Delle Sue Applicazioni. 1941; 5: 348-360.

[5] Gould V, Zenab R. Semigroups with inverse skeletons and Zappa-Szép products. Categories and General Algebraic Structures with Applications. 2013; 1: 59-89.

[6] Howie JM. Fundamentals of Semigroup Theory. Oxford, UK: Oxford University Press, 1995.

[7] Kaloujnine L, Krasner M. Produit complet des groupes de permutations et probléme d'extensions de groupes. II. Acta Scientiarum Mathematicarum, Szeged. 1950; 13: 208-230.

[8] Krohn K, Rhodes J. Algebraic theory of machines. I: Prime decomposition theorem for finite semigroups and machines. Transactions of American Mathematical Society. 1965; 116: 450-464.

[9] Kunze M. Zappa products. Acta Mathematica Hungarica. 1983; 41: 225-239.

[10] Lavers TG. Presentations of general products of monoids. Journal of Algebra. 1998; 204: 733-741.

[11] Mitchell B. Theory of categories. Academic Press, New York, 1965.

[12] Neumann BH. Decompositions of groups. Journal of London Mathematical Society. 1935; 10: 3-6.

[13] Neumann BH. Embedding theorems for semigroups. Journal of London Mathematical Society. 1960; 35: 184-192.

[14] Preston GB. Semidirect products of semigroups. Proceedings of the Royal Society of Edinburgh 1986; 102 A: 91-102.

[15] Rédei L. Das schiefe Produkt in der Gruppentheorie. Commentarii Mathematici Helvetici. 1947; 20: 225-264.

[16] Rédei L, Szép J. Die Verallgemeinerung der Theorie des Gruppenproduktes von Zappa-Casadio. Acta Scientiarum Mathematicarum, Szeged. 1955; 16: 165–170.

[17] Szép J. On factorizable, not simple groups. Acta Scientiarum Mathematicarum, Szeged. 1950; 13: 239-241.

[18] Szép J. Über eine neue Erweiterung von Ringen. Acta Scientiarum Mathematicarum, Szeged. 1958; 19: 51-62.

[19] Szép J. Sulle strutture fattorizzabili. Atti della Accademia Nazionale dei Lincei. Serie Ottava. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali. (1962); 32: 649–652.

[20] Wazzan S. The Zappa-Szép product of semigroups. PhD, Heriot-Watt University, Edinburgh, United Kingdom, 2008.

[21] Zappa G. Sulla construzione dei gruppi prodotto di due sottogruppi permutabili tra loro. Atti Secondo Congresso Università Italia Bologna. 1940; 119-125. Edizioni Rome, Cremonense (1942).

[22] Zenab R. Decomposition of semigroups into semidirect and Zappa-Szép products. PhD, University of York, United Kingdom, 2014.