Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2534 – 2593
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doi:10.3906/mat-2104-57

Higher cohomologies for presheaves of commutative monoids

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Received: 13.04.2021 • **Accepted/Published Online:** 04.10.2021 • **Final Version:** 29.11.2021

Abstract: We present an extension of the classical Eilenberg–MacLane higher order cohomology theories of abelian groups to presheaves of commutative monoids (and of abelian groups, then) over an arbitrary small category. These high-level cohomologies enjoy many desirable properties and the paper aims to explore them. The results apply directly in several settings such as presheaves of commutative monoids on a topological space, simplicial commutative monoids, presheaves of simplicial commutative monoids on a topological space, commutative monoids or simplicial commutative monoids on which a fixed monoid or group acts, and so forth. As a main application, we state and prove a precise cohomological classification both for braided and symmetric monoidal fibred categories whose fibres are abelian groupoids. The paper also includes a classification for extensions of commutative group coextensions of presheaves of commutative monoids, which is relevant to the study of $\mathcal{H}$-coextensions of presheaves of commutative regular monoids.

Key words: Commutative monoid, simplicial set, presheaf, cohomology, extension, Schützenberger kernel, fibration, monoidal category, braiding, symmetry

1. Introduction and summary

In the past fifties, Eilenberg and MacLane [16, 17] introduced what we now know as higher cohomology theories for abelian groups: For every integer $r \geq 2$, *the level-r cohomology groups* of an abelian group B with coefficients in an abelian group A, denoted by $H^n(B, r, A)$, are the cohomology groups of the Eilenberg–MacLane minimal simplicial sets $K(B, r)$ with coefficients in A, that is

$$H^n(B, r, A) = H^n(K(B, r), A), \quad n \geq 0.$$

At first, these high-level cohomology groups were studied mainly with interest in algebraic topology. But later, these cohomologies found applications in solving problems of a purely algebraic nature. For instance, in the classification of *Picard categories* $\mathcal{G} = (\mathcal{G}, \otimes, \iota, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c})$ (also known as symmetric categorical groups) by Grothendieck and Sinh (Sinh, H. X. Gr-catégories. Ph.D. Thesis, Université Paris VII, Paris, France, 1975. Aviable at https://pnp.mathematik.uni-stuttgart.de/lexmath/kuenzer/sinh.html); that is, categories $\mathcal{G}$ endowed with a tensor functor $\otimes : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$, a unit object $\iota \in \mathcal{G}$, and coherent constraints of associativity, left and right units and commutativity, in which every morphism is invertible (i.e. the underlying category is a groupoid) and every object is quasi-invertible. In that classification process, a complete invariant for the equivalence class of a Picard category $\mathcal{G}$ is given by a triple of data (B, A, c), where $B = \Pi_0\mathcal{G}$ is the abelian

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2010 *AMS Mathematics Subject Classification:* 18D30, 18F20, 18G90, 18M15, 18N50, 19D23, 20M50, 55N91

group of isomorphism classes of its objects, with multiplication induced by the tensor product, $\Pi_1\mathcal{G} = \mathrm{Aut}_{\mathcal{G}}(\iota)$ is the abelian group of automorphisms of its unit object, and c is a level-3 fifth-cohomology class $c \in H^5(B,3,A)$, which is canonically defined from the coherence pentagons and hexagons in $\mathcal{G}$. Subsequently, Joyal and Street in [24] stated an analogous classification result for *braided categorical groups* (defined similarly to Picard categories but without the symmetry requirement $\boldsymbol{c}_{Y,X}\boldsymbol{c}_{X,Y} = 1_{X\otimes Y}$) in terms of triples (B,A,c) as above, but now with $c \in H^4(B,2,A)$ a fourth-cohomology class of second-level.

Most of the work we present here was motivated by the challenge of establishing a precise classification for braided and symmetric C*-fibred categories* $\mathcal{G} = (\mathcal{G},\pi,\otimes,\iota,\boldsymbol{a},\boldsymbol{l},\boldsymbol{r},\boldsymbol{c})$, where C is any given small category; that is, categories $\mathcal{G}$ endowed with a (Grothendieck) fibration $\pi : \mathcal{G} \to \mathrm{C}$ and a braided or symmetric C-fibred monoidal structure by means of cartesian C-functors $\otimes : \mathcal{G}\times_{\mathrm{C}}\mathcal{G} \to \mathcal{G}$ and $\iota : \mathrm{C} \to \mathcal{G}$, and corresponding associativity, unit, and braiding or symmetry C-fibred constraints. Indeed, in this paper, we generalize the aforementioned results for the nonfibred case by Joyal–Street and Sinh by achieving a cohomological solution to that problem for those braided or symmetric C-fibred categories $\mathcal{G}$ in which every fibre category $\mathcal{G}_U = \pi^{-1}(1_U)$ is a groupoid whose automorphisms groups $\mathrm{Aut}_{\mathcal{G}_U}(X)$ are all abelian. Any such a braided or symmetric C-fibred category $\mathcal{G}$ produces, by taking isoclasses of objects in the fibre categories, a presheaf on C of commutative monoids $\Pi_0\mathcal{G} : \mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$, as well as, by taking automorphisms groups of objects in the fibre categories, an abelian group object $\Pi_1\mathcal{G}$ in the comma category of presheaves of commutative monoids on C over $\Pi_0\mathcal{G}$. Thus, we were naturally led to research for suitable cohomology theories for presheaves of commutative monoids $\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$ with coefficients in abelian group objects $\mathcal{A}$ in the slice category of presheaves of commutative monoids over them. In the paper, we call these coefficients $\mathbb{H}(\mathcal{M})$-modules, because they can be described in a simpler way as abelian group valued functors $\mathcal{A} : \mathbb{H}(\mathcal{M}) \to \mathsf{Ab}$ on a small category $\mathbb{H}(\mathcal{M})$ that the presheaf $\mathcal{M}$ suitably defines (this key observation is essentially due to Grillet [22, Chapter XII, §2]).

Presheaves on small categories are rather familiar objects and arise in many situations. The cohomology of presheaves of several algebraic structures (groups, rings, etc.) has been object of study with interest along the last decades. Particularly, we should refer here to the seminal work by Gerstenhaber–Shack (in deformation theory) on cohomology of presheaves of algebras (e.g., associative or Lie) [19–21], which greatly inspires part of this paper on higher cohomology of presheaves of commutative monoids. Also, our exposition is strongly influenced by several papers on cohomology of presheaves of simplicial sets (in equivariant homotopy theory). Particularly we should refer those by Dwyer–Kan [14, 15], Moerdijk–Svensson [28, 29], and Blanc–Johnson–Turner [4]. Particularly, following Moerdijk and Svensson, we define *the cohomology groups of a presheaf of simplicial sets* $\mathcal{S} : \mathrm{C}^{\mathrm{op}} \to \mathsf{SSet}$, denoted by $H^n(\mathcal{S},\mathcal{A})$, to be those of its category of simplices $\Delta(\mathcal{S})$, and following Gerstenhaber and Shack, we show suitable cochain complexes $C^\bullet(\mathcal{S},\mathcal{A})$ to compute them. The pointed case is relevant for our development, and we define *the pointed cohomology groups* of a presheaf of pointed simplicial sets $\mathcal{S}$, denoted by $H^n_{\mathrm{pt}}(\mathcal{S},\mathcal{A})$, as the relative cohomology groups to the subcategory $\Delta(\mathrm{pt}) \subseteq \Delta(\mathcal{S})$, where $\mathrm{pt} : \mathrm{C}^{\mathrm{op}} \to \mathsf{SSet}$ denote the constant presheaf on C defined by the constant simplicial set defined by the unitary set pt. These are computed by the subcomplex $C^\bullet(\mathcal{S},\mathrm{pt},\mathcal{A}) \subseteq C^\bullet(\mathcal{S},\mathcal{A})$ of cochains on $\mathcal{S}$ which vanish on pt. Both cohomology theories $H^n(\mathcal{S},\mathcal{A})$ and $H^n_{\mathrm{pt}}(\mathcal{S},\mathcal{A})$ base the higher cohomology theories of presheaves of commutative monoids we treat in the paper. We would like to emphasize that some of our results for presheaves of commutative monoids can also be proved in a more ad hoc way. However, our approach of using presheaves of simplicial sets opens further possibilities by considering other categories of presheaves of algebraic structures.

If $\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$ is a presheaf of commutative monoids, the classical $\overline{\mathrm{W}}$-construction by Eilenberg–

MacLane [17] and Moore (Moore, J.C. Algebraic homotopy theory, unpublished lecture notes, 1958) works recursively on it, giving rise to presheaves of simplicial commutative monoids $K(\mathcal{M},r):\mathrm{C}^{\mathrm{op}}\to\mathsf{SCMon}$, one for each integer $r\geq 0$. Furthermore, these come with a canonical funtor $\Delta(K(\mathcal{M},r))\to\mathbb{H}(\mathcal{M})$, through which every $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$ produces a system of coefficients on the underlying presheaf of pointed simplicial sets $K(\mathcal{M},r)$. This leads to the two definitions of *level-r cohomology groups of $\mathcal{M}$ with coefficients in $\mathcal{A}$* we consider. We define the first one as the cohomology of the presheaf of the simplicial sets $K(\mathcal{M},r)$, that is, by the cohomology groups

$$H^n(\mathcal{M},r,\mathcal{A})=H^n(K(\mathcal{M},r),\mathcal{A})\quad(n\geq 0),$$

while the second one, which following to Gerstenhaber and Shack [20] we call the *simple* level-r cohomology, is defined to be the pointed cohomology of $K(\mathcal{M},r)$, that is, by the cohomology groups

$$H^n_s(\mathcal{M},r,\mathcal{A})=H^n_{\mathrm{pt}}(K(\mathcal{M},r),\mathcal{A})\quad(n\geq 0).$$

Although both extensions of the higher order Eilenberg–MacLane cohomology theory are closely related, they are different in general. However, they coincide (for $n\geq 2$) in most of the relevant cases that we have in mind. For instance, this holds when a) $\mathrm{C}=O(X)$, the category defined by the partially ordered set of open subsets of a topological space X, that is, for presheaves of commutative monoids on topological spaces; b) $\mathrm{C}=[p]$, the category defined by the ordered set $\{0<1<\cdots<p\}$, that is, for p-ads of commutative monoids connected by a homomorphism; in particular, when $p=0$, for commutative monoids; c) $\mathrm{C}=\Delta$, the simplicial category of finite ordered sets $[p]$, with nondecreasing maps between them as its morphisms, that is, for simplicial commutative monoids; d) $\mathrm{C}=O(X)\times\Delta$, where X is a topological space, that is, for presheaves of simplicial commutative monoids on a topological space; e) $\mathrm{C}=Or(G)\times\Delta$, where $Or(G)$ is the orbit category of a group G, whose objects are the transitive left G-sets G/H, for any subgroup $H\subseteq G$, and whose morphisms are the G-equivariant maps between them, that is, for simplicial commutative monoids endowed with a left G-action by automorphisms. Here, one regards such a simplicial commutative monoid $\mathcal{M}$ as the presheaf of commutative monoids on $Or(G)\times\Delta$ such that $(G/H,[p])\mapsto\mathrm{Hom}_G(G/H,\mathcal{M}_p)\cong\mathcal{M}_p^H$; etc.

The first-level cohomology groups $H^n(\mathcal{M},1,\mathcal{A})$ and $H^n_s(\mathcal{M},1,\mathcal{A})$ are already known, since they are actually particular cases of the cohomology theories for presheaves of (not necessarily commutative) monoids recently studied by the authors in [10]. That is why here we focus mainly on those $H^n(\mathcal{M},r,\mathcal{A})$ and $H^n_s(\mathcal{M},r,\mathcal{A})$ with $r\geq 2$. Some special cases of these high-level cohomology groups for presheaves have been previously treated by the second author and collaborators in papers as [8, 9, 11, 12], where the reader can also find some results on the classification of monoidal fibred categories that are now particular cases of the classification that we carry out here for braided C-fibred categories (in abelian groupoids) $\mathcal{G}$, which we give in terms of triples $(\mathcal{M},\mathcal{A},c)$, with $\mathcal{M}=\Pi_0\mathcal{G}$, $\mathcal{A}=\Pi_1\mathcal{G}$ and c a cohomology class $c\in H^4_s(\mathcal{M},2,\mathcal{A})$, or $c\in H^5_s(\mathcal{M},3,\mathcal{A})$ in case $\mathcal{G}$ is symmetric, which is canonically defined from the coherence pentagons and hexagons in $\mathcal{G}$. But our result has more precedents. For instance, we should refer those for the classification of strictly commutative Picard stacks by Deligne [13] and of braided and symmetric categorical group stacks by Breen [6, 7]. The cohomology groups $H^2_s(\mathcal{M},2,\mathcal{A})$ and $H^3_s(\mathcal{M},3,\mathcal{A})$ are isomorphic and they have a natural interpretation in terms of *equivariant derivations* $d:\mathcal{M}\to\mathcal{A}$. The cohomology groups $H^3_s(\mathcal{M},2,\mathcal{A})$ and $H^4_s(\mathcal{M},3,\mathcal{A})$ are also isomorphic and they classify *commutative group coextensions* of $\mathcal{M}$ by $\mathcal{A}$, that is, locally surjective morphisms of presheaves of commutative monoids $\mathcal{E}\to\mathcal{M}$ equipped with a simply-transitive group action $\mathcal{A}\times_{\mathcal{M}}\mathcal{E}\to\mathcal{E}$

in the slice category of presheaves of commutative monoids over $\mathcal{M}$. We show how this classification is useful in the study of commutative coextensions of presheaves of commutative monoids $\mathcal{E} \to \mathcal{M}$ whose congruence kernel is contained in the presheaf of Green's congruence $\mathcal{H}$ of $\mathcal{E}$, particularly when $\mathcal{M}$ is valued in regular commutative monoids.

A brief outline of the plan of the paper is as follows. After this first introductory and summary section, Section 2 comprises a brief review of some standard definitions and constructions related to the cohomology of small categories and presheaves of small categories. Section 3 provides framework on the cohomology groups of presheaves of simplicial sets. Its content is pretty standard, except perhaps the linking long exact sequence in our Theorem 3.4. In Section 4 we mainly define the level-r cohomology groups of presheaves of commutative monoids $H^n(\mathcal{M}, r, \mathcal{A})$ and $H^n_s(\mathcal{M}, r, \mathcal{A})$, and we devote the following two sections to cochains, cocycles and coboundaries to compute them. Thus, we provide in Section 5 of suitable cochain complexes for computing the high-level cohomology groups of presheaves of commutative monoids, and in Section 6 we specify, for later reference in applications, what low dimensional cocycles of second and third level are. In Section 7 we establish de classification of commutative group coextensions of presheaves of commutative monoids by means of the cohomology groups $H^3_s(\mathcal{M}, 2, \mathcal{A})$, and we apply this result to the study of commutative $\mathcal{H}$-coextensions of presheaves of commutative monoids. The following long Section 8 is entirely dedicated to prove in great detail the classification of braided and symmetric fibred categories by means of the cohomology groups $H^4_s(\mathcal{M}, 3, \mathcal{A})$ and $H^5_s(\mathcal{M}, 2, \mathcal{A})$. Finally, we include Section 9, where we highlight how our previous results and constructions specialize when one limits the attention to presheaves of abelian groups, instead of general presheaves of commutative monoids.

2. Preliminaries

This section aims to make this paper as self-contained as possible; hence, at the same time as fixing some notations and terminology, we review some needed constructions and facts concerning cohomology of small categories and presheaves of small categories.

2.1. Cohomology of small categories

Let K be a small category. A (left) K-*module* is an abelian group valued functor $\mathcal{A} : \mathrm{K} \to \mathsf{Ab}$. For any such a K-module, we usually denote by $f_* : \mathcal{A}(k) \to \mathcal{A}(k')$, $a \mapsto f_*a$, to the homomorphism $\mathcal{A}(f)$ associated by $\mathcal{A}$ to a morphism $f : k \to k'$ of K. The category of K-modules, with morphisms the natural transformations, is denoted by K-Mod. We refer to Mac Lane [27, Chapter VIII, §3] for formalities, but recall that this is an abelian category where all limits and colimits are objectwise constructed.

There is a forgetful functor $\mathcal{U} : \mathrm{K}\text{-Mod} \to \mathsf{Set}\!\downarrow_{\mathrm{ObK}}$, from the category of K-modules to the slice category of sets over the set of objects of K, which carries a K-module $\mathcal{A}$ to the set $\mathcal{U}\mathcal{A} = \{(k, a) \mid k \in \mathrm{ObK},\, a \in \mathcal{A}(k)\}$, endowed with the projection map $\pi : \mathcal{U}\mathcal{A} \to \mathrm{ObK}$, given by $\pi(k, a) = k$. This functor $\mathcal{U}$ has a left adjoint, the *free* K-*module functor*

$$\mathcal{F} : \mathsf{Set}\!\downarrow_{\mathrm{ObK}} \to \mathrm{K}\text{-Mod},$$

which assigns to each $S = (S, \pi : S \to \mathrm{ObK})$ the K-module $\mathcal{F}S = \bigoplus_{s \in S} \mathbb{Z}\mathrm{Hom}_{\mathrm{K}}(\pi s, -)$, where $\mathbb{Z} : \mathsf{Set} \to \mathsf{Ab}$ denotes the free abelian group functor. We regard every element $s \in S$ as a generator of $\mathcal{F}S(\pi s)$ by writing $s = (s, id_{\pi s})$. Yoneda Lemma shows that the functor $\mathcal{F}$ is actually left adjoint to the functor $\mathcal{U}$. Thus, for

$S = (S, \pi)$ any set over ObK and any K-module $\mathcal{A}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{K}}(\mathcal{F}S, \mathcal{A}) \;\cong\; \prod_{s\in S} \mathcal{A}(\pi s), \quad \mathfrak{f} \mapsto \big(\mathfrak{f}_{\pi s}(s)\big)_{s\in S}. \tag{2.1}$$

From this, it is plain to see that any free K-module is projective and, moreover, the counit $\mathcal{FUA} \to \mathcal{A}$ is a projective presentation of any K-module $\mathcal{A}$. Hence, the category K-Mod has enough projective objects (it has also enough injective objects, but we will not use this fact).

The *cohomology groups* of the category K with coefficients in a K-module $\mathcal{A}$ (see Roos [31] and Watts [33]), denoted by $H^n(\mathrm{K}, \mathcal{A})$, are defined by

$$H^n(\mathrm{K}, \mathcal{A}) = \mathrm{Ext}^n_{\mathrm{K}}(\mathbb{Z}, \mathcal{A}) \qquad (n \geq 0),$$

where $\mathbb{Z} : \mathrm{K} \to \mathsf{Ab}$ is the constant K-module defined by the abelian group of integers.

A canonical cochain complex $C^\bullet(\mathrm{K}, \mathcal{A})$, which computes the cohomology groups $H^n(\mathrm{K}, \mathcal{A})$, is as follows. Recall that the nerve $N\mathrm{K}$ of a small category K is the simplicial set whose p-simplices are sequences

$$\sigma = (\sigma 0 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_p} \sigma p)$$

of p composable morphisms in K (objects $\sigma 0$ of K if $p = 0$), with faces $d_0\sigma = (\sigma_2, \dots, \sigma_p)$ ($\sigma 1$ if $p = 1$), $d_i\sigma = (\sigma_1, \dots, \sigma_{i+1}\sigma_i, \dots, \sigma_p)$ if $0 < i < p$, and $d_p\sigma = (\sigma_1, \dots, \sigma_{p-1})$ ($\sigma 0$ if $p = 1$), and degeneracies $s_i\sigma = (\sigma_1, \dots, \sigma_i, 1_{\sigma i}, \sigma_{i+1}, \dots, \sigma_p)$. The cochain complex $C^\bullet(\mathrm{K}, \mathcal{A})$ consists of the abelian groups

$$C^p(\mathrm{K}, \mathcal{A}) = \prod_{\sigma \in N_p\mathrm{K}} \mathcal{A}(\sigma p)$$

with $\partial : C^{p-1}(\mathrm{K}, \mathcal{A}) \to C^p(\mathrm{K}, \mathcal{A})$ given by $(\partial\xi)_\sigma = \sum_{i=0}^{p-1}(-1)^i \xi_{d_i\sigma} + (-1)^p\, \sigma_{p*}\xi_{d_p\sigma}$. By Gabriel–Zisman [18, Appendix II, Prop. 3.3], there are natural isomorphisms

$$H^n(\mathrm{K}, \mathcal{A}) \cong H^n C^\bullet(\mathrm{K}, \mathcal{A}) \qquad (n \geq 0).$$

2.2. The Grothendieck construction

Let C be a small category. If $\mathfrak{P} : \mathrm{C}^{\mathrm{op}} \to \mathsf{Cat}$ is a presheaf of small categories on C, for any morphism $\sigma : V \to U$ in C, we write $(\ )^\sigma : \mathfrak{P}(U) \to \mathfrak{P}(V)$, $x \mapsto x^\sigma$, to denote the functor $\mathfrak{P}(\sigma)$. The *Grothendieck construction* (see Thomason [32]) on such a presheaf $\mathfrak{P}$ is the category $\int_{\mathrm{C}} \mathfrak{P}$ whose objects are pairs (U, x) where U is an object of C and x is one of $\mathfrak{P}(U)$. A morphism $(\sigma, f) : (U, x) \to (V, y)$ in $\int_{\mathrm{C}} \mathfrak{P}$ is a pair where $\sigma : V \to U$ is a morphism in C and $f : x^\sigma \to y$ is a morphism in $\mathfrak{P}(V)$. Composition is defined by $(\tau, g)(\sigma, f) = (\sigma\tau, g f^\tau)$, and the identities are $1_{(U,x)} = (1_U, 1_x)$.

Note a morphism of presheaves $\pi : \mathfrak{P} \to \mathfrak{P}'$ (i.e. a natural transformation) induces a functor $\pi : \int_{\mathrm{C}} \mathfrak{P} \to \int_{\mathrm{C}} \mathfrak{P}'$ given by

$$\pi\big((\sigma, f) : (U, x) \to (V, y)\big) = (\sigma, \pi_V(f)) : (U, \pi_U(x)) \to (V, \pi_V(y)).$$

We set the following notations for a functor $\mathcal{A} : \int_{\mathrm{C}} \mathfrak{P} \to \mathsf{Ab}$, that is, for an $\int_{\mathrm{C}} \mathfrak{P}$-module:

(a) For any object U of C and any morphism $f : x \to y$ in $\mathfrak{P}(U)$, we write

$$f_* = \mathcal{A}(1_U, f) : \mathcal{A}(U, x) \to \mathcal{A}(U, y), \qquad a \mapsto f_* a.$$

(b) If $\sigma : V \to U$ is a morphism in C, for any object x of $\mathfrak{P}(U)$, we write

$$(\ )^\sigma = \mathcal{A}(\sigma, 1_{x^\sigma}) : \mathcal{A}(U, x) \to \mathcal{A}(V, x^\sigma), \qquad a \mapsto a^\sigma.$$

(c) For each object U of C, we denote by

$$\mathcal{A}(U) : \mathfrak{P}(U) \to \mathsf{Ab}$$

the $\mathfrak{P}(U)$-module obtained by restriction of $\mathcal{A}$ along the inclusion functor $\mathfrak{P}(U) \hookrightarrow \int_{\mathrm{C}} \mathfrak{P}$, which carries a morphism $f : x \to y$ of $\mathfrak{P}(U)$ to the morphism $(1_U, f) : (U, x) \to (U, y)$ of $\int_{\mathrm{C}} \mathfrak{P}$.

(d) If $\sigma : V \to U$ is a morphism in C, we denote by

$$\mathcal{A}(V)^\sigma : \mathfrak{P}(U) \to \mathsf{Ab}$$

the $\mathfrak{P}(U)$-module obtained by composing $\mathcal{A}(V)$ with the functor $(\ )^\sigma : \mathfrak{P}(U) \to \mathfrak{P}(V)$, that is,

$$\mathcal{A}(V)^\sigma(f : x \to y) = \mathcal{A}(1_V, f^\sigma) : \mathcal{A}(V, x^\sigma) \to \mathcal{A}(V, y^\sigma).$$

Thus, $(\ )^\sigma : \mathcal{A}(U) \to \mathcal{A}(V)^\sigma$ becomes a canonical morphism of $\mathfrak{P}(U)$-modules.

2.3. The category of factorizations

Let C be a small category. The *category of factorizations* of the category C^{op} (see Baues–Wirsching [3]), denoted by $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$, is the category whose objects are the morphisms σ of C and whose morphisms $(\tau, \tau') : \sigma \to \sigma'$ in $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$ are pairs of morphisms of C such that $\sigma' = \tau' \sigma \tau$, that is, making commutative the square

$$\begin{tikzcd} V \arrow[r, "\sigma"] & U \arrow[d, "\tau'"] \\ V' \arrow[u, "\tau"] \arrow[r, "\sigma'"] & U' \end{tikzcd}$$

Composition is given by $(\gamma, \gamma')(\tau, \tau') = (\tau\gamma, \gamma'\tau')$, and the identities are $1_{V \xrightarrow{\sigma} U} = (1_V, 1_U)$.

If $\mathcal{D} : \mathrm{F}(\mathrm{C}^{\mathrm{op}}) \to \mathsf{Ab}$ is an $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-module (i.e. a natural system on C^{op} in the terminology of [3]), for any morphisms $W \xrightarrow{\tau} V \xrightarrow{\sigma} U$ in C, we write $\sigma_* = \mathcal{D}(1_W, \sigma) : \mathcal{D}(\tau) \to \mathcal{D}(\sigma\tau)$ and $\tau^* = \mathcal{D}(\tau, 1_U) : \mathcal{D}(\sigma) \to \mathcal{D}(\sigma\tau)$. Thus, for any morphism $(\tau, \tau') : \sigma \to \sigma'$ in $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$ as above, we can write $\mathcal{D}(\tau, \tau') = \tau^* \tau'_* = \tau'_* \tau_* : \mathcal{D}(\sigma) \to \mathcal{D}(\sigma')$.

The *cohomology groups* of C^{op} with coefficients in a $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-module $\mathcal{D}$ [3] are defined by

$$H^n(\mathrm{C}^{\mathrm{op}}, \mathcal{D}) = H^n(\mathrm{F}(\mathrm{C}^{\mathrm{op}}), \mathcal{D}) \qquad (n \geq 0).$$

To compute these cohomology groups, we have the cochain complex $\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{D})$ where, for each integer $p \geq 0$,

$$\mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathcal{D}) = \prod_{\sigma \in N_p \mathrm{C}^{\mathrm{op}}} \mathcal{D}(\sigma_1 \cdots \sigma_p).$$

Here, we represent the p-simplices σ of the nerve of C^{op} as sequences $\sigma = (\sigma 0 \overset{\sigma_1}{\leftarrow} \cdots \overset{\sigma_p}{\leftarrow} \sigma p)$ of p composable arrows in C (objects $\sigma 0$ of C if $p=0$). The coboundary $\partial : \mathrm{F}^{p-1}(\mathrm{C}^{\mathrm{op}}, \mathcal{D}) \to \mathrm{F}^{p}(\mathrm{C}^{\mathrm{op}}, \mathcal{D})$ is given by $(\partial\xi)_\sigma = \sigma_{1*}\,\xi_{d_0\sigma} + \sum_{i=1}^{p-1}(-1)^i \xi_{d_i\sigma} + (-1)^p\,\sigma_p^*\,\xi_{d_p\sigma}$. By [3, Theorem (4.4)], there are natural isomorphisms

$$H^n(\mathrm{C}^{\mathrm{op}}, \mathcal{D}) \cong H^n\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{D}) \qquad (n \geq 0).$$

Example 2.1 Let $\mathfrak{P} : \mathrm{C}^{\mathrm{op}} \to \mathsf{Cat}$ be a presheaf of small categories on a small category C. Every pair of $\int_{\mathrm{C}}\mathfrak{P}$-modules $\mathcal{B}, \mathcal{A}$ gives rise to a $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-module $\mathcal{H}om(\mathcal{B}, \mathcal{A})$, which acts on morphisms $\sigma : V \to U$ of C by

$$\mathcal{H}om(\mathcal{B}, \mathcal{A})(\sigma) = \mathrm{Hom}_{\mathfrak{P}(U)}(\mathcal{B}(U), \mathcal{A}(V)^\sigma).$$

If $\tau : W \to V$ is also a morphism in C, the homomorphism $\sigma_* : \mathcal{H}om(\mathcal{B}, \mathcal{A})(\tau) \to \mathcal{H}om(\mathcal{B}, \mathcal{A})(\sigma\tau)$ carries a morphism of $\mathfrak{P}(V)$-modules $\mathfrak{f} : \mathcal{B}(V) \to \mathcal{A}(W)^\tau$ to the morphism of $\mathfrak{P}(U)$-modules $\sigma_*\mathfrak{f} : \mathcal{B}(U) \to \mathcal{A}(W)^{\sigma\tau}$ whose component at an object x of $\mathfrak{P}(U)$ is the composite

$$\mathcal{B}(U,x) \xrightarrow{(\ )^\sigma} \mathcal{B}(V, x^\sigma) \xrightarrow{\mathfrak{f}_{x^\sigma}} \mathcal{A}(W, x^{\sigma\tau}), \quad b \mapsto \mathfrak{f}_{x^\sigma} b^\sigma,$$

and the homomorphism $\tau^* : \mathcal{H}om(\mathcal{B}, \mathcal{A})(\sigma) \to \mathcal{H}om(\mathcal{B}, \mathcal{A})(\sigma\tau)$ carries a morphism of $\mathfrak{P}(U)$-modules $\mathfrak{f} : \mathcal{B}(U) \to \mathcal{A}(V)^\sigma$ to the morphism of $\mathfrak{P}(U)$-modules $\tau^*\mathfrak{f} : \mathcal{B}(U) \to \mathcal{A}(W)^{\sigma\tau}$ whose component at an object x of $\mathfrak{P}(U)$ is the composite

$$\mathcal{B}(U,x) \xrightarrow{\mathfrak{f}_x} \mathcal{A}(V, x^\sigma) \xrightarrow{(\ )^\tau} \mathcal{A}(W, x^{\sigma\tau}), \quad b \mapsto (f_x b)^\tau.$$

Hence, we have the cochain complex $\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}, \mathcal{A}))$, whose group of p-cochains is

$$\mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}, \mathcal{A})) = \prod_{\sigma \in N_p\mathrm{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathfrak{P}(\sigma 0)}\big(\mathcal{B}(\sigma 0), \mathcal{A}(\sigma p)^{\sigma_1 \cdots \sigma_p}\big).$$

For any $\xi \in \mathrm{F}^{p-1}(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}, \mathcal{A}))$, the component at each $\sigma \in N_p\mathrm{C}^{\mathrm{op}}$ of its coboundary $\partial\xi \in \mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}, \mathcal{A}))$ is the morphism of $\mathfrak{P}(\sigma 0)$-modules $(\partial\xi)_\sigma : \mathcal{B}(\sigma 0) \to \mathcal{A}(\sigma p)^{\sigma_1 \cdots \sigma_p}$ defined on each object x of $\mathfrak{P}(\sigma 0)$ by the homomorphism $(\partial\xi)_{\sigma,x} : \mathcal{B}(\sigma 0, x) \to \mathcal{A}(\sigma p, x^{\sigma_1 \cdots \sigma_p})$ given by

$$(\partial\xi)_{\sigma,x} b = \xi_{d_0\sigma, x^{\sigma_1}} b^{\sigma_1} + \sum_{i=1}^{p-1} (-1)^i\, \xi_{d_i\sigma,x} b + (-1)^p\, (\xi_{d_p\sigma,x} b)^{\sigma_p}.$$

The construction of $\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}, \mathcal{A}))$ is clearly functorial both in $\mathcal{B}$ and in $\mathcal{A}$. In particular, if $\mathcal{B}_\bullet$ is a chain complex of $\int_{\mathrm{C}}\mathfrak{P}$-modules, we have a cochain bicomplex

$$\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}_\bullet, \mathcal{A})) \tag{2.2}$$

where, for each $q \geq 0$, the horizontal complex $\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}_q, \mathcal{A}))$ is defined as above, and, for each $p \geq 0$, the vertical coboundary $\partial_v = \partial^* : \mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}_q, \mathcal{A})) \to \mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}_{q+1}, \mathcal{A}))$ is induced by the boundary $\partial : \mathcal{B}_{q+1} \to \mathcal{B}_q$ of $\mathcal{B}_\bullet$. Explicitly, for each $\xi \in \mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{B}_q, \mathcal{A}))$ and $\sigma \in N_p\mathrm{C}^{\mathrm{op}}$, $(\partial_v\xi)_\sigma = (-1)^p\,\xi_\sigma\partial_{\sigma 0}$, the signed composite of ξ_σ with $\partial_{\sigma 0} : \mathcal{B}_{q+1}(\sigma 0) \to \mathcal{B}_q(\sigma 0)$.

3. Cohomology of presheaves of (pointed) simplicial sets

Let Δ be the category of finite ordered sets $\mathbf{n} = \{0, \dots, n\}$ with weakly order-preserving maps between them. Recall that Δ is generated by the cofaces $\delta^i : \mathbf{n-1} \to \mathbf{n}$ (the monotone injections whose image omit i) and the codegeneracies $\varepsilon^i : \mathbf{n+1} \to \mathbf{n}$ (the monotone surjections which repeats i), $0 \leq i \leq n$, subject to the well-known cosimplicial identities: $\delta^j\delta^i = \delta^i\delta^{j-1}$ if $i < j$, etc. (see MacLane [26, V §5]). Let SSet denote the category of simplicial sets $S : \Delta^{op} \to \mathsf{Set}$. As usually, we write $\alpha^* : S_n \to S_m$ for the map $S(\alpha)$ induced by a map $\alpha : \mathbf{m} \to \mathbf{n}$ in Δ. In particular, $d_i = (\delta^i)^* : S_n \to S_{n-1}$ and $s_i = (\varepsilon^i)^* : S_n \to S_{n+1}$ are the face and degeneracy maps of S.

We will use the cohomology theory of simplicial sets by Gabriel–Zisman [18, Appendix II]. Briefly, recall that the *category of simplices* of a simplicial set S, denoted by $\Delta(S)$, is the category whose objects are the simplices $u \in S$. If $u \in S_n$ and $v \in S_m$, then a morphism $\alpha : u \to v$ is a map $\alpha : \mathbf{n} \to \mathbf{m}$ in Δ such that $u = \alpha^* v$. This category is generated by the morphisms

$$\delta^i : d_i u \to u, \qquad \varepsilon^i : s_i u \to u \qquad (u \in S_n,\ 0 \leq i \leq n), \tag{3.1}$$

with relations the cosimplicial identities. A *coefficient system* on a simplicial set S is a $\Delta(S)$-module, that is, an abelian group valued functor $\mathcal{A} : \Delta(S) \to \mathsf{Ab}$, and the *cohomology groups of S with coefficients in $\mathcal{A}$*, denoted by $H^n(S, \mathcal{A})$, are defined to be those of its category of simplices:

$$H^n(S, \mathcal{A}) = H^n(\Delta(S), \mathcal{A}) \qquad (n \geq 0).$$

Now, let C be a fixed small category. A presheaf of simplicial sets on C is a contravariant functor $\mathcal{S} : \mathrm{C}^{\mathrm{op}} \to \mathsf{SSet}$; thus, $\mathcal{S}$ provides to each object U of C of a simplicial set $\mathcal{S}(U)$ and to each arrow $\sigma : V \to U$ of a simplicial map $\mathcal{S}(\sigma)$ which we denote by

$$(\ )^\sigma : \mathcal{S}(U) \to \mathcal{S}(V), \qquad u \mapsto u^\sigma.$$

We define the *category of simplices* of the presheaf $\mathcal{S}$,

$$\Delta(\mathcal{S}) = \textstyle\int_{\mathrm{C}} \Delta\mathcal{S},$$

to be the category obtained by applying the Grothendieck construction on the presheaf of categories $\Delta\mathcal{S} : \mathrm{C}^{\mathrm{op}} \to \mathsf{Cat}$ obtained by composing $\mathcal{S}$ with the functor $\Delta : \mathsf{SSet} \to \mathsf{Cat}$, which assigns to each simplicial set its category of simplices. Explicitly, $\Delta(\mathcal{S})$ has objects the pairs (U, u) where $U \in \mathrm{Ob}\mathrm{C}$ and $u \in \mathcal{S}(U)$. If $u \in \mathcal{S}(U)_m$ and $v \in \mathcal{S}(V)_n$, then a morphism $(\sigma, \alpha) : (U, u) \to (V, v)$ in $\Delta(\mathcal{S})$ consists of a morphism $\sigma : V \to U$ in C and a map $\alpha : \mathbf{m} \to \mathbf{n}$ in Δ such that $u^\sigma = \alpha^* v$. The composition in $\Delta(\mathcal{S})$ of two morphisms $(\sigma, \alpha) : (U, u) \to (V, v)$ and $(\tau, \beta) : (V, v) \to (W, w)$ is the morphism $(\sigma\tau, \beta\alpha) : (U, u) \to (W, w)$, and the identity of an object (U, u) with $u \in \mathcal{S}(U)_n$ is $1_{(U,u)} = (1_U, 1_{\mathbf{n}})$.

For the following definition, c.f. Moerdijk–Svensson [29, Definition 2.1].

Definition 3.1 *A system of coefficients on a presheaf of simplicial sets $\mathcal{S}$ is a $\Delta(\mathcal{S})$-module, that is, an abelian group valued functor on its category of simplices $\mathcal{A} : \Delta(\mathcal{S}) \to \mathbf{Ab}$. The cohomology groups $H^n(\mathcal{S}, \mathcal{A})$ are defined to be those of the category $\Delta(\mathcal{S})$:*

$$H^n(\mathcal{S}, \mathcal{A}) = H^n(\Delta(\mathcal{S}), \mathcal{A}) \qquad (n \geq 0)\,. \tag{3.2}$$

3.1. The cochain complex $C^\bullet(\mathcal{S},\mathcal{A})$

We show below a workable cochain complex to compute the cohomology groups of a presheaf of simplicial sets, which comes from the homotopy colimit construction by Bousfield–Kan [5]. First, recall that, to compute the cohomology groups of a simplicial set S with coefficients in a $\Delta(S)$-module $\mathcal{A}$, we have the cochain complex $C^\bullet(S,\mathcal{A})$ which consists of the abelian groups

$$C^q(S,\mathcal{A}) = \prod_{u\in S_q}\mathcal{A}(u)$$

with coboundary $\partial : C^{q-1}(S,\mathcal{A}) \to C^q(S,\mathcal{A})$ defined by $(\partial\xi)_u = \sum_{j=0}^{q}(-1)^j\,\delta_*^j\,\xi_{d_ju}$. By Gabriel–Zisman [18, Appendix II, Prop. 4.2], there are natural isomorphisms $H^n(S,\mathcal{A})\cong H^nC^\bullet(S,\mathcal{A})$.

Now, suppose $\mathcal{S}:\mathrm{C}^{\mathrm{op}}\to\mathsf{SSet}$ is a presheaf of simplicial sets and let $\mathcal{A}$ be a $\Delta(\mathcal{S})$-module. Using the notations (a)–(d) in subsection 2.2, for each integer $q\geq 0$, let $\mathfrak{C}^q(\mathcal{S},\mathcal{A})$ be $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-module which carries a morphism $\sigma : V\to U$ of C to the abelian group

$$\mathfrak{C}^q(\mathcal{S},\mathcal{A})(\sigma) = C^q(\mathcal{S}(U),\mathcal{A}(V)^\sigma) = \prod_{u\in\mathcal{S}(U)_q}\mathcal{A}(V,u^\sigma). \tag{3.3}$$

If $\tau : W\to V$ is also a morphism in C, then $\sigma_* : \mathfrak{C}^q(\mathcal{S},\mathcal{A})(\tau)\to\mathfrak{C}^q(\mathcal{S},\mathcal{A})(\sigma\tau)$ is defined by $(\sigma_*\xi)_u = \xi_{u^\sigma}$, and $\tau^* : \mathfrak{C}^q(\mathcal{S},\mathcal{A})(\sigma)\to\mathfrak{C}^q(\mathcal{S},\mathcal{A})(\sigma\tau)$ by $(\tau^*\xi)_u = \xi_u^{\tau}$. As q varies, we have a cochain complex $\mathfrak{C}^\bullet(\mathcal{S},\mathcal{A})$ of $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-modules which carries each morphism $\sigma : V\to U$ of C to the cochain complex $C^\bullet(S(U),\mathcal{A}(V)^\sigma)$. Hence, as in subsection 2.3, we have the cochain bicomplex

$$\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^\bullet(\mathcal{S},\mathcal{A})), \tag{3.4}$$

whose group of (p,q)-cochains is

$$\mathrm{F}^p(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^q(\mathcal{S},\mathcal{A})) = \prod_{\sigma\in N_p\mathrm{C}^{\mathrm{op}}} C^q(\mathcal{S}(\sigma0),\mathcal{A}(\sigma p)^{\sigma_1\cdots\sigma_p}) = \prod_{\sigma\in N_p\mathrm{C}^{\mathrm{op}}}\ \prod_{u\in\mathcal{S}(\sigma0)_q}\mathcal{A}(\sigma p, u^{\sigma_1\cdots\sigma_p}),$$

and whose horizontal and vertical coboundaries

$$\mathrm{F}^{p-1}(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^q(\mathcal{S},\mathcal{A}))\xrightarrow{\ \partial_h\ }\mathrm{F}^p(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^q(\mathcal{S},\mathcal{A}))\xleftarrow{\ \partial_v\ }\mathrm{F}^p(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^{q-1}(\mathcal{S},\mathcal{A}))$$

are respectively given by

$$(\partial_h\xi)_{\sigma,u} = \xi_{d_0\sigma,u^{\sigma_1}} + \sum_{i=1}^{p-1}(-1)^i\,\xi_{d_i\sigma,u} + (-1)^p\,\xi^{\sigma_p}_{d_p\sigma,u},$$

$$(\partial_v\xi)_{\sigma,u} = (-1)^p\sum_{j=0}^{q}(-1)^j\,\delta_*^j\xi_{\sigma,d_ju}.$$

We define the *complex of cochains of the presheaf $\mathcal{S}$ with coefficients in a $\Delta(\mathcal{S})$-module $\mathcal{A}$ as*

$$C^\bullet(\mathcal{S},\mathcal{A}) = \operatorname{Tot}\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^\bullet(\mathcal{S},\mathcal{A})), \tag{3.5}$$

the total cochain complex of the bicomplex $\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathfrak{C}^\bullet(\mathcal{S},\mathcal{A}))$. That is,

$$C^n(\mathcal{S},\mathcal{A}) = \bigoplus_{p+q=n} \mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathfrak{C}^q(\mathcal{S},\mathcal{A})), \qquad \partial = \partial_h + \partial_v : C^{n-1}(\mathcal{S},\mathcal{A}) \to C^n(\mathcal{S},\mathcal{A}).$$

Theorem 3.2 *For any coefficient system $\mathcal{A}$ on a presheaf of simplicial sets $\mathcal{S}$, there are natural isomorphisms*

$$H^n(\mathcal{S},\mathcal{A}) \cong H^n C^\bullet(\mathcal{S},\mathcal{A}) \qquad (n \geq 0).$$

Proof Let $\Psi(\mathcal{S})$ be the simplicial replacement of the presheaf $\mathcal{S}$ (see Bousfield–Kan [5]), that is, the bisimplicial set whose set of (p,q)-bisimplices is

$$\Psi_{p,q}(\mathcal{S}) = \{(\sigma,u) \mid \sigma \in N_p\mathrm{C}^{\mathrm{op}}, u \in \mathcal{S}(\sigma 0)_q\}.$$

The horizontal face and degeneracy maps on a bisimplex $(\sigma,u) \in \Psi_{p,q}(\mathcal{S})$ are defined by

$$d_i^h(\sigma,u) = \begin{cases} (d_0\sigma, u^{\sigma_1}) & i=0, \\ (d_i\sigma, u) & i>0, \end{cases} \qquad s_i^h(\sigma,u) = (s_i\sigma,u),$$

and the vertical ones by $d_j^v(\sigma,u) = (\sigma, d_j u)$ and $s_j^v(\sigma,u) = (\sigma, s_j u)$. We endow each set $\Psi_{p,q}(\mathcal{S})$ with the map $\pi : \Psi_{p,q}(\mathcal{S}) \to \mathrm{Ob}\Delta(\mathcal{S})$ given by $\pi(\sigma,u) = (\sigma p, u^{\sigma_1\cdots\sigma_p})$, and let $\mathcal{P}_{p,q}$ be the free $\Delta(\mathcal{S})$-module on $(\Psi_{p,q}(\mathcal{S}),\pi)$. These $\mathcal{P}_{p,q}$ provide a bisimplicial $\Delta(Sc)$-module $\mathcal{P}$ in which the horizontal and vertical face and degeneracy operators

$$\mathcal{P}_{p+1,q} \xleftarrow{s_i^h} \mathcal{P}_{p,q} \xrightarrow{d_i^h} \mathcal{P}_{p-1,q}, \qquad \mathcal{P}_{p,q+1} \xleftarrow{s_j^v} \mathcal{P}_{p,q} \xrightarrow{d_j^v} \mathcal{P}_{p,q-1},$$

are respectively defined on generators $(\sigma,u) \in \Psi_{p,q}(\mathcal{S})$ by

$$d_i^h(\sigma,u) = \begin{cases} (d_0\sigma, u^{\sigma_1}) & i=0, \\ (d_i\sigma,u) & 0<i<p, \\ (d_p\sigma,u)^{\sigma_p} & i=p, \end{cases} \qquad \begin{array}{l} s_i^h(\sigma,u) = (s_i\sigma,u), \\ d_j^v(\sigma,u) = \ \delta_*^j(\sigma, d_j u), \\ s_j^v(\sigma,u) = \varepsilon_*^j(\sigma, s_j u). \end{array}$$

A brief analysis of $\mathcal{P}$ tell us that, for any object (V,v) of $\Delta(\mathcal{S})$ where $v \in \mathcal{S}(V)_n$, the bisimplicial abelian group $\mathcal{P}(V,v)$ can be described as follows: $\mathcal{P}_{p,q}(V,v) = \mathbb{Z}\{(\sigma,u,\alpha)\}$ is the free abelian group on the set of triples (σ,u,α) where $\sigma \in N_{p+1}\mathrm{C}^{\mathrm{op}}$, $u \in \mathcal{S}(\sigma 0)_q$, and $\alpha : \mathbf{q} \to \mathbf{n}$ is a map in Δ satisfying that $\alpha^* v = u^{\sigma_1\cdots\sigma_{p+1}}$. The horizontal and vertical face and degeneracy homomorphisms

$$\mathcal{P}_{p+1,q}(V,v) \xleftarrow{s_i^h} \mathcal{P}_{p,q}(V,v) \xrightarrow{d_i^h} \mathcal{P}_{p-1,q}(V,v), \quad \mathcal{P}_{p,q+1}(V,v) \xleftarrow{s_j^v} \mathcal{P}_{p,q}(V,v) \xrightarrow{d_j^v} \mathcal{P}_{p,q-1}(V,v),$$

are respectively defined on generators (σ,u,α) of $\mathcal{P}_{p,q}$ by

$$d_i^h(\sigma,u,\alpha) = \begin{cases} (d_0\sigma, u^{\sigma_1}, \alpha) & i=0, \\ (d_i\sigma,u,\alpha) & 0<i\leq p, \end{cases} \qquad d_j^v(\sigma,u,\alpha) = \ (\sigma, d_j u, \alpha\delta^j),$$

$$s_i^h(\sigma,u,\alpha) = (s_i\sigma,u,\alpha), \qquad s_j^v(\sigma,u,\alpha) = (\sigma, s_j u, \alpha\varepsilon^j).$$

Let us now write $\mathcal{P}_{\bullet,\bullet}$ for the alternating faces sum chain bicomplex associated to $\mathcal{P}$, that is, the bicomplex with $\mathcal{P}_{p,q}$ as $\Delta(\mathcal{S})$-module of bidegree (p,q) and the horizontal and vertical boundaries are $\partial^h = \sum_{i=0}^{p}(-1)^i d_i^h : \mathcal{P}_{p,q} \to \mathcal{P}_{p-1,q}$ and $\partial^v = (-1)^p \sum_{j=0}^{q}(-1)^j d_j^v : \mathcal{P}_{p,q} \to \mathcal{P}_{p,q-1}$.

We claim that the total complex $\mathrm{Tot}\mathcal{P}_{\bullet,\bullet}$ is a projective resolution of the constant $\Delta(\mathcal{S})$-module $\mathbb{Z}$, with augmentation the morphism of $\Delta(\mathcal{S})$-modules $\mu : \mathcal{P}_{0,0} \twoheadrightarrow \mathbb{Z}$ defined on generators $(U,u) \in \Psi_{0,0}(\mathcal{S})$ by $\mu(U,u) = 1 \in \mathbb{Z} = \mathbb{Z}(U,u)$. In effect, as every $\Delta(\mathcal{S})$-module $\bigoplus_{p+q=n}\mathcal{P}_{p,q}$ is free, whence projective, it suffices to prove that, for any object (V,v) of $\Delta(\mathcal{S})$, the chain complex of abelian groups $\mathrm{Tot}\mathcal{P}_{\bullet,\bullet}(V,v) \xrightarrow{\mu} \mathbb{Z} = \mathbb{Z}(V,v) \to 0$ is exact. To do this, let us fix such a (V,v), say with $v \in \mathcal{S}(V)_n$, and proceed as follows: Let $\Delta^n = \mathrm{Hom}_\Delta(-,\mathbf{n})$ be the standard simplicial n-simplex and let $C_\bullet(\Delta^n)$ denote the alternating faces sum chain complex associated to the simplicial abelian group $\mathbb{Z}\Delta^n : \Delta^{\mathrm{op}} \to \mathsf{Ab}$, $\mathbf{q} \mapsto \mathbb{Z}\mathrm{Hom}_\Delta(\mathbf{q},\mathbf{n})$. Regarding $C_\bullet(\Delta^n)$ as a bicomplex concentrated in horizontal degree zero, there is a bicomplex morphism $\mu : \mathcal{P}_{\bullet,\bullet}(V,v) \to C_\bullet(\Delta^n)$ which is defined by the homomorphisms $\mu : \mathcal{P}_{0,q} \to \mathbb{Z}\Delta^n_q$ given on generators by $\mu(\sigma,u,\alpha) = \alpha$. For every $q \geq 0$, the augmented chain complex $\mathcal{P}_{\bullet,q} \to \mathbb{Z}\Delta^n_q \to 0$ admits a contracting homotopy h, which is given by the homomorphisms $h_{-1} : \mathbb{Z}\Delta^n_q \to \mathcal{P}_{0,q}$ and $h_p : \mathcal{P}_{p,q}(V,v) \to \mathcal{P}_{p+1,q}(V,v)$ defined respectively on generators by $h_{-1}(\alpha) = (1_V, \alpha^* v, \alpha)$ and $h_p(\sigma,u,\alpha) = (-1)^{p+1}((\sigma_1,\ldots,\sigma_{p+1},1_{\sigma(p+1)}),u,\alpha)$. It follows that the induced

$$\mu : \mathrm{Tot}\mathcal{P}_{\bullet,\bullet}(V,v) \to \mathrm{Tot}C_\bullet(\Delta^n) = C_\bullet(\Delta^n)$$

is a homology isomorphism. Hence, the result follows from the well-known fact that $H_p(\Delta^n) = 0$ for $p < 0$ and $H_0(\Delta^n) \cong \mathbb{Z}$. Indeed, the augmented complex $C_\bullet(\Delta^n) \xrightarrow{\mu} \mathbb{Z} \to 0$, where $\mu : \mathbb{Z}\Delta^n_0 \to \mathbb{Z}$ is defined on any generator $\alpha \in \Delta^n_0$ by $\mu(\alpha) = 1$, has a contracting homotopy given by the homomorphisms $k_{-1} : \mathbb{Z} \to \mathbb{Z}\Delta^n_0$ and $k_q : \mathbb{Z}\Delta^n_q \to \mathbb{Z}\Delta^n_{q+1}$, where $k_{-1}(1) : \mathbf{0} \to \mathbf{n}$ is the map $0 \mapsto 0$ and, for any $\alpha : \mathbf{q} \to \mathbf{n}$ in Δ, $k_p(\alpha) : \mathbf{q+1} \to \mathbf{n}$ is the map $0 \mapsto 0$ and $i+1 \mapsto \alpha(i)$.

Therefore, for any $\Delta(\mathcal{S})$-module $\mathcal{A}$, the cohomology groups $H^n(\mathcal{S},\mathcal{A})$ can be computed as those of the cochain complex $\mathrm{Hom}_{\Delta(\mathcal{S})}(\mathrm{Tot}\mathcal{P}_{\bullet,\bullet},\mathcal{A})$. To complete the proof, note the isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\Delta(\mathcal{S})}(\mathcal{P}_{p,q},\mathcal{A}) &\overset{(2.1)}{\cong} \prod_{(\sigma,u)\in\Psi_{p,q}(\mathcal{S})} \mathcal{A}(\sigma p, u^{\sigma_1\cdots\sigma_p}) \\ &= \prod_{\sigma\in N_p\mathrm{C}^{\mathrm{op}}}\ \prod_{u\in\mathcal{S}(\sigma 0)_q} \mathcal{A}(\sigma p, u^{\sigma_1\cdots\sigma_p}) = \mathrm{F}^p(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^q(\mathcal{S},\mathcal{A})),\end{aligned}$$

which provide an isomorphism of bicomplexes $\mathrm{Hom}_{\Delta(\mathcal{S})}(\mathcal{P}_{\bullet,\bullet},\mathcal{A}) \cong \mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}},\mathcal{C}^\bullet(\mathcal{S},\mathcal{A}))$, whence an isomorphism of complexes

$$\mathrm{Hom}_{\Delta(\mathcal{S})}(\mathrm{Tot}\mathcal{P}_{\bullet,\bullet},\mathcal{A}) = \mathrm{Tot}\mathrm{Hom}_{\Delta(\mathcal{S})}(\mathcal{P}_{\bullet,\bullet},\mathcal{A}) \cong \mathrm{Tot}\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^\bullet(\mathcal{S},\mathcal{A})) = C^\bullet(\mathcal{S},\mathcal{A}).$$

□

Remark 3.3 *The cochain bicomplex* $\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^\bullet(\mathcal{S},\mathcal{A}))$ *in* (3.4) *is actually an instance of the bicomplex* (2.2) *in Example 2.1. With a little more generality, suppose* $\mathcal{S} : \mathrm{C}^{\mathrm{op}} \to \mathsf{SSet}$ *a presheaf of simplicial sets,* $\mathfrak{P} : \mathrm{C}^{\mathrm{op}} \to \mathsf{Cat}$ *a presheaf of small categories, and* $\pi : \Delta\mathcal{S} \to \mathfrak{P}$ *a morphism of presheaves of small categories. Let, for each*

$q \geq 0$ and $U \in \mathrm{ObC}$, the set $\mathcal{S}(U)_q$ be endowed with the composite map

$$\mathcal{S}(U)_q \xhookrightarrow{inc.} \mathrm{Ob}\Delta(\mathcal{S}(U)) \xrightarrow{\pi} \mathrm{Ob}\mathfrak{P}(U),$$

and let $\mathcal{F}_q(\mathcal{S}(U))$ be the corresponding free $\mathfrak{P}(U)$-module. These $\mathcal{F}_q(\mathcal{S}(U))$ provide a chain complex of $\Delta(\mathcal{S}(U))$-modules $\mathcal{F}_\bullet(\mathcal{S}(U))$, whose boundary $\partial : \mathcal{F}_q(\mathcal{S}(U)) \to \mathcal{F}_{q-1}(\mathcal{S}(U))$ is given on generators by $\partial(u) = \sum_{j=0}^{q}(-1)^j \delta_*^j d_j u$. Denote by

$$\mathcal{F}_\bullet(\mathcal{S}) \tag{3.6}$$

the chain complex of $\int_{\mathrm{C}}\mathfrak{P}$-modules such that $\mathcal{F}_\bullet(S)(U) = \mathcal{F}_\bullet(S(U))$, for each object U of C, and, for every morphism $\sigma : V \to U$ in C, the morphism of chain complexes of $\mathfrak{P}(U)$-modules $(\ )^\sigma : \mathcal{F}_\bullet(S(U)) \to \mathcal{F}_\bullet(S(V))^\sigma$ acts, at each degree $q \geq 0$, on the generators $u \in S(U)_q$ of $\mathcal{F}_q(S(U))$ by $u \mapsto u^\sigma$. If $\mathcal{A}$ is an $\int_{\mathrm{C}}\mathfrak{P}$-module and we regard $\mathcal{A}$ also as a $\Delta(\mathcal{S})$-module by restriction along the induced functor $\pi : \Delta(\mathcal{S}) = \int_{\mathrm{C}}\Delta\mathcal{S} \to \int_{\mathrm{C}}\mathfrak{P}$, then there is a natural isomorphism of cochain complexes of $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-modules

$$\mathcal{H}om(\mathcal{F}_\bullet(\mathcal{S}), \mathcal{A}) \cong \mathfrak{C}^\bullet(\mathcal{S}, \mathcal{A}), \tag{3.7}$$

whose component at a morphism $\sigma : V \to U$ of C consists of the isomorphism of chain complexes of abelian groups $\mathrm{Hom}_{\mathfrak{P}(U)}(\mathcal{F}_\bullet(\mathcal{S}(U)), \mathcal{A}(V)^\sigma) \cong C^\bullet(\mathcal{S}(U), \mathcal{A}(V)^\sigma)$ which are given by the isomorphisms (2.1)

$$\mathrm{Hom}_{\mathfrak{P}(U)}(\mathcal{F}_q(\mathcal{S}(U)), \mathcal{A}(V)^\sigma)) \cong \prod_{u \in \mathcal{S}(U)_q} \mathcal{A}(V, \pi(u)^\sigma).$$

Therefore, there is an induced isomorphism of cochain bicomplexes

$$\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{F}_\bullet(\mathcal{S}), \mathcal{A}) \cong \mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathfrak{C}^\bullet(\mathcal{S}, \mathcal{A})).$$

3.2. The pointed case

The pointed case is relevant for our development. Let $\mathrm{pt} : \Delta^{\mathrm{op}} \to \mathsf{SSet}$ denote the simplicial set with only one n-simplex pt_n, for any $n \geq 0$, and $\alpha^*\mathrm{pt}_n = \mathrm{pt}_m$, for any map $\alpha : \mathbf{m} \to \mathbf{n}$ in Δ, and let also $\mathrm{pt} : \mathrm{C}^{\mathrm{op}} \to \mathsf{SSet}$ denote the constant presheaf on C defined by the simplicial set pt. Then, a *pointed presheaf of simplicial sets* means a presheaf $\mathcal{S} : \mathrm{C}^{\mathrm{op}} \to \mathsf{SSet}$ endowed with a presheaf map $\mathrm{pt} \to \mathcal{S}$. This is the same thing as a presheaf on C with values in the category of pointed simplicial sets. For any such a pointed presheaf, $\Delta(\mathrm{pt}) \subseteq \Delta(\mathcal{S})$ is a full subcategory and every coefficient system $\mathcal{A}$ on $\mathcal{S}$ gives by restriction a coefficient system on pt, say $\mathcal{A}|_{\mathrm{pt}}$. We write $C^n(\mathcal{S}, \mathrm{pt}, \mathcal{A})$ for the group of n-cochains on $\mathcal{S}$ with values in $\mathcal{A}$ which vanish on pt. These cochains form a subcomplex $C^\bullet(\mathcal{S}, \mathrm{pt}, \mathcal{A})$ which fits into a short exact sequence

$$0 \to C^\bullet(\mathcal{S}, \mathrm{pt}, \mathcal{A}) \to C^\bullet(\mathcal{S}, \mathcal{A}) \to C^\bullet(\mathrm{pt}, \mathcal{A}|_{\mathrm{pt}}) \to 0.$$

In cohomology this gives a long exact sequence

$$\cdots \to H^n_{\mathrm{pt}}(\mathcal{S}, \mathcal{A}) \to H^n(\mathcal{S}, \mathcal{A}) \to H^n(\mathrm{pt}, \mathcal{A}|_{\mathrm{pt}}) \to H^{n+1}_{\mathrm{pt}}(\mathcal{S}, \mathcal{A}) \to \cdots \tag{3.8}$$

where we refer to the relative cohomology groups

$$H^n_{\mathrm{pt}}(\mathcal{S}, \mathcal{A}) = H^n C^\bullet(\mathcal{S}, \mathrm{pt}, \mathcal{A}) \tag{3.9}$$

as the cohomology *groups of the pointed presheaf of simplicial sets* $\mathcal{S}$.

Note that $\Delta(\mathrm{pt}) \cong \mathrm{C}^{\mathrm{op}} \times \Delta$, $(U, \mathrm{pt}_n) \leftrightarrow (U, \mathbf{n})$, so that a coefficient system on the presheaf pt identifies with a functor $\Delta \to \mathrm{C}^{\mathrm{op}}$-Mod, that is, with a cosimplicial C^{op}-module. We say that a coefficient system $\mathcal{A}$ on a pointed presheaf of simplicial set $\mathcal{S}$ is *pointed* whenever the cosimplicial C^{op}-module $\mathcal{A}|_{\mathrm{pt}} : \Delta \to \mathrm{C}^{\mathrm{op}}$-Mod is constant. This means that, for any object U of C and $n \geq 0$, $\mathcal{A}(U, \mathrm{pt}_n) = \mathcal{A}(U, \mathrm{pt}_0)$ and, for any map $\alpha : \mathbf{m} \to \mathbf{n}$ in Δ, $\alpha_* = id_{\mathcal{A}(U,\mathrm{pt}_0)}$. Also, if we denote by

$$\mathcal{A}|_{\mathrm{C}^{\mathrm{op}}} : \mathrm{C}^{\mathrm{op}} \to \mathsf{Ab}$$

the C^{op}-module obtained by restriction of $\mathcal{A}$ via the canonical inclusion functor

$$\mathrm{C}^{\mathrm{op}} \hookrightarrow \Delta(\mathcal{S}), \quad (U \overset{\sigma}{\leftarrow} V) \mapsto \big((U, \mathrm{pt}_0) \overset{(\sigma, 1_0)}{\longrightarrow} (V, \mathrm{pt}_0)\big), \tag{3.10}$$

then we can say that a coefficient system $\mathcal{A}$ on a pointed presheaf of simplicial set $\mathcal{S}$ is pointed if the diagram below commutes.

$$\begin{tikzcd} \Delta(\mathrm{pt}) \arrow[r, hook] \arrow[d, equal, "\cong"'] & \Delta(\mathcal{S}) \arrow[dr, "\mathcal{A}"] & \\ & & \mathsf{Ab} \\ \mathrm{C}^{\mathrm{op}} \times \Delta \arrow[r, "pr"] & \mathrm{C}^{\mathrm{op}} \arrow[ur, "\mathcal{A}|_{\mathrm{C}^{\mathrm{op}}}"'] & \end{tikzcd}$$

For a pointed coefficient system $\mathcal{A}$ on a pointed presheaf of simplicial set $\mathcal{S}$, *we define the complex of pointed cochains of* $\mathcal{S}$ *with coefficients in* $\mathcal{A}$ as the subcomplex

$$C^\bullet_{\mathrm{pt}}(\mathcal{S}, \mathcal{A}) \subseteq C^\bullet(\mathcal{S}, \mathcal{A}),$$

where $C^0_{\mathrm{pt}}(\mathcal{S}, \mathcal{A}) = 0$ and, for $n \geq 1$,

$$C^n_{\mathrm{pt}}(\mathcal{S}, \mathcal{A}) = \bigoplus_{\substack{p+q=n \\ q \geq 1}} \mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathfrak{C}^q(\mathcal{S}, \mathcal{A})) = \bigoplus_{p+q=n-1} \mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathfrak{C}^{q+1}(\mathcal{S}, \mathcal{A})).$$

Theorem 3.4 *Let* $\mathrm{pt} \to \mathcal{S}$ *be a pointed presheaf of simplicial sets on a category* C*. For any pointed* $\Delta(\mathcal{S})$*-module* $\mathcal{A}$*,*

(a) there are natural isomorphisms $H^n_{\mathrm{pt}}(\mathcal{S}, \mathcal{A}) \cong H^n C^\bullet_{\mathrm{pt}}(\mathcal{S}, \mathcal{A})$*,*

(b) there are natural isomorphisms $H^n(\mathrm{pt}, \mathcal{A}|_{\mathrm{pt}}) \cong H^n(\mathrm{C}^{\mathrm{op}}, \mathcal{A})$*,*

(c) there is a natural long exact sequence

$$\cdots \to H^n_{\mathrm{pt}}(\mathcal{S}, \mathcal{A}) \to H^n(\mathcal{S}, \mathcal{A}) \to H^n(\mathrm{C}^{\mathrm{op}}, \mathcal{A}) \to H^{n+1}_{\mathrm{pt}}(\mathcal{S}, \mathcal{A}) \to \cdots$$

Proof For any integer $q \geq 0$, the $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-module $\mathfrak{C}^q(\mathrm{pt}, \mathcal{A}|_{\mathrm{pt}})$ carries a morphism $\sigma : V \to U$ of C to the abelian group $\mathfrak{C}^q(\mathrm{pt}, \mathcal{A}|_{\mathrm{pt}})(\sigma) = \mathcal{A}(V, \mathrm{pt}_0)$ and, if $\tau : W \to V$ is in C^{op}, then

$$\begin{aligned} \sigma_* = id : \mathfrak{C}^q(\mathrm{pt}, \mathcal{A}|_{\mathrm{pt}})(\tau) = \mathcal{A}(W, \mathrm{pt}_0) &\longrightarrow \mathfrak{C}^q(\mathrm{pt}, \mathcal{A}|_{\mathrm{pt}})(\sigma\tau) = \mathcal{A}(W, \mathrm{pt}_0), \\ \tau^* = (\ )^\tau : \mathfrak{C}^q(\mathrm{pt}, \mathcal{A}|_{\mathrm{pt}})(\sigma) = \mathcal{A}(V, \mathrm{pt}_0) &\longrightarrow \mathfrak{C}^q(\mathrm{pt}, \mathcal{A}|_{\mathrm{pt}})(\sigma\tau) = \mathcal{A}(W, \mathrm{pt}_0). \end{aligned}$$

This way, we see that $\mathfrak{C}^\bullet(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}})$ is the cochain complex of $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-modules

$$\mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \xrightarrow{0} \mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \xrightarrow{id} \mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \xrightarrow{0} \mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \xrightarrow{id} \cdots$$

and there is a cochain homotopy equivalence

$$\begin{tikzcd}
\mathfrak{C}^\bullet(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \arrow[d] & \mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \arrow[r, "0"] \arrow[d, equal] & \mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \arrow[r, "id"] \arrow[d] & \mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \arrow[r, "0"] \arrow[d] & \mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \arrow[r, "id"] \arrow[d] & \cdots \\
\mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}): & \mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which induces a cohomology isomorphism cochain map

$$C^\bullet(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) = \mathrm{TotF}^\bullet(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^\bullet(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}})) \to \mathrm{TotF}^\bullet(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}})) = \mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}})).$$

Since a direct comparison shows that $\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}},\mathfrak{C}^0(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}})) = C^\bullet(\mathrm{C}^{\mathrm{op}},\mathcal{A}|_{\mathrm{C}^{\mathrm{op}}})$, we actually have a natural cohomology isomorphism cochain map $C^\bullet(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \to C^\bullet(\mathrm{C}^{\mathrm{op}},\mathcal{A}|_{\mathrm{C}^{\mathrm{op}}})$. This gives part (b) of the Theorem, whence the exact sequence in part (c) follows from that in (3.8). Furthermore, since the complex $C^\bullet_{\mathrm{pt}}(\mathcal{S},\mathcal{A})$ occurs into the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathcal{S},\mathrm{pt},\mathcal{A}) \arrow[r] \arrow[d, hook] & C^\bullet(\mathcal{S},\mathcal{A}) \arrow[r] \arrow[d, equal] & C^\bullet(\mathrm{pt},\mathcal{A}|_{\mathrm{pt}}) \arrow[r] \arrow[d, "\mathfrak{f}"] & 0 \\
0 \arrow[r] & C^\bullet_{\mathrm{pt}}(\mathcal{S},\mathcal{A}) \arrow[r] & C^\bullet(\mathcal{S},\mathcal{A}) \arrow[r] & C^\bullet(\mathrm{C}^{\mathrm{op}},\mathcal{A}|_{\mathrm{C}^{\mathrm{op}}}) \arrow[r] & 0
\end{tikzcd}$$

we conclude that the inclusion $C^\bullet(\mathcal{S},\mathrm{pt},\mathcal{A}) \hookrightarrow C^\bullet_{\mathrm{pt}}(\mathcal{S},\mathcal{A})$ is also a cohomology isomorphism cochain map. Hence, part (a) of the theorem follows. □

4. Cohomologies for presheaves of commutative monoids

Throughout the paper, the monoids are denoted multiplicatively and the letter e is used to designate their units. Also $e = \{e\}$ denotes the trivial monoid.

Let C be a fixed small category. A presheaf of commutative monoids on C is a contravariant functor $\mathcal{M}:\mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$ from C into the category CMon of commutative monoids. Thus, $\mathcal{M}$ provides a commutative monoid $\mathcal{M}(U)$ to each object U of C, and a homomorphism

$$(\ )^\sigma : \mathcal{M}(U) \to \mathcal{M}(V), \quad x \mapsto x^\sigma,$$

to each arrow $\sigma : V \to U$ in C. If $\mathcal{M}$ and $\mathcal{M}'$ are presheaves of commutative monoids on C, then a morphism $\mathfrak{f}:\mathcal{M}\to\mathcal{M}'$ is a natural transformation, so it consists of homomorphisms $\mathfrak{f} = \mathfrak{f}_U : \mathcal{M}(U) \to \mathcal{M}'(U)$, one for each object U of C, such that $\mathfrak{f}x^\sigma = (\mathfrak{f}x)^\sigma$ for any $\sigma : V \to U$ in C and $x \in \mathcal{M}(U)$.

4.1. The coefficients: $\mathbb{H}(\mathcal{M})$-modules

For a commutative monoid M, let $\mathbb{H}(M)$ be the category whose objects are the elements $x \in M$ and an arrow $z : x \to y$ is an element $z \in M$ such that $zx = y$. Composition in $\mathbb{H}(M)$ is given by multiplication in M and the identity of an object x is $e : x \to x$. In [22, Chapter XII, §2], Grillet observes that there is an equivalence $\mathsf{Ab}(\mathsf{CMon}\!\downarrow_M) \simeq \mathbb{H}(M)$-Mod between the category of abelian group objects in the comma category of commutative monoids over M and the category of $\mathbb{H}(M)$-modules. The construction of Grillet's category $\mathbb{H}(M)$ defines a functor $\mathbb{H} : \mathsf{CMon} \to \mathsf{Cat}$, which assigns to a homomorphism $f : M \to M'$ the functor $\mathbb{H}(f) : \mathbb{H}(M) \to \mathbb{H}(M')$ defined by $\mathbb{H}(f)(z : x \to y) = f(z) : f(x) \to f(y)$.

Definition 4.1 *If $\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$ is a presheaf of commutative monoids, we define*

$$\mathbb{H}(\mathcal{M}) = \textstyle\int_{\mathrm{C}} \mathbb{H}\mathcal{M}$$

to be the category obtained by applying the Grothendieck construction on the presheaf of categories obtained by composing $\mathcal{M}$ with the functor $\mathbb{H}$.

Explicitly, the objects of $\mathbb{H}(\mathcal{M})$ are pairs (U, x) where $U \in \mathrm{ObC}$ and $x \in \mathcal{M}(U)$, and a morphism $(\sigma, z) : (U, x) \to (V, y)$ in $\mathbb{H}(\mathcal{M})$ consists of a morphism $\sigma : V \to U$ in C together with an element $z \in \mathcal{M}(V)$ such that $zx^\sigma = y$. Composition is defined by $(\tau, t)(\sigma, z) = (\sigma\tau, t\, z^\tau)$, and the identities are $1_{(U,x)} = (1_U, e)$.

After Grillet's equivalence, it is quite straightforward to see that there is an equivalence

$$\mathsf{Ab}\big(\mathrm{Psh}(\mathrm{C}, \mathsf{CMon})\!\downarrow_{\mathcal{M}}\big) \simeq \mathbb{H}(\mathcal{M})\text{-Mod} \tag{4.1}$$

between the category of abelian group objects in the comma category of presheaves of commutative monoids on C over $\mathcal{M}$ and the category of $\mathbb{H}(\mathcal{M})$-modules. Hence, $\mathbb{H}(\mathcal{M})$-modules naturally arise as coefficients for the cohomology of the presheaf of commutative monoids $\mathcal{M}$.

4.2. The rth-level cohomology groups $H^n(\mathcal{M}, r, \mathcal{A})$ and $H^n_s(\mathcal{M}, r, \mathcal{A})$

For a commutative monoid M, the simplicial commutative monoids $K(M, r)$, for $r \geq 0$, are defined by iterating the classifying (or delooping) $\overline{\mathrm{W}}$-construction by Eilenberg–MacLane [17] and Moore (Moore, J.C. Algebraic homotopy theory, unpublished lecture notes, 1958) on the constant simplicial monoid given by M. Thus $K(M, 0)$ is the simplicial monoid with $K_n(M, 0) = M$ and $d_i = s_i = id_M : M \to M$ for all $0 \leq i \leq n$, and $K(M, r+1)$ is constructed from $K(M, r)$ by taking

$$\begin{cases} K_0(M, r+1) = e, \\ K_{n+1}(M, r+1) = K_n(M, r) \times K_{n-1}(M, r) \times \cdots \times K_0(M, r). \end{cases}$$

The face and degeneracy maps on a simplex $u \in K_{n+1}(M, r+1)$, written as $u = (u_n, \ldots, u_0)$ with $u_j \in K_j(M, r)$, are defined by

$$d_i u = \begin{cases} (u_{n-1}, \ldots, u_0), & i = 0, \\ (d_{i-1}u_n, \ldots, d_1 u_{n-i+2}, (d_0 u_{n-i+1})u_{n-i}, u_{n-i-1}, \ldots, u_0), & 1 \leq i \leq n, \\ (d_n u_n, \ldots, d_1 u_1), & i = n+1, \end{cases}$$

$$s_i u = \ (s_{i-1}u_n, \ldots, s_0 u_{n-i+1}, e, u_{n-i}, \ldots, u_0).$$

The category $\mathbb{H}(M)$ has a natural structure of commutative monoid object in Cat, where the product is given by $(z : x \to y)(z' : x' \to y') = (zz' : xx' \to yy')$, and there is a canonical functor

$$\pi = \pi_M : \Delta(K(M,r)) \to \mathbb{H}(M)$$

satisfying that, for any simplices $u, v \in K_n(M,r)$ and any map $\alpha : \mathbf{m} \to \mathbf{n}$ in Δ, the equations

$$\pi(\alpha : \alpha^* u \to u)\,\pi(\alpha : \alpha^* v \to v) = \pi(\alpha : \alpha^*(uv) \to uv), \quad \pi(\alpha : \alpha^* e \to e) = e : e \to e.$$

hold (see Calvo–Cegarra [9, §4.1]). This functor π is recursively defined: If $r = 0$, it acts on objects by $\pi(u) = u$, and on the generator morphisms (3.1) by $\pi(\delta^i : d_i u \to u) = \pi(\varepsilon^i : s_i u \to u) = e : u \to u$. Then, the functor $\pi : \Delta(K(M, r+1)) \to \mathbb{H}(M)$ is constructed from $\pi : \Delta(K(M,r)) \to \mathbb{H}(M)$ as follows: On the 0-simplex e of $K(M, r+1)$ the functor π acts by $\pi e = e$, on an $(n+1)$-simplex $u = (u_n, \dots, u_0)$ with $u_j \in K_j(M,r)$, by $\pi u = \prod_{j=0}^{n} \pi u_j$, and on the generating morphisms by

$$\pi(\delta^i : d_i u \to u) = \begin{cases} \displaystyle\prod_{j=0}^{n-1} (e : \pi u_j \to \pi u_j)(\pi u_n : e \to \pi u_n), & i = 0, \\ \displaystyle\prod_{j=0}^{n-i} (e : \pi u_j \to \pi u_j) \prod_{j=n-i+1}^{n} \pi(\delta^{n-j} : d_{n-j} u_{j-i+1} \to u_{j-i+1}), & 1 \le i \le n, \\ \displaystyle(\pi u_0 : e \to \pi u_0) \prod_{j=1}^{n} \pi(\delta^j : d_j u_j \to u_j), & i = n+1, \end{cases}$$

$$\pi(\varepsilon^i : s_i u \to u) = \ e : \pi u \to \pi u.$$

By composition with the functor π, every $\mathbb{H}(M)$-module $\mathcal{A}$ provides a coefficient system on each simplicial set $K(M,r)$, which we also denote by $\mathcal{A}$, and therefore the cohomology groups $H^n(K(M,r),\mathcal{A})$ are defined. These are the *r th-level cohomology groups $H^n(M,r,\mathcal{A})$ of the commutative monoid M* (see [9, Theorem 4.5]), that is, $H^n(M,r,\mathcal{A}) = H^n(K(M,r),\mathcal{A})$, for $n \ge 0$.

Note the construction of $K(M,r)$ from M defines a functor $K(-,r) : \mathsf{CMon} \to \mathsf{SCMon}$ from the category of commutative monoids to the category of simplicial commutative monoids, and that the functor $\pi_M : \Delta(K(M,r)) \to \mathbb{H}(M)$ is natural in M, so we have actually a natural transformation

$$\begin{tikzcd} \mathsf{CMon} \arrow[r, bend left, "\Delta K(-,r)"] \arrow[r, bend right, "\mathbb{H}"'] \arrow[r, phantom, "\downarrow\pi"] & \mathsf{Cat}. \end{tikzcd} \tag{4.2}$$

Now, let $\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$ be a presheaf of commutative monoids on a small category C. For each integer $r \ge 1$, by composing $\mathcal{M}$ with $K(-,r)$, we have a presheaf of simplicial commutative monoids, denoted by

$$K(\mathcal{M}, r) : \mathrm{C}^{\mathrm{op}} \to \mathsf{SCMon},$$

which carries an object U of C to the simplicial commutative monoid $K(\mathcal{M}(U), r)$, and a morphism $\sigma : V \to U$ of C to the simplicial homomorphism $(\ )^\sigma : K(\mathcal{M}(U), r) \to K(\mathcal{M}(V), r)$, $u \mapsto u^\sigma$. Let us stress that $\mathrm{K}(\mathcal{M}, r)$ is obviously pointed by the presheaf morphism

$$e : \mathrm{pt} \to K(\mathcal{M}, r)$$

defined by the unit elements $e \in K_n(\mathcal{M}(U), r)$, for every $U \in \mathrm{ObC}$ and $n \geq 0$.

The natural transformation (4.2) defines a morphism of presheaves of small categories

$$\begin{tikzcd} \mathrm{C}^{\mathrm{op}} \arrow[r, bend left, "\Delta K(\mathcal{M},r)"] \arrow[r, bend right, "\mathbb{H}\mathcal{M}"'] \arrow[r, phantom, "\downarrow\pi"] & \mathsf{Cat}. \end{tikzcd}$$

and thus, we have a canonical functor

$$\pi : \Delta(K(\mathcal{M}, r)) = \textstyle\int_{\mathrm{C}} \Delta K(\mathcal{M}, r) \longrightarrow \int_{\mathrm{C}} \mathbb{H}\mathcal{M} \ = \mathbb{H}(\mathcal{M}). \tag{4.3}$$

Hence, by composition with it, every $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$ defines a coefficient system on the pointed presheaf of simplicial sets $K(\mathcal{M}, r) : \mathrm{C}^{\mathrm{op}} \to \mathsf{SSet}$, which we also denote by $\mathcal{A}$. Let us stress that this is always a pointed system of coefficients on $K(\mathcal{M}, r)$. In effect, if

$$\mathcal{A}|_{\mathrm{C}^{\mathrm{op}}} : \mathrm{C}^{\mathrm{op}} \to \mathsf{Ab}$$

denote the C^{op}-module obtained by restriction of $\mathcal{A}$ via the canonical inclusion functor

$$\mathrm{C}^{\mathrm{op}} \hookrightarrow \mathbb{H}(\mathcal{M}), \quad (U \overset{\sigma}{\leftarrow} V) \mapsto \big((U, e) \xrightarrow{(\sigma,e)} (V, e)\big),$$

we have the commutative diagram of functors

$$\begin{tikzcd} \Delta(\mathrm{pt}) \arrow[r, "\Delta(e)"] \arrow[d, equal, "\cong"'] & \Delta(K(\mathcal{M},r)) \arrow[r, "\pi"] & \mathbb{H}(\mathcal{M}) \arrow[d, "\mathcal{A}"] \\ \mathrm{C}^{\mathrm{op}} \times \Delta \arrow[r, "pr"] & \mathrm{C}^{\mathrm{op}} \arrow[r, "\mathcal{A}|_{\mathrm{C}^{\mathrm{op}}}"] & \mathsf{Ab} \end{tikzcd}$$

Thus, the restriction of $\mathcal{A}$ to $\Delta(\mathrm{pt})$, $\mathcal{A}|_{\mathrm{pt}}$, identifies with constant cosimplicial C^{op}-module defined by $\mathcal{A}|_{\mathrm{C}^{\mathrm{op}}}$.

Definition 4.2 *Let $\mathcal{M}$ be a presheaf of commutative monoids on a small category C. For each integer $r \geq 1$, we define the (respect. simple) rth-level cohomology groups of $\mathcal{M}$ with coefficients in an $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$, denoted by $H^n(\mathcal{M}, r, \mathcal{A})$ (respect. $H^n_s(\mathcal{M}, r, \mathcal{A})$), to be those of the presheaf of simplicial sets $\mathrm{K}(\mathcal{M}, r)$, that is,*

$$H^n(\mathcal{M}, r, \mathcal{A}) = H^n(K(\mathcal{M}, r), \mathcal{A}), \quad H^n_s(\mathcal{M}, r, \mathcal{A}) = H^n_{\mathrm{pt}}(K(\mathcal{M}, r), \mathcal{A}) \qquad (n \geq 0).$$

The cohomology groups $H^n(\mathcal{M}, r, \mathcal{A})$ and the simple ones $H^n_s(\mathcal{M}, r, \mathcal{A})$ are closely related.

Proposition 4.3 *Let $\mathcal{M}$ be a presheaf of commutative monoids on C. For any $\mathbb{H}(\mathcal{M})$ module $\mathcal{A}$, there is a natural long exact sequence*

$$\cdots \to H^n_s(\mathcal{M}, r, \mathcal{A}) \to H^n(\mathcal{M}, r, \mathcal{A}) \to H^n(\mathrm{C}^{\mathrm{op}}, \mathcal{A}|_{\mathrm{C}^{\mathrm{op}}}) \to H^{n+1}_s(\mathcal{M}, r, \mathcal{A}) \to \cdots \tag{4.4}$$

Proof This follows from Theorem 3.4 (c). $\square$

Corollary 4.4 *If* C *has a final object, for any presheaf of commutative monoids* $\mathcal{M}$ *on* C*, any* $\mathbb{H}(\mathcal{M})$*-module* $\mathcal{A}$*, and any integer* $r \geq 1$*, there are natural isomorphisms*

$$H^n_s(\mathcal{M}, r, \mathcal{A}) \cong H^n(\mathcal{M}, r, \mathcal{A}) \qquad (n \geq 2).$$

As we said in the introduction, the first-level cohomology cohomology groups $H^n(\mathcal{M}, 1, \mathcal{A})$ and $H^n_s(\mathcal{M}, 1, \mathcal{A})$ are already known. Recall that, if M is a (not necessarily commutative) monoid, the Leech category $\mathbb{D}(M)$ is the category of factorization of M regarded as an only-one object category. Thus, the objects of $\mathbb{D}(M)$ are the elements $x \in M$ and an arrow $(z, z') : x \to y$ is a pair of elements $z, z' \in M$ such that $z'xz = y$. The cohomology groups of M with coefficients in a $\mathbb{D}(M)$-module $\mathcal{A}$ (see Leech, J., The cohomology of monoids, unpublished 1976 lecture notes) are defined as $H^n(M, \mathcal{A}) = \operatorname{Ext}^n_{\mathbb{D}(M)}(\mathbb{Z}, \mathcal{A})$. In [10], the authors present two extensions of Leech cohomology theory of monoids to presheaves of (not necessarily commutative) monoids $\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \mathsf{Mon}$, by introducing cohomology groups $H^n(\mathcal{M}, \mathcal{A}) = \operatorname{Ext}^n_{\mathbb{D}(\mathcal{M})}(\mathbb{Z}, \mathcal{A})$ and simple cohomology groups $H^n_s(\mathcal{M}, \mathcal{A}) = \operatorname{Ext}^{n-1}_{\mathbb{D}(\mathcal{M})}(I\mathcal{M}, \mathcal{A})$. Here, $\mathbb{D}(\mathcal{M}) = \int_{\mathrm{C}} \mathbb{D}\mathcal{M}$ is the Grothendieck construction on the presheaf of small categories obtained by composition of $\mathcal{M}$ with the functor $\mathbb{D} : \mathsf{Mon} \to \mathsf{Cat}$, the coefficients $\mathcal{A}$ are $\mathbb{D}(\mathcal{M})$-modules, and $I\mathcal{M}$ is the ideal augmentation of the presheaf $\mathcal{M}$. When a monoid M is commutative, there is a canonical functor $\mathbb{D}(M) \to \mathbb{H}(M)$ which is the identity on objects and carries a morphism $(z, z') : x \to y$ of $\mathbb{D}(M)$ to the morphism $zz' : x \to y$ of $\mathbb{H}(M)$. This is natural in M, so for any presheaf of commutative monoids $\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$ there is an induced functor $\mathbb{D}(\mathcal{M}) \to \mathbb{H}(\mathcal{M})$ through which every $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$ is regarded as a $\mathbb{D}(\mathcal{M})$-module, and therefore, the groups $H^n(\mathcal{M}, \mathcal{A})$ and $H^n_s(\mathcal{M}, \mathcal{A})$ are defined.

Proposition 4.5 *Let* $\mathcal{M}$ *be a presheaf of commutative monoids on a small category* C*. For any* $\mathbb{H}(\mathcal{M})$*-module* $\mathcal{A}$ *there are natural isomorphisms*

$$H^n(\mathcal{M}, 1, \mathcal{A}) \cong H^n(\mathcal{M}, \mathcal{A}), \quad H^n_s(\mathcal{M}, 1, \mathcal{A}) \cong H^n_s(\mathcal{M}, \mathcal{A}).$$

Proof For any commutative monoid M, $K(M, 1)$ is the classifying simplicial monoid of M, that is, the simplicial monoid whose q-simplices are sequences $x = (x_1, \cdots, x_q)$ of n elements in M (e, if $n = 0$), with faces $d_0 x = (x_2, \ldots, x_q)$, $d_i x = (x_1, \ldots, x_i x_{i+1}, \ldots, x_q)$ if $0 < i < q$, and $d_q x = (x_1, \ldots, x_{q-1})$, and degeneracies $s_i x = (x, \ldots, x_i, e, x_{i+1}, \ldots, x_q)$. In other words, $K(M, 1)$ is the nerve of the only-one object category defined by the monoid M. Then, for any presheaf $\mathcal{M} : \mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$ and any $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$, the cochain bicomplex (3.4), $\mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathfrak{C}^\bullet(K(\mathcal{M}, 1), \mathcal{A}))$, consists of the groups

$$\mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathfrak{C}^q(K(\mathcal{M}, 1), \mathcal{A})) = \prod_{\sigma \in N_p \mathrm{C}^{\mathrm{op}}} \prod_{x \in \mathcal{M}(\sigma 0)_q} \mathcal{A}\big(\sigma p, (x_1 \cdots x_q)^{\sigma_1 \cdots \sigma_p}\big),$$

with horizontal and vertical coboundaries respectively given by

$$(\partial_h \xi)_{\sigma, x} = \xi_{d_0 \sigma, x^{\sigma_1}} + \sum_{i=1}^{p-1} (-1)^i \, \xi_{d_i \sigma, x} + (-1)^p \, \xi^{\sigma_p}_{d_p \sigma, x},$$

$$(\partial_v \xi)_{\sigma, x} = (-1)^p \big(x_{1*}^{\sigma_1 \cdots \sigma_p} \xi_{\sigma, d_0 x} + \sum_{j=1}^{q-1} (-1)^j \xi_{\sigma, d_j x} + (-1)^q x_{q*}^{\sigma_1 \cdots \sigma_p} \xi_{\sigma, d_q x} \big).$$

By a direct comparison, we see that its total complex, that is, the complex $C^\bullet(K(\mathcal{M},1),\mathcal{A})$ of cochains of the presheaf of simplicial sets $K(\mathcal{M},1)$ with coefficients in $\mathcal{A}$, agrees with the complex of cochains of the presheaf of monoids $C^\bullet(\mathcal{M},\mathcal{A})$ defined in [10, Definition 1], as well as the subcomplex $C^\bullet_{\mathrm{pt}}(K(\mathcal{M},1),\mathcal{A})\subseteq C^\bullet(K(\mathcal{M},1),\mathcal{A})$ of pointed cochains agrees with the subcomplex of simple cochains $C^\bullet_s(\mathcal{M},\mathcal{A})\subseteq C^\bullet(\mathcal{M},\mathcal{A})$ defined in [10, Definition 2]. The claimed isomorphisms in the proposition follow from Theorems 3.2 and 3.4(a) above and [10, Theorem 2]. □

5. The complexes $C^\bullet(\mathcal{M},r,\mathcal{A})$ and $C^\bullet_s(\mathcal{M},r,\mathcal{A})$

If M is a commutative monoid, there is a cochain complex more lucid than $C^\bullet(K(M,r),\mathcal{A})$ to compute its cohomology groups $H^n(M,r,\mathcal{A})$. This is defined by Calvo–Cegarra [9, §4.2] as $\mathrm{Hom}_{\mathbb{H}(M)}(\mathcal{A}_\bullet(M,r),\mathcal{A})$, from a certain chain complex of $\mathbb{H}(M)$-modules $\mathcal{A}_\bullet(M,r)$ which we describe below. Previously, recall from [9, §1.1] that the category $\mathbb{H}(M)$-Mod is symmetric monoidal. Here, the tensor product of two $\mathbb{H}(M)$-modules $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A}\otimes_{\mathbb{H}(M)}\mathcal{B}$, is the $\mathbb{H}(M)$-module whose component at any $x\in M$, $(\mathcal{A}\otimes_{\mathbb{H}(M)}\mathcal{B})(x)$, is the quotient of the abelian group $\bigoplus_{zt=x}\mathcal{A}(z)\otimes_{\mathbb{Z}}\mathcal{B}(t)$ by the subgroup generated by the elements $y_*a\otimes b-a\otimes y_*b$, for $y,z,t\in M$ with $yzt=x$, $a\in\mathcal{A}(z)$ and $b\in\mathcal{B}(t)$. For any $y\in M$, the induced homomorphism $y_*:(\mathcal{A}\otimes_{\mathbb{H}(M)}\mathcal{B})(x)\to(\mathcal{A}\otimes_{\mathbb{H}(M)}\mathcal{B})(xy)$ is given on generators by $y_*\big(a\otimes b\big)=y_*a\otimes b=a\otimes y_*b$. For the following needed proposition, recall from subsection 2.1 we have the free $\mathbb{H}(M)$-module functor $\mathcal{F}:\mathbf{Set}\!\downarrow_M\to\mathbb{H}(M)$-Mod, which assigns to every set over M, say $S=(S,\pi:S\to M)$, the $\mathbb{H}(M)$-module $\mathcal{F}S$ whose component at each $x\in M$ is the free abelian group $\mathcal{F}S(x)=\mathbb{Z}\{(s,u)\mid s\in S,\,u\in M,\,u\,\pi(s)=x\}$.

Proposition 5.1 *For $S=(S,\pi)$ and $T=(T,\pi)$ two sets over M, there is an isomorphism of $\mathbb{H}(M)$-modules*

$$\mathcal{F}(S\times T)\cong\mathcal{F}S\otimes_{\mathbb{H}(M)}\mathcal{F}T,$$

where the product set $S\times T$ is over M with $\pi(s,t)=\pi(s)\pi(t)$.

Proof This is in [9, Proposition 1.3]. The isomorphism acts on generators by $(s,t)\mapsto s\otimes t$. □

The chain complexes of $\mathbb{H}(M)$-modules $\mathcal{A}_\bullet(M,r)$, for $r\geq 1$, are recursively defined as follows.

♦1: *$\mathcal{A}_n(M,1)$ is the free $\mathbb{H}(M)$-module on the set over M consisting of n-tuples of elements of M*

$$\alpha_n=[x_1|\cdots|x_n],\quad \textit{with}\;\;\pi(\alpha_n)=x_1\cdots x_n,$$

(the empty 0-tuple $[\;]$ with $\pi[\;]=e$ if $n=0$) which are called generic n-cells of $\mathcal{A}(M,1)$, with the relations $\alpha_n=0$ whenever some $x_i=e$.

For $r\geq 1$, $\mathcal{A}_n(M,r+1)$ is constructed from $\mathcal{A}_\bullet(M,r)$ by taking $\mathcal{A}_0(M,r+1)=\mathcal{A}_0(M,r)$ and, for $n\geq 1$,

$$\mathcal{A}_n(M,r+1)=\bigoplus_{\substack{n_1,\dots,n_p\geq r\\ p+\sum n_i=n}}\mathcal{A}_{n_1}(M,r)\otimes_{\mathbb{H}(M)}\cdots\otimes_{\mathbb{H}(M)}\mathcal{A}_{n_p}(M,r),$$

where we shall use the bar notation: If $x_1,\dots,x_p\in M$ are elements with $x_1\cdots x_p=x$, then for any $c_{n_i}\in\mathcal{A}_{n_i}(M,r)(x_i)$, we write

$$[c_{n_1}|_{r+1}\cdots|_{r+1}c_{n_p}|$$

for the generator $c_{n_1}\otimes\cdots\otimes c_{n_p}$ of the abelian group $\mathcal{A}_n(M,r+1)(x)$. It follows from Proposition 5.1 and induction that

♦2: *Each $\mathcal{A}_n(M,r+1)$ is the free $\mathbb{H}(M)$-module generated by the set over M consisting of all p-tuples, which we call* generic n-cells *of $\mathcal{A}_\bullet(M,r+1)$,*

$$\alpha_n=[\alpha_{n_1}|_{r+1}\alpha_{n_2}|_{r+1}\cdots|_{r+1}\alpha_{n_p}],\qquad \textit{with }\ \pi(\alpha_n)=\pi(\alpha_{n_1})\cdots\pi(\alpha_{n_p}),$$

of generic n_i-cells α_{n_i} of $\mathcal{A}_\bullet(M,r)$, such that $n_i\geq r$ and $p+\sum n_i=n$, with the relations $\alpha_n=0$ whenever some $\alpha_{n_i}=0$.

♦3: *The boundary $\partial:\mathcal{A}_n(M,1)\to\mathcal{A}_{n-1}(M,1)$ is the morphism of $\mathbb{H}(M)$-modules whose action on generic cells is given by*

$$\begin{aligned}\partial[x_1|\cdots|x_n]=x_{1*}[x_2|\cdots|x_n]+\sum_{i=1}^{n-1}(-1)^i[x_1|\cdots|x_ix_{i+1}|\cdots|x_n]\\ +(-1)^n\,x_{n*}[x_1|\cdots|x_{n-1}].\end{aligned}$$

To describe recursively the differential of $\mathcal{A}_\bullet(M,r+1)$, for $r\geq 1$, we need the *shuffle products*

$$\circ_r:\mathcal{A}_n(M,r)\otimes_{\mathbb{H}(M)}\mathcal{A}_m(M,r)\to\mathcal{A}_{n+m}(M,r),$$

which are the morphisms of $\mathbb{H}(M)$-modules whose effect on generic cells is given by

$$[\beta_{m_1}|_r\cdots|_r\beta_{m_p}]\circ_r[\beta_{m_{p+1}}|_r\cdots|_r\beta_{m_{p+q}}]=\sum_\sigma(-1)^{e(\sigma)}[\beta_{m_{\sigma^{-1}(1)}}|_r\cdots|_r\beta_{m_{\sigma^{-1}(p+q)}}],$$

where the sum is taken over all (p,q)-shuffles σ and $e(\sigma)=\sum(1+m_i)(1+m_{p+j})$ summed over all pairs $(i,p+j)$ such that $\sigma(i)>\sigma(p+j)$.

♦4: *The boundary $\partial:\mathcal{A}_n(M,r+1)\to\mathcal{A}_{n-1}(M,r+1)$ is the morphism of $\mathbb{H}(M)$-modules which acts on a generic n-cell $\alpha_n=[\alpha_{n_1}|_{r+1}\cdots|_{r+1}\alpha_{n_p}]$ by*

$$\begin{aligned}\partial\alpha_n=\;&-\textstyle\sum_{i=1}^p(-1)^{e_{i-1}}[\alpha_{n_1}|_{r+1}\cdots|_{r+1}\alpha_{n_{i-1}}|_{r+1}\,\partial\alpha_{n_i}\,|_{r+1}\alpha_{n_{i+1}}|_{r+1}\cdots|_{r+1}\alpha_{n_p}]\\ &+\textstyle\sum_{i=1}^{p-1}(-1)^{e_i}[\alpha_{n_1}|_{r+1}\cdots|_{r+1}\alpha_{n_{i-1}}|_{r+1}\alpha_{n_i}\circ_r\alpha_{n_{i+1}}|_{r+1}\alpha_{n_{i+2}}|_{r+1}\cdots|_{r+1}\alpha_{n_p}],\end{aligned}$$

where the exponents of the signs are $e_i=i+\sum n_i$.

Now, suppose $\mathcal{M}:\mathrm{C}^{\mathrm{op}}\to\mathsf{CMon}$ a presheaf of commutative monoids on a small category C. For each integer $r\geq 1$, let $\mathcal{A}_\bullet(\mathcal{M},r)$ be the chain complex of $\mathbb{H}(\mathcal{M})$-modules such that, for each object U of C,

$$\mathcal{A}_\bullet(\mathcal{M},r)(U)=\mathcal{A}_\bullet(\mathcal{M}(U),r),$$

and, for each morphism $\sigma:V\to U$, the action morphism of chain complexes of $\mathbb{H}(\mathcal{M}(U))$-modules

$$(\ )^\sigma:\mathcal{A}_\bullet(\mathcal{M}(U),r)\to\mathcal{A}_\bullet(\mathcal{M}(V),r)^\sigma$$

on generic cells of $\mathcal{A}_\bullet(\mathcal{M}(U),r)$ is given by the (recursive on r) formula

$$[\alpha_{n_1}|_r \cdots |_r \alpha_{n_k}]^\sigma = [\alpha^\sigma_{n_1}|_r \cdots |_r \alpha^\sigma_{n_k}].$$

Then, as in (2.2), for any $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$, we have a cochain bicomplex

$$C^{\bullet,\bullet}(\mathcal{M},r,\mathcal{A}) = \mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{A}_\bullet(\mathcal{M},r),\mathcal{A}))$$

whose group of (p,q)-cochains is

$$C^{p,q}(\mathcal{M},r,\mathcal{A}) = \prod_{\sigma\in N_p\mathrm{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathbb{H}(\mathcal{M}(\sigma 0))}\big(\mathcal{A}_q(\mathcal{M}(\sigma 0),r), \mathcal{A}(\sigma p)^{\sigma_1\cdots\sigma_p}\big).$$

For any $\xi \in C^{p-1,q}(\mathcal{M},r,\mathcal{A})$, its horizontal coboundary $\partial_h\xi \in C^{p,q}(\mathcal{M},r,\mathcal{A})$ at any $\sigma \in N_p\mathrm{C}^{\mathrm{op}}$ is the morphism of $\mathbb{H}(\mathcal{M}(\sigma 0))$-modules $(\partial_h\xi)_\sigma : \mathcal{A}_q(\mathcal{M}(\sigma 0),r) \to \mathcal{A}(\sigma p)^{\sigma_1\cdots\sigma_p}$ which acts on generic cells by

$$\begin{aligned}(\partial_h\xi)_\sigma[\alpha_{n_1}|_r \cdots |_r \alpha_{n_k}] = \xi_{d_0\sigma}[\alpha^{\sigma_1}_{n_1}|_r \cdots |_r \alpha^{\sigma_1}_{n_k}] + \sum_{i=1}^{p-1}(-1)^i \xi_{d_i\sigma}[\alpha_{n_1}|_r \cdots |_r \alpha_{n_k}]\\ + (-1)^p\,(\xi_{d_p\sigma}[\alpha_{n_1}|_r \cdots |_r \alpha_{n_k}])^{\sigma_p},\end{aligned}$$

and for any $\xi \in C^{p,q-1}(\mathcal{M},r,\mathcal{A})$ its vertical coboundary $\partial_v\xi \in C^{p,q}(\mathcal{M},r,\mathcal{A})$ at any $\sigma \in N_p\mathrm{C}^{\mathrm{op}}$ is the morphism of $\mathbb{H}(\mathcal{M}(\sigma 0))$-modules

$$(\partial_v\xi)_\sigma = (-1)^p\,\xi_\sigma\,\partial : \mathcal{A}_q(\mathcal{M}(\sigma 0),r) \to \mathcal{A}(\sigma p)^{\sigma_1\cdots\sigma_p},$$

the signed composite of ξ_σ with the coboundary $\partial : \mathcal{A}_q(\mathcal{M}(\sigma 0),r) \to \mathcal{A}_{q-1}(\mathcal{M}(\sigma 0),r)$.

We define the *complex of reduced level-r cochains of the presheaf $\mathcal{M}$ with coefficients in a $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$ as*

$$C^\bullet(\mathcal{M},r,\mathcal{A}) = \mathrm{Tot}C^{\bullet,\bullet}(\mathcal{M},r,\mathcal{A}), \tag{5.1}$$

and the *complex of simple reduced level-r cochains* as the subcomplex

$$C^\bullet_s(\mathcal{M},r,\mathcal{A}) \subseteq C^\bullet(\mathcal{M},r,\mathcal{A}), \tag{5.2}$$

where $C^0_s(\mathcal{M},r,\mathcal{A}) = 0$ and, for $n \geq 1$,

$$C^n_s(\mathcal{M},r,\mathcal{A}) = \bigoplus_{\substack{p+q=n\\ q\geq 1}} C^{p,q}(\mathcal{M},r,\mathcal{A}) = \bigoplus_{p+q=n-1} C^{p,q+1}(\mathcal{M},r,\mathcal{A}).$$

Theorem 5.2 *Let $\mathcal{M}$ be a presheaf of commutative monoids on C. For any $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$ and integers $r,n \geq 0$, there are natural isomorphisms*

$$H^n(\mathcal{M},r,\mathcal{A}) \cong H^nC^\bullet(\mathcal{M},r,\mathcal{A}), \tag{5.3}$$

$$H^n_s(\mathcal{M},r,\mathcal{A}) \cong H^nC^\bullet_s(\mathcal{M},r,\mathcal{A}). \tag{5.4}$$

Proof Let $\mathcal{F}_\bullet(K(\mathcal{M},r))$ be the chain complex of $\mathbb{H}(\mathcal{M})$-modules defined as in (3.6). By [9, Theorems 3.1 and 3.3], for every object U of C, there is a chain homotopy equivalence

$$g_U : \mathcal{A}_\bullet(\mathcal{M}(U),r) \to \mathcal{F}_\bullet(K(\mathcal{M}(U),r)),$$

which is natural in the monoid $\mathcal{M}(U)$. In particular, for any morphism $\sigma : V \to U$ in C, the square

$$\begin{array}{ccc} \mathcal{A}_\bullet(\mathcal{M}(U), r) & \xrightarrow{g_U} & \mathcal{F}_\bullet(K(\mathcal{M}(U), r)) \\ {\scriptstyle (\ )^\sigma}\downarrow & & \downarrow{\scriptstyle (\ )^\sigma} \\ \mathcal{A}_\bullet(\mathcal{M}(V), r)^\sigma & \xrightarrow{g_V^\sigma} & \mathcal{F}_\bullet(K(\mathcal{M}(V), r)^\sigma \end{array}$$

commutes. Thus, we have a homotopy equivalence between chain complexes of $\mathbb{H}(\mathcal{M})$-modules $g : \mathcal{A}_\bullet(\mathcal{M}, r) \to \mathcal{F}_\bullet(K(\mathcal{M}, r))$ which, as in Example 2.1, induces a chain homotopy equivalence of $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-modules

$$g^* : \mathcal{H}om(\mathcal{F}_\bullet(K(\mathcal{M}, r)), \mathcal{A}) \to \mathcal{H}om(\mathcal{A}_\bullet(\mathcal{M}, r), \mathcal{A}).$$

Since there is a natural isomorphism of cochain complexes of $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-modules

$$\mathcal{H}om(\mathcal{F}_\bullet(K(\mathcal{M}, r)), \mathcal{A}) \overset{(3.7)}{\cong} \mathcal{C}^\bullet(K(\mathcal{M}, r), \mathcal{A}),$$

we also have a chain homotopy equivalence of complexes of $\mathrm{F}(\mathrm{C}^{\mathrm{op}})$-modules

$$g^* : \mathcal{C}^\bullet(K(\mathcal{M}, r), \mathcal{A})) \to \mathcal{H}om(\mathcal{A}_\bullet(\mathcal{M}, r), \mathcal{A}), \tag{5.5}$$

which induces a cochain bicomplex map

$$g^* : \mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{C}^\bullet(K(\mathcal{M}, r), \mathcal{A})) \longrightarrow \mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{A}_\bullet(\mathcal{M}, r), \mathcal{A}),$$

where every vertical chain map $\mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathcal{C}^\bullet(K(\mathcal{M}, r), \mathcal{A})) \longrightarrow \mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{A}_\bullet(\mathcal{M}, r), \mathcal{A}))$ is a homotopy equivalence. Therefore, the induced map on the total complexes

$$\mathrm{Tot}(g^*) : C^\bullet(K(\mathcal{M}, r), \mathcal{A}) \to C^\bullet(\mathcal{M}, r, \mathcal{A})$$

is a cohomology isomorphism. Then, the isomorphisms (5.3) follow from Theorem 3.2.

For the remaining isomorphisms (5.4), note that g^* in (5.5) induces a homotopy equivalence on the corresponding suspension cochain complexes (i.e. those obtained by shifting the degrees up by one and changing the sign in the original coboundary)

$$\Sigma\, g^* : \Sigma\, \mathcal{C}^\bullet(K(\mathcal{M}, r), \mathcal{A})) \to \Sigma\, \mathcal{H}om(\mathcal{A}_\bullet(\mathcal{M}, r), \mathcal{A}).$$

This induces a cochain bicomplex map

$$\Sigma g^* : \mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \Sigma\, \mathcal{C}^\bullet(K(\mathcal{M}, r), \mathcal{A})) \longrightarrow \mathrm{F}^\bullet(\mathrm{C}^{\mathrm{op}}, \Sigma\, \mathcal{H}om(\mathcal{A}_\bullet(\mathcal{M}, r), \mathcal{A})),$$

where every vertical chain map is a homotopy equivalence. Therefore, the induced map on the total complexes

$$\mathrm{Tot}(\Sigma g^*) : \Sigma\, C^\bullet_{\mathrm{pt}}(K(\mathcal{M}, r), \mathcal{A}) \to \Sigma\, C^\bullet_s(\mathcal{M}, r, \mathcal{A})$$

is a cohomology isomorphism. As both $C^0_{\mathrm{pt}}(K(\mathcal{M}, r), \mathcal{A})$ and $C^0_s(\mathcal{M}, r, \mathcal{A})$ are zero, the isomorphisms (5.4) follow from Theorem 3.4 (a). $\square$

6. Low dimensional simple cocycles of second and third level

Let $\mathcal{M}$ be a presheaf of commutative monoids on C, and let $\mathcal{A}$ be any given $\mathbb{H}(\mathcal{M})$-module. As we pointed out in Proposition 4.5, the level-1 simple cohomology groups $H^n_s(\mathcal{M},1,\mathcal{A})$ are the same as the already known simple cohomology groups $H^n_s(\mathcal{M},\mathcal{A})$, which are studied in detail by the authors in [10]. In the rest of this paper, we focus on the cohomology groups $H^n_s(\mathcal{M},2,\mathcal{A})$, for $n\leq 4$, and $H^n_s(\mathcal{M},3,\mathcal{A})$, for $n\leq 5$. Since, by (5.4), there are natural isomorphisms $H^n_s(\mathcal{M},r,\mathcal{A})\cong H^nC^\bullet_s(\mathcal{M},r,\mathcal{A})$, we specify below the relevant truncated subcomplexes of the complexes $C^\bullet_s(\mathcal{M},2,\mathcal{A})$ and $C^\bullet_s(\mathcal{M},3,\mathcal{A})$.

Previously, a brief analysis of the chain complexes of $\mathbb{H}(\mathcal{M})$-modules $\mathcal{A}_\bullet(\mathcal{M},2)$ and $\mathcal{A}_\bullet(\mathcal{M},3)$ in low dimensions is needed. There are *suspension* monomorphisms

$$\mathrm{S}:\Sigma\mathcal{A}_\bullet(\mathcal{M},r)\hookrightarrow\mathcal{A}_\bullet(\mathcal{M},r+1), \tag{6.1}$$

by means of which, for any $U\in\mathrm{Ob}\mathrm{C}$, we identify each generic cell α of $\mathcal{A}_{n-1}(\mathcal{M}(U),r)$ with the generic cell $\mathrm{S}\alpha=[\alpha]$ of $\mathcal{A}_n(\mathcal{M}(U),r+1)$ (see Calvo–Cegarra [9, (21)]). Note that, by induction, one proves that any generic non-zero n-cell of $\mathcal{A}_\bullet(\mathcal{M}(U),r+1)$ can be written uniquely in the form

$$\alpha_n=[x_1|_{k_1}x_2|_{k_2}\cdots|_{k_{m-1}}x_m]$$

with $x_i\in\mathcal{M}(U)$, $x_i\neq e$, $1\leq m$, $1\leq k_i\leq r+1$, and $r+1+\sum_{i=1}^{m-1}k_i=n$. So written,

$$\pi\alpha_n=x_1\cdots x_m.$$

It follows that $\mathcal{A}_n(\mathcal{M}(U),r+1)=0$ for $0<n\leq r$. Furthermore, if some $k_i=r+1$, then $n\geq 2r+2$. Indeed, the generic n-cells of lowest n appearing in $\mathcal{A}_\bullet(\mathcal{M}(U),r+1)$ but not in $\mathcal{A}_\bullet(\mathcal{M}(U),r)$ are those generic $(2r+2)$-cells of the form $[x|_{r+1}y]$. Hence, via the suspension morphism, $\mathcal{A}_{n-1}(\mathcal{M},r)$ is identified with $\mathcal{A}_n(\mathcal{M},r+1)$ for $r+1\leq n<2r+2$, whereas $\mathcal{A}_{n-1}(\mathcal{M},r)\subsetneq\mathcal{A}_n(\mathcal{M},r+1)$ for $n\geq 2r+2$. In particular,

$$\mathcal{A}_1(\mathcal{M},2)=0=\mathcal{A}_1(\mathcal{M},3)=\mathcal{A}_2(\mathcal{M},3)$$

and we have the anticommutative (i.e. with $\partial\mathrm{S}=-\mathrm{S}\partial$) diagram of suspensions

$$\begin{tikzcd}
\mathcal{A}_\bullet(\mathcal{M},1): & \cdots & \mathcal{A}_4(\mathcal{M},1) \arrow[r] \arrow[d, hook, "\mathrm{S}"'] & \mathcal{A}_3(\mathcal{M},1) \arrow[r] \arrow[d, hook, "\mathrm{S}"'] & \mathcal{A}_2(\mathcal{M},1) \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathcal{A}_1(\mathcal{M},1) \arrow[d, equal, "\mathrm{S}"'] \\
\mathcal{A}_\bullet(\mathcal{M},2): & \cdots & \mathcal{A}_5(\mathcal{M},2) \arrow[r] \arrow[d, hook, "\mathrm{S}"'] & \mathcal{A}_4(\mathcal{M},2) \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathcal{A}_3(\mathcal{M},2) \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathcal{A}_2(\mathcal{M},2) \arrow[d, equal, "\mathrm{S}"'] \\
\mathcal{A}_\bullet(\mathcal{M},3): & \cdots & \mathcal{A}_6(\mathcal{M},3) \arrow[r] & \mathcal{A}_5(\mathcal{M},3) \arrow[r] & \mathcal{A}_4(\mathcal{M},3) \arrow[r] & \mathcal{A}_3(\mathcal{M},3)
\end{tikzcd}$$

where, for each $U\in\mathrm{Ob}\mathrm{C}$, the complex $\mathcal{A}_\bullet(\mathcal{M}(U),1)$ is explicitly described in ♦1 and ♦3, and, by ♦2 and ♦4, we have that

- ♦ $\mathcal{A}_4(\mathcal{M}(U),2)$ is the free $\mathbb{H}(\mathcal{M}(U))$-module on the set over $\mathcal{M}(U)$ consisting of the suspensions of the nonzero generic cells $[x|y|z]$ of $\mathcal{A}_3(\mathcal{M}(U),1)$ together the nonzero generic cells $[x|_2y]$, whose coboundary is

$$\partial[x|_2y]=[x|y]-[y|x].$$

- $\mathcal{A}_5(\mathcal{M}(U),2)$ is the free $\mathbb{H}(\mathcal{M}(U))$-module on the set over $\mathcal{M}(U)$ consisting of the suspensions of the nonzero generic cells $[x|y|z|t]$ of $\mathcal{A}_4(\mathcal{M}(U),1)$ together the nonzero generic 5-cells $[x|_2y|z]$ and $[x|y|_2z]$, whose coboundaries are

$$\partial[x|_2y|z] = -y_*[x|_2z] + [x|_2yz] - z_*[x|_2y] + [x|y|z] - [y|x|z] + [y|z|x]$$
$$\partial[x|y|_2z] = -x_*[y|_2z] + [xy|_2z] - y_*[x|_2z] - [x|y|z] + [x|z|y] - [z|x|y].$$

- $\mathcal{A}_6(\mathcal{M}(U),3)$ is the free $\mathbb{H}(\mathcal{M}(U))$-module on the set over $\mathcal{M}(U)$ consisting of the suspensions of the nonzero generic cells $[x|y|z|t]$ of $\mathcal{A}_4(\mathcal{M}(U),1)$ and the nonzero generic cells $[x|_2y|z]$ and $[x|y|_2z]$ of $\mathcal{A}_5(\mathcal{M}(U),2)$, together with noncero generic 6-cells $[x|_3y]$, whose coboundary is

$$\partial[x|_3y] = -[x|_2y] - [y|_2x].$$

Then, recalling that $C^\bullet_s(\mathcal{M},r,\mathcal{A})$ is the subcomplex of $C^\bullet(\mathcal{M},r,\mathcal{A})$ with

$$\begin{aligned} C^n_s(\mathcal{M},r,\mathcal{A}) &= \bigoplus_{p+q=n-1} \mathrm{F}^p(\mathrm{C}^{\mathrm{op}}, \mathcal{H}om(\mathcal{A}_{q+1}(\mathcal{M},r),\mathcal{A})) \\ &= \bigoplus_{p+q=n-1} \prod_{\sigma\in N_p\mathrm{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathbb{H}(\mathcal{M}(\sigma 0))}(\mathcal{A}_{q+1}(\mathcal{M}(\sigma 0),r), \mathcal{A}(\sigma p)^{\sigma_1\cdots\sigma_p}) \end{aligned}$$

and using the isomorphisms (2.1), we get the following description of the complexes $C^\bullet_s(\mathcal{M},2,\mathcal{A})$ and $C^\bullet_s(\mathcal{M},3,\mathcal{A})$ in low dimensions:

$$C^1_s(\mathcal{M},2,\mathcal{A}) = 0 = C^1_s(\mathcal{M},3,\mathcal{A}) = C^2_s(\mathcal{M},3,\mathcal{A}). \tag{6.2}$$

There is an anticommutative diagram of chain complexes

$$\begin{tikzcd} C^2_s(\mathcal{M},2,\mathcal{A}) \arrow[r] & C^3_s(\mathcal{M},2,\mathcal{A}) \arrow[r] & C^4_s(\mathcal{M},2,\mathcal{A}) \arrow[r] & C^5_s(\mathcal{M},2,\mathcal{A}) \\ C^3_s(\mathcal{M},3,\mathcal{A}) \arrow[r] \arrow[u, equal, "\mathrm{S}^*"] & C^4_s(\mathcal{M},3,\mathcal{A}) \arrow[r] \arrow[u, equal, "\mathrm{S}^*"] & C^5_s(\mathcal{M},3,\mathcal{A}) \arrow[r] \arrow[u, equal, "\mathrm{S}^*"] & C^6_s(\mathcal{M},3,\mathcal{A}) \arrow[u, "\mathrm{S}^*"'] \end{tikzcd} \tag{6.3}$$

where:

- A pointed 2-cochain $f\in C^2_s(\mathcal{M},2,\mathcal{A})$ is a map assigning an element

$$f(U;x)\in\mathcal{A}(U,x), \quad \text{to each } U\in\mathrm{ObC} \text{ and } x\in\mathcal{M}(U), \tag{6.4}$$

such that $f(U;e)=0$.

- a pointed 3-cochain $g\in C^3_s(\mathcal{M},2,\mathcal{A})$ is a map assigning elements

$$\begin{cases} g(U;x|y)\in\mathcal{A}(U,xy), & \text{to each } U\in\mathrm{ObC} \text{ and } x,y\in\mathcal{M}(U), \\ g(\sigma;x)\in\mathcal{A}(U_1,x^\sigma), & \text{to each } U_0\xleftarrow{\sigma}U_1 \text{ of C and } x\in\mathcal{M}(U_0), \end{cases} \tag{6.5}$$

such that $g(U;x|y)=0$ if $x=e$ or $y=e$, and $g(\sigma;e)=0$.

♦ The coboundary $\partial : C^2_s(\mathcal{M},2,\mathcal{A}) \to C^3_s(\mathcal{M},2,\mathcal{A})$ acts on an 2-cochain f by

$$\partial f(U;x|y) = -x_* f(U;y) + f(U;xy) - y_* f(U;x), \tag{6.6}$$

$$\partial f(\sigma;x) = f(U_0;x)^\sigma - f(U_1;x^\sigma). \tag{6.7}$$

♦ A pointed 4-cochain $h \in C^4_s(\mathcal{M},2,\mathcal{A})$ is a map assigning elements

$$\begin{cases} h(U;x|y|z) \in \mathcal{A}(U,xyz), & \text{to each } U \in \mathrm{ObC} \text{ and } x,y,z \in \mathcal{M}(U), \\ h(U;x|_2 y) \in \mathcal{A}(U,xy), & \text{to each } U \in \mathrm{ObC} \text{ and } x,y \in \mathcal{M}(U), \\ h(\sigma;x|y) \in \mathcal{A}(U_1,x^\sigma y^\sigma), & \text{to each } U_0 \xleftarrow{\sigma} U_1 \text{ of C and } x,y \in \mathcal{M}(U_0), \\ h(\sigma,\tau;x) \in \mathcal{A}(U_2,x^{\sigma\tau}), & \text{to each } U_0 \xleftarrow{\sigma} U_1 \xleftarrow{\tau} U_2 \text{ of C and } x \in \mathcal{M}(U_0), \end{cases} \tag{6.8}$$

such that $h(U;x|y|z) = 0$ if some of x,y,z is e, $h(U;x|_2 y) = 0 = h(\sigma;x|y)$ if $x = e$ or $y = e$, and $h(\sigma,\tau;e) = 0$.

♦ The coboundary $\partial : C^3_s(\mathcal{M},2,\mathcal{A}) \to C^4_s(\mathcal{M},2,\mathcal{A})$ acts on a 3-cochain g by

$$\partial g(U;x|y|z) = -x_* g(U;y|z) + g(U;xy|z) - g(U;x|yz) + z_* g(U;x|y), \tag{6.9}$$

$$\partial g(U;x|_2 y) = g(U;x|y) - g(U;y|x), \tag{6.10}$$

$$\partial g(\sigma;x|y) = g(U_0;x|y)^\sigma - g(U_1;x^\sigma|y^\sigma) + x^\sigma_* g(\sigma;y) - g(\sigma;xy) + y^\sigma_* g(\sigma;x), \tag{6.11}$$

$$\partial g(\sigma,\tau;x) = -g(\tau;x^\sigma) + g(\sigma\tau;x) - g(\sigma;x)^\tau. \tag{6.12}$$

♦ A pointed 5-cochain $\xi \in C^5_s(\mathcal{M},2,\mathcal{A})$ is a map assigning elements

$$\begin{cases} \xi(U;x|y|z|t) \in \mathcal{A}(U,xyzt), & \text{to each } U \in \mathrm{ObC} \text{ and } x,y,z,t \in \mathcal{M}(U), \\ \xi(U;x|_2 y|z) \in \mathcal{A}(U,xyz), & \text{to each } U \in \mathrm{ObC} \text{ and } x,y,z \in \mathcal{M}(U), \\ \xi(U;x|y|_2 z) \in \mathcal{A}(U,xyz), & \text{to each } U \in \mathrm{ObC} \text{ and } x,y,z \in \mathcal{M}(U), \\ \xi(\sigma;x|y|z) \in \mathcal{A}(U_1,x^\sigma y^\sigma z^\sigma), & \text{to each } U_0 \xleftarrow{\sigma} U_1 \text{ of C and } x,y,z \in \mathcal{M}(U_0), \\ \xi(\sigma;x|_2 y) \in \mathcal{A}(U_1,x^\sigma y^\sigma), & \text{to each } U_0 \xleftarrow{\sigma} U_1 \text{ of C and } x,y \in \mathcal{M}(U_0), \\ \xi(\sigma,\tau;x|y) \in \mathcal{A}(U_2,x^{\sigma\tau} y^{\sigma\tau}), & \text{to each } U_0 \xleftarrow{\sigma} U_1 \xleftarrow{\tau} U_2 \text{ of C and } x,y \in \mathcal{M}(U_0), \\ \xi(\sigma,\tau,\gamma;x) \in \mathcal{A}(U_3,x^{\sigma\tau\gamma}), & \text{to each } U_0 \xleftarrow{\sigma} U_1 \xleftarrow{\tau} U_2 \xleftarrow{\gamma} U_3 \text{ of C and } x \in \mathcal{M}(U_0). \end{cases} \tag{6.13}$$

such that $\xi(U;x|y|z|t) = 0$ if some of x,y,z or t is e, $\xi(\sigma;x|y|z) = 0 = \xi(U;x|_2 y|z) = \xi(U;x|y|_2 z)$ if some of x,y or z is e, $\xi(\sigma;x|_2 y) = 0 = \xi(\sigma,\tau;x|y)$ if $x = e$ or $y = e$, and $\xi(\sigma,\tau,\gamma;e) = 0$.

♦ The coboundary $\partial : C^4_s(\mathcal{M},2,\mathcal{A}) \to C^5_s(\mathcal{M},2,\mathcal{A})$ acts on a 4-cochain h by

$$\begin{aligned} \partial h(U;x|y|z|t) = &- x_* h(U;y|z|t) + h(U;xy|z|t) - h(U;x|yz|t) + h(U;x|y|zt) \\ &- t_* h(U;x|y|z), \end{aligned} \tag{6.14}$$

$$\partial h(U;x|_2y|z) = -\,y_*h(U;x|_2z) + h(U;x|_2yz) - z_*h(U;x|_2y) + h(U;x|y|z) \tag{6.15}$$
$$-\,h(U;y|x|z) + h(U;y|z|x),$$

$$\partial h(U;x|y|_2z) = -\,x_*h(U;y|_2z) + h(U;xy|_2z) - y_*h(U;x|_2z) - h(U;x|y|z) \tag{6.16}$$
$$+\,h(U;x|z|y) - h(U;z|x|y),$$

$$\partial h(\sigma;x|y|z) = h(U_0;x|y|z)^\sigma - h(U_1;x^\sigma|y^\sigma|z^\sigma) + x_*^\sigma h(\sigma;y|z) - h(\sigma;xy|z) \tag{6.17}$$
$$+\,h(\sigma;x|yz) - z_*^\sigma h(\sigma;x|y),$$

$$\partial h(\sigma;x|_2y) = h(U_0;x|_2y)^\sigma - h(U_1;x^\sigma|_2y^\sigma) - h(\sigma;x|y) + h(\sigma;y|x), \tag{6.18}$$

$$\partial h(\sigma,\tau;x|y) = -\,h(\tau;x^\sigma|y^\sigma) + h(\sigma\tau;x|y) - h(\sigma;x|y)^\tau - x_*^{\sigma\tau}h(\sigma,\tau;y) \tag{6.19}$$
$$+\,h(\sigma,\tau;xy) - y_*^{\sigma\tau}h(\sigma,\tau;x),$$

$$\partial h(\sigma,\tau,\gamma;x) = -h(\tau,\gamma;x^\sigma) + h(\sigma\tau,\gamma;x) - h(\sigma,\tau\gamma;x) + h(\sigma,\tau;x)^\gamma. \tag{6.20}$$

- ♦ A pointed 6-cochain $\xi \in C_s^6(\mathcal{M},3,\mathcal{A})$ is a map assigning elements such as a pointed 5-cocycle does in (6.13), together with the assignment of an element

$$\xi(U;x|_3y) \in \mathcal{A}(U,xy), \quad \text{to each } U \in \mathrm{ObC} \text{ and } x,y \in \mathcal{M}(U), \tag{6.21}$$

  such that $\xi(U;x|_3y) = 0$ whenever $x = e$ or $y = e$.

- ♦ The coboundary $\partial : C_s^5(\mathcal{M},3,\mathcal{A}) \to C_s^6(\mathcal{M},3,\mathcal{A})$ acts on a 5-cochain h such as $-\partial h$ does in the formulas (6.14)-(6.20) above and, in addition, by

$$\partial h(U;x|_3y) = -h(U;x|_2y) - h(U;y|_2x). \tag{6.22}$$

As we said at the beginning of this section, in the rest of the paper we are going to use only the cohomology groups $H_s^n(\mathcal{M},2,\mathcal{A})$, for $n \leq 4$, and $H_s^n(\mathcal{M},3,\mathcal{A})$, for $n \leq 5$, and for them we have the following first consequences of (6.2) and (6.3).

Theorem 6.1 *Let $\mathcal{M}$ be a presheaf of commutative monoids. For any $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$*

$$H_s^1(\mathcal{M},2,\mathcal{A}) = 0 = H_s^1(\mathcal{M},3,\mathcal{A}) = H_s^2(\mathcal{M},3,\mathcal{A}),$$

there are natural isomorphisms

$$H_s^2(\mathcal{M},2,\mathcal{A}) \cong H_s^3(\mathcal{M},3,\mathcal{A}), \quad H_s^3(\mathcal{M},2,\mathcal{A}) \cong H_s^4(\mathcal{M},3,\mathcal{A}),$$

and a natural monomorphism $H_s^5(\mathcal{M},3,\mathcal{A}) \hookrightarrow H_s^4(\mathcal{M},2,\mathcal{A})$.

The cohomology groups $H^2_s(\mathcal{M},2,\mathcal{A})$ have a natural description in terms of *equivariant derivations* $d:\mathcal{M}\to\mathcal{A}$, that is, functions that assigns to each pair (U,x), where U is an object of C and x an element of $\mathcal{M}(U)$, an element $d(U;x)\in\mathcal{A}(U,x)$ satisfying

$$d(U;xy)=x_*d(U;y)+y_*d(U;x),\quad \text{for any object } U \text{ of C and } x,y\in\mathcal{M}(U), \tag{6.23}$$

$$d(U,x)^\sigma=d(V;x^\sigma),\quad \text{for any morphism } \sigma:V\to U \text{ of C and } x\in\mathcal{M}(U). \tag{6.24}$$

Under pointwise addition, the set of all derivations $d:\mathcal{M}\to\mathcal{A}$ has an abelian group structure. We denote this abelian group by $\mathrm{Der}(\mathcal{M},\mathcal{A})$.

Theorem 6.2 *Let $\mathcal{M}$ be a presheaf of commutative monoids. For any $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$ there is a natural isomorphism $H^2_s(\mathcal{M},2,\mathcal{A})\cong\mathrm{Der}(\mathcal{M},\mathcal{A})$.*

Proof $H^2_s(\mathcal{M},2,\mathcal{A})\cong Z^2_s(\mathcal{M},2,\mathcal{A})$, since $C^1_s(\mathcal{M},2,\mathcal{A})=0$. As the derivation requirements (6.23) and (6.24) just read the 2-cocycle conditions $\partial f=0$ in (6.6) and (6.7), and the normalization condition $f(U;e)=0$ for a simple 2-cochain $f\in C^2_s(\mathcal{M},2,\mathcal{A})$ is always satisfied by a derivation (from the equality $d(U;e)=d(U;e)+d(U;e)$), we see that $Z^2_s(\mathcal{M},2,\mathcal{A})=\mathrm{Der}(\mathcal{M},\mathcal{A})$. □

In the next sections, we show natural realizations of cohomology classes in $H^3_s(\mathcal{M},2,\mathcal{A})$, $H^4_s(\mathcal{M},2,\mathcal{A})$ and $H^5_s(\mathcal{M},3,\mathcal{A})$.

7. Commutative group coextensions

If $\mathcal{M}$ is a presheaf of commutative monoids on C, by a *commutative coextension* (or *extension*) of $\mathcal{M}$ we mean a morphism of presheaves of commutative monoids $\mathfrak{f}:\mathcal{E}\to\mathcal{M}$ such that, for any $U\in\mathrm{ObC}$, the homomorphism $\mathfrak{f}:\mathcal{E}(U)\to\mathcal{M}(U)$ is surjective. If $\mathcal{A}$ is an $\mathbb{H}(\mathcal{M})$-module, a *commutative group coextension $\mathcal{E}=(\mathcal{E},\mathfrak{f},+)$ of $\mathcal{M}$ by $\mathcal{A}$* is a commutative coextension $\mathfrak{f}:\mathcal{E}\to\mathcal{M}$ endowed, for every $U\in\mathrm{ObC}$ and $x\in\mathcal{M}(U)$, with a simply transitive action group action $+$ of $\mathcal{A}(U,x)$ on the fibre set

$$\mathcal{E}(U,x)=\{u\in\mathcal{E}(U)\mid \mathfrak{f}u=x\},$$

satisfying

$$u'(a+u)=x'_*a+u'u \qquad \text{for all } a\in\mathcal{A}(U,x),\ u\in\mathcal{E}(U,x),\ u'\in\mathcal{E}(U,x'), \tag{7.1}$$

$$(a+u)^\sigma=a^\sigma+u^\sigma \qquad \text{for all } \sigma:V\to U \text{ in C},\ a\in\mathcal{A}(U,x),\ u\in\mathcal{E}(U,x). \tag{7.2}$$

Two such commutative group coextensions of $\mathcal{M}$ by $\mathcal{A}$, say $\mathcal{E}$ and $\mathcal{E}'$, are *equivalent* in case there is an isomorphism of presheaves of monoids $\mathfrak{g}:\mathcal{E}\cong\mathcal{E}'$ which preserves projection to $\mathcal{M}$ and the action of $\mathcal{A}$, that is, $\mathfrak{f}'\mathfrak{g}=\mathfrak{f}$, so that $\mathfrak{g}\mathcal{E}(U,x)=\mathcal{E}'(U,x)$ for all $U\in\mathrm{ObC}$ and $x\in\mathcal{M}(U)$, and $\mathfrak{g}(a+u)=a+\mathfrak{g}u$ whenever $u\in\mathcal{E}(U,x)$ and $a\in\mathcal{A}(U,x)$. Let $\mathrm{Ext}(\mathcal{M},\mathcal{A})$ denote the set of equivalence classes $[\mathcal{E}]$ of commutative group coextensions $\mathcal{E}$ of $\mathcal{M}$ by $\mathcal{A}$.

Theorem 7.1 *Let $\mathcal{M}$ be a presheaf of commutative monoids on C. For any $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$, there is a natural bijection $\mathrm{Ext}(\mathcal{M},\mathcal{A})\cong H^3_s(\mathcal{M},2,\mathcal{A})$.*

Proof We first define a map $\mathrm{Ext}(\mathcal{M},\mathcal{A}) \to H^3_s(\mathcal{M},2,\mathcal{A})$. Let $\mathcal{E} = (\mathcal{E},\mathfrak{f},+)$ be a commutative group coextension of $\mathcal{M}$ by $\mathcal{A}$. For each object U of C and each $x \in \mathcal{M}(U)$, let us choose an element $sx \in \mathcal{E}(U,x)$ (in particular, we select $se = e$). Then, a 3-cocycle

$$g^{\mathcal{E}} \in Z^3_s(\mathcal{M},2,\mathcal{A})$$

is constructed as follows: For each object U of C and $x,y \in \mathcal{M}(U)$, let $g^{\mathcal{E}}(U;x|y) \in \mathcal{A}(U,xy)$ be the element such that $s(xy) = g^{\mathcal{E}}(U;x|y) + sx\,sy$. Also, for every morphism $\sigma : V \to U$ of C and each $x \in \mathcal{M}(U)$, let $g^{\mathcal{E}}(\sigma;x) \in \mathcal{A}(V,x^\sigma)$ be the element such that $(sx)^\sigma = g^{\mathcal{E}}(\sigma;x) + sx^\sigma$. Clearly, we have $g^{\mathcal{E}}(U;x|y) = 0$ if $x = e$ or $y = e$, and $g^{\mathcal{E}}(\sigma;e) = 0$. So we have a cochain $g \in C^3_{\mathrm{pt}}(\mathcal{M},2,\mathcal{A})$. It is well-known that the cocycle conditions $\partial g^{\mathcal{E}}(U;x|y|z) = 0$ in (6.9) and $\partial g^{\mathcal{E}}(U;x|_2 y) = 0$ in (6.10) follow, respectively, from the associativity and commutativity laws $(sx\,sy)sz = sx(sy\,sz)$ and $sx\,sy = sx\,sy$ in each monoid $\mathcal{E}(U)$ (see the proof of Theorem 4.1 by Grillet in [22, Chaper V]). To see that the cocycle condition $\partial g(\sigma;x|y) = 0$ in (6.7) holds, let us compute $(sx\,sy)^\sigma = (sx)^\sigma(sy)^\sigma$ in two ways

$$\begin{aligned}
(sx\,sy)^\sigma =& (-g^{\mathcal{E}}(U;x|y) + s(xy))^\sigma \overset{(7.2)}{=} -g^{\mathcal{E}}(U;x|y)^\sigma + (s(xy))^\sigma \\
&= -g^{\mathcal{E}}(U;x|y)^\sigma + g^{\mathcal{E}}(\sigma;xy) + s(xy)^\sigma = -g^{\mathcal{E}}(U;x|y)^\sigma + g^{\mathcal{E}}(\sigma;xy) + s(x^\sigma y^\sigma), \\
(sx)^\sigma(sy)^\sigma =& (g^{\mathcal{E}}(\sigma;x) + sx^\sigma)(g^{\mathcal{E}}(\sigma;y) + sy^\sigma) \overset{(7.1)}{=} x^\sigma_* g^{\mathcal{E}}(\sigma;y) + y^\sigma_* g^{\mathcal{E}}(\sigma;x) + sx^\sigma sy^\sigma \\
=& x^\sigma_* g^{\mathcal{E}}(\sigma;y) + y^\sigma_* g^{\mathcal{E}}(\sigma;x) - g^{\mathcal{E}}(V;x^\sigma|y^\sigma) + s(x^\sigma y^\sigma).
\end{aligned}$$

Hence, comparison gives $(\partial g)(\sigma;x|y) = 0$. Similarly, the cocycle condition $(\partial g)(\sigma,\tau;x) = 0$ in (6.12) follows from the equality $(sx)^{\sigma\tau} = ((sx)^\sigma)^\tau$, since

$$\begin{aligned}
((sx)^\sigma)^\tau =& (g^{\mathcal{E}}(\sigma;x) + sx^\sigma)^\tau = g^{\mathcal{E}}(\sigma;x)^\tau + g^{\mathcal{E}}(\tau;x^\sigma) + s(x^\sigma)^\tau = g^{\mathcal{E}}(\sigma;x)^\tau + g^{\mathcal{E}}(\tau;x^\sigma) + sx^{\sigma\tau}, \\
(sx)^{\sigma\tau} =& g^{\mathcal{E}}(\sigma\tau;x) + sx^{\sigma\tau}.
\end{aligned}$$

The cohomology class $[g^{\mathcal{E}}] \in H^3_s(\mathcal{M},2,\mathcal{A})$ does not depend on the choice of the crossed section s. In effect, suppose s' is another one, and let g' be the 3-cocycle defined as above but from s'. If we take, for each $U \in \mathrm{ObC}$ and $x \in \mathcal{M}(U)$, the element $f(U;x) \in \mathcal{A}(U,x)$ which is determined by the equation $f(U;x) + s'x = sx$, then we have a 2-cochain $f \in C^2_s(\mathcal{M},2,\mathcal{A})$. Since, for any $x,y \in \mathcal{M}(U)$,

$$\begin{aligned}
s(xy) =& g^{\mathcal{E}}(U;x|y) + sx\,sy = g^{\mathcal{E}}(U;x|y) + (f(U;x) + s'x)(f(U;y) + s'y) \\
\overset{(7.1)}{=}& g^{\mathcal{E}}(U;x|y) + y_* f(U;x) + x_* f(U;y) + s'x\,s'y, \\
s(xy) =& f(U;xy) + s'(xy) = f(U;xy) + g'(U;x|y) + s'x\,s'y,
\end{aligned}$$

it follows, by comparison, that $g^{\mathcal{E}}(U;x|y) = g'(U;x|y) + \partial f(U;x|y)$, see (6.6). Similarly, for any arrow $\sigma : V \to U$ in C and any $x \in \mathcal{M}(U)$, we have

$$\begin{aligned}
(sx)^\sigma =& g^{\mathcal{E}}(\sigma;x) + sx^\sigma = g^{\mathcal{E}}(\sigma;x) + f(V;x^\sigma) + s'x^\sigma, \\
(sx)^\sigma =& (f(U;x) + s'x)^\sigma \overset{(7.2)}{=} f(U;x)^\sigma + (s'x)^\sigma = f(U;x)^\sigma + g'(\sigma;x) + s'x^\sigma,
\end{aligned}$$

whence $g^{\mathcal{E}}(\sigma;x)=g'(\sigma;x)+\partial f(\sigma;x)$, see (6.7). Thus, $g^{\mathcal{E}}=g'+\partial f$ and $[g^{\mathcal{E}}]=[g']$.

Furthermore, for an equivalence of extensions $\mathfrak{g}:\mathcal{E}\cong\mathcal{E}'$ we easily see that $g^{\mathcal{E}'}=g^{\mathcal{E}}$ if we take the representative elements $s'x=\mathfrak{g}sx\in\mathcal{E}'(U,x)$ for the construction of $g^{\mathcal{E}'}$, and therefore we have a well-defined map

$$\mathrm{H}:\mathrm{Ext}(\mathcal{M},\mathcal{A})\to H^3_s(\mathcal{M},2,\mathcal{A}),\qquad \text{given by } [\mathcal{E}]\mapsto[g^{\mathcal{E}}].$$

To see that H is surjective, for every $g\in Z^3_s(\mathcal{M},2,\mathcal{A})$ we can construct a commutative group coextension of $\mathcal{M}$ by $\mathcal{A}$

$$\mathcal{E}^g=(\mathcal{E}^g,\mathfrak{f},+)\tag{7.3}$$

such that $\mathrm{H}[\mathcal{E}^g]=[g]$ as follows. For each object U of C, $\mathcal{E}^g(U)=\big\{(x,a)\mid x\in\mathcal{M}(U),a\in\mathcal{A}(U,x)\big\}$ with multiplication $(x,a)(y,b)=(xy,-g(U;x|y)+x_*b+y_*a)$. A straightforward verification shows that this multiplication is associative, commutative, and unitary thanks to the cocycle conditions $(\partial g)(U;x|y|z)=0$ in (6.9), $\partial g(U;x|_2y)=0$ in (6.10), and the condition $g(U;x|e)=e$ (see [22, Chapter V, Theorem 4.1]). For every arrow $\sigma:V\to U$ of C, the map $(\ )^\sigma:\mathcal{E}^g(U)\to\mathcal{E}^g(V)$ is given by $(x,a)^\sigma=(x^\sigma,g(\sigma,x)+a^\sigma)$. This is multiplicative and unitary due the cocycle condition $(\partial g)(\sigma;x|y)=0$ in (6.11) and the equality $g(\sigma,e)=0$. If $\tau:W\to V$ is also an arrow in C, the equality $((x,a)^\sigma)^\tau=(x,a)^{\sigma\tau}$, for any $(x,a)\in\mathcal{E}^g(U)$, follows from the cocycle condition $(\partial g)(\sigma,\tau;x)=0$ in (6.12). Furthermore, for any object U of C, the cocycle condition $(\partial g)(1_U,1_U;x)=0$ gives $g(1_U,x)=0$, which implies the equality $(x,a)^{1_U}=(x,a)$, for any $(x,a)\in\mathcal{E}^g(U)$. Thus, $\mathcal{E}^g$ is a presheaf of monoids on C. The locally surjective morphism $\mathfrak{f}:\mathcal{E}^g\to\mathcal{M}$ is, at each $U\in\mathrm{ObC}$, the projection $\mathfrak{f}:\mathcal{E}^g(U)\to\mathcal{M}(U)$, $(x,a)\mapsto x$. For any $x\in\mathcal{M}(U)$, the simply transitive action $+:\mathcal{A}(U,x)\times\mathcal{E}^g(U,x)\to\mathcal{E}^g(U,x)$ is given by $b+(x,a)=(x,b+a)$. Conditions (7.1) and (7.2) are easily verified, so that $\mathcal{E}_g=(\mathcal{E}^g,\mathfrak{f},+)$ is actually a commutative group coextension of $\mathcal{M}$ by $\mathcal{A}$. Moreover, $\mathrm{H}[\mathcal{E}^g]=[g]$, since $g^{\mathcal{E}^g}=g$ if we take the obvious crossed section for this extension which is given by $sx=(x,0)$, for each $U\in\mathrm{ObC}$ and $x\in\mathcal{M}(U)$.

Finally, we see that the map H is injective: For any commutative group coextension $\mathcal{E}$ of $\mathcal{M}$ by $\mathcal{A}$ and any of its crossed sections s, there is an equivalence $\mathcal{E}^{g^{\mathcal{E}}}\cong\mathcal{E}$ which is locally defined by the isomorphisms of monoids $\mathcal{E}^{g^{\mathcal{E}}}(U)\cong\mathcal{E}(U)$ given by $(x,a)\mapsto a+sx$. Furthermore, if $g,g'\in Z^3_s(\mathcal{M},2,\mathcal{A})$ are cohomologous, say $g=g'+\partial f$ for some $f\in C^2_s(\mathcal{M},2,\mathcal{A})$, then there is an equivalence $\mathcal{E}^g\cong\mathcal{E}^{g'}$ which is locally defined by the isomorphisms of monoids $\mathcal{E}^g(U)\cong\mathcal{E}_{g'}(U)$ given by $(x,a)\mapsto(x,f(U;x)+a)$. Hence, the injectivity of H follows. □

Theorem 7.1 is useful to study commutative $\mathcal{H}$-*coextensions* of a presheaf of commutative monoids $\mathcal{M}$ on a small category C; that is, commutative coextensions $\mathfrak{f}:\mathcal{E}\to\mathcal{M}$ such that, for every object U of C, the congruence kernel of $\mathfrak{f}:\mathcal{E}(U)\to\mathcal{M}(U)$ is contained in the Green's relation (congruence in this case) $\mathcal{H}$ of the monoid $\mathcal{E}(U)$. For any such commutative $\mathcal{H}$-coextension, every element $x\in\mathcal{M}(U)$ determines an abelian group

$$\mathcal{K}_{\mathfrak{f}}(U,x),$$

the so-called Schützenberger group of the $\mathcal{H}$-class $\mathcal{E}(U,x)$ of the monoid $\mathcal{E}(U)$. Explicitly, see Grillet [22, Chapter I, Proposition 4.6], $\mathcal{K}_{\mathfrak{f}}(U,x)$ consists of equivalence classes $[a]_x$ of elements $a\in\mathcal{E}(U)$ such that $\mathfrak{f}a\,x=x$, where $[a]_x=[a']_x$ whenever $au=a'u$ for some (then, for any) $u\in\mathcal{E}(U,x)$. The addition in $\mathcal{K}_{\mathfrak{f}}(U,x)$ is induced by the multiplication in $\mathcal{E}(U)$: $[a]_x+[a']_x=[aa']_x$. The assignment $(U,x)\mapsto\mathcal{K}_{\mathfrak{f}}(U,x)$ is the function on objects of an $\mathbb{H}(\mathcal{M})$-module $\mathcal{K}_{\mathfrak{f}}$ (*the Schützenberger kernel of the $\mathcal{H}$-coextension*) where, for any object U of C and $x,y\in\mathcal{M}(U)$, the homomorphism

$$y_*:\mathcal{K}_{\mathfrak{f}}(U,x)\to\mathcal{K}_{\mathfrak{f}}(U,xy),\qquad \text{is given by }\; y_*[a]_x=[a]_{xy},$$

(see [22, Chapter V, Lemma 2.1]) and, for any morphism $\sigma:V\to U$ in C and $x\in\mathcal{M}(U)$, the homomorphism

$$(\ )^\sigma:\mathcal{K}_{\mathfrak{f}}(U,x)\to\mathcal{K}_{\mathfrak{f}}(V,x^\sigma),\qquad \text{is given by }\; ([a]_x)^\sigma=[a^\sigma]_{x^\sigma}.$$

Furthermore, for every $U\in\mathrm{Ob}\mathrm{C}$ and $x\in\mathcal{M}(U)$, there is a canonical simply transitive group action $+$ of $\mathcal{K}_{\mathfrak{f}}(U,x)$ on $\mathcal{E}(U,x)$, which is given by $[a]_x+u=au$. Conditions (7.1) and (7.2) are easily verified, so we are in presence of a commutative group coextension $(\mathcal{E},\mathfrak{f},+)$ of $\mathcal{M}$ by $\mathcal{K}_{\mathfrak{f}}$.

Notice the Schützenberger kernel of the $\mathcal{H}$-coextension $\mathcal{K}_{\mathfrak{f}}$ is an $\mathbb{H}(\mathcal{M})$-module having an additional property: It is *thin*. An $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$ is thin if, for any object U of C and $x,y,z\in\mathcal{M}(U)$ such that $xz=yz$, the equality $x_*=y_*:\mathcal{A}(U,z)\to\mathcal{A}(U,xz)$ holds (see Grillet [22, page 118]). Therefore:

Proposition 7.2 *Every commutative $\mathcal{H}$-coextension of a presheaf of commutative monoids $\mathcal{M}$ is a commutative group coextension of $\mathcal{M}$ by a thin $\mathbb{H}(\mathcal{M})$-module.*

In general, the converse of the above proposition fails. The following is a sufficient condition for a commutative group coextension of a presheaf of commutative monoids $\mathcal{M}$ on a small category C to be an $\mathcal{H}$-coextension of $\mathcal{M}$. We say that an $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$ is *surjecting* if, for every object U of C and $x\in\mathcal{M}(U)$, there exists $y\in\mathcal{M}(U)$ such that $xy=x$ and the homomorphism $x_*:\mathcal{A}(y)\to\mathcal{A}(x)$ is surjective (cf. Grillet [22, page 119] and Leech (Leech, J., The cohomology of monoids, Chapter V,3.21, unpublished 1976 lecture notes). This, for instance, holds whenever $x_*:\mathcal{A}(U,e)\to\mathcal{A}(U,x)$ is surjective for every $U\in\mathrm{Ob}\mathrm{C}$ and $x\in\mathcal{M}(U)$.

Lemma 7.3 *Every commutative group coextension of a presheaf of commutative monoids $\mathcal{M}$ by a surjecting $\mathbb{H}(\mathcal{M})$-module is a commutative $\mathcal{H}$-coextension of $\mathcal{M}$.*

Proof Let $\mathcal{M}$ be a presheaf of commutative monoids on a small category C, and let $\mathcal{A}$ be any thin $\mathbb{H}(\mathcal{M})$-module. By Theorem 7.1, every commutative group coextension of $\mathcal{M}$ by $\mathcal{A}$ is, up to equivalence, of the form $\mathcal{E}^g=(\mathcal{E}^g,\mathfrak{f},+)$, as in (7.3), for some $g\in Z^3_s(\mathcal{M},2,\mathcal{A})$. Then, it suffices to prove that, for any $U\in\mathrm{Ob}\mathrm{C}$, the congruence kernel of $\mathfrak{f}:\mathcal{E}_g(U)\to\mathcal{M}(U)$ is contained in $\mathcal{H}$. That is, to prove that for any $x\in\mathcal{M}(U)$ and $(x,a),(x,b)\in\mathcal{E}_g(U,x)$ there is $(y,c)\in\mathcal{E}_g(U)$ such that $(x,a)=(x,b)(y,c)$: By hypothesis, there is $y\in\mathcal{M}(U)$ with $xy=x$ and $x_*:\mathcal{A}(U,y)\to\mathcal{A}(U,x)$ surjective. Taking $c\in\mathcal{A}(U,y)$ such that $x_*c=a-y_*b+g(U,x|y)$, we have $(x,b)(y,c)=(xy,x_*c+y_*b-g(U;x|y))=(x,a)$. □

As a relevant consequence, we have the following (cf. Leech [25, Theorems 3.9, 5.18]).

Proposition 7.4 *Every commutative group coextension of a presheaf of commutative regular monoids $\mathcal{M}$ by a thin $\mathbb{H}(\mathcal{M})$-module is a commutative $\mathcal{H}$-coextension of $\mathcal{M}$.*

Proof We prove that every thin $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$ is surjecting: Let $U\in\mathrm{Ob}\mathrm{C}$ and $x\in\mathcal{M}(U)$. As $\mathcal{M}(U)$ is regular, there is $y\in\mathcal{M}(U)$ such that $x=xyx$. Since $\mathcal{A}$ is thin, the equality $(xy)x=ex$ yields $(xy)_*=e_*:\mathcal{A}(U,x)\to\mathcal{A}(U,x)$; that is, $x_*y_*=id_{\mathcal{A}(U,x)}$ whence $x_*:\mathcal{A}(U,xy)\to\mathcal{A}(U,x)$ is surjective. □

8. Braided and symmetric fibred categories

To fix notation, we recall from Grothendieck [23] some terminology about fibred categories.

Let $\pi : \mathcal{G} \to \mathrm{C}$ be a functor. For any object U of C, $\mathcal{G}_U$ denotes the *fibre category* of π over U. Its objects, called U*-objects*, are those objects X of $\mathcal{G}$ which are mapped to U by π and its morphisms, termed U*-morphisms*, are those mapped to 1_U by π. More generally, if $\sigma : V \to U$ is a morphism in C, then those morphisms f in $\mathcal{G}$ such that $\pi(f) = \sigma$ are termed σ*-morphisms*. A σ-morphism $f : Y \to X$ is deemed cartesian if, for any σ-morphism $g : Y' \to X$, there exists a unique V-morphism $g' : Y' \to Y$ such that $fg' = g$. The functor π is called a *fibration* provided that, for any morphism $\sigma : V \to U$ of C and any object $X \in \mathcal{G}_U$, there exists a cartesian σ-morphism $f : Y \to X$ and, moreover, the composition of cartesian morphisms is also cartesian.

A C-*fibred category* $\mathcal{G} = (\mathcal{G}, \pi)$ is a category $\mathcal{G}$ endowed with a fibration $\pi : \mathcal{G} \to \mathrm{C}$. The C-fibred categories are the objects of a 2-category, whose 1-morphisms are the C-*fibred functors* between them, that is, functors $F : \mathcal{G} \to \mathcal{G}'$ such that $\pi' F = \pi$ and carry cartesian morphisms in $\mathcal{G}$ to cartesian morphisms in $\mathcal{G}'$, and whose 2-morphisms are the C-*fibred transformations* between them, that is, natural transformations $\theta : F \Rightarrow F'$ such that $\pi'\theta = id_\pi$. An equivalence $F : \mathcal{G} \to \mathcal{G}'$ in this 2-category is termed a C-*fibred equivalence*.

In a C-fibred category $\mathcal{G}$ occurs that all morphisms are cartesian if and only if it is fibred in groupoids, that is, in every fibre category $\mathcal{G}_U$ all morphisms are invertible. In this paper, we deal only with C-*fibred categories in abelian groupoids*, that is, C-fibred categories $\mathcal{G}$ in which the fiber categories $\mathcal{G}_U$ are groupoids whose automorphism groups $\mathrm{Aut}_{\mathcal{G}_U}(X)$ are all abelian. Hence, from now on, $\underline{\text{C-fibred category means C-fibred categories in abelian groupoids}}$. In any such C-fibred category the following basic property holds.

Lemma 8.1 *Let $\mathcal{G}$ be a C-fibred category. Suppose $\sigma : V \to U$ is a morphism in C and the squares in $\mathcal{G}$ below commute, where f and g are σ-morphisms, a is a U-morphism, and b and b' are V-morphisms. Then $b = b'$.*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{f} & X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{g} & X \end{array}$$

Proof Since f is cartesian, $g = fc$ for some V-morphism $c : Y \to Y$. As $\mathrm{Aut}_{\mathcal{G}_U}(Y)$ is abelian, $bc = cb$. Then, $fcb' = afc = fbc = fcb$. Since fc is cartesian, it follows that $b' = b$. $\square$

For necessary notational accuracy, we detail below the monoidal fibred categories, in the sense of Saavedra [30, Chapter I, §4.5], we are going to work with.

Let C be a category. A *braided* C*-fibred category* $\mathcal{G} = (\mathcal{G}, \pi, \otimes, \iota, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{l}, \boldsymbol{c})$ consists of a C-fibred category $\mathcal{G} = (\mathcal{G}, \pi)$ endowed with C-fibred functors $\otimes : \mathcal{G} \times_{\mathrm{C}} \mathcal{G} \to \mathcal{G}$ and $\iota : \mathrm{C} \to \mathcal{G}$, and C-fibred transformations

$$\boldsymbol{a} = \big\{\boldsymbol{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\big\}_{(X,Y,Z) \in \mathrm{Ob}(\mathcal{G} \times_{\mathrm{C}} \mathcal{G} \times_{\mathrm{C}} \mathcal{G})} \tag{8.1}$$

$$\boldsymbol{c} = \big\{\boldsymbol{c}_{X,Y} : X \otimes Y \to Y \otimes X\big\}_{(X,Y) \in \mathrm{Ob}(\mathcal{G} \times_{\mathrm{C}} \mathcal{G})} \tag{8.2}$$

$$\boldsymbol{l} = \big\{\boldsymbol{l}_X : \iota\pi X \otimes X \to X\big\}_{X \in \mathrm{Ob}\mathcal{G}}, \quad \boldsymbol{r} = \big\{\boldsymbol{r}_X : X \otimes \iota\pi X \to X\big\}_{X \in \mathrm{Ob}\mathcal{G}} \tag{8.3}$$

(called the *associativity, braiding, and unit constraints*, respectively) satisfying, for all objects $U \in \mathrm{Ob}\mathrm{C}$ and

$X, Y, Z, T \in \mathrm{Ob}\mathcal{G}_U$, the commutativity of the following coherence diagrams (see Joyal–Street [24]):

$$\begin{tikzcd} ((X\otimes Y)\otimes Z)\otimes T \arrow[r, "\boldsymbol{a}_{X,Y,Z}\otimes 1_T"] \arrow[d, "\boldsymbol{a}_{X\otimes Y,Z,T}"'] & (X\otimes(Y\otimes Z))\otimes T \arrow[r, "\boldsymbol{a}_{X,Y\otimes Z,T}"] & X\otimes((Y\otimes Z)\otimes T) \arrow[d, "1_X\otimes \boldsymbol{a}_{Y,Z,T}"] \\ (X\otimes Y)\otimes(Z\otimes T) \arrow[rr, "\boldsymbol{a}_{X,Y,Z\otimes T}"] & & X\otimes(Y\otimes(Z\otimes T)) \end{tikzcd} \tag{8.4}$$

$$\begin{tikzcd} (X\otimes \iota U)\otimes Y \arrow[rr, "\boldsymbol{a}_{X,\iota U,Y}"] \arrow[dr, "\boldsymbol{r}_X\otimes 1_Y"'] & & X\otimes(\iota U\otimes Y) \arrow[dl, "1_X\otimes \boldsymbol{l}_Y"] \\ & X\otimes Y & \end{tikzcd} \tag{8.5}$$

$$\begin{tikzcd} (X\otimes Y)\otimes Z \arrow[r, "\boldsymbol{c}_{X,Y}\otimes 1_Z"] \arrow[d, "\boldsymbol{a}_{X,Y,Z}"'] & (Y\otimes X)\otimes Z \arrow[r, "\boldsymbol{a}_{Y,X,Z}"] & Y\otimes(X\otimes Z) \arrow[d, "1_Y\otimes \boldsymbol{c}_{X,Z}"] \\ X\otimes(Y\otimes Z) \arrow[r, "\boldsymbol{c}_{X,Y\otimes Z}"] & (Y\otimes Z)\otimes X \arrow[r, "\boldsymbol{a}_{Y,Z,X}"] & Y\otimes(Z\otimes X) \end{tikzcd} \tag{8.6}$$

$$\begin{tikzcd} X\otimes(Y\otimes Z) \arrow[r, "1_X\otimes \boldsymbol{c}_{Y,Z}"] \arrow[d, "\boldsymbol{a}^{-1}_{X,Y,Z}"'] & X\otimes(Z\otimes Y) \arrow[r, "\boldsymbol{a}^{-1}_{X,Z,Y}"] & (X\otimes Z)\otimes Y \arrow[d, "\boldsymbol{c}_{X,Z}\otimes 1_Y"] \\ (X\otimes Y)\otimes Z \arrow[r, "\boldsymbol{c}_{X\otimes Y,Z}"] & Z\otimes(X\otimes Y) \arrow[r, "\boldsymbol{a}^{-1}_{Z,X,Y}"] & (Z\otimes X)\otimes Y \end{tikzcd} \tag{8.7}$$

A *symmetric* C*-fibred category* is a braided C-fibred category $\mathcal{G}$, as above, whose braiding is a symmetry, that is, the equation

$$\boldsymbol{c}_{Y,X}\,\boldsymbol{c}_{X,Y} = 1_{X\otimes Y} \tag{8.8}$$

holds for any pair of objects (X,Y) of $\mathcal{G}\times_{\mathrm{C}}\mathcal{G}$.

Let $\mathcal{G}$, $\mathcal{G}'$ be braided C-fibred categories. A *braided* C*-fibred functor* $F = (F, \boldsymbol{\varphi}, \boldsymbol{\varphi}^*) : \mathcal{G} \to \mathcal{G}'$ is a C-fibred functor $F : \mathcal{G} \to \mathcal{G}'$ endowed with C-fibred transformations

$$\boldsymbol{\varphi} = \big\{\boldsymbol{\varphi}_{X,Y} : F(X\otimes Y) \to FX \otimes' FY\big\}_{(X,Y)\in \mathrm{Ob}(\mathcal{G}\times_{\mathrm{C}}\mathcal{G})}, \tag{8.9}$$

$$\boldsymbol{\varphi}^* = \big\{\boldsymbol{\varphi}^*_U : F\iota U \to \iota' U\big\}_{U\in \mathrm{ObC}}, \tag{8.10}$$

such that, for any object U of C and U-objects X, Y, Z of $\mathcal{G}$, the following four diagrams commute

$$\begin{tikzcd} F((X\otimes Y)\otimes Z) \arrow[r, "\boldsymbol{\varphi}_{X\otimes Y,Z}"] \arrow[d, "F\boldsymbol{a}_{X,Y,Z}"'] & F(X\otimes Y)\otimes' FZ \arrow[r, "\boldsymbol{\varphi}_{X,Y}\otimes' 1_{FZ}"] & (FX\otimes' FY)\otimes' FZ \arrow[d, "\boldsymbol{a}'_{FX,FY,FZ}"] \\ F(X\otimes(Y\otimes Z)) \arrow[r, "\boldsymbol{\varphi}_{X,Y\otimes Z}"] & FX\otimes' F(Y\otimes Z) \arrow[r, "1_{FX}\otimes' \boldsymbol{\varphi}_{Y,Z}"] & FX\otimes'(FY\otimes' FZ) \end{tikzcd} \tag{8.11}$$

$$\begin{tikzcd} F(X\otimes Y) \arrow[r, "\boldsymbol{\varphi}_{X,Y}"] \arrow[d, "F\boldsymbol{c}_{X,Y}"'] & FX\otimes' FY \arrow[d, "\boldsymbol{c}'_{FX,FY}"] \\ F(Y\otimes X) \arrow[r, "\boldsymbol{\varphi}_{Y,X}"] & FY\otimes' FX \end{tikzcd} \tag{8.12}$$

$$\begin{tikzcd} F(X\otimes \iota U) \arrow[r, "\varphi_{X,\iota U}"] \arrow[d, "Fr_X"'] & FX\otimes' F\iota U \arrow[d, "1_{FX}\otimes'\varphi^*_U"] \\ FX & FX\otimes' \iota' U \arrow[l, "r'_{FX}"'] \end{tikzcd} \qquad \begin{tikzcd} F(\iota U\otimes X) \arrow[r, "\varphi_{\iota U,X}"] \arrow[d, "Fl_X"'] & F\iota U\otimes' FX \arrow[d, "\varphi^*_U\otimes' 1_{FX}"] \\ FX & \iota' U\otimes' FX \arrow[l, "l'_{FX}"'] \end{tikzcd} \tag{8.13}$$

If $F, F' : \mathcal{G} \to \mathcal{G}'$ are braided C-fibred functors between braided C-fibred categories, then a *monoidal* C*-fibred transformation* between them is a C-fibred transformation on the underlying C-fibred functors $\theta : F \Rightarrow F'$ such that, for any object U of C and U-objects X, Y of $\mathcal{G}$, the following two diagrams commute.

$$\begin{tikzcd} F(X\otimes Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "\theta_{X\otimes Y}"'] & FX\otimes' FY \arrow[d, "\theta_X\otimes'\theta_Y"] \\ F'(X\otimes Y) \arrow[r, "\varphi'_{X,Y}"] & F'X\otimes' F'Y \end{tikzcd} \qquad \begin{tikzcd} F\iota U \arrow[dr, "\varphi^*_U"] \arrow[dd, "\theta_{\iota U}"'] & \\ & \iota' U \\ F'\iota U \arrow[ur, "\varphi'^*_U"'] & \end{tikzcd} \tag{8.14}$$

The composite $F'F : \mathcal{G} \to \mathcal{G}''$ of two braided fibred functors $F : \mathcal{G} \to \mathcal{G}'$ and $F' : \mathcal{G}' \to \mathcal{G}''$ is also braided with structure data (8.9) and (8.10) respectively given by the compositions

$$F'F(X\otimes Y) \xrightarrow{F'\varphi_{X,Y}} F'(FX\otimes' FY) \xrightarrow{\varphi'_{FX,FY}} F'FX\otimes'' F'FY,$$
$$F'F\iota U \xrightarrow{F'\varphi^*_U} F'\iota' U \xrightarrow{\varphi'^*_U} \iota'' U.$$

This composition is associative and unitary, so that the category of braided C-fibred categories is defined. Actually, this is the underlying category of the 2*-category of braided* C*-fibred categories*, whose 2-cells are the monoidal C-fibred transformations. In this 2-category, both the horizontal and vertical composition of 2-cells are defined as the ordinary vertical and horizontal composition of natural transformations. Note every 2-cell is invertible, and therefore this 2-category is a *track category* in the sense of Baues–Jibladze [2] (also known as $(2,1)$-categories). An equivalence $F : \mathcal{G} \to \mathcal{G}'$ in this 2-category is called a *braided* C*-fibred equivalence*. We say that two braided (or symmetric) C-fibred categories are *equivalent* if they are connected by a braided C-fibred equivalence.

In this section, our goal is to give a cohomological classification for equivalence classes of braided and symmetric C-fibred categories. The following is an useful basic fact.

Lemma 8.2 *Let $F : \mathcal{G} \to \mathcal{G}'$ be a braided C-fibred functor between braided C-fibred categories. The following conditions are equivalent.*

1. *F is a braided C-fibred equivalence.*

2. *F is a C-fibred equivalence.*

3. *F is fully faithful, and for any $U \in \mathrm{ObC}$ and $X' \in \mathcal{G}'_U$, there is an $X \in \mathrm{Ob}\mathcal{G}_U$ and a U-morphism $FX \to X'$.*

4. *For any object U of C, the induced functor $F_U : \mathcal{G}_U \to \mathcal{G}'_U$ is an equivalence.*

Proof (1) $\Rightarrow$ (2) is obvious and (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) is proved by Grothendieck in [23, Propositions 4.2 and 6.10]. (2) $\Rightarrow$ (1): If F is a C-fibred equivalence, there exists a C-fibred functor $F' : \mathcal{G}' \to \mathcal{G}$ and C-fibred transformations $\theta : FF' \Rightarrow 1_{\mathcal{G}'}$ and $\theta' : F'F \Rightarrow 1_{\mathcal{G}}$ such that $\theta F = F\theta'$ and $F'\theta = \theta' F'$. Then, F' can be provided in a unique way with a structure of braided C-fibred functor such that θ and θ' become monoidal C-fibred transformations. Indeed, the structure constraints are given, for each $U \in \mathrm{ObC}$, by the U-morphisms $\boldsymbol{\varphi}'_{X',Y'} : F'(X' \otimes' Y') \to F'X' \otimes F'Y'$, for $X', Y' \in \mathcal{G}'_U$, and $\boldsymbol{\varphi}'^{*}_U : F'\iota' U \to \iota U$ which make the diagrams below commutative.

$$
\begin{tikzcd}
FF'(X' \otimes' Y') \arrow[r, "F\boldsymbol{\varphi}'_{X',Y'}"] \arrow[d, "\theta_{X' \otimes' Y'}"'] & F(F'X' \otimes F'Y') \arrow[d, "\boldsymbol{\varphi}_{F'X',F'Y'}"] \\
X' \otimes' Y' & FF'X' \otimes' FF'Y' \arrow[l, "\theta_{X'} \otimes' \theta_{Y'}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FF'\iota' U \arrow[dr, "F\boldsymbol{\varphi}'^{*}_U"] \arrow[dd, "\theta_{\iota' U}"'] & \\
 & F\iota U \arrow[dl, "\boldsymbol{\varphi}^{*}_U"] \\
\iota' U &
\end{tikzcd}
$$

The verification of all requirements is entirely parallel to the given by Sinh (Sinh, H. X. Gr-catégories. Ph.D. Thesis, Université Paris VII, Paris, France, 1975, Chapitre I, §5. Aviable at https://pnp.mathematik.uni-stuttgart.de/lexmath/kuenzer/sinh.html) for the nonfibred case, so we omit the details. □

8.1. The classifying invariants of a braided (symmetric) C-fibred category

For the following definition, recall from Theorem 6.1 that, for any presheaf of commutative monoids $\mathcal{M}$ and and $\mathbb{H}(\mathcal{M})$-module $\mathcal{A}$, the pointed cohomology group $H^5_s(\mathcal{M},3,\mathcal{A})$ is canonically a subgroup of $H^4_s(\mathcal{M},2,\mathcal{A})$.

Definition 8.3 *A "classifying system" for braided (resp. symmetric) C-fibred categories is a triple $(\mathcal{M},\mathcal{A},c)$, where $\mathcal{M}$ is a presheaf of commutative monoids, $\mathcal{A}$ is an $\mathbb{H}(\mathcal{M})$-module, and c is a cohomology class $c \in H^4_s(\mathcal{M},2,\mathcal{A})$ (resp. $c \in H^5_s(\mathcal{M},3,\mathcal{A})$). Two such classifying systems $(\mathcal{M},\mathcal{A},c)$ and $(\mathcal{M}',\mathcal{A}',c')$ are isomorphic whenever there exists an isomorphism of presheaves of commutative monoids $\mathfrak{f} : \mathcal{M} \cong \mathcal{M}'$ and an isomorphism of $\mathbb{H}(\mathcal{M})$-modules $\mathfrak{F} : \mathcal{A} \cong \mathfrak{f}^*\mathcal{A}'$ such that $\mathfrak{f}^*(c') = \mathfrak{F}_*(c)$, where*

$$
\mathfrak{f}^* : H^4_s(\mathcal{M}',2,\mathcal{A}') \cong H^4_s(\mathcal{M},2,\mathfrak{f}^*\mathcal{A}'), \quad \mathfrak{F}_* : H^4_s(\mathcal{M},2,\mathcal{A}) \cong H^4_s(\mathcal{M},2,\mathfrak{f}^*\mathcal{A}')
$$

are the corresponding induced isomorphisms in cohomology. Let $[\mathcal{M},\mathcal{A},c]$ denote the isomorphism class of a classifying system $(\mathcal{M},\mathcal{A},c)$.

Let $\mathcal{G}$ be a braided C-fibred category. We associate to $\mathcal{G}$ a classifying system $(\Pi_0\mathcal{G},\Pi_1\mathcal{G},[h^{\mathcal{G}}])$ as follows.

8.1.1. The presheaf $\Pi_0\mathcal{G} : \mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$

For each $U \in \mathrm{ObC}$, let

$$
\Pi_0\mathcal{G}(U) = \{[X] \mid X \in \mathrm{Ob}\mathcal{G}_U\}
$$

be the commutative monoid of U-isomorphism classes $[X]$ of U-objects X of $\mathcal{G}$, with multiplication $[X][Y] = [X \otimes Y]$ and unit $e = [\iota U]$. If $\sigma : V \to U$ is a morphism in C and $[X] \in \pi_0\mathcal{G}(U)$, then we write $[X]^\sigma = [X'] \in \pi_0\mathcal{G}(V)$ whenever there exists a σ-morphism $X' \to X$. To see that the class $[X]^\sigma$ is well defined, suppose that $f : X' \to X$ and $g : Y' \to Y$ are both σ-morphisms and $h : X \to Y$ is a U-morphism.

Then, as g is cartesian there exits a V-morphism $h' : X' \to Y'$ such that $hf = gh'$, and therefore, $[X'] = [Y']$ in $\Pi_0\mathcal{G}(V)$. Furthermore, as $\pi : \mathcal{G} \to \mathrm{C}$ is a fibration, x^σ is defined for all $x \in \pi_0\mathcal{G}(U)$ and so we have a map

$$(\ )^\sigma : \Pi_0\mathcal{G}(U) \to \Pi_0\mathcal{G}(V), \qquad x \mapsto x^\sigma,$$

which is actually a homomorphism of monoids: If $x = [X], y = [Y] \in \Pi_0\mathcal{G}(U)$ and $f : X' \to X$ and $g : Y' \to Y$ are σ-morphisms, then $f \otimes g : X' \otimes Y' \to X \otimes Y$ is also a σ-morphism; whence, $([X][Y])^\sigma = [X \otimes Y]^\sigma = [X' \otimes Y'] = [X'][Y'] = [X]^\sigma[Y]^\sigma$. Moreover, as $\iota\sigma : \iota V \to \iota U$ is a σ-morphism, $e^\sigma = [\iota U]^\sigma = [\iota V] = e$.

So defined, $\Pi_0\mathcal{G}$ is a presheaf of commutative monoids. Note that if $\sigma : V \to U$ and $\sigma' : W \to V$ are any two composable morphisms of C. For any $[X] \in \Pi_0\mathcal{G}(U)$, any σ-morphism $f : X' \to X$ and any σ'-morphism $f' : X'' \to X'$, the composite $ff' : X'' \to X$ is a $\sigma\sigma'$-morphism, and therefore $[X]^{\sigma\sigma'} = [X''] = [X']^{\sigma'} = ([X]^\sigma)^{\sigma'}$.

8.1.2. The module $\Pi_1\mathcal{G} : \mathbb{H}(\Pi_0\mathcal{G}) \to \mathsf{Ab}$

Let U be any object of C. For each $x \in \Pi_0\mathcal{G}(U)$, we choose a representative U-object of x, say

$$sx = s(U, x) \in \mathrm{Ob}\mathcal{G}_U, \tag{8.15}$$

that is, such that $x = [sx]$ (in particular, we select $se = \iota U$) and define

$$\Pi_1\mathcal{G}(U, x) = \mathrm{Aut}_{\mathcal{G}_U}(sx),$$

the abelian group of U-automorphisms of sx in $\mathcal{G}$. For any pair of elements $x, y \in \Pi_0\mathcal{G}(U)$, the homomorphism

$$x_* : \Pi_1\mathcal{G}(U, y) \to \Pi_1\mathcal{G}(U, xy), \qquad a \mapsto x_*a,$$

carries each U-automorphism $a : sy \to sy$ to the U-automorphism $x_*a : s(xy) \to s(xy)$ determined by the commutativity of either of the following two diagrams

$$\begin{tikzcd} s(xy) \arrow[r, "\Gamma_{x,y}"] \arrow[d, dotted, "x_*a"'] & sx \otimes sy \arrow[d, "1\otimes a"] \\ s(xy) \arrow[r, "\Gamma_{x,y}"] & sx \otimes sy \end{tikzcd} \qquad \begin{tikzcd} s(xy) \arrow[r, "\Gamma_{y,x}"] \arrow[d, dotted, "x_*a"'] & sy \otimes sx \arrow[d, "a\otimes 1"] \\ s(xy) \arrow[r, "\Gamma_{y,x}"] & sy \otimes sx \end{tikzcd} \tag{8.16}$$

where

$$\Gamma_{x,y} : s(xy) \to sx \otimes sy \tag{8.17}$$

is a U-morphism (any, by Lemma 8.1). For instance, we take

$$\Gamma_{e,x} = \boldsymbol{l}_{sx}^{-1} : sx \to \iota U \otimes sx \ \text{ and } \ \Gamma_{x,e} = \boldsymbol{r}_{sx}^{-1} : sx \to sx \otimes \iota U. \tag{8.18}$$

Since we have the commutative diagram

$$\begin{tikzcd} s(xy) \arrow[r, "\Gamma"] \arrow[d, dotted, "x_*a"'] & sx \otimes sy \arrow[r, "\boldsymbol{c}"] \arrow[d, "1\otimes a"] & sy \otimes sx \arrow[d, "a\otimes 1"] \\ s(xy) \arrow[r, "\Gamma"] & sx \otimes sy \arrow[r, "\boldsymbol{c}"] & sy \otimes sx, \end{tikzcd}$$

it follows from Lemma 8.1 that the square on the left in (8.16) commutes if and only if the one on the right does. The equality $e_* = 1_{\Pi_1\mathcal{G}(U,y)}$ holds, since the left unit constraint $\boldsymbol{l} : \iota U \otimes sy \to sy$ is natural. For any $x, y, z \in \Pi_0\mathcal{G}(U)$, the equality $(xy)_* = x_*y_* : \Pi_1\mathcal{G}(U,z) \to \Pi_1\mathcal{G}(U,xyz)$ follows from Lemma 8.1, since, for any $a \in \Pi_1\mathcal{G}(U,z)$, we have the commutative diagrams in $\mathcal{G}_U$

$$\begin{tikzcd}
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "(xy)_*a"'] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] \arrow[d, "1\otimes a"'] & (sx\otimes sy)\otimes sz \arrow[r, "\boldsymbol{a}"] \arrow[d, "(1\otimes 1)\otimes a"'] & sx\otimes(sy\otimes sz) \arrow[d, "1\otimes(1\otimes a)"] \\
s(xyz) \arrow[r, "\Gamma"] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] & (sx\otimes sy)\otimes sz \arrow[r, "\boldsymbol{a}"] & sx\otimes(sy\otimes sz),
\end{tikzcd}$$

$$\begin{tikzcd}
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "x_*y_*a"'] & sx\otimes s(yz) \arrow[r, "1\otimes\Gamma"] \arrow[d, "1\otimes y_*a"] & sx\otimes(sy\otimes sz) \arrow[d, "1\otimes(1\otimes a)"] \\
s(xyz) \arrow[r, "\Gamma"] & sx\otimes s(yz) \arrow[r, "1\otimes\Gamma"] & sx\otimes(sy\otimes sz).
\end{tikzcd}$$

If $\sigma : V \to U$ is a morphism in C, for any $x \in \Pi_0\mathcal{G}(U)$, the homomorphism

$$(\ )^\sigma : \Pi_1\mathcal{G}(U,x) \to \Pi_1\mathcal{G}(V,x^\sigma), \qquad a \mapsto a^\sigma,$$

carries each U-automorphism $a : sx \to sx$ to the V-automorphism $a^\sigma : sx^\sigma \to sx^\sigma$ determined by the commutativity of the diagram

$$\begin{tikzcd}
sx^\sigma \arrow[r, "\Theta_{\sigma,x}"] \arrow[d, dotted, "a^\sigma"'] & sx \arrow[d, "a"] \\
sx^\sigma \arrow[r, "\Theta_{\sigma,x}"] & sx
\end{tikzcd} \tag{8.19}$$

where

$$\Theta_{\sigma,x} : sx^\sigma \to sx \tag{8.20}$$

is a σ-morphism (any, by Lemma 8.1). For instance, we take

$$\Theta_{\sigma,e} = \iota\sigma : \iota V \to \iota U \ \text{ and } \ \Theta_{1_U,x} = 1_{sx}. \tag{8.21}$$

For any other $y \in \Pi_0\mathcal{G}(U)$, the equality $(y_*a)^\sigma = y_*^\sigma a^\sigma$ follows, by Lemma 8.1, from the commutativity of the diagrams

$$\begin{tikzcd}
s(yx)^\sigma \arrow[r, "\Theta"] \arrow[d, "(y_*a)^\sigma"'] & s(yx) \arrow[r, "\Gamma"] \arrow[d, "y_*a"'] & sy\otimes sx \arrow[d, "1\otimes a"] \\
s(yx)^\sigma \arrow[r, "\Theta"] & s(yx) \arrow[r, "\Gamma"] & sy\otimes sx
\end{tikzcd}
\qquad
\begin{tikzcd}
s(yx)^\sigma \arrow[r, "\Gamma"] \arrow[d, "y_*^\sigma a^\sigma"'] & sy^\sigma\otimes sx^\sigma \arrow[r, "\Theta\otimes\Theta"] \arrow[d, "1\otimes a^\sigma"] & sy\otimes sx \arrow[d, "1\otimes a"] \\
s(yx)^\sigma \arrow[r, "\Gamma"] & sy^\sigma\otimes sx^\sigma \arrow[r, "\Theta\otimes\Theta"] & sy\otimes sx
\end{tikzcd}$$

The equalities $a^{1_U} = a$ and $(a^\sigma)^\tau = a^{\sigma\tau}$, for any morphism $\tau : W \to V$ in C, are quite obviously verified. Thus, $\Pi_1\mathcal{G}$ is actually an $\mathbb{H}(\Pi_0\mathcal{G})$-module.

8.1.3. The cohomology class $[h^{\mathcal{G}}] \in H^4_s(\Pi_0\mathcal{G}, 2, \Pi_1\mathcal{G})$

We define a 2th-level pointed 4-cocycle

$$h^{\mathcal{G}} \in Z^4_s(\Pi_0\mathcal{G}, 2, \Pi_1\mathcal{G}), \tag{8.22}$$

as follows: For each $U \in \mathrm{Ob}\mathrm{C}$ and $x,y,z \in \Pi_0\mathcal{G}(U)$, the elements $h^{\mathcal{G}}(U;x|y|z) \in \Pi_1\mathcal{G}(U,xyz)$ and $h^{\mathcal{G}}(U;x|_2y) \in \Pi_1\mathcal{G}(U,xy)$ are, respectively, those U-automorphisms in $\mathcal{G}$ uniquely determined by the commutativity of the diagrams

$$\begin{tikzcd} s(xyz) \arrow[r, "\Gamma"] \arrow[d, dotted, "h^{\mathcal{G}}(U;x|y|z)"'] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] & (sx\otimes sy)\otimes sz \arrow[d, "\boldsymbol{a}"] \\ s(xyz) \arrow[r, "\Gamma"] & sx\otimes s(yz) \arrow[r, "1\otimes\Gamma"] & sx\otimes(sy\otimes sz) \end{tikzcd} \qquad \begin{tikzcd} s(xy) \arrow[r, "\Gamma"] \arrow[d, dotted, "h^{\mathcal{G}}(U;x|_2y)"'] & sx\otimes sy \arrow[d, "\boldsymbol{c}"] \\ s(yx) \arrow[r, "\Gamma"] & sy\otimes sx \end{tikzcd} \tag{8.23}$$

Also, for any morphism $\sigma: V\to U$, we define $h^{\mathcal{G}}(\sigma;x|y) \in \Pi_1\mathcal{G}(V,x^\sigma y^\sigma)$ to be the V-automorphism in $\mathcal{G}$ making commutative the diagram on the left below; and similarly, for any $\tau: W\to V$, the element $h^{\mathcal{G}}(\sigma,\tau;x) \in \Pi_1\mathcal{G}(W,x^{\sigma\tau})$ is the W-automorphism in $\mathcal{G}$ determined by the commutativity of the diagram on the right below.

$$\begin{tikzcd} s(xy)^\sigma \arrow[rr, "\Theta"] \arrow[d, dotted, "h^{\mathcal{G}}(\sigma;x|y)"'] & & s(xy) \arrow[d, "\Gamma"] \\ s(xy)^\sigma \arrow[r, "\Gamma"] & sx^\sigma\otimes sy^\sigma \arrow[r, "\Theta\otimes\Theta"] & sx\otimes sy \end{tikzcd} \qquad \begin{tikzcd} sx^{\sigma\tau} \arrow[r, "\Theta"] \arrow[d, dotted, "h^{\mathcal{G}}(\sigma,\tau;x)"'] & sx^\sigma \arrow[d, "\Theta"] \\ sx^{\sigma\tau} \arrow[r, "\Theta"] & sx \end{tikzcd} \tag{8.24}$$

Let us point out that $h^{\mathcal{G}}$ satisfies extra normalization properties, namely

$$\begin{array}{ll} h^{\mathcal{G}}(1_U,1_U;x)=1 & \text{for any } U\text{-object } x, \\ h^{\mathcal{G}}(1_U;x,y)=1 & \text{for any } U\text{-objects } x,y. \end{array} \tag{8.25}$$

The required normalization conditions in (6.8) follow from the commutativity of the diagrams below, where the inner regions labeled with (A) commute by naturality of the unit constraints $\boldsymbol{l}$ or $\boldsymbol{r}$ and the other by the references therein:

$\underline{h^{\mathcal{G}}(U;e|x|y)=1=h^{\mathcal{G}}(U;x|e|y)=h^{\mathcal{G}}(U;x|y|z)}$, for any $U\in\mathrm{Ob}\mathrm{C}$ and $y,z\in\Pi_{\mathcal{G}}(U)$:

$$\begin{tikzcd} s(exy)=s(xy) \arrow[r, "\Gamma"] \arrow[d, equal, "1"'] \arrow[dr, phantom, "(A)"] & sx\otimes sy \arrow[r, "\boldsymbol{l}^{-1}\otimes 1"] \arrow[dr, "\boldsymbol{l}^{-1}"'] & (\iota U\otimes sx)\otimes sy \arrow[d, "\boldsymbol{a}"] \arrow[d, phantom, "{[24,2.1]}"'] \\ s(exy)=s(xy) \arrow[r, "\boldsymbol{l}^{-1}"] & \iota U\otimes s(xy) \arrow[r, "1\otimes\Gamma"] & \iota U\otimes(sx\otimes sy) \end{tikzcd}$$

$$\begin{tikzcd} s(xey)=s(xy) \arrow[r, "\Gamma"] \arrow[d, equal, "1"'] & sx\otimes sy \arrow[r, "\boldsymbol{r}^{-1}\otimes 1"] & (sx\otimes\iota U)\otimes sy \arrow[d, "\boldsymbol{a}"] \\ s(xey)=s(xy) \arrow[r, "\Gamma"] & sx\otimes sy \arrow[r, "1\otimes\boldsymbol{l}^{-1}"] \arrow[ur, "\boldsymbol{r}^{-1}\otimes 1"] \arrow[ur, phantom, "(8.5)"', shift right=3] & sx\otimes(\iota U\otimes sy) \end{tikzcd}$$

$$\begin{tikzcd} s(xye)=s(xy) \arrow[r, "\boldsymbol{r}^{-1}"] \arrow[d, equal, "1"'] \arrow[dr, phantom, "(A)"] & s(xy)\otimes\iota U \arrow[r, "\Gamma\otimes 1"] & (sx\otimes sy)\otimes\iota U \arrow[d, "\boldsymbol{a}"] \\ s(xye)=s(xy) \arrow[r, "\Gamma"] & sx\otimes sy \arrow[r, "1\otimes\boldsymbol{r}^{-1}"] \arrow[ur, "\boldsymbol{r}^{-1}"] \arrow[ur, phantom, "{[24,2.1]}"', shift right=3] & sx\otimes(sy\otimes\iota U) \end{tikzcd}$$

$\underline{h^{\mathcal{G}}(\sigma;e|x)=1=h^{\mathcal{G}}(\sigma;x|e)}$, for any $\sigma:V\to U$ in ObC and $x\in\Pi_{\mathcal{G}}(U)$:

$$
\begin{tikzcd}
s(ex)^\sigma = sx^\sigma \arrow[rr, "\Theta"] \arrow[d, equal, "1"'] \arrow[drr, phantom, "(A)"] & & sx \arrow[d, "\boldsymbol{l}^{-1}"] \\
s(ex)^\sigma = sx^\sigma \arrow[r, "\boldsymbol{l}^{-1}"] & \iota V\otimes sx^\sigma \arrow[r, "\iota\sigma\otimes\Theta"] & \iota U\otimes sx
\end{tikzcd}
\qquad
\begin{tikzcd}
s(xe)^\sigma = sx^\sigma \arrow[rr, "\Theta"] \arrow[d, equal, "1"'] \arrow[drr, phantom, "(A)"] & & sx \arrow[d, "\boldsymbol{r}^{-1}"] \\
s(xe)^\sigma = sx^\sigma \arrow[r, "\boldsymbol{r}^{-1}"] & sx^\sigma\otimes\iota V \arrow[r, "\Theta\otimes\iota\sigma"] & sx\otimes\iota U
\end{tikzcd}
$$

$\underline{h^{\mathcal{G}}(\sigma,\tau;e)=1}$, for any $W\overset{\tau}{\to}V\overset{\sigma}{\to}U$ in ObC :

$$
\begin{tikzcd}
\iota W \arrow[r, "\iota\tau"] \arrow[d, equal, "1"'] & \iota V \arrow[d, "\iota\sigma"] \\
\iota W \arrow[r, "\iota(\sigma\tau)"] & \iota U
\end{tikzcd}
$$

To verify the cocycle conditions $\partial h=1$ in (6.14)-(6.20), we observe the commutativity of the outside of the seven diagrams below, where the inner regions labeled with (A) commute by the naturality of $\boldsymbol{a}$ or $\boldsymbol{c}$, those with (B) commute since $\otimes$ is a functor, and the other ones by the references therein. In all cases the composite vertical arrow on the right of the diagram is an identity, and therefore Lemma 8.1 implies that the composite vertical arrow on the left must be also an identity, which reads the required cocycle condition.

$\underline{\partial h^{\mathcal{G}}(U;x|y|z|t)=1}$, for any $U\in$ ObC and $x,y,z,t\in\Pi_0\mathcal{G}(U)$:

$$
\begin{tikzcd}[column sep=small]
s(xyzt) \arrow[r, "\Gamma"] \arrow[d, "t_*h^{\mathcal{G}}(U;x|y|z)"'] \arrow[dr, phantom, "(8.16)"] & s(xyz)\otimes st \arrow[r, "\Gamma\otimes 1"] \arrow[d, "h^{\mathcal{G}}(U;x|y|z)\otimes 1"] \arrow[drr, phantom, "(8.23)"] & (s(xy)\otimes sz)\otimes st \arrow[r, "(\Gamma\otimes 1)\otimes 1"] & ((sx\otimes sy)\otimes sz)\otimes st \arrow[d, "\boldsymbol{a}\otimes 1"] \\
s(xyzt) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|yz|t)"'] \arrow[drr, phantom, "(8.23)"] & s(xyz)\otimes st \arrow[r, "\Gamma\otimes 1"] & (sx\otimes s(yz))\otimes st \arrow[r, "(1\otimes\Gamma)\otimes 1"] \arrow[d, "\boldsymbol{a}"] \arrow[dr, phantom, "(A)"] & (sx\otimes(sy\otimes sz))\otimes st \arrow[d, "\boldsymbol{a}"] \\
s(xyzt) \arrow[r, "\Gamma"] \arrow[d, "x_*h^{\mathcal{G}}(U;y|z|t)"'] \arrow[dr, phantom, "(8.16)"] & sx\otimes s(yzt) \arrow[r, "1\otimes\Gamma"] \arrow[d, "1\otimes h^{\mathcal{G}}(U;y|z|t)"] \arrow[drr, phantom, "(8.23)"] & sx\otimes(s(yz)\otimes st) \arrow[r, "1\otimes(\Gamma\otimes 1)"] & sx\otimes((sy\otimes sz)\otimes st) \arrow[d, "1\otimes\boldsymbol{a}"] \\
s(xyzt) \arrow[r, "\Gamma"] \arrow[dd, "h^{\mathcal{G}}(U;x|y|zt)^{-1}"'] \arrow[ddr, phantom, "(8.23)"] & sx\otimes s(yzt) \arrow[r, "1\otimes\Gamma"] & sx\otimes(sy\otimes s(zt)) \arrow[r, "1\otimes(1\otimes\Gamma)"] \arrow[d, "\boldsymbol{a}^{-1}"] \arrow[dr, phantom, "(A)"] & sx\otimes(sy\otimes(sz\otimes st)) \arrow[dd, "\boldsymbol{a}^{-1}"] \\
& & (sx\otimes sy)\otimes s(zt) \arrow[dr, "1\otimes\Gamma"] \arrow[d, phantom, "(B)"] & \\
s(xyzt) \arrow[r, "\Gamma"] \arrow[dd, "h^{\mathcal{G}}(U;xy|z|t)^{-1}"'] \arrow[ddr, phantom, "(8.23)"] & s(xy)\otimes s(zt) \arrow[ur, "\Gamma\otimes 1"] \arrow[dr, "1\otimes\Gamma"'] & & (sx\otimes sy)\otimes(sz\otimes st) \arrow[dd, "\boldsymbol{a}^{-1}"] \\
& & s(xy)\otimes(sz\otimes st) \arrow[ur, "\Gamma\otimes 1"'] \arrow[d, "\boldsymbol{a}^{-1}"] \arrow[dr, phantom, "(A)"] & \\
s(xyzt) \arrow[r, "\Gamma"] & s(xyz)\otimes st \arrow[r, "\Gamma\otimes 1"] & (s(xy)\otimes sz)\otimes st \arrow[r, "(\Gamma\otimes 1)\otimes 1"] & ((sx\otimes sy)\otimes sz)\otimes st
\end{tikzcd}
$$

$\underline{\partial h^{\mathcal{G}}(U; x|_2 y|z) = 1}$, for any $U \in \mathrm{Ob}\mathrm{C}$ and $x, y, z \in \Pi_0\mathcal{G}(U)$:

$$
\begin{tikzcd}
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "z_* h^{\mathcal{G}}(U;x|_2 y)"'] \arrow[dr, phantom, "(8.16)"] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] \arrow[d, "h^{\mathcal{G}}(U;x|_2 y)\otimes 1"] \arrow[dr, phantom, "(8.23)"] & (sx\otimes sy)\otimes sz \arrow[d, "\boldsymbol{c}\otimes 1"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;y|x|z)"'] \arrow[drr, phantom, "(8.23)"] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] & (sy\otimes sx)\otimes sz \arrow[d, "\boldsymbol{a}"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "y_* h^{\mathcal{G}}(U;x|_2 z)"'] \arrow[dr, phantom, "(8.16)"] & sy\otimes s(xz) \arrow[r, "1\otimes\Gamma"] \arrow[d, "1\otimes h^{\mathcal{G}}(U;x|_2 z)"] \arrow[dr, phantom, "(8.23)"] & sy\otimes(sx\otimes sz) \arrow[d, "1\otimes\boldsymbol{c}"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;y|z|x)^{-1}"'] \arrow[drr, phantom, "(8.23)"] & sy\otimes s(xz) \arrow[r, "1\otimes\Gamma"] & sy\otimes(sz\otimes sx) \arrow[d, "\boldsymbol{a}^{-1}"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|_2 yz)^{-1}"'] \arrow[dr, phantom, "(8.23)"] & s(yz)\otimes sx \arrow[r, "\Gamma\otimes 1"] \arrow[d, "\boldsymbol{c}^{-1}"] \arrow[dr, phantom, "(A)"] & (sy\otimes sz)\otimes sx \arrow[d, "\boldsymbol{c}^{-1}"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|y|z)^{-1}"'] \arrow[drr, phantom, "(8.23)"] & sx\otimes s(yz) \arrow[r, "1\otimes\Gamma"] & sx\otimes(sy\otimes sz) \arrow[d, "\boldsymbol{a}^{-1}"] \\
s(xyz) \arrow[r, "\Gamma"] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] & (sx\otimes sy)\otimes sz
\end{tikzcd}
$$

$\underline{\partial h^{\mathcal{G}}(U; x|y|_2 z) = 1}$, for any $U \in \mathrm{Ob}\mathrm{C}$ and $x, y, z \in \Pi_0\mathcal{G}(U)$:

$$
\begin{tikzcd}
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|y|z)^{-1}"'] \arrow[drr, phantom, "(8.23)"] & sx\otimes s(yz) \arrow[r, "1\otimes\Gamma"] & sx\otimes(sy\otimes sz) \arrow[d, "\boldsymbol{a}^{-1}"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;xy|_2 z)"'] \arrow[dr, phantom, "(8.23)"] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] \arrow[d, "\boldsymbol{c}"] \arrow[dr, phantom, "(A)"] & (sx\otimes sy)\otimes sz \arrow[d, "\boldsymbol{c}"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;z|x|y)^{-1}"'] \arrow[drr, phantom, "(8.23)"] & sz\otimes s(xy) \arrow[r, "\Gamma\otimes 1"] & sz\otimes(sx\otimes sy) \arrow[d, "\boldsymbol{a}^{-1}"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "y_* h^{\mathcal{G}}(U;x|_2 z)^{-1}"'] \arrow[dr, phantom, "(8.16)"] & s(xz)\otimes sy \arrow[r, "\Gamma\otimes 1"] \arrow[d, "h^{\mathcal{G}}(U;x|_2 z)^{-1}\otimes 1"] \arrow[dr, phantom, "(8.23)"] & (sz\otimes sx)\otimes sy \arrow[d, "\boldsymbol{c}\otimes 1"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|z|y)"'] \arrow[drr, phantom, "(8.23)"] & s(xz)\otimes sy \arrow[r, "\Gamma\otimes 1"] & (sx\otimes sz)\otimes sy \arrow[d, "\boldsymbol{a}^{-1}"] \\
s(xyz) \arrow[r, "\Gamma"] \arrow[d, "x_* h^{\mathcal{G}}(U;y|_2 z)^{-1}"'] \arrow[dr, phantom, "(8.16)"] & sx\otimes s(yz) \arrow[r, "1\otimes\Gamma"] \arrow[d, "1\otimes h^{\mathcal{G}}(U;y|_2 z)^{-1}"] \arrow[dr, phantom, "(8.23)"] & sx\otimes(sz\otimes sy) \arrow[d, "1\otimes\boldsymbol{c}"] \\
s(xyz) \arrow[r, "\Gamma"] & sx\otimes s(yz) \arrow[r, "1\otimes\Gamma"] & sx\otimes(sy\otimes sz)
\end{tikzcd}
$$

$\underline{\partial h^{\mathcal{G}}(\sigma; x|y|z) = 1}$, for any $\sigma : V \to U$ in C and $x, y, z \in \Pi_0\mathcal{G}(U)$:

$$
\begin{tikzcd}
s(xyz)^\sigma \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(U;x|y|z)^\sigma"'] \arrow[dr, phantom, "(8.19)"] & s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|y|z)"] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] \arrow[d, phantom, "(8.23)"] & (sx\otimes sy)\otimes sz \arrow[ddd, "\boldsymbol{a}"] \\
s(xyz)^\sigma \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\sigma;x|yz)"'] \arrow[dr, phantom, "(8.24)"] & s(xyz) \arrow[dr, "\Gamma"] & & \\
s(xyz)^\sigma \arrow[r, "\Gamma"] \arrow[d, "x^\sigma_* h^{\mathcal{G}}(\sigma;y|z)"'] \arrow[dr, phantom, "(8.16)"] & sx^\sigma\otimes s(yz)^\sigma \arrow[r, "\Theta\otimes\Theta"] \arrow[d, "1\otimes h^{\mathcal{G}}(\sigma;y|z)"] \arrow[dr, phantom, "{(8.24),(B)}"] & sx\otimes s(yz) \arrow[dr, "1\otimes\Gamma"] & \\
s(xyz)^\sigma \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(V;x^\sigma|y^\sigma|z^\sigma)^{-1}"'] \arrow[dr, phantom, "(8.23)"] & sx^\sigma\otimes s(yz)^\sigma \arrow[r, "1\otimes\Gamma"] & sx^\sigma\otimes(sy^\sigma\otimes sz^\sigma) \arrow[r, "\Theta\otimes(\Theta\otimes\Theta)"] \arrow[d, "\boldsymbol{a}^{-1}"] & sx\otimes(sy\otimes sz) \arrow[ddd, "\boldsymbol{a}^{-1}"] \\
s(xyz)^\sigma \arrow[r, "\Gamma"] \arrow[d, "z^\sigma_* h^{\mathcal{G}}(\sigma;x|y)^{-1}"'] \arrow[dr, phantom, "(8.16)"] & s(xy)^\sigma\otimes sz^\sigma \arrow[r, "\Gamma\otimes 1"] \arrow[d, "h^{\mathcal{G}}(\sigma;x|y)^{-1}\otimes 1"] \arrow[dr, phantom, "{(8.24),(B)}"] & (sx^\sigma\otimes sy^\sigma)\otimes sz^\sigma \arrow[ddr, "(\Theta\otimes\Theta)\otimes\Theta"] \arrow[r, phantom, "(A)"] & \\
s(xyz)^\sigma \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(\sigma;xy|z)^{-1}"'] \arrow[dr, phantom, "(8.24)"] & s(xy)^\sigma\otimes sz^\sigma \arrow[dr, "\Theta\otimes\Theta"] & & \\
s(xyz)^\sigma \arrow[r, "\Theta"] & s(xyz) \arrow[r, "\Gamma"] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] & (sx\otimes sy)\otimes sz
\end{tikzcd}
$$

$\underline{\partial h^{\mathcal{G}}(\sigma; x|_2 y) = 1}$, for any $\sigma : V \to U$ in C and $x, y \in \Pi_0\mathcal{G}(U)$:

$$
\begin{tikzcd}
s(xy)^\sigma \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(U;x|_2 y)^\sigma"'] \arrow[dr, phantom, "(8.19)"] & s(xy) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|_2 y)"] \arrow[dr, phantom, "(8.23)"] & sx\otimes sy \arrow[dd, "\boldsymbol{c}"] \\
s(xy)^\sigma \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\sigma;y|x)"'] \arrow[dr, phantom, "(8.24)"] & s(xy) \arrow[dr, "\Gamma"] & \\
s(xy)^\sigma \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(V;x^\sigma|_2 y^\sigma)^{-1}"'] \arrow[dr, phantom, "(8.23)"] & sy^\sigma\otimes sx^\sigma \arrow[r, "\Theta\otimes\Theta"] \arrow[d, "\boldsymbol{c}^{-1}"] \arrow[dr, phantom, "(A)"] & sy\otimes sx \arrow[dd, "\boldsymbol{c}^{-1}"] \\
s(xy)^\sigma \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(\sigma;x|y)^{-1}"'] \arrow[dr, phantom, "(8.24)"] & sx^\sigma\otimes sy^\sigma \arrow[dr, "\Theta\otimes\Theta"] & \\
s(xy)^\sigma \arrow[r, "\Theta"] & s(xy) \arrow[r, "\Gamma"] & sx\otimes sy
\end{tikzcd}
$$

$\underline{\partial h^{\mathcal{G}}(\sigma,\tau;x|y)=1}$, for any $W \xrightarrow{\tau} V \xrightarrow{\sigma} U$ in C and $x,y\in\Pi_0\mathcal{G}(U)$:

$$
\begin{tikzcd}
s(xy)^{\sigma\tau} \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\sigma;x|y)^{\tau}"'] \arrow[dr, phantom, "(8.19)"] & s(xy)^{\sigma} \arrow[rr, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\sigma;x|y)"] \arrow[drr, phantom, "(8.24)"] & & s(xy) \arrow[ddd, "\Gamma"] \\
s(xy)^{\sigma\tau} \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\tau;x^{\sigma}|y^{\sigma})"'] \arrow[dr, phantom, "(8.24)"] & s(xy)^{\sigma} \arrow[dr, "\Gamma"] & & \\
s(xy)^{\sigma\tau} \arrow[r, "\Gamma"] \arrow[d, "x_*^{\sigma\tau}h^{\mathcal{G}}(\sigma,\tau;y)"'] \arrow[dr, phantom, "(8.16)"] & sx^{\sigma\tau}\otimes sy^{\sigma\tau} \arrow[r, "\Theta\otimes\Theta"] \arrow[d, "1\otimes h^{\mathcal{G}}(\sigma,\tau;y)"] \arrow[dr, phantom, "{(8.24),(B)}"] & sx^{\sigma}\otimes sy^{\sigma} \arrow[dr, "\Theta\otimes\Theta"] & \\
s(xy)^{\sigma\tau} \arrow[r, "\Gamma"] \arrow[d, "y_*^{\sigma\tau}h^{\mathcal{G}}(\sigma,\tau;x)"'] \arrow[dr, phantom, "(8.16)"] & sx^{\sigma\tau}\otimes sy^{\sigma\tau} \arrow[d, "h^{\mathcal{G}}(\sigma,\tau;x)\otimes 1"] & & sx\otimes sy \arrow[ddd, "\Gamma^{-1}"] \\
s(xy)^{\sigma\tau} \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(\sigma\tau;x|y)^{-1}"'] & sx^{\sigma\tau}\otimes sy^{\sigma\tau} \arrow[urr, "\Theta\otimes\Theta"] \arrow[dr, phantom, "(8.24)"] & & \\
s(xy)^{\sigma\tau} \arrow[d, "h^{\mathcal{G}}(\sigma,\tau;xy)^{-1}"'] \arrow[drrr, "\Theta"] \arrow[dr, phantom, "(8.24)"] & & & \\
s(xy)^{\sigma\tau} \arrow[r, "\Theta"] & s(xy)^{\sigma} \arrow[rr, "\Theta"] & & s(xy)
\end{tikzcd}
$$

$\underline{\partial h^{\mathcal{G}}(\sigma,\tau,\gamma;x)=1}$, for any $T \xrightarrow{\gamma} W \xrightarrow{\tau} V \xrightarrow{\sigma} U$ in C and $x\in\Pi_0\mathcal{G}(U)$:

$$
\begin{tikzcd}
sx^{\sigma\tau\gamma} \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\sigma,\tau;x)^{\gamma}"'] \arrow[dr, phantom, "(8.19)"] & sx^{\sigma\tau} \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\sigma,\tau;x)"] \arrow[dr, phantom, "(8.24)"] & sx^{\sigma} \arrow[r, "\Theta"] & sx \arrow[dddd, equal, "1"] \\
sx^{\sigma\tau\gamma} \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\sigma\tau,\gamma;x)"'] \arrow[dr, phantom, "(8.24)"] & sx^{\sigma\tau} \arrow[dddrr, "\Theta"] & & \\
sx^{\sigma\tau\gamma} \arrow[d, "h^{\mathcal{G}}(\sigma,\tau\gamma;x)^{-1}"'] \arrow[ddrrr, "\Theta"] \arrow[dr, phantom, "(8.24)"] & & & \\
sx^{\sigma\tau\gamma} \arrow[d, "h^{\mathcal{G}}(\tau,\gamma;x^{\sigma})^{-1}"'] \arrow[drr, "\Theta"] \arrow[dr, phantom, "(8.24)"] & & & \\
sx^{\sigma\tau\gamma} \arrow[r, "\Theta"] & sx^{\sigma\tau} \arrow[r, "\Theta"] & sx^{\sigma} \arrow[r, "\Theta"] & sx
\end{tikzcd}
$$

Let us remark that if $\mathcal{G}$ is a symmetric C-fibred category, then

$$h^{\mathcal{G}} \in Z^5_s(\Pi_0\mathcal{G},3,\Pi_1\mathcal{G}), \tag{8.26}$$

that is, for any $U\in\mathrm{ObC}$ and $x,y\in\Pi_0(U)$, the 5-cycle condition $\partial h^{\mathcal{G}}(U;x|_3y)=1$ in (6.22) holds. This follows from Lemma 8.1, since we have the commutative diagram below, where $\boldsymbol{c}^2 = 1$.

$$\begin{tikzcd} s(xy) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|_2 y)"'] \arrow[dr, phantom, "(8.24)"] & sx \otimes sy \arrow[d, "\boldsymbol{c}"] \\ s(xy) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;y|_2 x)"'] \arrow[dr, phantom, "(8.24)"] & sy \otimes sx \arrow[d, "\boldsymbol{c}"] \\ s(xy) \arrow[r, "\Gamma"] & sx \otimes sy. \end{tikzcd}$$

Let $[h^{\mathcal{G}}]$ denote the cohomology class of $h^{\mathcal{G}}$ in $H^4_s(\Pi_0\mathcal{G}, 2, \Pi_1\mathcal{G})$ (or in $H^5_s(\Pi_0\mathcal{G}, 3, \Pi_1\mathcal{G})$ if $\mathcal{G}$ is symmetric).

Proposition 8.4 *The class $[\Pi_0\mathcal{G}, \Pi_1\mathcal{G}, [h^{\mathcal{G}}]]$ only depends on the equivalence class $[\mathcal{G}]$.*

Proof First, we observe that $[\Pi_0\mathcal{G}, \Pi_1\mathcal{G}, [h^{\mathcal{G}}]]$ does not depend on the selections we have made for its construction. Suppose we have another choice of representative elements $s'x = s'(U,x)$ (also with $s'e = \iota U$), U-morphisms $\Gamma' : s'(xy) \to s'x \otimes s'y$ (with $\Gamma'_{e,x} = \boldsymbol{l}^{-1}$ and $\Gamma'_{x,e} = \boldsymbol{r}^{-1}$) and σ-morphisms $\Theta' : s'x^\sigma \to s'x$ (with $\Theta'_{\sigma,e} = \iota\sigma$ and $\Theta'_{1_U,x} = 1_{s'x}$), and let $\Pi'_1\mathcal{G}$ and h' denote the corresponding $\mathbb{H}(\Pi_0\mathcal{G})$-module and cocycle in $Z^4_s(\Pi_0\mathcal{G}, 2, \Pi'_1\mathcal{G})$ constructed from this choice. We show an isomorphism of $\mathbb{H}(\Pi_0\mathcal{G})$-modules $\mathfrak{F} : \Pi_1\mathcal{G} \cong \Pi'_1\mathcal{G}$ such that $\mathfrak{F}_*[h^{\mathcal{G}}] = [h']$ as follows. Let us select U-morphisms $\psi = \psi_{U,x} : sx \to s'x$ (with $\psi_{U,e} = 1_{\iota U}$). Then, we have isomorphisms $\mathfrak{F} : \Pi_1\mathcal{G}(U,x) \cong \Pi'_1\mathcal{G}(U,x)$ defined by the commutativity of the squares

$$\begin{tikzcd} s'x \arrow[r, "\psi"] \arrow[d, "\mathfrak{F}a"'] & sx \arrow[d, "a"] \\ s'x \arrow[r, "\psi"] & sx. \end{tikzcd} \tag{8.27}$$

If $x, y \in \Pi_0\mathcal{G}(U)$, for any $a \in \Pi_1\mathcal{G}(U,x)$ we see that $y_*\mathfrak{F}a = \mathfrak{F}(y_*a)$ by applying Lemma 8.1 to the commutative diagrams

$$\begin{tikzcd} s'(yx) \arrow[r, "\psi"] \arrow[d, "\mathfrak{F}(y_*a)"'] & s(yx) \arrow[r, "\Gamma"] \arrow[d, "y_*a"] & sy \otimes sx \arrow[d, "1\otimes a"] \\ s'(yx) \arrow[r, "\psi"] & s(yx) \arrow[r, "\Gamma"] & sy \otimes sx \end{tikzcd} \qquad \begin{tikzcd} s'(yx) \arrow[r, "\Gamma'"] \arrow[d, "y_*\mathfrak{F}(a)"'] & s'y \otimes s'x \arrow[r, "\psi\otimes\psi"] \arrow[d, "1\otimes\mathfrak{F}a"] & sy \otimes sx \arrow[d, "1\otimes a"] \\ s'(yx) \arrow[r, "\Gamma'"] & s'y \otimes s'x \arrow[r, "\psi\otimes\psi"] & sy \otimes sx \end{tikzcd}$$

Similarly, if $\sigma : V \to U$ is a morphism in C, we see that $\mathfrak{F}a^\sigma = (\mathfrak{F}a)^\sigma$ by the commutative diagrams

$$\begin{tikzcd} s'x^\sigma \arrow[r, "\psi"] \arrow[d, "\mathfrak{F}a^\sigma"'] & sx^\sigma \arrow[r, "\Theta"] \arrow[d, "a^\sigma"] & sx \arrow[d, "a"] \\ s'x^\sigma \arrow[r, "\psi"] & sx^\sigma \arrow[r, "\Theta"] & sx \end{tikzcd} \qquad \begin{tikzcd} s'x^\sigma \arrow[r, "\Theta'"] \arrow[d, "(\mathfrak{F}a)^\sigma"'] & s'x \arrow[r, "\psi"] \arrow[d, "\mathfrak{F}a"] & sx \arrow[d, "a"] \\ s'x^\sigma \arrow[r, "\Theta'"] & s'x \arrow[r, "\psi"] & sx. \end{tikzcd}$$

Thus, $\mathfrak{F} : \Pi_1\mathcal{G} \cong \Pi'_1\mathcal{G}$ is an isomorphism of $\mathbb{H}(\Pi_0\mathcal{G})$-modules. We now define a pointed 3-cochain $g \in C^3_s(\Pi_0\mathcal{G}, 2, \Pi'_1\mathcal{G})$ such that $h' = \mathfrak{F}_*h^{\mathcal{G}} \cdot \partial g$ by taking, for each $U \in \mathrm{ObC}$ and $x, y \in \Pi_0\mathcal{G}(U)$, $g(U; x|y) \in$

$\Pi_1'\mathcal{G}(U,xy)$ the U-automorphism in $\mathcal{G}$ determined by the commutativity of the square on the left below and, for each morphism $\sigma: V \to U$ in C, $g(\sigma;x) \in \Pi_1'\mathcal{G}(V,x^\sigma)$ the V-automorphism which makes commutative the diagram on the right.

$$
\begin{tikzcd}
s'(xy) \arrow[rr, "\Gamma'"] \arrow[d, dotted, "g(U;x|y)"'] & & s'x\otimes s'y \arrow[d, "\psi\otimes\psi"] \\
s'(xy) \arrow[r, "\psi"] & s(xy) \arrow[r, "\Gamma"] & sx\otimes sy
\end{tikzcd}
\qquad
\begin{tikzcd}
s'x^\sigma \arrow[rr, "\psi"] \arrow[d, dotted, "g(\sigma;x)"'] & & sx^\sigma \arrow[d, "\Theta"] \\
s'x^\sigma \arrow[r, "\Theta'"] & s'x \arrow[r, "\psi"] & sx
\end{tikzcd}
\tag{8.28}
$$

The required normalization conditions $g(U;x|e) = 1 = g(U;e|y)$ follow from the naturality of the unit constraints $\boldsymbol{l}$ and $\boldsymbol{r}$, while the equality $g(\sigma;e) = 1$ becomes obvious. To verify that $h' = \mathfrak{F}_*h^{\mathcal{G}} \cdot \partial g$, we observe the commutativity of the outside of the four diagrams below, where the inner regions labeled with (A) commute by the naturality of $\boldsymbol{a}$ or $\boldsymbol{c}$, those with (B) commute since $\otimes$ is a functor, and the other ones by the references therein. In each case, the composite vertical arrow on the left of the diagram is the corresponding value of $\mathfrak{F}_*h^{\mathcal{G}} \cdot \partial g$ while, by definition, it must be the corresponding value of h':

$\underline{h'(U;x|y|z) = \mathfrak{F}h^{\mathcal{G}}(U;x|y|z) \cdot \partial g(U;x|y|z)}$, for any $U \in \mathrm{ObC}$ and $x,y,z \in \Pi_0\mathcal{G}(U)$:

$$
\begin{tikzcd}
s'(xyz) \arrow[rr, "\Gamma'"] \arrow[d, "z_*g(U;x|y)"'] \arrow[drr, phantom, "(8.16)"] & & s'(xy)\otimes s'z \arrow[rr, "\Gamma'\otimes 1"] \arrow[d, "g(U;x|y)\otimes 1"] & & (s'x\otimes s'y)\otimes s'z \arrow[dl, "(\psi\otimes\psi)\otimes\psi"'] \arrow[ddddd, "\boldsymbol{a}"] \\
s'(xyz) \arrow[rr, "\Gamma'"] \arrow[d, "g(U;xy|z)"'] \arrow[drr, phantom, "(8.28)"] & & s'(xy)\otimes s'z \arrow[d, "\psi\otimes\psi"] \arrow[r, phantom, "(8.28){,}(B)"] & {} & \\
s'(xyz) \arrow[r, "\psi"] \arrow[d, "\mathfrak{F}h^{\mathcal{G}}(U;x|y|z)"'] \arrow[dr, phantom, "(8.27)"] & s(xyz) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|y|z)"] \arrow[dr, phantom, "(8.23)"] & s(xy)\otimes sz \arrow[r, "\Gamma\otimes 1"] & (sx\otimes sy)\otimes sz \arrow[d, "\boldsymbol{a}"] \arrow[r, phantom, "(A)"] & {} \\
s'(xyz) \arrow[r, "\psi"] \arrow[d, "g(U;x|yz)^{-1}"'] \arrow[drr, phantom, "(8.28)"] & s(xyz) \arrow[r, "\Gamma"] & sx\otimes s(yz) \arrow[r, "1\otimes\Gamma"] \arrow[d, "\psi^{-1}\otimes\psi^{-1}"] & sx\otimes(sy\otimes sz) \arrow[ddr, "\psi^{-1}\otimes(\psi^{-1}\otimes\psi^{-1})"'] & \\
s'(xyz) \arrow[rr, "\Gamma'"] \arrow[d, "x_*g(U;y|z)^{-1}"'] \arrow[drr, phantom, "(8.16)"] & & s'x\otimes s'(yz) \arrow[d, "1\otimes g(U;y|z)^{-1}"] \arrow[r, phantom, "(8.28){,}(B)"] & {} & \\
s'(xyz) \arrow[rr, "\Gamma'"] & & s'x\otimes s'(yz) \arrow[rr, "1\otimes\Gamma'"] & & s'x\otimes(s'y\otimes s'z)
\end{tikzcd}
$$

$\underline{h'(U;x|_2 y) = \mathfrak{F}h^{\mathcal{G}}(U;x|_2 y)\cdot \partial g(U;x|_2 y)}$, for any $U \in \mathrm{Ob}\mathrm{C}$ and $x, y \in \Pi_0\mathcal{G}(U)$:

$$\begin{tikzcd}
s'(xy) \arrow[rrr, "\Gamma'"] \arrow[d, "g(U;x|y)"'] & & \arrow[d, phantom, "(8.28)"] & s'x\otimes s'y \arrow[dl, "\psi\otimes\psi"'] \arrow[ddd, "\boldsymbol{c}"] \\
s'(xy) \arrow[r, "\psi"] \arrow[d, "\mathfrak{F}h^{\mathcal{G}}(U;x|_2 y)"'] \arrow[dr, phantom, "(8.27)"] & s(xy) \arrow[r, "\Gamma"] \arrow[d, "h^{\mathcal{G}}(U;x|_2 y)"] \arrow[dr, phantom, "(8.23)"] & sx\otimes sy \arrow[d, "\boldsymbol{c}"] \arrow[r, phantom, "(A)"] & {} \\
s'(xy) \arrow[r, "\psi"] \arrow[d, "g(U;y|x)^{-1}"'] & s(xy) \arrow[r, "\Gamma"] \arrow[d, phantom, "(8.28)"] & sy\otimes sx \arrow[dr, "\psi^{-1}\otimes\psi^{-1}"] & \\
s'(xy) \arrow[rrr, "\Gamma'"] & {} & & s'y\otimes s'x
\end{tikzcd}$$

$\underline{h'(\sigma,\tau;x) = \mathfrak{F}h^{\mathcal{G}}(\sigma,\tau;x)\cdot \partial g(\sigma,\tau;x)}$, for any $W \xrightarrow{\tau} V \xrightarrow{\sigma} U$ in C and $x \in \Pi_0\mathcal{G}(U)$:

$$\begin{tikzcd}
s'x^{\sigma\tau} \arrow[rrr, "\Theta'"] \arrow[d, "(g(\sigma,x)^{\tau})^{-1}"'] & \arrow[d, phantom, "(8.19)"] & & s'x^{\sigma} \arrow[dl, "g(\sigma,x)^{-1}"'] \arrow[dddd, "\Theta'"] \\
s'x^{\sigma\tau} \arrow[rr, "\Theta'"] \arrow[d, "g(\tau,x^{\sigma})^{-1}"'] & \arrow[d, phantom, "(8.28)"] & s'x^{\sigma} \arrow[d, "\psi"] & \\
s'x^{\sigma\tau} \arrow[r, "\psi"] \arrow[d, "\mathfrak{F}h^{\mathcal{G}}(\sigma,\tau;x)"'] \arrow[dr, phantom, "(8.27)"] & sx^{\sigma\tau} \arrow[r, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\sigma,\tau;x)"] \arrow[dr, phantom, "(8.24)"] & sx^{\sigma} \arrow[d, "\Theta"] \arrow[r, phantom, "(8.28)"] & {} \\
s'x^{\sigma\tau} \arrow[r, "\psi"] \arrow[d, "g(\sigma\tau;x)"'] & sx^{\sigma\tau} \arrow[r, "\Theta"] \arrow[d, phantom, "(8.28)"] & sx \arrow[dr, "\psi^{-1}"] & \\
s'x^{\sigma\tau} \arrow[rrr, "\Theta'"] & {} & & s'x
\end{tikzcd}$$

$\underline{h'(\sigma;x|y) = \mathfrak{F}h^{\mathcal{G}}(\sigma;x|y)\cdot \partial g(\sigma;x|y)}$, for any $\sigma : V \to U$ in C and $x, y \in \Pi_0\mathcal{G}(U)$:

$$\begin{tikzcd}
s'(xy)^\sigma \arrow[rrrr, "\Theta'"] \arrow[d, "g(U;x|y)^\sigma"'] \arrow[drr, phantom, "(8.19)"] & & & & s'(xy) \arrow[dl, "g(U;x|y)"'] \arrow[dddddd, "\Gamma'"] \\
s'(xy)^\sigma \arrow[rrr, "\Theta'"] \arrow[d, "g(\sigma;xy)^{-1}"'] \arrow[drr, phantom, "(8.28)"] & & & s'(xy) \arrow[d, "\psi"] & \\
s'(xy)^\sigma \arrow[r, "\psi"] \arrow[d, "\mathfrak{F}\,h^{\mathcal{G}}(\sigma;x|y)"'] \arrow[dr, phantom, "(8.27)"] & s(xy)^\sigma \arrow[rr, "\Theta"] \arrow[d, "h^{\mathcal{G}}(\sigma;x|y)"] \arrow[drr, phantom, "(8.24)"] & & s(xy) \arrow[d, "\Gamma"] \arrow[dr, phantom, "(8.28)"] & \\
s'(xy)^\sigma \arrow[r, "\psi"] \arrow[d, "g(V;x^\sigma|y^\sigma)^{-1}"'] \arrow[dr, phantom, "(8.28)"] & s(xy)^\sigma \arrow[r, "\Gamma"] & sx^\sigma \otimes sy^\sigma \arrow[r, "\Theta\otimes\Theta"] \arrow[d, "\psi^{-1}\otimes\psi^{-1}"] & sx\otimes sy \arrow[dddr, "\psi^{-1}\otimes\psi^{-1}"] & \\
s'(xy)^\sigma \arrow[rr, "\Gamma'"] \arrow[d, "x^\sigma_* g(\sigma;y)"'] \arrow[dr, phantom, "(8.16)"] & & s'x^\sigma \otimes s'y^\sigma \arrow[d, "1\otimes g(\sigma;y)"] \arrow[dr, phantom, "(8.28){,}(B)"] & & \\
s'(xy)^\sigma \arrow[rr, "\Gamma'"] \arrow[d, "y_* g(\sigma;x)"'] \arrow[dr, phantom, "(8.16)"] & & s'x^\sigma \otimes s'y^\sigma \arrow[d, "g(\sigma;x)\otimes 1"] & & \\
s'(xy)^\sigma \arrow[rr, "\Gamma'"] & & s'x^\sigma \otimes s'y^\sigma \arrow[rr, "\Theta'\otimes\Theta'"] & & s'x\otimes s'y
\end{tikzcd}$$

Suppose now that $F : \mathcal{G} \to \mathcal{G}'$ is a braided C-fibred functor establishing an equivalence between the braided C-fibred categories $\mathcal{G}$ and $\mathcal{G}'$, and suppose that the construction of $(\Pi_0\mathcal{G}, \Pi_1\mathcal{G}, h^{\mathcal{G}})$ has been made by means of representative objects U-objects sx of $\mathcal{G}$, U-morphisms $\Gamma : s(xy) \to sx \otimes sy$ and σ-morphisms $\Theta : sx^\sigma \to sx$. There is an induced isomorphism of presheaves of commutative monoids

$$\mathfrak{f} : \Pi_0\mathcal{G} \cong \Pi_0\mathcal{G}',$$

where, for every object U of C, the bijection $\mathfrak{f} : \Pi_0\mathcal{G}(U) \to \Pi_0\mathcal{G}'(U)$ is given by $\mathfrak{f}[X] = [FX]$. This is a homomorphism of monoids since, for any U-objects X, Y of $\mathcal{G}$

$$\mathfrak{f}([X][Y]) = \mathfrak{f}[X \otimes Y] = [F(X\otimes Y)] \overset{(8.9)}{=} [FX \otimes' FY] = [FX][FY] = \mathfrak{f}[X]\mathfrak{f}[Y],$$

$$\mathfrak{f}e = \mathfrak{f}[U] = [F\iota U] \overset{(8.10)}{=} [\iota' U] = e.$$

Moreover, if $\sigma : V \to U$ is a morphism in C and $f : X' \to X$ is any σ-morphism in $\mathcal{G}$, then $Ff : FX' \to FX$ is a σ-morphism of $\mathcal{G}'$ and therefore

$$(\mathfrak{f}[X])^\sigma = [FX]^\sigma = [FX'] = \mathfrak{f}[X'] = \mathfrak{f}([X]^\sigma),$$

so that $\mathfrak{f} : \Pi_0\mathcal{G} \to \Pi_0\mathcal{G}'$ is actually an isomorphism of presheaves. Now, every element of $\Pi_0\mathcal{G}'(U)$ writes uniquely as $\mathfrak{f}x$ for a unique $x \in \Pi_0\mathcal{G}(U)$ and, for the construction of the triple $(\Pi_0\mathcal{G}', \Pi_1\mathcal{G}', h^{\mathcal{G}'})$, we can take the U-object $s'\mathfrak{f}x = Fsx$ of $\mathcal{G}'$ as representative of $\mathfrak{f}x$ (but $s'e = \iota' U$), the composite of the U-morphisms

$$Fs(xy) \xrightarrow{F\Gamma} F(sx\otimes sy) \xrightarrow{\varphi} Fsx \otimes' Fsy$$

as the U-morphism $\Gamma'_{\mathfrak{f}x,\mathfrak{f}y} : s'\mathfrak{f}(xy) \to s'\mathfrak{f}x \otimes s'\mathfrak{f}y$ (but $\Gamma'_{e,y} = \boldsymbol{l'}^{-1}$ and $\Gamma'_{x,e} = \boldsymbol{r'}^{-1}$), and the σ-morphisms $\Theta'_{\sigma,\mathfrak{f}x} = F\Theta_{\sigma,x} : s'\mathfrak{f}x^\sigma \to s'\mathfrak{f}x$. Then, an isomorphism of $\mathbb{H}(\Pi_0\mathcal{G})$-modules

$$\mathfrak{F} : \Pi_1\mathcal{G} \cong \mathfrak{f}^*\Pi_1\mathcal{G}'$$

such that $\mathfrak{F}_* h^{\mathcal{G}} = \mathfrak{f}^* h^{\mathcal{G}'}$, where

$$\mathfrak{f}^* : Z^4_s(\Pi_0\mathcal{G}', 2, \Pi_1\mathcal{G}') \to Z^4_s(\Pi_0\mathcal{G}, 2, \mathfrak{f}^*\Pi_1\mathcal{G}'), \quad \mathfrak{F}_* : Z^4_s(\Pi_0\mathcal{G}, 2, \Pi_1\mathcal{G}) \to Z^4_s(\Pi_0\mathcal{G}, 2, \mathfrak{f}^*\Pi_1\mathcal{G}')$$

are the corresponding induced homomorphisms on cocycles, is given by the isomorphisms of abelian groups

$$\mathfrak{F} : \Pi_1\mathcal{G}(U,x) = \mathrm{Aut}_{\mathcal{G}_U}(sx) \cong \mathrm{Aut}_{\mathcal{G}'_U}(Fsx) = \Pi_1\mathcal{G}'(U,\mathfrak{f}x), \quad a \mapsto Fa,$$

defined by restriction of the functor F. All the needed requirements for $\mathfrak{F}$ follow from the commutativity of the outside of the diagrams below, where the inner regions labeled with (A) commute by the naturality of $\boldsymbol{\varphi}$ and the other ones by the references therein (and using that F is a functor):

$\underline{\mathfrak{f}y_*\mathfrak{F}a = \mathfrak{F}(y_*a)}$, for any object U of C, $x, y \in \Pi_0\mathcal{G}(U)$ and $a \in \Pi_1\mathcal{G}(U,x)$:

$$\begin{tikzcd}
Fs(yx) \arrow[r, "F\Gamma"] \arrow[d, "F(y_*a)"'] \arrow[dr, phantom, "(8.16)"] & F(sy \otimes sx) \arrow[r, "\boldsymbol{\varphi}"] \arrow[d, "F(1\otimes a)"] \arrow[dr, phantom, "(A)"] & Fsy \otimes' Fsx \arrow[d, "1\otimes' Fa"] \\
Fs(yx) \arrow[r, "F\Gamma"] & F(sy \otimes sx) \arrow[r, "\boldsymbol{\varphi}"] & Fsy \otimes' Fsx
\end{tikzcd}$$

$\underline{(\mathfrak{F}a)^\sigma = \mathfrak{F}a^\sigma}$, for any morphism $\sigma : V \to U$ of C, $x \in \Pi_0\mathcal{G}(U)$ and $a \in \Pi_1\mathcal{G}(U,x)$:

$$\begin{tikzcd}
Fsx^\sigma \arrow[r, "F\Theta"] \arrow[d, "Fa^\sigma"'] \arrow[dr, phantom, "(8.19)"] & Fsx \arrow[d, "Fa"] \\
Fsx^\sigma \arrow[r, "F\Theta"] & Fsx.
\end{tikzcd}$$

$\underline{h^{\mathcal{G}'}(U; \mathfrak{f}x|\mathfrak{f}y|\mathfrak{f}z) = \mathfrak{F}h^{\mathcal{G}}(U; x|y|z)}$, for any $U \in \mathrm{ObC}$ and $x, y, z \in \Pi_0\mathcal{G}(U)$:

$$\begin{tikzcd}
Fs(xyz) \arrow[r, "F\Gamma"] \arrow[ddd, "Fh^{\mathcal{G}}(U;x|y|z)"'] & F(s(xy)\otimes sz) \arrow[r, "\boldsymbol{\varphi}"] \arrow[dr, "F(\Gamma\otimes 1)"'] & Fs(xy)\otimes' Fsz \arrow[r, "F\Gamma\otimes' 1"] \arrow[d, phantom, "(A)"] & F(sx\otimes sy)\otimes' Fz \arrow[r, "\boldsymbol{\varphi}\otimes' 1"] & (Fsx\otimes' Fsy)\otimes' Fsz \arrow[ddd, "\boldsymbol{a}'"] \\
\arrow[r, phantom, "(8.23)"] & {} & F((sx\otimes sy)\otimes sz) \arrow[ur, "\boldsymbol{\varphi}"'] \arrow[d, "F\boldsymbol{a}"'] \arrow[r, phantom, "(8.11)"] & {} & \\
& & F(sx\otimes(sy\otimes sz)) \arrow[dr, "\boldsymbol{\varphi}"] \arrow[d, phantom, "(A)"] & & \\
Fs(xyz) \arrow[r, "F\Gamma"] & F(sx\otimes s(yz)) \arrow[r, "\boldsymbol{\varphi}"] \arrow[ur, "F(1\otimes\Gamma)"] & Fsx\otimes' Fs(yz) \arrow[r, "1\otimes' F\Gamma"] & Fsx\otimes' F(sy\otimes sz) \arrow[r, "1\otimes'\boldsymbol{\varphi}"] & Fsx\otimes'(Fsy\otimes' Fsz)
\end{tikzcd}$$

$\underline{h^{\mathcal{G}'}(U;\mathfrak{f}x|_2\mathfrak{f}y)=\mathfrak{F}h^{\mathcal{G}}(U;x|_2y)}$, for any $U\in\mathrm{Ob}\mathrm{C}$ and $x,y\in\Pi_0\mathcal{G}(U)$:

$$\begin{tikzcd}
Fs(xy) \arrow[r, "F\Gamma"] \arrow[d, "Fh^{\mathcal{G}}(U;x|_2y)"'] \arrow[dr, phantom, "(8.23)"] & F(sx\otimes sy) \arrow[r, "\varphi"] \arrow[d, "F\boldsymbol{c}"] \arrow[dr, phantom, "(8.12)"] & Fsx\otimes' Fsy \arrow[d, "\boldsymbol{c}'"] \\
Fs(xy) \arrow[r, "F\Gamma"] & F(sy\otimes sx) \arrow[r, "\varphi"] & Fsy\otimes' Fsx
\end{tikzcd}$$

$\underline{h^{\mathcal{G}'}(\sigma;\mathfrak{f}x|\mathfrak{f}y)=\mathfrak{F}h^{\mathcal{G}}(\sigma;x|y)}$, for any $\sigma:V\to U$ in C and $x,y\in\Pi_0\mathcal{G}(U)$:

$$\begin{tikzcd}
Fs(xy)^{\sigma} \arrow[rrr, "F\Theta"] \arrow[dd, "Fh^{\mathcal{G}}(\sigma;x|y)"'] & & & Fs(xy) \arrow[d, "F\Gamma"] \\
 & (8.24) & & F(sx\otimes sy) \arrow[d, "\varphi"] \arrow[dl, phantom, "(A)"] \\
Fs(xy)^{\sigma} \arrow[r, "F\Gamma"] & F(sx^{\sigma}\otimes sy^{\sigma}) \arrow[r, "\varphi"] \arrow[urr, "F(\Theta\otimes\Theta)"] & Fsx^{\sigma}\otimes' Fsy^{\sigma} \arrow[r, "F\Theta\otimes' F\Theta"] & Fsx\otimes' Fsy
\end{tikzcd}$$

$\underline{h^{\mathcal{G}'}(\sigma,\tau;\mathfrak{f}x)=\mathfrak{F}h^{\mathcal{G}}(\sigma,\tau;x)}$, for any $W\xrightarrow{\tau}V\xrightarrow{\sigma}U$ in C and $x\in\Pi_0\mathcal{G}(U)$:

$$\begin{tikzcd}
Fsx^{\sigma\tau} \arrow[r, "F\Theta"] \arrow[d, "Fh^{\mathcal{G}}(\sigma,\tau;x)"'] \arrow[dr, phantom, "(8.24)"] & Fsx^{\sigma} \arrow[d, "F\Theta"] \\
Fsx^{\sigma\tau} \arrow[r, "F\Theta"] & Fsx
\end{tikzcd}$$

$\square$

Below is our main result in this section.

Theorem 8.5 *The mapping* $[\mathcal{G}]\mapsto[\Pi_0\mathcal{G},\Pi_1\mathcal{G},[h^{\mathcal{G}}]]$ *establishes a natural one-to-one correspondence between equivalence classes of braided or symmetric* C*-fibred categories and isomorphism classes of classifying systems for them.*

Proof The result follows from Propositions 8.6, 8.8 and 8.9 below. $\square$

Proposition 8.6 *Let* $\mathcal{M}$ *a presheaf of commutative monoids on* C *and* $\mathcal{A}$ *an* $\mathbb{H}(\mathcal{M})$*-module. For every* $h\in Z^4_s(\mathcal{M},2,\mathcal{A})$ *there exists a braided* C*-fibred category* $\mathcal{G}^h$ *such that*

$$[\Pi_0\mathcal{G}^h,\Pi_1\mathcal{G}^h,[h^{\mathcal{G}^h}]]=[\mathcal{M},\mathcal{A},[h]].$$

Furthermore, $\mathcal{G}^h$ *is symmetric whenever* $h\in Z^5_s(\mathcal{M},3,\mathcal{A})$*.*

Proof Previously to construct $\mathcal{G}^h$, we state the following useful technical lemma.

Lemma 8.7 *Suppose* $h\in Z^4_s(\mathcal{M},2,\mathcal{A})$*. For any morphism* $\sigma:V\to U$ *in* C *and* $x\in\mathcal{M}(U)$*,*

$$h(1_U,1_U,x)^{\sigma}=h(1_U,\sigma;x),\tag{8.29}$$

$$h(1_V,1_V;x^{\sigma})=h(\sigma,1_V;x),\tag{8.30}$$

and, for any $U \in \mathrm{ObC}$ *and* $x, y \in \mathcal{M}(U)$,

$$h(1_U, 1_U; xy) = x_* h(1_U, 1_U; y) + y_* h(1_U, 1_U; x) + h(1_U; x|y). \tag{8.31}$$

Proof (8.29) and (8.30) follow from the cocycle conditions $\partial h(1_U, 1_U, \sigma; x) = 0$ and $\partial h(\sigma, 1_V, 1_V; x) = 0$ in (6.20), respectively, while (8.31) follows from the cocycle condition $\partial h(1_U, 1_U; x|y) = 0$ in (6.19). □

The claimed braided C-fibred category $\mathcal{G}^h$ is as follows. Its objects are pairs (U, x) with U an object of C and x an element of the monoid $\mathcal{M}(U)$. A morphism $(V, x') \to (U, x)$ in $\mathcal{G}^h$ is a pair (σ, a) where $\sigma : V \to U$ is a morphism in C such that $x' = x^\sigma$ and $a \in \mathcal{A}(V, x')$. The composition of two morphisms $(\sigma, a) : (V, x') \to (U, x)$ and $(\tau, b) : (W, x'') \to (V, x')$ is defined by

$$(\sigma, a)(\tau, b) = (\sigma\tau, b + a^\tau + h(\sigma, \tau; x)) : (W, x'') \to (U, x).$$

If $(T, x''') \xrightarrow{(\gamma,c)} (W, x'') \xrightarrow{(\tau,b)} (V, x') \xrightarrow{(\sigma,a)} (U, x)$ are any three composible morphisms, then

$$\begin{aligned}
\big((\sigma, a)(\tau, b)\big)(\gamma, c) &= \big(\sigma\tau\gamma, c + b^\gamma + a^{\tau\gamma} + h(\sigma, \tau; x)^\gamma + h(\sigma\tau, \gamma; x)\big),\\
(\sigma, a)\big((\tau, b)(\gamma, c)\big) &= \big(\sigma\tau\gamma, c + b^\gamma + a^{\tau\gamma} + h(\tau, \gamma; x') + h(\sigma, \tau\gamma; x)\big),
\end{aligned}$$

and, by comparison, we see that the composition in $\mathcal{G}^h$ is associative thanks to the cocycle condition $\partial h(\sigma, \tau, \gamma; x) = 0$ in (6.14). Furthermore, the identity of an object (U, x) is

$$1_{(U,x)} = (1_U, -h(1_U, 1_U; x)) : (U, x) \to (U, x). \tag{8.32}$$

In effect, for any morphism $(\sigma, a) : (V, x') \to (U, x)$ in $\mathcal{G}^h$,

$$\begin{aligned}
1_{(U,x)}(\sigma, a) &= \big(\sigma, a - h(1_U, 1_U; x)^\sigma + h(1_U, \sigma; x)\big) \overset{(8.29)}{=} (\sigma, a),\\
(\sigma, a)1_{(V,x')} &= \big(\sigma, a - h(1_V, 1_V; x') + h(\sigma, 1_V; x)\big) \overset{(8.30)}{=} (\sigma, a).
\end{aligned}$$

There is a canonical functor $\pi : \mathcal{G}^h \to \mathrm{C}$ that sends an object (U, x) to U and a morphism (σ, a) to σ. This is a fibration. Indeed, every σ-morphism $(\sigma, a) : (V, x') \to (U, x)$ is cartesian since any other σ-morphism with target (U, x) writes as $(\sigma, b) : (V, x') \to (U, x)$, for some element $b \in \mathcal{A}(V, x')$, and then $(1_V, b - a - h(\sigma, 1_V; x)) : (V, x') \to (V, x')$ is the unique 1_V-morphism such that $(\sigma, a)(1_V, b - a - h(\sigma, 1_V; x)) = (\sigma, b)$. Let us observe that every fibre category $\mathcal{G}^h_U$ is the groupoid whose objects are the pairs (U, x) with $x \in \mathcal{M}(U)$, there are no morphisms between different objects, and the group of automorphisms of each object (U, x) is the abelian group

$$\mathrm{Aut}_{\mathcal{G}^h_U}(U, x) = \{(1_U, a),\, a \in \mathcal{A}(U, x)\}$$

where the composition is defined by $(1_U, a)(1_U, b) = (1_U, a + b + h(1_U, 1_U; x))$. Its unit is (8.32) and inverses are $(1_U, a)^{-1} = (1_U, -a - 2\,h(1_U, 1_U; x))$. Thus, the functor π makes $\mathcal{G}^h$ into a C-fibred category.

The C-fibred tensor product $\otimes : \mathcal{G}^h \times_{\mathrm{C}} \mathcal{G}^h \to \mathcal{G}^h$ is defined on objects by $(U, x) \otimes (U, y) = (U, xy)$ and on morphisms by

$$\big((V, x') \xrightarrow{(\sigma,a)} (U, x)\big) \otimes \big((V, y') \xrightarrow{(\sigma,b)} (U, y)\big) \;=\; \big((V, x'y') \xrightarrow{(\sigma, x'_* b + y'_* a - h(\sigma; x|y))} (U, xy)\big).$$

For any morphisms $(W,x'')\xrightarrow{(\tau,b)}(V,x')\xrightarrow{(\sigma,a)}(U,x)$ and $(W,y'')\xrightarrow{(\tau,d)}(V,y')\xrightarrow{(\sigma,c)}(U,y)$,

$$\begin{aligned}&\big((\sigma,a)(\tau,b)\big)\otimes\big((\sigma,c)(\tau,d)\big)\\&\qquad=\big(\sigma\tau, x''_*d+x''_*c^\tau+y''_*a^\tau+y''_*b+x''_*h(\sigma,\tau;y)+y''_*h(\sigma,\tau;x)-h(\sigma\tau;x|y)\big),\\&\big((\sigma,a)\otimes(\sigma,c)\big)\big((\tau,b)\otimes(\tau,d)\big)\\&\qquad=\big(\sigma\tau, x''_*d+x''_*c^\tau+y''_*a^\tau+y''_*b-h(\tau;x'|y')-h(\sigma;x|y)^\tau+h(\sigma,\tau;xy)\big),\end{aligned}$$

whence, by comparison, it follows that $\otimes$ preserves compositions due to the cocycle condition $\partial h(\sigma,\tau;x|y)=0$ in (6.19). Besides, $\otimes$ preserves identities since, for any objects (U,x) and (U,y) of $\mathcal{G}^h$,

$$\begin{aligned}1_{(U,x)}\otimes 1_{(U,y)}&=\big(1_U,-x_*h(1_U,1_U,y)-y_*h(1_U,1_U;x)-h(1_U;x|y)\big)\\&\overset{(8.31)}{=}\big(1_U,-h(1_U,1_U;xy)\big)=1_{(U,xy)}=1_{(U,x)\otimes(U,y)}.\end{aligned}$$

The C-fibred associativity constraint $\boldsymbol{a}$ in $\mathcal{G}^h$ is given by the family of morphisms

$$\boldsymbol{a}_{(U,x),(U,y),(U,z)}=(1_U,h(U;x|y|z)-h(1_U,1_U;xyz)):(U,(xy)z)\to(U,x(yz)).$$

These are natural: for any σ-morphisms $(\sigma,a):(V,x')\to(U,x)$, $(\sigma,b):(V,y')\to(U,y)$, and $(\sigma,c):(V,z')\to(U,z)$, a direct computation gives

$$\begin{aligned}&\big((\sigma,a)\otimes((\sigma,b))\otimes(\sigma,c)\big)\,\boldsymbol{a}_{(V,x'),(V,y'),(V,z')}=\\&\quad(\sigma,x'_*y'_*c+x'_*z'_*b+y'_*z'_*a-x'_*h(\sigma;y|z)-h(\sigma;x|yz)+h(V;x'|y'|z')-h(1_V,1_V;x'y'z')+h(\sigma,1_V;xyz))\\&\quad\overset{(8.30)}{=}(\sigma,x'_*y'_*c+x'_*z'_*b+y'_*z'_*a-x'_*h(\sigma;y|z)-h(\sigma;x|yz)+h(V;x'|y'|z')),\end{aligned}$$

$$\begin{aligned}&\boldsymbol{a}_{(U,x),(U,y),(U,z)}\,\big((\sigma,a)\otimes((\sigma,b))\otimes(\sigma,c))\big)=\\&\quad(\sigma,x'_*y'_*c+x'_*z'_*b+y'_*z'_*a-z'_*h(\sigma;x|y)-h(\sigma;xy|z)+h(U;x|y|z)^\sigma-h(1_U,1_U;xyz)^\sigma+h(1_U,\sigma;xyz))\\&\quad\overset{(8.29)}{=}(\sigma,x'_*y'_*c+x'_*z'_*b+y'_*z'_*a-z'_*h(\sigma;x|y)-h(\sigma;xy|z)+h(U;x|y|z)^\sigma).\end{aligned}$$

Hence, by comparison, we see that the naturality requirement for $\boldsymbol{a}$ follows from the cocycle condition $\partial h(\sigma;x|y|z)=0$ in (6.17). Furthermore, it satisfies the coherence condition (8.4). To see that, first we observe that, for any $U\in\mathrm{ObC}$ and $x,y,z,t\in\mathcal{M}(U)$,

$$\begin{aligned}1_{(U,x)}\otimes\boldsymbol{a}_{(U,y),(U,z),(U,t)})&=(1_U,-h(1_U,1_U;x))\otimes(1_U,h(U;y|z|t)-h(1_U,1_U;yzt))\\&=(1_U,x_*h(U;y|z|t)-x_*h(1_U,1_U;yzt)-y_*z_*t_*h(1_U,1_U;x)-h(1_U;x|yzt))\\&\overset{(8.31)}{=}(1_U,x_*h(U;y|z|t)-h(1_U;xyzt)),\end{aligned}$$

$$\begin{aligned}\boldsymbol{a}_{(U,x),(U,y),(U,z)})\otimes 1_{(U,t)}&=(1_U,h(U;x|y|z)-h(1_U,1_U;xyz))\otimes(1_U,-h(1_U,1_U;t))\\&=(1_U,t_*h(U;x|y|z)-t_*h(1_U,1_U;xyz)-x_*y_*z_*h(1_U,1_U;t)-h(1_U;xyz|t))\\&\overset{(8.31)}{=}(1_U;t_*h(U;x|y|z)-h(1_U;xyzt)).\end{aligned}$$

Then, thanks to the cocycle condition $\partial h(U,x|y|z|t)=0$ in (6.14), we have

$$\begin{aligned}
&(1_{(U,x)}\otimes \boldsymbol{a}_{(U,y),(U,z),(U,t)})\,\boldsymbol{a}_{(U,x),(U,yz),(U,t)}(\boldsymbol{a}_{(U,x),(U,y),(U,z)}\otimes 1_{(U,t)})\\
&\qquad = (1_U, x_*h(U;y|z|t)-h(1_U;xyzt))(1_U, h(U;x|yz|t)-h(1_U,1_U;xyzt))\\
&\qquad\quad (1_U;t_*h(U;x|y|z)-h(1_U;xyzt))\\
&\qquad = (1_U;x_*h(U;y|z|t)+h(U;x|yz|t)+t_*h(U;x|y|z)-h(1_U,1_U;xyzt))\\
&\qquad \overset{(6.14)}{=} ((1_U,h(U;x|y|zt)+h(U;xy|z|t)-h(1_U,1_U;xyzt))\\
&\qquad = (1_U;t_*h(U;x|y|zt)-h(1_U;xyzt))(1_U;t_*h(U;xy|z|t)-h(1_U;xyzt))\\
&\qquad = \boldsymbol{a}_{(U,x),(U,y),(U,zt)}\,\boldsymbol{a}_{(U,xy),(U,z),(U,t)}.
\end{aligned}$$

The C-fibred braiding $\boldsymbol{c}$ consists of the morphisms

$$\boldsymbol{c}_{(U,x),(U,y)}=\big(1_U, h(U;x|_2y)-h(1_U,1_U,xy)\big):(U,xy)\to(U,yx). \tag{8.33}$$

If $(\sigma,a):(V,x')\to(U,x)$, $(\sigma,b):(V,y')\to(U,y)$ are σ-morphisms in $\mathcal{G}^h$, then

$$\begin{aligned}
&\boldsymbol{c}_{(U,x),(U,y)}\,\big((\sigma,a)\otimes(\sigma,b)\big)\\
&\qquad = (1_U,h(U;x|_2y)-h(1_U,1_U,xy))\,(\sigma,x'_*b+y'_*a-h(\sigma;x|y))\\
&\qquad = (\sigma,x'_*b+y'_*a-h(\sigma;x|y)+h(U;x|_2y)^\sigma-h(1_U,1_U;xy)^\sigma+h(1_U,\sigma;xy))\\
&\qquad \overset{(8.29)}{=} (\sigma,x'_*b+y'_*a-h(\sigma;x|y)+h(U;x|_2y)^\sigma)\\
&\qquad \overset{(6.18)}{=} (\sigma,x'_*b+y'_*a-h(\sigma;y|x)+h(V;x'|_2y'))\\
&\qquad \overset{(8.30)}{=} (\sigma,x'_*b+y'_*a-h(\sigma;y|x)+h(V;x'|_2y')-h(1_V,1_V;x'y')+h(\sigma,1_V;xy))\\
&\qquad = (\sigma,x'_*b+y'_*a-h(\sigma;y|x))\,(1_V,h(V;x'|_2y'))-h(1_V,1_V;x'y'))\\
&\qquad = \big((\sigma,b)\otimes(\sigma,a)\big)\,\boldsymbol{c}_{(V,x'),(V,y')},
\end{aligned}$$

whence we see that the naturality of $\boldsymbol{c}$ holds owing to cocycle condition $\partial h(\sigma;x|_2y)=0$ in (6.18). Moreover, the coherence requirement (8.6) holds. To prove this, first observe that, for any $U\in\mathrm{Ob}\mathrm{C}$ and $x,y,z\in\mathcal{M}(U)$, we have

$$\begin{aligned}
1_{(U,x)}\otimes\boldsymbol{c}_{(U,y),(U,z)} &= (1_U,x_*h(U;y|_2z)-x_*h(1_U,1_U;yz)-y_*z_*h(1_U,1_U;x)-h(1_U;x|yz))\\
&\overset{(8.31)}{=} (1_U,x_*h(U;y|_2z)-h(1_U,1_U;xyz)),\\
\boldsymbol{c}_{(U,x),(U,y)}\otimes 1_{(U,z)} &= (1_U,z_*h(U;x|_2y)-z_*h(1_U,1_U;xy)-x_*y_*h(1_U,1_U;x)-h(1_U;xy|z)\\
&\overset{(8.31)}{=} (1_U,x_*h(U;x|_2y)-h(1_U,1_U;xyz)).
\end{aligned}$$

Then,

$$
\begin{aligned}
&(1_{(U,y)} \otimes \boldsymbol{c}_{(U,x),(U,z)})\, \boldsymbol{a}_{(U,y),(U,x),(U,z)}\, (\boldsymbol{c}_{(U,x),(U,y)} \otimes 1_{(U,z)}) \\
&\qquad = (1_U, y_* h(U; y|_2 z) - h(1_U, 1_U; xyz))(1_U, h(U; y|x|z) - h(1_U, 1_U; xyz)) \\
&\qquad\quad (1_U, z_*(U; x|_2 y) - h(1_U, 1_U; xyz)) \\
&\qquad = (1_U, y_* h(U; x|_2 z) + h(U; y|x|z) + z_* h(U; x|_2 y) - h(1_U, 1_U; xyz)), \\
&\boldsymbol{a}_{(U,y),(U,z),(U,x)}\, \boldsymbol{c}_{(U,x),(U,yz)}\, \boldsymbol{a}_{(U,x),(U,y),(U,z)} \\
&\qquad = (1_U, h(U; y|z|x) - h(1_U, 1_U; xyz))(1_U, h(U; x|_2 yz) - h(1_U, 1_U; xyz)) \\
&\qquad\quad (1_U, h(U; x|y|z) - h(1_U, 1_U; xyz)) \\
&\qquad = (1_U, h(U; y|z|x) + h(U; x|_2 yz) + h(U; x|y|z) - h(1_U, 1_U; xyz)),
\end{aligned}
$$

whence, by comparison, (8.6) follows from the cocycle condition $\partial h(U; x|_2 y|z) = 0$ in (6.15). Similarly,

$$
\begin{aligned}
&(\boldsymbol{c}_{(U,x),(U,z)} \otimes 1_{(U,y)})\, \boldsymbol{a}^{-1}_{(U,x),(U,z),(U,y)}\, (1_{(U,x)} \otimes \boldsymbol{c}_{(U,y),(U,z)}) \\
&\qquad = (1_U, y_* h(U; x|_2 z) - h(1_U, 1_U; xyz))(1_U, -h(U; x|z|y) - h(1_U, 1_U; xyz)) \\
&\qquad\quad (1_U, x_* h(U; y|_2 z) - h(1_U, 1_U; xyz)) \\
&\qquad = (1_U, y_* h(U; x|_2 z) - h(U; x|z|y) + x_* h(U; y|_2 z) - h(1_U, 1_U; xyz)), \\
&\boldsymbol{a}^{-1}_{(U,z),(U,x),(U,y)}\, \boldsymbol{c}_{(U,xy),(U,z)}\, \boldsymbol{a}^{-1}_{(U,x),(U,y),(U,z)} \\
&\qquad = (1_U, -h(U; z|x|y) - h(1_U, 1_U; xyz))(1_U, h(U; xy|_2 z) - h(1_U, 1_U; xyz)) \\
&\qquad\quad (1_U, -h(U; x|y|z) - h(1_U, 1_U; xyz)) \\
&\qquad = (1_U, -h(U; z|x|y) + h(U; xy|_2 z) - h(U; x|y|z) - h(1_U, 1_U; xyz)),
\end{aligned}
$$

and the coherence condition (8.7) follows from the cocycle condition $\partial h(U; x|y|_2 z) = 0$ in (6.16).

The unit C-functor $\iota : \mathrm{C} \to \mathcal{G}^h$ is defined on objects by $\iota(U) = (U, e)$ and on morphisms by $\iota(\sigma : V \to U) = (\sigma, 0) : (V, e) \to (U, e)$. This is plainly recognized as a functor thanks to the normalization condition $h(\sigma, \tau; e) = 0$. The C-fibred unit constraints $\boldsymbol{l}$ and $\boldsymbol{r}$ in $\mathcal{G}^h$ are both identities, that is, for any $U \in \mathrm{ObC}$ and $x \in \mathcal{M}(U)$,

$$
\boldsymbol{l}_{(U,x)} = 1_{(U,x)} = \boldsymbol{r}_{(U,x)} : (U, x) \to (U, x).
$$

Notice that $(U, x) \otimes (U, e) = (U, x) = (U, e) \otimes (U, x)$ and, for any morphism $(\sigma, a) : (V, x') \to (U, x)$ in $\mathcal{G}^h$,

$$
\begin{aligned}
(\sigma, a) \otimes (\sigma, 0) &= (\sigma, e_* a + x_* 0 - h(\sigma; x|e)) = (\sigma, a), \\
(\sigma, 0) \otimes (\sigma, a) &= (\sigma, e_* a + x_* 0 - h(\sigma; e|x)) = (\sigma, a),
\end{aligned}
$$

since $e_* a = a$, $x_* 0 = 0$, and $h(\sigma; x|e) = 0 = h(\sigma; e|x)$. Furthermore, the coherence requirement (8.5) trivially holds since, for any $U \in \mathrm{ObC}$ and $x, y \in \mathcal{M}(U)$,

$$
\begin{aligned}
&1_{(U,x)} \otimes \boldsymbol{l}_{(U,y)} = 1_{(U,x)} \otimes 1_{(U,y)} = 1_{(U,xy)}, \\
&\boldsymbol{a}_{(U,x),(U,e),(U,y)} = (1_U, h(U; x|e|y) - h(1_U, 1_U; xy)) = (1_U, -h(1_U, 1_U; xy)) = 1_{(U,xy)}, \\
&\boldsymbol{r}_{(U,x)} \otimes 1_{(U,y)} = 1_{(U,x)} \otimes 1_{(U,y)} = 1_{(U,xy)}.
\end{aligned}
$$

All in all, we have that $\mathcal{G}^h$ is actually a braided C-fibred category. Furthermore, in case that $h \in Z^5_s(\mathcal{M},3,\mathcal{A})$, then the 5-cocycle condition $\partial h = 0$ in (6.22) says us that, for any $U \in \mathrm{Ob}\mathrm{C}$ and $x, y \in \mathcal{M}(U)$, $h(U; x|_2 y) + h(U; y|_2 x) = 0$, which implies the symmetry condition (8.8) for the braiding $\boldsymbol{c}$ of $\mathcal{G}^h$:

$$\boldsymbol{c}_{(U,y),(U,x)}\boldsymbol{c}_{(U,x),(U,y)} = (1_U, h(U; x|_2 y) + h(U; y|_2 x) - h(1_U, 1_U; xy)) = 1_{(U,xy)} = 1_{(U,x)\otimes(U,y)}.$$

We now prove that $[\Pi_0\mathcal{G}^h, \Pi_1\mathcal{G}^h, [h^{\mathcal{G}^h}]] = [\mathcal{M}, \mathcal{A}, [h]]$: If, for each object U of C and $x \in \mathcal{M}(U)$, we denote by $[U,x]$ the U-isomorphism class of (U,x) in $\mathcal{G}^h$, then $[U,x] = [U,x'] \Leftrightarrow x = x'$, $[U,x][U,y] = [U,xy]$ and, for any morphism $\sigma : V \to U$ in C, $[U,x]^\sigma = [V, x^\sigma]$. Hence, an isomorphism of presheaves of commutative monoids

$$\mathfrak{f} : \mathcal{M} \cong \Pi_0\mathcal{G}^h$$

is defined by the monoid isomorphisms

$$\mathfrak{f} : \mathcal{M}(U) \cong \Pi_0\mathcal{G}^h(U) \quad \text{given by } \ x \mapsto \mathfrak{f}x = [U,x].$$

Moreover, there is an isomorphism of $\mathbb{H}(\mathcal{M})$-modules

$$\mathfrak{F} : \mathcal{A} \cong \mathfrak{f}^*\Pi_1\mathcal{G}^h$$

such that $\mathfrak{F}_*h = \mathfrak{f}^*h^{\mathcal{G}^h}$, which is defined by the isomorphisms of abelian groups

$$\mathfrak{F} : \mathcal{A}(U,x) \cong \mathrm{Aut}_{\mathcal{G}^h_U}(U,x) = \Pi_1\mathcal{G}^h(U,\mathfrak{f}x), \qquad \text{given by } \ a \mapsto \mathfrak{F}a = (1_U, a - h(1_U, 1_U; x)).$$

This proves the result. The verification of all the needed requirements for $\mathfrak{F}$ is as follows.

$\underline{\mathfrak{f}y_*\mathfrak{F}a = \mathfrak{F}(y_*a)}$, for any object U of C, $x, y \in \mathcal{M}(U)$, and $a \in \mathcal{A}(U,x)$: From the commutativity of the diagram

$$\begin{tikzcd}
(U,xy) \arrow[r, "\Gamma=1"] \arrow[d, "\mathfrak{f}y_*\mathfrak{F}a"'] & (U,y)\otimes(U,x) \arrow[d, "1\otimes\mathfrak{F}a"] \\
(U,xy) \arrow[r, "\Gamma=1"] & (U,y)\otimes(U,x),
\end{tikzcd}$$

it follows that

$$\begin{aligned}
\mathfrak{f}y_*\mathfrak{F}a &= 1_{(U,y)} \otimes \mathfrak{F}a = (1_U, -h(1_U, 1_U; y) \otimes (1_U, a - h(1_U, 1_U, x)) \\
&= (1_U, y_*a - y_*h(1_U, 1_U; x) - x_*h(1_U, 1_U, y) - h(1_U; y|x)) \\
&\overset{(8.31)}{=} (1_U, y_*a - h(1_U, 1_U; xy)) = \mathfrak{F}(y_*a).
\end{aligned}$$

$\underline{(\mathfrak{F}a)^\sigma = \mathfrak{F}a^\sigma}$, for any morphism $\sigma : V \to U$ of C, $x \in \mathcal{M}(U)$ and $a \in \mathcal{A}(U,x)$: By Lemma 8.1, it suffices to check the commutativity of the square in $\mathcal{G}^h$

$$\begin{tikzcd}
(V,x^\sigma) \arrow[r, "\Theta=(\sigma,0)"] \arrow[d, "\mathfrak{F}a^\sigma"'] & (U,x) \arrow[d, "\mathfrak{F}a"] \\
(V,x^\sigma) \arrow[r, "\Theta=(\sigma,0)"] & (U,x)
\end{tikzcd}$$

$$(\sigma,0)\,\mathfrak{F}a^{\sigma} = (\sigma,0)(1_V, a^{\sigma} - h(1_V,1_V;x^{\sigma})) = (\sigma, a^{\sigma} - h(1_V,1_V;x^{\sigma}) + h(\sigma,1_V;x)) \overset{(8.30)}{=} (\sigma, a^{\sigma})$$

$$\mathfrak{F}a\,(\sigma,0) = (1_U, a - h(1_U,1_U;x))(\sigma,0) = (\sigma, a^{\sigma} - h(1_U,1_U;x)^{\sigma} + h(1_U,\sigma;x)) \overset{(8.29)}{=} (\sigma, a^{\sigma}).$$

$\underline{\mathfrak{F}h(U;x|y|z) = h^{\mathcal{G}^h}(U;\mathfrak{f}x|\mathfrak{f}y|\mathfrak{f}z)}$, for any $U \in \mathrm{ObC}$ and $x,y,z \in \mathcal{M}(U)$: By the commutativity of the diagram

$$\begin{tikzcd}
(U,xyz) \arrow[r, "\Gamma=1"] \arrow[d, "h^{\mathcal{G}^h}(U;\mathfrak{f}x|\mathfrak{f}y|\mathfrak{f}z)"'] & (U,xy)\otimes(U,z) \arrow[r, "\Gamma\otimes 1=1\otimes 1=1"] & ((U,x)\otimes(U,y))\otimes(U,z) \arrow[d, "\boldsymbol{a}"] \\
(U,xyz) \arrow[r, "\Gamma=1"] & (U,x)\otimes(U,yz) \arrow[r, "1\otimes\Gamma=1\otimes 1=1"] & (U,x)\otimes((U,y))\otimes(U,z))
\end{tikzcd}$$

it follows that $h^{\mathcal{G}^h}(U;\mathfrak{f}x|\mathfrak{f}y|\mathfrak{f}z) = \boldsymbol{a} = (1_U, h(U;x|y|z) - h(1_U,1_U;xyz)) = \mathfrak{F}h(U;x|y|z)$.

$\underline{\mathfrak{F}h(U;x|_2 y) = h^{\mathcal{G}^h}(U;\mathfrak{f}x|_2\mathfrak{f}y)}$, for any $U \in \mathrm{ObC}$ and $x,y \in \mathcal{M}(U)$: From the commutative diagram

$$\begin{tikzcd}
(U,xy) \arrow[r, "\Gamma=1"] \arrow[d, "h^{\mathcal{G}^h}(U;\mathfrak{f}x|_2\mathfrak{f}y)"'] & (U,x)\otimes(U,y) \arrow[d, "\boldsymbol{c}"] \\
(U,xy) \arrow[r, "\Gamma=1"] & (U,y)\otimes(U,x)
\end{tikzcd}$$

it follows that $h^{\mathcal{G}^h}(U;\mathfrak{f}x|_2\mathfrak{f}y) = \boldsymbol{c} = (1_U, h(U;x|_2 y) - h(1_U,1_U;xy)) = \mathfrak{F}h(U;x|_2 y)$.

$\underline{\mathfrak{F}h(\sigma;x|y)) = h^{\mathcal{G}^h}(\sigma;\mathfrak{f}x|\mathfrak{f}y)}$, for any morphism $\sigma : V \to U$ of C and $x,y \in \mathcal{M}(U)$: By Lemma 8.1, it suffices to check the commutativity of the square in $\mathcal{G}^h$

$$\begin{tikzcd}
(V,x^{\sigma}y^{\sigma}) \arrow[rr, "\Theta=(\sigma,0)"] \arrow[d, "\mathfrak{F}h(\sigma;x|y)"'] & & (U,xy) \arrow[d, "\Gamma=1"] \\
(V,x^{\sigma}y^{\sigma}) \arrow[r, "\Gamma=1"] & (V,x^{\sigma})\otimes(V,y^{\sigma}) \arrow[r, "\Theta\otimes\Theta=(\sigma,0)\otimes(\sigma,0)"] & (U,x)\otimes(U,y)
\end{tikzcd}$$

$$((\sigma,0)\otimes(\sigma,0))\,\mathfrak{F}h(\sigma;x|y) = (\sigma, -h(\sigma,x|y))(1_V, h(\sigma;x|y) - h(1_V,1_V;x^{\sigma}y^{\sigma}) = (\sigma,0).$$

$\underline{\mathfrak{F}h(\sigma,\tau;x)) = h^{\mathcal{G}^h}(\sigma,\tau;\mathfrak{f}x)}$, for any $W \xrightarrow{\tau} V \xrightarrow{\sigma} U$ in C and $x \in \mathcal{M}(U)$: By Lemma 8.1, it suffices to check the commutativity of the square in $\mathcal{G}^h$

$$\begin{tikzcd}
(W,x^{\sigma\tau}) \arrow[r, "\Theta=(\tau,0)"] \arrow[d, "\mathfrak{F}h(\sigma,\tau;x)"'] & (V,x^{\sigma}) \arrow[d, "\Theta=(\sigma,0)"] \\
(W,x^{\sigma\tau}) \arrow[r, "\Theta=(\sigma\tau,0)"] & (U,x)
\end{tikzcd}$$

$$\begin{aligned}
(\sigma\tau,0)\,\mathfrak{F}h(\sigma,\tau;x) &= (\sigma\tau,0)(1_W, h(\sigma,\tau;x) - h(1_W,1_W;x^{\sigma\tau})) \\
&= (\sigma\tau, h(\sigma,\tau;x) - h(1_W,1_W;x^{\sigma\tau}) + h(\sigma\tau,1_W;x)) \overset{(8.30)}{=} (\sigma\tau, h(\sigma,\tau;x) \\
&= (\sigma,0)(\tau,0).
\end{aligned}$$

$\square$

Proposition 8.8 *If* $[\mathcal{M},\mathcal{A},[h]]=[\mathcal{M}',\mathcal{A}',[h']]$, *then* $[\mathcal{G}^h]=[\mathcal{G}^{h'}]$.

Proof Suppose we have an isomorphism of presheaves of commutative monoids $\mathfrak{f}:\mathcal{M}\cong\mathcal{M}'$, an isomorphism of $\mathbb{H}(\mathcal{M})$-modules $\mathfrak{F}:\mathcal{A}\cong\mathfrak{f}^*\mathcal{A}'$ and a 3-cochain $g\in C^3_s(\mathcal{M},2,\mathfrak{f}^*\mathcal{A}')$ such that $\mathfrak{F}h=\mathfrak{f}^*h'+\partial g$. Then, a braided C-fibred isomorphism $F:\mathcal{G}^h\cong\mathcal{G}^{h'}$ is defined by

$$(V,x')\xrightarrow{(\sigma,a)}(U,x)\ \overset{F}{\mapsto}\ (V,\mathfrak{f}x')\xrightarrow{(\sigma,\mathfrak{F}a-g(\sigma;x))}(U,\mathfrak{f}x),$$

together with the structure isomorphisms

$$\begin{cases} F((U,x)\otimes(U,y))=(U,\mathfrak{f}x\mathfrak{f}y)\xrightarrow{\varphi=(1_U,g(U;x|y)-h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y))}F(U,x)\otimes F(U,y)=(U,\mathfrak{f}x\mathfrak{f}y)\\ F\iota U=(U,e)\xrightarrow{\varphi^*=1_{(U,e)}}\iota U=(U,e).\end{cases}$$

So defined, we see that F is a C-fibred functor because of the coboundary equation $\mathfrak{F}h(\sigma,\tau;x)=h'(\sigma,\tau;\mathfrak{f}x)+\partial g(\sigma,\tau;x)$:

$$F(1_{(U,x)})=(1_U,-\mathfrak{F}h(1_U,1_U;x)-g(1_U;x))=(1_U,-h'(1_U,1_U;\mathfrak{f}x))=1_{F(U,x)}$$

and, for $(W,x'')\overset{(\tau,b)}{\longrightarrow}(V,x')\overset{(\sigma,a)}{\longrightarrow}(U,x)$ any two composable morphisms in $\mathcal{G}^h$,

$$\begin{aligned}F((\sigma,a)(\tau,b))=F(\sigma\tau,b+a^\tau+h(\sigma,\tau;x))&=(\sigma\tau,\mathfrak{F}b+\mathfrak{F}a^\tau+\mathfrak{F}h(\sigma,\tau;x)-g(\sigma\tau;x))\\ &=(\sigma\tau,\mathfrak{F}b+\mathfrak{F}a^\tau-g(\tau;x')-g(\sigma;x)^\tau+h'(\sigma,\tau;\mathfrak{f}x))\\ &=(\sigma,\mathfrak{F}a-g(\sigma;x))(\tau,\mathfrak{F}b-g(\tau;x'))=F(\sigma,a)\,F(\tau,b).\end{aligned}$$

The structure constraints φ of F are natural: for any two σ-morphisms $(\sigma,a):(V,x')\to(U,x)$ and $(\sigma,b):(V,y')\to(U,y)$ in $\mathcal{G}^h$, a direct computation gives

$$\begin{aligned}&\varphi_{(U,x),(U,y)}\,F((\sigma,a)\otimes(\sigma,b))\\ &\qquad=(1_U,g(U;x|y)-h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y))(\sigma,\mathfrak{F}(x'_*b)+\mathfrak{F}(y'_*a)-\mathfrak{F}h(\sigma;x|y)-g(\sigma;xy))\\ &\qquad=(1_U,g(U;x|y)-h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y))(\sigma,\mathfrak{f}x'_*\mathfrak{F}b+\mathfrak{f}y'_*\mathfrak{F}a-\mathfrak{F}h(\sigma;x|y)-g(\sigma;xy))\\ &\qquad\overset{(8.29)}{=}(\sigma,\mathfrak{f}x'_*\mathfrak{F}b+\mathfrak{f}y'_*\mathfrak{F}a-\mathfrak{F}h(\sigma;x|y)-g(\sigma;xy)+g(U;x|y)^\sigma),\end{aligned}$$

$$\begin{aligned}&(F(\sigma,a)\otimes F(\sigma,b))\varphi_{(V,x'),(V,y')}\\ &\quad=((\sigma,\mathfrak{F}a-g(\sigma,x))\otimes(\sigma,\mathfrak{F}b-g(\sigma,y))(1_V,g(V,x'|y')-h'(1_V,1_V;\mathfrak{f}x'\mathfrak{f}y'))\\ &\quad=(\sigma,\mathfrak{f}x'_*\mathfrak{F}b-\mathfrak{f}x'_*g(\sigma,y)+\mathfrak{f}y'_*\mathfrak{F}a-\mathfrak{f}y'_*g(\sigma,x)-h'(\sigma;\mathfrak{f}x|\mathfrak{f}y))(1_V,g(V;x'|y')+h'(1_V,1_V;\mathfrak{f}x'\mathfrak{f}y'))\\ &\quad\overset{(8.30)}{=}(\sigma,g(V;x'|y')+\mathfrak{f}x'_*\mathfrak{F}b-\mathfrak{f}x'_*g(\sigma,y)+\mathfrak{f}y'_*\mathfrak{F}a-\mathfrak{f}y'_*g(\sigma,x)-h'(\sigma;\mathfrak{f}x|\mathfrak{f}y)).\end{aligned}$$

Hence, by comparison, we see that the coboundary equation $\mathfrak{F}h(\sigma;x|y)=h'(\sigma;\mathfrak{f}x,\mathfrak{f}y)+\partial g(\sigma;x|y)$ implies the naturality requirement.

The coherence condition (8.11) holds. To prove it, we first observe that, for any $U\in \mathrm{ObC}$ and $x,y,z\in\mathcal{M}(U)$,

$$F\boldsymbol{a}_{(U,x),(U,y),(U,z)} = (1_U, \mathfrak{F}h(U;x|y|z) - \mathfrak{F}h(1_U,1_U;xyz) - g(1_U;xyz)) \overset{(6.12)}{=} (1_U, \mathfrak{F}h(U;x|y|z) - h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y\mathfrak{f}z)).$$

$$\begin{aligned}1_{F(U,x)}\otimes\boldsymbol{\varphi}_{(U,y),(U,z)} &= (1_U, -h'(1_U,1_U;\mathfrak{f}x)\otimes(1_U, g(U;y|z) - h'(1_U,1_U;\mathfrak{f}y\mathfrak{f}z))\\ &= \big(1_U, \mathfrak{f}x_*g(U;y|z) - \mathfrak{f}x_*h'(1_U,1_U,\mathfrak{f}y\mathfrak{f}z) - \mathfrak{f}y_*\mathfrak{f}z_*h'(1_U,1_U;\mathfrak{f}x) - h'(1_U;\mathfrak{f}x|\mathfrak{f}y\mathfrak{f}z)\big)\\ &\overset{(8.31)}{=} (1_U, \mathfrak{f}x_*g(U;y|z) - h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y\mathfrak{f}z)),\end{aligned}$$

$$\begin{aligned}\boldsymbol{\varphi}_{(U,x),(U,y)}\otimes 1_{F(U,z)} &= (1_U, g(U;xy) - h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y))\otimes(1_U,-h'(1_U,1_U;\mathfrak{f}z))\\ &= (1_U, -\mathfrak{f}x_*\mathfrak{f}y_*h'(1_U,1_U;\mathfrak{f}z) + \mathfrak{f}z_*g(U;x|y) - \mathfrak{f}z_*h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y) - h'(1_U;\mathfrak{f}x\mathfrak{f}y|\mathfrak{f}z))\\ &\overset{(8.31)}{=} (1_U, \mathfrak{f}z_*g(U;x|y) - h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y\mathfrak{f}z)),\end{aligned}$$

Then, a direct computation gives

$$\begin{aligned}&(1_{F(U,x)}\otimes\boldsymbol{\varphi}_{(U,x),(U,y)})\,\boldsymbol{\varphi}_{(U,x),(U,yz)})\,F\boldsymbol{a}_{(U,x),(U,y),(U,z)}\\ &\qquad = (1_U, \mathfrak{f}x_*g(U;y|z) + g(U;x|yz) + \mathfrak{F}h(U;x|y|z) - h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y\mathfrak{f}z))),\\ &\boldsymbol{a}_{F(U,x),F(U,y),F(U,z)}(\boldsymbol{\varphi}_{(U,x),(U,y)}\otimes 1_{F(U,z)})\boldsymbol{\varphi}_{(U,xy),(U,z)}\\ &\qquad = (1_U, h'(U;\mathfrak{f}x|\mathfrak{f}y|\mathfrak{f}z) - \mathfrak{f}x_*g(U;y|z) - g(U;x|yz) - h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y\mathfrak{f}z)),\end{aligned}$$

so that (8.11) holds thanks to the equality $\mathfrak{F}h(U;x|y|z) = h'(U;\mathfrak{f}x|\mathfrak{f}y|\mathfrak{f}z) + \partial g(U;x|y|z)$.

With respect to the verification of the coherence condition (8.12), first we observe that, for any $U\in\mathrm{ObC}$ and $x,y\in\mathcal{M}(U)$,

$$F\boldsymbol{c}_{(U,x),(U,y)} = (1_U, \mathfrak{F}h(U;x|_2y) - \mathfrak{F}h(1_U,1_U;xy) - g(1_U;xy)) \overset{(6.12)}{=} (1_U, \mathfrak{F}h(U;x|_2y) - h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y)).$$

Then, a direct computation gives

$$\begin{aligned}\boldsymbol{\varphi}_{(U,y),(U,x)}\,F\boldsymbol{c}_{(U,x),(U,y)} &= (1_U, g(U;y|x) + \mathfrak{F}h(U;x|_2y) - h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y)),\\ \boldsymbol{c}_{F(U,x),F(U,y)}\,\boldsymbol{\varphi}_{(U,x),(U,y)} &= (1_U, h'(U;\mathfrak{f}x|_2\mathfrak{f}y) + g(U;x|y) - h'(1_U,1_U;\mathfrak{f}x\mathfrak{f}y)),\end{aligned}$$

whence the result follows from the coboundary equation $\mathfrak{F}h(U;x|_2y) = h'(U;\mathfrak{f}x|_2\mathfrak{f}y) + \partial g(U;x|_2y)$.

Finally, the coherence conditions in (8.13) trivially hold since $\boldsymbol{\varphi}_{(U,x),(U,e)} = 1_{(U,\mathfrak{f}x)} = \boldsymbol{\varphi}_{(U,e),(U,x)}$, for any $U\in\mathrm{ObC}$ and $x\in\mathcal{M}(U)$, due to the normalization condition $g(U;x|e) = 0 = g(U;e|x)$. $\square$

Proposition 8.9 *For any braided* C*-fibred category* $\mathcal{G}$, $[\mathcal{G}] = [\mathcal{G}^{h^{\mathcal{G}}}]$.

Proof Suppose that the construction of $(\Pi_0\mathcal{G},\Pi_1\mathcal{G},h^{\mathcal{G}})$ has been made by means of representative U-objects $s(U,x)$, as in (8.15), U-morphisms $\Gamma_{x,y}: s(U,xy)\to s(U,x)\otimes s(U,y)$, (8.17), and σ-morphisms $\Theta_{\sigma,x}: s(V,x^\sigma)\to s(U,x)$, (8.20). Then, a braided C-fibred equivalence $s:\mathcal{G}^{h^{\mathcal{G}}}\to\mathcal{G}$ is defined by the functor given by

$$(V,x^\sigma)\xrightarrow{(\sigma,a)}(U,x)\ \overset{s}{\mapsto}\ s(V,x^\sigma)\xrightarrow{\Theta_{\sigma,x}\,a}s(U,x), \tag{8.34}$$

together with the structure isomorphisms

$$\begin{cases}\boldsymbol{\varphi}_{(U,x),(U,y)}=\Gamma_{x,y}: s((U,x)\otimes(U,y))=s(U,xy)\longrightarrow s(U,x)\otimes s(U,y),\\ \boldsymbol{\varphi}^*_U=1_{\iota U}: s(U,e)=\iota U\longrightarrow \iota U.\end{cases}$$

Since

$$s(1_{(U,x)})=s(1_U,h^{\mathcal{G}}(1_U,1_U;x)^{-1})\overset{(8.25)}{=}s(1_U,1_{s(U,x)})=\Theta_{1_U,x}1_{s(U,x)}\overset{(8.21)}{=}1_{s(U,x)},$$

and, for $(W,x^{\sigma\tau})\overset{(\tau,b)}{\longrightarrow}(V,x^\sigma)\overset{(\sigma,a)}{\longrightarrow}(U,x)$ any two composable morphisms in $\mathcal{G}^{h^{\mathcal{G}}}$,

$$\begin{aligned}s((\sigma,a)(\tau,b))&=s(\sigma\tau,b\,a^\tau h^{\mathcal{G}}(\sigma,\tau;x))=\Theta_{\sigma\tau,x}b\,a^\tau h^{\mathcal{G}}(\sigma,\tau;x)\\ &\overset{(8.24)}{=}\Theta_{\sigma,x}\Theta_{\tau,x^\sigma}b\,a^\tau\overset{(8.19)}{=}\Theta_{\sigma,x}\,a\,\Theta_{\tau,x^\sigma}b=s(\sigma,a)(\tau,b).\end{aligned}$$

s is actually a functor, which is clearly C-fibred. The structure constraints $\boldsymbol{\varphi}$ of s are natural since, for any two σ-morphisms $(\sigma,a):(V,x')\to(U,x)$ and $(\sigma,b):(V,y')\to(U,y)$ in $\mathcal{G}^{h^{\mathcal{G}}}$,

$$\begin{aligned}\boldsymbol{\varphi}_{(U,x),(U,y)}\ s((\sigma,a)\otimes(\sigma,b))&=\Gamma_{x,y}\,s(\sigma,x'_*b\ \ y'_*a\ \ h^{\mathcal{G}}(\sigma;x|y)^{-1})=\Gamma_{x,y}\ \Theta_{\sigma,xy}\ x'_*b\ \ y'_*a\ \ h^{\mathcal{G}}(\sigma;x|y)^{-1}\\ &\overset{(8.24)}{=}(\Theta_{\sigma,x}\otimes\Theta_{\sigma,y})\ \Gamma_{x',y'}\ x'_*b\ \ y'_*a\overset{(8.16)}{=}(\Theta_{\sigma,x}\otimes\Theta_{\sigma,y})\ (1\otimes b)\ (a\otimes 1)\ \Gamma_{x',y'}\\ &=(\Theta_{\sigma,x}\,a\otimes\Theta_{\sigma,y}\,b)\,\Gamma_{x',y'}=(s(\sigma,a)\otimes s(\sigma,b))\ \boldsymbol{\varphi}_{(V,x'),(V,y')}.\end{aligned}$$

Furthermore, the coherence conditions (8.13) for s trivially follow from the equalities in (8.18) whereas those in (8.11) and (8.12) are, respectively, a direct consequence of the own definition of $h^{\mathcal{G}}(U;x|y|z)$ and $h^{\mathcal{G}}(U;x|_2y)$ in (8.23), since

$$\begin{aligned}s\,\boldsymbol{a}_{(U,x),(U,y),(U,z)}&\overset{(8.25)}{=}s(1_U,h^{\mathcal{G}}(U;x|y|z))=\Theta_{1_U,xyz}h^{\mathcal{G}}(U;x|y|z)\overset{(8.21)}{=}h^{\mathcal{G}}(U;x|y|z),\\ s\,\boldsymbol{c}_{(U,x),(U,y)}&\overset{(8.25)}{=}s(1_U,h^{\mathcal{G}}(U;x|_2y))=\Theta_{1_U,xy}h^{\mathcal{G}}(U;x|_2y)\overset{(8.21)}{=}h^{\mathcal{G}}(U;x|_2y).\end{aligned}$$

If U is any object of C, then every U-object X of $\mathcal{G}$ is U-isomorphic to $s(U,[X])$, and therefore the functor $s:\mathcal{G}^{h^{\mathcal{G}}}\to\mathcal{G}$ is essentially surjective. If $\sigma:V\to U$ is a morphism in C, for any objects (V,x') and (U,x) of $\mathcal{G}^{h^{\mathcal{G}}}$, there is a σ-morphism in $\mathcal{G}^{h^{\mathcal{G}}}$ from (V,x') to (U,x) if and only if $x'=x^\sigma$, that is, if and only if there is a σ-morphism in $\mathcal{G}$ from $s(V,x')$ to $s(U,x)$. Moreover, in such a case, since $\Theta_{\sigma,x}$ is cartesian, the mapping (8.34) establishes a bijection between the set of σ-morphisms in $\mathcal{G}^{h^{\mathcal{G}}}$ from (V,x') to (U,x) and the set of σ-morphisms in $\mathcal{G}$ from $s(V,x')$ to $s(U,x)$. Therefore, the functor s is full and faithful. Hence, by Lemma 8.2, s is a braided C-fibred equivalence. $\square$

9. The special case of presheaves of abelian groups

In this section, we review how our results specialize when we limit our attention to presheaves of abelian groups $\mathrm{C}^{\mathrm{op}} \to \mathsf{Ab}$ (i.e. C^{op}-modules) instead of general presheaves of commutative monoids $\mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$.

Let $\mathcal{B}$ be a C^{op}-module (written multiplicatively). The canonical embedding

$$\mathrm{C}^{\mathrm{op}} \hookrightarrow \mathbb{H}(\mathcal{B}), \quad (U \overset{\sigma}{\leftarrow} V) \mapsto \big((U,e) \xrightarrow{(\sigma,e)} (V,e)\big),$$

is an equivalence of categories, since for any object (U,x) of $\mathbb{H}(\mathcal{B})$, the morphisms $(1_U,x):(U,e)\to(U,x)$ and $(1_U,x^{-1}):(U,x)\to(U,e)$ are mutually inverse of each other. Then, there is an induced equivalence of categories

$$\mathbb{H}(\mathcal{B})\text{-Mod} \simeq \mathrm{C}^{\mathrm{op}}\text{-Mod}, \qquad \mathcal{A} \mapsto \mathcal{A}|_{\mathrm{C}^{\mathrm{op}}}, \tag{9.1}$$

whose quasi-inverse sends a C^{op}-module $\mathcal{A}$ to the $\mathbb{H}(\mathcal{B})$-module, equally denoted by $\mathcal{A}$, such that $\mathcal{A}(U,x)=\mathcal{A}(U)$ and $y_*a=a$ for any $U\in \mathrm{ObC}$, $x,y\in\mathcal{B}(U)$, and $a\in\mathcal{A}(U)$. For any $\mathbb{H}(\mathcal{B})$-module $\mathcal{A}$, the isomorphism of $\mathbb{H}(\mathcal{B})$-modules

$$\mathfrak{F}:\mathcal{A}\cong\mathcal{A}|_{\mathrm{C}^{\mathrm{op}}}$$

is given by the isomorphisms of abelian groups $x_*^{-1}:\mathcal{A}(U,x)\cong\mathcal{A}(U,e)$.

Hence, there is no loss of generality in assuming that the coefficients $\mathcal{A}$ for the rth-level cohomology groups $H^n_s(\mathcal{B},r,\mathcal{A})$ of a C^{op}-module $\mathcal{B}$ are simply C^{op}-modules, regarded as $\mathbb{H}(\mathcal{B})$-modules as above. For these, most of our constructions and results are rewritten in a simpler way and revise many of the previously established facts on the topic by Calvo–Cegarra–Quang in [8] and by Cegarra-Khmaladze in [11] and [12]. Notice that, when we plug a C^{op}-module $\mathcal{A}$ into the complex of cochains $C^\bullet_s(\mathcal{B},r,\mathcal{A})$ of Section 5, we just obtain (up to normalization) the cochain complex shown in [8, §3] to compute the cohomology groups of $\mathcal{B}$ with coefficients in $\mathcal{A}$.

For instance, Theorem 6.2 specializes by giving, for any C^{op}-modules $\mathcal{A},\mathcal{B}$, a natural isomorphism

$$H^2_s(\mathcal{B},2,\mathcal{A})\cong \mathrm{Hom}_{\mathrm{C}^{\mathrm{op}}}(\mathcal{B},\mathcal{A}).$$

The main result in Section 7 particularizes here by giving the cohomological classification of isomorphism classes of *coextensions of* C^{op}*-modules* of $\mathcal{B}$ by $\mathcal{A}$, so that there is a natural isomorphism

$$H^3_s(\mathcal{B},2,\mathcal{A})\cong \mathrm{Ext}_{\mathrm{C}^{\mathrm{op}}}(\mathcal{B},\mathcal{A}).$$

Notice that in a commutative group coextension $\mathcal{E}=(\mathcal{E},\mathfrak{f},+)$ of $\mathcal{B}$ by $\mathcal{A}$, as defined in Section 7, every monoid $\mathcal{E}(U)$ is a group, so that $\mathcal{E}$ is actually a C^{op}-module. In effect, let $u\in\mathcal{E}(U)$ and suppose that $\mathfrak{f}u=x$. Let us choose any $u'\in\mathcal{E}(U)$ such that $\mathfrak{f}u'=x^{-1}$. Since $uu'\in\mathcal{E}(U,e)$, we can write $uu'=a+e$ for some $a\in\mathcal{A}(U)$. Then, by (7.1), $u(-a+u')=-a+a+e=e$ and u is invertible in $\mathcal{E}(U)$. Furthermore, the isomorphisms $\mathfrak{i}:\mathcal{A}(U)\cong\mathcal{E}(U,e)$, $a\mapsto \mathfrak{i}a=a+e$, define an isomorphism of C^{op}-modules $\mathfrak{i}:\mathcal{A}\cong\mathrm{Ker}(\mathfrak{f})$, so that we are in presence of a short exact sequence of C^{op}-modules $0\to\mathcal{A}\xrightarrow{\mathfrak{i}}\mathcal{E}\xrightarrow{\mathfrak{f}}\mathcal{B}\to 1$. And conversely, any such short exact sequence can be regarded as a group coextension, where the simply-transitive actions $+:\mathcal{A}(U)\times\mathcal{E}(U,x)\to\mathcal{E}(U,x)$ are given by $a+u=\mathfrak{i}a\,u$. This way, we see that a group coextension of $\mathcal{B}$ by $\mathcal{A}$ is the same thing as a coextension of $\mathcal{B}$ by $\mathcal{A}$ in the abelian category of C^{op}-modules.

The results in Section 8 specialize here by giving a cohomological classification of *braided* C*-fibred categorical groups and Picard* C*-fibred categories*, that is, braided or symmetric C-fibred categories $\mathcal{G}$ which are locally compact in the sense that, for any object U of C and any U-object X of $\mathcal{G}$, the endofuntor $X\otimes - : \mathcal{G}_U \to \mathcal{G}_U$ is an autoequivalence. We should mention here the work of Breen in [6, 7] (see also Aldrovandi and Noohi [1]), where he offers an excellent discussion about the cohomology classification of symmetric and braided categorical group stacks over an arbitrary site C. However, Breen's results are far from being as explicit as ours in this work in terms of cocycles when C is discrete.

Notice that, if $\mathcal{G}$ is a braided C-fibred categorical group, the presheaf $\Pi_0\mathcal{G} : \mathrm{C}^{\mathrm{op}} \to \mathsf{CMon}$ is actually a presheaf of abelian groups, that is, a C^{op}-module, since for any U-object X of $\mathcal{G}$ there is a U-object X' with a U-morphism $X\otimes X' \to \iota U$, and therefore $[X'] = [X]^{-1}$ in $\Pi_0\mathcal{G}(U)$, that is, $\Pi_0\mathcal{G}(U)$ is an abelian group for any object U of C. By (9.1), the $\mathbb{H}(\Pi_0\mathcal{G})$-module $\Pi_1\mathcal{G}$ is isomorphic to the C^{op}-module $\Pi_1\mathcal{G}|_{\mathrm{C}^{\mathrm{op}}}$, which is simply given by $U \mapsto \mathrm{Aut}_{\mathcal{G}_U}(\iota U)$, and the classifying system $(\Pi_0\mathcal{G}, \Pi_1\mathcal{G}, [h^{\mathcal{G}}])$ becomes isomorphic to $(\Pi_0\mathcal{G}, \Pi_1\mathcal{G}|_{\mathrm{C}^{\mathrm{op}}}, [\mathfrak{F}_* h^{\mathcal{G}}])$. Furthermore, for any C^{op}-modules $\mathcal{A}$, $\mathcal{B}$, and any cocycle $h \in Z^4_s(\mathcal{B},2,\mathcal{A})$, the associated braided C-fibred category $\mathcal{G}^h$ in Proposition 8.6 is easily recognized to be a braided C-fibred categorical group, which is a Picard C-fibred category if $h \in Z^5_s(\mathcal{B},3,\mathcal{A})$. Then, Theorem 8.5 particularizes as follows.

Theorem 9.1 *The mapping* $[\mathcal{G}] \mapsto [\Pi_0\mathcal{G}, \Pi_1\mathcal{G}|_{\mathrm{C}^{\mathrm{op}}}, [\mathfrak{F}_* h^{\mathcal{G}}]]$ *establishes a natural one-to-one correspondence between equivalence classes of braided* C*-fibred categorical groups (resp. Picard* C*-fibred categories) and isomorphism classes of classifying systems* $[\mathcal{B},\mathcal{A},c]$*, where* $\mathcal{A}$ *and* $\mathcal{B}$ *are* C^{op}*-modules and* $c \in H^4_s(\mathcal{B},2,\mathcal{A})$ *(resp.* $c \in H^5_s(\mathcal{B},3,\mathcal{A})$*).*

To finish, let us recall that a Picard C-fibred category $\mathcal{G}$ is termed *strictly commutative* whenever its symmetry constraint satisfies $\boldsymbol{c}_{X,X} = 1_{X\otimes X}$, for any $X \in \mathrm{Ob}\mathcal{G}$. If $\mathcal{B}$ and $\mathcal{A}$ are C^{op}-modules and $h \in Z^5_s(\mathcal{B},3,\mathcal{A})$, then the associated Picard C-fibred category $\mathcal{G}^h$ becomes strictly commutative if and only if $h(U;x|_2x) = 0$, for any object U of C and $x \in \mathcal{B}(U)$. Then, thanks to the universal coefficient exact sequence $0 \to \mathrm{Ext}^2_{\mathrm{C}^{\mathrm{op}}}(\mathcal{B},\mathcal{A}) \to H^5_s(\mathcal{B},3,\mathcal{A}) \to \mathrm{Hom}_{\mathrm{C}^{\mathrm{op}}}(\mathcal{B}/\mathcal{B}^2,\mathcal{A})$, where the last homomorphism carries a cohomology class $[h] \in H^5_s(\mathcal{B},3,\mathcal{A})$ to the homomorphism of C^{op}-modules which, at any object U of C, is the homomorphism $q_h : \mathcal{B}(U)/\mathcal{B}(U)^2 \to \mathcal{A}(U)$ given by $q_h[x] = h(U;x|_2x)$, we see that que $\mathcal{G}^h$ is a strictly commutative Picard C-fibred category if and only if $[h] \in \mathrm{Ext}^2_{\mathrm{C}^{\mathrm{op}}}(\mathcal{B},\mathcal{A})$. Thus, we recover from Theorem 9.1 the well-known result by Deligne [13, Proposition 1.4.15] showing that equivalence classes of strictly commutative Picard C-fibred categories are classified by isomorphism classes of classifying systems $[\mathcal{B},\mathcal{A},c]$, where $\mathcal{A}$ and $\mathcal{B}$ are C^{op}-modules and $c \in \mathrm{Ext}^2_{\mathrm{C}^{\mathrm{op}}}(\mathcal{B},\mathcal{A})$.

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