Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2623 – 2645

doi:10.3906/mat-2108-88

Generating finite Coxeter groups with elements of the same order

Sarah HART[1], **Veronica KELSEY**[2,*], **Peter ROWLEY**[3]
[1]Department of Economics, Mathematics and Statistics, Birkbeck University of London, UK
[2]School of Mathematics and Statistics, University of St Andrews, U.K.
[3]Department of Mathematics, University of Manchester, U.K.

Received: 18.08.2021 • **Accepted/Published Online:** 11.10.2021 • **Final Version:** 29.11.2021

Abstract: Supposing G is a group and k a natural number, $d_k(G)$ is defined to be the minimal number of elements of G of order k which generate G (setting $d_k(G) = 0$ if G has no such generating sets). This paper investigates $d_k(G)$ when G is a finite Coxeter group either of type B_n or D_n, or of exceptional type. Together with the work of Garzoni and Yu, this determines $d_k(G)$ for all finite irreducible Coxeter groups G when $2 \leq k \leq \mathrm{rank}(G)$ ($\mathrm{rank}(G)+1$ when G is of type A_n).

Key words: Coxeter group, generating set, generators, order

1. Introduction

There is a substantial and varied literature devoted to the question of which groups can be generated by two elements. That the symmetric group of degree n can be generated by an n-cycle and a transposition is already documented in Jordan's Traité [6] (though probably known earlier to Cauchy). At the beginning of the 20th century, Miller [8] showed that, except for some small cases, the alternating group of degree n can be generated by an element of order two and an element of order three, and he later obtained similar results for symmetric groups in [9]. A common feature of many of the familiar generating sets for the symmetric group of degree n is that either the number of generators, or the order of the elements, grows as n does. This led Lanier [7] to investigate ways in which symmetric groups and alternating groups can be generated by elements which all have the same order. For a group G and a positive integer k, let $d_k(G)$ denote the smallest number of elements of G of order k which generate G, with the convention that $d_k(G) = 0$ if there are no such generating sets. Also, we denote the symmetric group (respectively the alternating group) of degree n, by $\mathrm{Sym}(n)$ (respectively $\mathrm{Alt}(n)$). In [7], Lanier proved that for $3 \leq k \leq n$, $d_k(\mathrm{Sym}(n)) \leq 3$ if k is even, and $d_k(\mathrm{Alt}(n)) \leq 3$ if k is odd. Further, he showed that if $3 \leq k \leq n-2$, with k even, then $d_k(\mathrm{Alt}(n)) \leq 4$. Earlier results on generating $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ using k-cycles were obtained by Annin and Maglione [1].

Recently, Garzoni [3] extended the results of Lanier to show that, for $3 \leq k \leq n$, with specified exceptions, $d_k(\mathrm{Sym}(n)) = d_k(\mathrm{Alt}(n)) = 2$ when k is even, and $d_k(\mathrm{Alt}(n)) = 2$ when k is odd. We will state the main results from [3] in Section 2 in more detail, as they have a bearing on our main theorem. As is well known, $\mathrm{Sym}(n)$ is the Coxeter group of type A_{n-1}, and so this raises the question of finding $d_k(G)$ for the other finite irreducible Coxeter groups. In fact, the case when $k = 2$ has already been settled by Yu in [10]. Our main

*Correspondence: vk49@st-andrews.ac.uk

result, which we now state, answers this question for Coxeter groups of types B_n and D_n, and $3 \leq k \leq n$.

Theorem 1.1 *Suppose $3 \leq k \leq n$ and G is a Coxeter group of type either B_n or D_n. Then the following hold.*

(i) Suppose k is odd. If $n = 2k-1$ and k is prime, or if $n = 4$ and $k = 3$, then $d_k(G) = 0$ and $d_k(G') = 3$. In all other cases, $d_k(G) = 0$ and $d_k(G') = 2$.

(ii) If k is even, then $d_k(G) = 2$.

We remark that the proof of Theorem 1.1 is constructive, in that specific generating elements of order k are found.

In the rest of this section, we outline the structure of this paper. In Section 2, we begin by setting up the notation that we will use. Elements of groups of types B_n and D_n can be thought of as signed permutations in a way that will be explained in Section 2, where we also give some basic information about these groups that we will need. The proof in [3] that $d_k(\mathrm{Sym}(n)) = 2$ for even k, and that $d_k(\mathrm{Alt}(n)) = 2$ for odd k, is split into several subcases depending on k and n, defining a pair a and b of elements of order k in each case that generate $\mathrm{Sym}(n)$ or $\mathrm{Alt}(n)$. The definitions of these a and b are given at the end of Section 2. The reason for giving this detailed description is that, in many cases, it is possible, by a careful addition of signs to the appropriate a and b, to find signed permutations x and y of order k that generate groups of type B_n and D_n (or their derived subgroups). There is, however, a significant minority of cases in which this is not possible. For example, suppose that k divides n. Then, usually, the given a and b are products of k-cycles. Adding an odd number of minus signs to a k-cycle results in an element of order $2k$, so we cannot do this. However, if we add an even number of minus signs to each cycle, then the resulting elements x and y will themselves contain an even number of minus signs, meaning that we cannot generate the whole of a group of type B_n in this way. Therefore, we are forced to use different elements, which requires us additionally to show, for the elements we use, that the underlying pair of permutations in $\mathrm{Sym}(n)$ do still generate $\mathrm{Sym}(n)$ (or $\mathrm{Alt}(n)$, as appropriate). In particular, it is not possible to use the elements given in [3] in the following cases (where we write $n = tk + j$ with $t \geq 1$ and $0 \leq j < k$):

(i) k even, t even, $k > 4$, $j = 1$;

(ii) k even, t even, j odd, $j \notin \{1, k-1\}$;

(iii) $n \geq 2k$, k even, t odd, $j \neq k-1$;

(iv) $n \geq 2k$, $k \equiv 2 \mod 4$, $j = 2$;

(v) $n = 2k-1$; and

(vi) $n = k$.

When k is odd, $d_k(G) = d_k(G) = 0$. This is because G' has index 4 in G, and index 2 in G^+, and so every element of odd order in G is contained in G'. The proof of Theorem 1.1 is split into two sections. Section 3 deals with the case $n \geq 2k$, and Section 4 with the case $n < 2k$. For completeness, in Section 5, we determine $d_k(G)$ when $3 \leq k \leq n$ and G is an irreducible exceptional finite Coxeter group.

2. Background and basic results

Let $n \in \mathbb{N}$ with $n \geq 3$. We use $W(B_n)$ and $W(D_n)$ to denote, respectively, the Coxeter groups of type B_n and D_n. In our proof of Theorem 1.1, we shall view $W(B_n)$ as the group of signed permutations of the set $\Omega = \{1, 2, \dots, n\}$. That is, setting $\tilde{\Omega} = \{1, 2, \dots, n, -1, -2, \dots, -n\}$, the group $W(B_n)$ is the subgroup of $\mathrm{Sym}(\tilde{\Omega})$ consisting of those permutations σ of $\tilde{\Omega}$ for which $(-\alpha)^\sigma = -\alpha^\sigma$ for all $\alpha \in \Omega$. Thus, we may describe an element of $W(B_n)$ by giving it as a permutation of Ω, equipped with a plus or minus sign above each element of Ω. Therefore, for example, $\sigma = (\overset{-}{1}, \overset{+}{2}, \overset{-}{3}) \in W(B_n)$ is the permutation $1^\sigma = -2$, $2^\sigma = 3$, $3^\sigma = -1$, $(-1)^\sigma = 2$, $(-2)^\sigma = -3$ and $(-3)^\sigma = 1$. We note here that we will write our permutations on the right (of elements of Ω) and so multiply permutations from left to right. For a permutation τ of $\mathrm{Sym}(n)$, we will write $\overset{+}{\tau}$ to mean the signed permutation obtained by adding a plus sign above each number. For example, if $\tau = (1, 2, 3)$, then $\overset{+}{\tau} = (\overset{+}{1}, \overset{+}{2}, \overset{+}{3})$.

For $\sigma \in W(B_n)$, we may express σ as a product of pairwise disjoint signed cycles. A signed cycle is said to be positive if there are an even number of minus signs above the elements of the cycle and negative if there are an odd number of minus signs. For example $(\overset{+}{1}, \overset{+}{3}, \overset{-}{2})$ is a negative 3-cycle and $(\overset{-}{4}, \overset{-}{5})$ is a positive 2-cycle. Now, $W(B_n)$ contains a subgroup of index 2 which consists of all those signed permutations of Ω with an even number of minus signs. This subgroup is isomorphic to $W(D_n)$, and we will use this version of $W(D_n)$ in our later calculations. Thus, for example, $(\overset{-}{1}, \overset{+}{2}, \overset{-}{3})(\overset{-}{4}, \overset{-}{5}) \in W(D_5)$. For an element $\sigma \in W(B_n)$, $\hat{\sigma}$ is the permutation of $\mathrm{Sym}(\Omega) = \mathrm{Sym}(n)$ obtained by removing all the signs from σ. As an example, if $\sigma = (\overset{-}{1}, \overset{+}{2}, \overset{-}{3})(\overset{-}{4}, \overset{-}{5})$, then $\hat{\sigma} = (1, 2, 3)(4, 5)$. We remark that if σ (when expressed in disjoint signed cycle form) contains the cycle $(\overset{\varepsilon_1}{a_1}, \dots, \overset{\varepsilon_m}{a_m})$, where $a_i \in \Omega$ and $\varepsilon_i \in \{+, -\}$, then σ^{-1} contains the cycle $(\overset{\varepsilon_m}{a_1}, \overset{\varepsilon_{m-1}}{a_m}, \dots, \overset{\varepsilon_1}{a_2})$.

Now put $G = W(B_n)$ and let G^+ be all the signed permutations in G with an even number of minus signs. Thus, $G^+ = W(D_n)$. We briefly discuss some of the structural features of G and G^+. Letting N denote the subgroup of G consisting of all σ for which $\hat{\sigma}$ is the identity permutation of Ω, we have that N is a normal elementary abelian subgroup of G of order 2^n. Also let N^+ be the set of elements in N with an even number of minus signs. Then N^+ has order 2^{n-1} and is a normal subgroup of both G and G^+. We have that $G/N \cong \mathrm{Sym}(n) \cong G^+/N^+$. Also G', the derived subgroup of G, has index 4 in G, with $G' \leq G^+$, $G' \cap N = N^+$ and $G'/N^+ \cong \mathrm{Alt}(n)$. Note that $(G^+)' = G'$. Let z be the element of N sending α to $-\alpha$ for all $\alpha \in \Omega$. Then $Z(G) = \langle z \rangle$. Note that $Z(G^+) = \langle z \rangle$ when n is even and $Z(G^+) = 1$ when n is odd. For details on the above, see [4].

Lemma 2.1 *(i) Suppose that $H \leq G$ (respectively $H \leq G'$) and $H \cap N \nsubseteq Z(G) \cup N^+$ (respectively $H \cap N \not\leq Z(G)$). Set $\bar{G} = G/N$ (respectively $\bar{G'} = G'/N^+$). If $\bar{H} = \bar{G}$ (respectively $\bar{H} = \bar{G'}$), then $H = G$ (respectively $H = G'$).*

(ii) Suppose that $H \leq G^+$ with $H \cap N^+ \not\leq Z(G^+)$, and set $\bar{G^+} = G^+/N^+$. If $\bar{H} = \bar{G^+}$, then $H = G^+$.

Proof

(i) We begin by recalling that the only nontrivial proper normal subgroups of G contained in N are $Z(G)$

and N^+. Suppose that $\bar{H} = \bar{G}$. Then $G = HN$. Since $H \cap N$ is normal in H and N is abelian, $H \cap N$ is a normal subgroup of G. Because $H \cap N \not\subseteq Z(G) \cup N^+$ by assumption, $H \cap N = N$, whence $H = G$. A similar argument applies when $H \leq G'$.

(ii) The proof is similar to part (i). □

In order to write down permutations in an efficient way, we introduce the following arrow notation.

Definition 2.2 *Let $\alpha, \beta \in \Omega$ and $r \in \mathbb{N}$, where $\beta = \alpha + rs$. Then*

$$(\ldots, \alpha \nearrow_r \beta, \ldots) = (\ldots, \alpha \nearrow_r \alpha + rs, \ldots)$$

describes the permutation

$$(\ldots, \alpha, \alpha + r, \alpha + 2r, \ldots, \alpha + rs, \ldots).$$

When $r = 1$, we omit the subscript r. We analogously have the notations $\searrow_r$ and $\searrow$.

As an example of the arrow notation, we have $(1, 2 \nearrow_2 6, 7) = (1, 2, 4, 6, 7)$ and $(1, 7 \searrow 3) = (1, 7, 6, 5, 4, 3)$. Later, we shall encounter such cycles as $(1, 2, k+3 \nearrow 2k-1)$, which we take to mean $(1, 2)$ if $k = 3$, whereas, for example, it is $(1, 2, 8, 9)$ if $k = 5$. In other words $\alpha \nearrow \beta$ is null if $\alpha > \beta$.

Following [3], we now define, for $k \geq 3$, elements a, b of $\mathrm{Sym}(n)$ that generate $\mathrm{Alt}(n)$ when k is odd, and $\mathrm{Sym}(n)$ when k is even, except in some small exceptional cases. Specifically, the elements given do not generate these groups when (n, k) is one of $(6, 3), (6, 4), (7, 3), (8, 3)$. Throughout, we require that $3 \leq k \leq n$ and we write $n = tk + j$, where $0 \leq j < k$ and $t \geq 1$. We begin by listing, in Definition 2.3, the different a and b that are used in Theorem 2.5. Then, in Definition 2.4, we will explain which of these different a and b are selected for each value of k and n. We note that the context in which each choice of a and b are used ensures that they are indeed elements of $\mathrm{Sym}(n)$, that they each have order k, and that the expressions given are in disjoint cycle form. For example, the element a_1 given in Definition 2.3 contains the transposition $(tk{+}1, tk{+}2)$, but this is not an issue because, as will be seen in Definition 2.4, it is only chosen for the element a in a case when k is even and $j > 1$.

Definition 2.3 *Suppose k is a positive integer with $3 \leq k \leq n$. We write $n = tk + j$, where $0 \leq j < k$ and $t \geq 1$, and define the following permutations a_0, ..., a_4 of $\mathrm{Sym}(n)$. Note that a_2 is only required when k is even.*

$$a_0 = (1 \nearrow k)(k{+}1 \nearrow 2k) \cdots ((t-1)k{+}1 \nearrow tk);$$

$$a_1 = (1 \nearrow k)(k{+}1 \nearrow 2k) \cdots ((t-1)k{+}1 \nearrow tk)(tk{+}1, tk{+}2);$$

$$a_2 = (1 \nearrow k) \cdots ((t{-}2)k{+}1 \nearrow (t{-}1)k)((t{-}1)k{+}1 \nearrow (t{-}1)k{+}\tfrac{k}{2})((t{-}1)k{+}\tfrac{k}{2}{+}1 \nearrow tk);$$

$$a_3 = (1, 2, 3, 4)(5, 6, 7, 8) \cdots (4t{-}7, 4t{-}6, 4t{-}5, 4t{-}4)(4t{-}3, 4t{-}2)(4t{-}1, 4t);$$

$$a_4 = (1 \nearrow k).$$

Next, we define permutations b_0, $\ldots$, b_7 of $\mathrm{Sym}(n)$*. Note that b_4 is only used when k is even.*

$$b_0 = (k \nearrow 2k-1)\cdots((t-1)k \nearrow tk-1)(tk \nearrow n);$$

$$b_1 = (k-2, k-1)(k \nearrow 2k-1)\cdots((t-1)k \nearrow tk-1)(tk \nearrow n);$$

$$b_2 = (k, k+2, k+1, k+3 \nearrow 2k-1)(2k \nearrow 3k-1)\cdots((t-1)k \nearrow tk-1)\cdot \cdot (tk \nearrow n, 1 \nearrow k-j-1);$$

$$b_3 = (k-2, k-1)(k, k+2, k+1, k+3 \nearrow 2k-1)(2k \nearrow 3k-1)\cdots((t-1)k \nearrow tk-1)\cdot \cdot (tk \nearrow n, 1 \nearrow k-j-1);$$

$$b_4 = (k, k+2, k+1, k+3 \nearrow 2k-1)(2k \nearrow 3k-1)\cdots((t-1)k \nearrow tk-1)\cdot \cdot (tk \nearrow n, 1 \nearrow \tfrac{k}{2}-j-1)(\tfrac{k}{2}-j \nearrow k-j-1);$$

$$b_5 = (1,2)(4,5,6,7)(8,9,10,11)\cdots(4t-4, 4t-3, 4t-2, 4t-1)(n-1, n);$$

$$b_6 = (1 \nearrow k-j, k+1 \nearrow n);$$

$$b_7 = (1, 3 \nearrow n, 2).$$

For each of the possible configurations of n, k, t, and j, we next assign the relevant a and b from the list given in Definition 2.3.

Definition 2.4 *Suppose k is a positive integer with $3 \le k \le n$. We write $n = tk + j$, where $0 \le j < k$, and $t \ge 1$. If $n \ge 2k$, then a and b are as shown in Table 1. If $k < n < 2k$, then set $a = a_4$ and $b = b_6$. Finally, if $n = k$, then set $a = a_4$ and $b = b_7$.*

Table 1. Choice of a and b when $n \ge 2k$

				a	b
$j = k-1$	k odd			a_0	b_0
	k even	t odd		a_0	b_0
		t even		a_1	b_1
$j \neq k-1$	k odd			a_0	b_2
	k even	t odd		a_0	b_2
		t even	$j \notin \{0,1\}$	a_1	b_3
			$j = 0$	a_2	b_4
			$j = 1$, $k \neq 4$	a_2	b_4
			$j = 1$, $k = 4$	a_3	b_5

We note that, in all cases, it is quick to check that a and b each have order k.

Theorem 2.5 (see [3]) *Let k be a positive integer with $3 \le k \le n$.*

(i) Suppose k is odd. If $k = 3$ and $n \in \{6,7,8\}$, then $d_3(\mathrm{Alt}(n)) = 2$. In all other cases, if a and b are as given in Definition 2.4, then $\langle a, b\rangle = \mathrm{Alt}(n)$. Hence, $d_k(\mathrm{Alt}(n)) = 2$ (unless $n = k = 3$ in which case $d_3(\mathrm{Alt}(3)) = 1$).

(ii) Suppose k is even. If $k=4$ and $n=6$, then $d_4(\mathrm{Sym}(n))=2$. In all other cases, if a and b are as given in Definition 2.4, then $\langle a,b\rangle=\mathrm{Sym}(n)$. Hence, $d_k(\mathrm{Sym}(n))=2$.

We remark that Garzoni [3] additionally proves that $d_k(\mathrm{Alt}(n))=2$ whenever k is even and $4\le k\le n$.

3. The case $n\ge 2k$

In this section, and in Section 4, $G=W(B_n)$ and $G^+=W(D_n)$ where $n\ge 3$, and $3\le k\le n$. Also, we write $n=kt+j$ where $0\le j<k$. We first examine the case $k=3$.

Lemma 3.1 *Suppose $k=3$ and $n\ge 6$. Then $d_3(G')=2$ and $d_3(G)=d_3(G^+)=0$.*

Proof When $6\le n\le 11$, it can easily be checked using MAGMA [2] that G' can be generated by two elements of order 3. Hence, we now assume that $n\ge 12$, so we have $n=3t+j$ for some $t\ge 4$ and $0\le j<3$. Suppose first that $n=3t$, and define x and y as follows.

$$x=(\overset{+}{1},\overset{+}{2},\overset{+}{3})(\overset{+}{4},\overset{+}{5},\overset{+}{6})\cdots(\overset{+}{3t-5},\overset{+}{3t-4},\overset{+}{3t-3})(\overset{-}{3t-2},\overset{+}{3t-1},\overset{-}{3t});$$

$$y=(\overset{+}{3},\overset{+}{5},\overset{+}{4})(\overset{+}{6},\overset{+}{7},\overset{+}{8})\cdots(\overset{+}{3t-3},\overset{+}{3t-2},\overset{+}{3t-1})(\overset{+}{3t},\overset{+}{1},\overset{+}{2}).$$

Note that $x,y\in G'$. Then $\hat{x}=a_0$ and $\hat{y}=b_2$, following the notation of Definition 2.3. Hence, $\langle\hat{x},\hat{y}\rangle=\mathrm{Alt}(n)$, by Theorem 2.5. That is, writing $H=\langle x,y\rangle$, we have $\bar{H}=\bar{G}'\cong\mathrm{Alt}(n)$. We now use Lemma 2.1 to show that $H=G'$. Calculation gives

$$x^{-1}=(\overset{+}{1},\overset{+}{3},\overset{+}{2})(\overset{+}{4},\overset{+}{6},\overset{+}{5})\cdots(\overset{+}{3t-5},\overset{+}{3t-3},\overset{+}{3t-4})(\overset{-}{3t-2},\overset{+}{3t},\overset{-}{3t-1})\text{ and}$$

$$x^{-1}y=(\overset{+}{1},\overset{+}{5},\overset{+}{3},\overset{+}{3t}\ {}_{3}{\searrow}\ \overset{+}{6},\overset{+}{4}\nearrow_{3}\overset{+}{3t-5},\overset{-}{3t-2})(\overset{-}{3t-1}).$$

Thus, $x^{-1}y$ consists of one negative cycle of length $2t+1$, one negative 1-cycle, and $t-2$ fixed points. Since $t\ge 4$, this means there are at least two fixed points. Hence, $(x^{-1}y)^{2t+1}\in H\cap N^+\setminus Z(G)$. Therefore, by Lemma 2.1, we have $\langle x,y\rangle=G'$.

Next, suppose $n=3t+1$. We keep the same x as for the case $n=3t$, but now choose

$$y=(\overset{+}{3},\overset{+}{5},\overset{+}{4})(\overset{+}{6},\overset{+}{7},\overset{+}{8})\cdots(\overset{+}{3t-3},\overset{+}{3t-2},\overset{+}{3t-1})(\overset{+}{3t},\overset{+}{3t+1},\overset{+}{1}).$$

As in the case $n=3t$, we have $\hat{x}=a_0$ and $\hat{y}=b_2$. Thus, by Theorem 2.5, $\bar{H}=\langle\hat{x},\hat{y}\rangle=\bar{G}'$. Now,

$$x^{-1}y=(\overset{+}{1},\overset{+}{5},\overset{+}{3},\overset{+}{2},\overset{+}{3t}\ {}_{3}{\searrow}\ \overset{+}{6},\overset{+}{4}\nearrow_{3}\overset{+}{3t-5},\overset{-}{3t-2},\overset{+}{3t+1})(\overset{-}{3t-1}).$$

This time we have a negative cycle of length $2t+3$, a negative 1-cycle, and $t-3$ fixed points. Therefore, since $t>3$, we see that $(x^{-1}y)^{2t+3}\in H\cap N^+\setminus Z(G)$, and hence again $H=G'$.

Finally, when $n=3t+2$, we set

$$x=(\overset{+}{1},\overset{+}{2},\overset{+}{3})(\overset{-}{4},\overset{+}{5},\overset{-}{6})(\overset{+}{7},\overset{+}{8},\overset{+}{9})\cdots(\overset{+}{3t-2},\overset{+}{3t-1},\overset{+}{3t})\text{ and}$$

$$y=(\overset{+}{3},\overset{+}{4},\overset{+}{5})\cdots(\overset{+}{3t-3},\overset{+}{3t-2},\overset{+}{3t-1})(\overset{+}{3t},\overset{+}{3t+1},\overset{+}{3t+2}).$$

Here, because we are in the '$j = k-1$' case of Table 1, the required elements to generate $\mathrm{Alt}(n)$ are a_0 and b_0. Since $\hat{x} = a_0$ and $\hat{y} = b_0$, Theorem 2.5 implies that $\bar{H} = \langle \hat{x}, \hat{y} \rangle = \bar{G'}$. Now,

$$x^{-1} = (\overset{+}{1}, \overset{+}{3}, \overset{+}{2})(\overset{-}{4}, \overset{+}{6}, \overset{-}{5}) \cdots (\overset{+}{3t-5}, \overset{+}{3t-3}, \overset{+}{3t-4})(\overset{+}{3t-2}, \overset{+}{3t}, \overset{+}{3t-1}) \text{ and}$$

$$x^{-1}y = (\overset{+}{1}, \overset{-}{4}, \overset{+}{7} \nearrow_{3}, \overset{+}{3t+1}, \overset{+}{3t+2}, \overset{+}{3t} \searrow_{3} \overset{+}{3}, \overset{+}{2})(\overset{-}{5}).$$

In this case, $x^{-1}y$ has a negative $(2t+3)$-cycle, a negative 1-cycle, and $t-2$ fixed points. Thus, $(x^{-1}y)^{2t+3}$ consists of $2t+4$ negative 1-cycles and $t-2$ fixed points. Therefore, since $t > 3$, we have $(x^{-1}y)^{2t+3} \in H \cap N^+ \setminus Z(G)$, and hence again $H = G'$, thus completing the proof of Lemma 3.1. $\square$

Lemma 3.2 *Suppose $k > 3$ and $n \geq 2k$, with $j = k-1$. If k is odd, then $d_k(G) = d_k(G^+) = 0$, but $d_k(G') = 2$. If k is even and t is odd, then $d_k(G) = d_k(G^+) = 2$.*

Proof We set

$$x = (\overset{+}{1} \nearrow \overset{+}{k})(\overset{-}{k+1}, \overset{+}{k+2} \nearrow \overset{+}{2k-1}, \overset{-}{2k})(\overset{+}{2k+1} \nearrow \overset{+}{3k}) \cdots (\overset{+}{(t-1)k+1} \nearrow \overset{+}{tk}) \text{ and}$$

$$y = (\overset{+}{k} \nearrow \overset{+}{2k-1}) \cdots (\overset{+}{tk} \nearrow \overset{+}{n}).$$

Now $\hat{x} = a_0$ and $\hat{y} = b_0$, as in Table 1. Writing $H = \langle x, y \rangle$, we therefore have that $\bar{H} = \mathrm{Alt}(n)$ when k is odd, and $\bar{H} = \mathrm{Sym}(n)$ when k is even and t is odd by Theorem 2.5. We note also that $x, y \in G^+$. Now

$$x^{-1} = (\overset{+}{k} \searrow \overset{+}{1})(\overset{+}{2k} \searrow \overset{+}{k+3}, \overset{-}{k+2}, \overset{-}{k+1})(\overset{+}{3k} \searrow \overset{+}{2k+1}) \cdots (\overset{+}{tk} \searrow \overset{+}{(t-1)k+1}) \text{ and}$$

$$x^{-1}y = (\overset{+}{1}, \overset{-}{k+1}, \overset{+}{2k+1} \nearrow_{k} \overset{+}{tk+1}, \overset{+}{tk+2} \nearrow \overset{+}{n}, \overset{+}{tk} \searrow_{k} \overset{+}{k}, \overset{+}{k-1} \searrow \overset{+}{2})(\overset{-}{k+2}).$$

Observe that $x^{-1}y$ has a negative cycle of length $2t+2k-3$, a negative 1-cycle, and at least one fixed point, namely $k+3$. Thus, $h = (x^{-1}y)^{2(t+k)-3}$ has $2(t+k-1)$ negative 1-cycles and $n - 2(t+k-1)$ fixed points. This means that $h \in (H \cap N^+) \setminus Z(G)$. Therefore, by Lemma 2.1, $H = G'$ when k is odd, and $H = G^+$ when k is even and t is odd.

It remains to show that, when k is even and t is odd, we have $d_k(G) = 2$. Let $x_e = x(\overset{-}{tk+1})$ and set $H_e = \langle x_e, y \rangle$. Then $\hat{x}_e = \hat{x} = a_0$, so $\bar{H}_e = \langle \hat{x}_e, \hat{y} \rangle = \mathrm{Sym}(n)$. We have $x_e^{-1} = x^{-1}(\overset{-}{tk+1})$, and thus

$$x_e^{-1}y = (\overset{+}{1}, \overset{-}{k+1}, \overset{+}{2k+1} \nearrow_{k} \overset{+}{(t-1)k+1}, \overset{-}{tk+1}, \overset{+}{tk+2} \nearrow \overset{+}{n}, \overset{+}{tk} \searrow_{k} \overset{+}{k}, \overset{+}{k-1} \searrow \overset{+}{2})(\overset{-}{k+2}).$$

This time, $x_e^{-1}y$ has a positive cycle of length $2(t+k)-3$, a negative 1-cycle, and a number of fixed points (possibly zero). Therefore, $h = (x_e^{-1}y)^{2(t+k)-3}$ consists of exactly one negative 1-cycle and $n-1$ fixed points. Hence, $h \in (H_e \cap N) \setminus (N^+ \cup Z(G))$. Therefore, by Lemma 2.1, $H_e = G$. $\square$

Lemma 3.3 *Suppose k is even and $n \geq 2k$. If t is even and $j = k-1$, then $d_k(G) = d_k(G^+) = 2$.*

Proof Let a_1 and b_1 be as in Table 1. Then $\langle a_1, b_1\rangle = \mathrm{Sym}(n)$, by Theorem 2.5. Let $x = \overset{+}{a}_1$, $y = (\overset{-}{1})\overset{+}{b}_1$, and $y_e = (\overset{-}{k-2})(\overset{-}{k-1})\overset{+}{b}_1$, setting $H = \langle x, y\rangle$ and $H_e = \langle x, y_e\rangle$. Note that $x, y_e \in G^+$, whereas $y \in G \setminus G^+$. For example,

$$x = (\overset{+}{1} \nearrow \overset{+}{k})(\overset{+}{k+1} \nearrow \overset{+}{2k}) \dots (\overset{+}{(t-1)k+1} \nearrow \overset{+}{tk})(\overset{+}{tk+1}, \overset{+}{tk+2}) \text{ and}$$

$$y = (\overset{-}{1})(\overset{+}{k-2}, \overset{+}{k-1})(\overset{+}{k} \nearrow \overset{+}{2k-1})(\overset{+}{2k} \nearrow \overset{+}{3k-1}) \dots (\overset{+}{tk} \nearrow \overset{+}{n}),$$

so that

$$xy^{-1} = (\overset{+}{1} \nearrow \overset{+}{k-3}, \overset{+}{k-1} \nearrow_k \overset{+}{tk-1}, \overset{+}{n} \searrow \overset{+}{tk+2}, \overset{+}{tk} \searrow_k \overset{+}{2k}, \overset{-}{k}).$$

Therefore, xy^{-1} is a negative cycle of length $2(t+k)-5$. The number of fixed points is

$$n - (2(t+k)-5) = kt + (k-1) - 2t - 2k + 5 = (k-2)(t-1) + 2 \geq 4.$$

Hence, $(xy^{-1}) \in (H \cap N) \setminus (N^+ \cup Z(G))$. Therefore, $H = G$. On the other hand,

$$xy_e^{-1} = (\overset{+}{1} \nearrow \overset{+}{k-4}, \overset{-}{k-3}, \overset{+}{k-1} \nearrow_k \overset{+}{tk-1}, \overset{+}{n} \searrow \overset{+}{tk+2}, \overset{+}{tk} \searrow_k \overset{+}{k})(\overset{-}{k-2}),$$

an element consisting of a negative cycle of length $2(t+k)-5$, one negative 1-cycle, and at least three fixed points. Consequently, $(xy_e^{-1})^{(2(t+k)-5)} \in (H_e \cap N^+) \setminus Z(G)$. Therefore, $H_e = G^+$. Thus, $d_k(G^+) = 2$. □

Lemma 3.4 *Suppose $k = 4$, t is even, $j = 1$ and $n \geq 2k$. Then $d_4(G) = d_4(G^+) = 2$.*

Proof We have $n = 4t+1$ for some even t. From Definition 2.4 and Theorem 2.5, if we have elements x, y of G such that $\hat{x} = a_3$ and $\hat{y} = b_5$, and setting $H = \langle x, y\rangle$ as usual, then $\bar{H} = \mathrm{Sym}(n)$. With this in mind, let

$$x = (\overset{+}{1}, \overset{+}{2}, \overset{+}{3}, \overset{+}{4})(\overset{+}{5}, \overset{+}{6}, \overset{+}{7}, \overset{+}{8}) \cdots (\overset{+}{4t-7}, \overset{+}{4t-6}, \overset{+}{4t-5}, \overset{+}{4t-4})(\overset{+}{4t-3}, \overset{+}{4t-2})(\overset{+}{4t-1}, \overset{+}{4t}) \text{ and}$$

$$y = (\overset{\varepsilon}{1}, \overset{-}{2})(\overset{+}{4}, \overset{+}{5}, \overset{+}{6}, \overset{+}{7})(\overset{+}{8}, \overset{+}{9}, \overset{+}{10}, \overset{+}{11}) \cdots (\overset{+}{4t-4}, \overset{+}{4t-3}, \overset{+}{4t-2}, \overset{+}{4t-1})(\overset{+}{4t}, \overset{+}{4t+1}),$$

where $\varepsilon \in \{+, -\}$. Then $\bar{H} = \mathrm{Sym}(n)$. Moreover,

$$x^{-1} = (\overset{+}{4}, \overset{+}{3}, \overset{+}{2}, \overset{+}{1})(\overset{+}{8}, \overset{+}{7}, \overset{+}{6}, \overset{+}{5}) \cdots (\overset{+}{4t-4}, \overset{+}{4t-5}, \overset{+}{4t-6}, \overset{+}{4t-7})(\overset{+}{4t-3}, \overset{+}{4t-2})(\overset{+}{4t-1}, \overset{+}{4t}) \text{ and}$$

$$x^{-1}y = (\overset{+}{1} \nearrow_4 \overset{+}{4t-3}, \overset{+}{4t-1}, \overset{+}{4t+1}, \overset{+}{4t} \searrow_4 \overset{+}{4}, \overset{-}{3})(\overset{\varepsilon}{2}).$$

If $\varepsilon = +$, then $(x^{-1}y)^{2t+3}$ consists of $2t+3$ negative 1-cycles (an odd number) and $2t-2$ fixed points. Hence, $(x^{-1}y)^{2t+3} \in (H \cap N) \setminus (N^+ \cup Z(G))$. Therefore, by Lemma 2.1, $H = G$. On the other hand, if $\varepsilon = -$, then x and y are elements of G^+, and $(x^{-1}y)^{2t+3}$ consists of $2t+4$ negative 1-cycles and $2t-3$ fixed points. Thus, $(x^{-1}y)^{2t+3} \in H \cap N^+ \setminus Z(G^+)$, which implies that $H = G^+$. □

Lemma 3.5 *Suppose k and t are even, with $k > 4$, and let $n = kt+1$, with $n \geq 2k$. Then the elements a and b given by*

$$a = (1 \nearrow k) \cdots ((t-2)k+1 \nearrow (t-1)k)((t-1)k+1 \nearrow (t-1)k+\tfrac{k}{2})((t-1)k+\tfrac{k}{2}+1 \nearrow tk),$$

$$b = (k, k+2, k+1, k+3 \nearrow 2k-1)(2k \nearrow 3k-1) \cdots ((t-1)k \nearrow tk-1)\cdot$$

$$\cdot\, (tk, n, 2 \nearrow \tfrac{k}{2}-1)(\tfrac{k}{2} \nearrow k-1)$$

generate $\mathrm{Sym}(n)$.

Proof Let $H = \langle a, b\rangle$. Clearly, H is a transitive group. We claim that H is a primitive group. Assume, by way of a contradiction, that H preserves a nontrivial block Δ containing n. Then $\Delta^a = \Delta$, because n is a fixed point of a. For any other point $\lambda \in \Delta$, $\lambda^{\langle a\rangle}$ is therefore contained in Δ. If λ is in the first cycle of a, then $2 \in \Delta$. Hence, $\Delta^b = \Delta$ since $n^b = 2$. Thus, $\Delta^H = \Delta$, contradicting the transitivity of H. If λ is in the last cycle of a, then Δ contains tk. Hence, $\Delta^b = \Delta$, because $(tk)^b = n$. Thus, $\Delta^H = \Delta$, again contradicting the transitivity of H. Therefore, λ must be in a cycle of a other than the first and last cycle, meaning that Δ contains $ik+1$ and $ik+2$ for some i with $2 \leq i \leq t-1$. Since $(ik+1)^b = ik+2$, it follows that $\Delta^b = \Delta$. Thus, $\Delta^H = \Delta$, another contradiction. Hence, $\Delta \subseteq \Omega\backslash\mathrm{supp}(a)$, which means that $|\Delta| = 1$, contradicting the fact that Δ is a nontrivial block. Therefore, H is primitive.

Observe that $a^{-1}b$ is the product of a $(t+5)$-cycle and a $(t+1)$-cycle. Specifically,

$$a^{-1}b = (1, k+2, k+3, k+1 \nearrow_k (t-1)k+1, (t-1)k+\tfrac{k}{2}+1, n, 2)(\tfrac{k}{2}, tk \searrow_k k).$$

Therefore, $(a^{-1}b)^{t+5}$ is a $(t+1)$-cycle, hence, by [5], $\mathrm{Alt}(n) \leq H$. Finally, because $a \in \mathrm{Sym}(n)\backslash\mathrm{Alt}(n)$, we find $H = \mathrm{Sym}(n)$. □

Lemma 3.6 *Suppose k and t are even with $n \geq 2k$. If $k \geq 6$ and $j = 1$, or if $k \geq 4$ and $j = 0$, then* $d_k(G) = d_k(G^+) = 2$.

Proof Suppose first that $k \geq 4$ and $j = 0$. Let

$$x = (\overset{+}{1} \nearrow \overset{+}{k}) \cdots (\overset{+}{(t-2)k+1} \nearrow \overset{+}{(t-1)k})(\overset{+}{(t-1)k+1} \nearrow \overset{+}{(t-1)k+\tfrac{k}{2}})(\overset{+}{(t-1)k+\tfrac{k}{2}+1} \nearrow \overset{+}{tk}),$$

$$y = (\overset{+}{k}, \overset{+}{k+2}, \overset{+}{k+1}, \overset{+}{k+3} \nearrow \overset{+}{2k-1})(\overset{+}{2k} \nearrow \overset{+}{3k-1}) \cdots (\overset{+}{(t-1)k} \nearrow \overset{+}{n-1})\cdot$$

$$\cdot\, (\overset{-}{tk}, \overset{+}{1} \nearrow \overset{+}{\tfrac{k}{2}-1})(\overset{+}{\tfrac{k}{2}} \nearrow \overset{+}{k-2}, \overset{\varepsilon}{k-1}),$$

where $\varepsilon \in \{+,-\}$, and set $H = \langle x, y\rangle$. Then $\hat{x} = a_2$, $\hat{y} = b_4$, and so $\bar{H} = \mathrm{Sym}(n)$ by Theorem 2.5. When either $t > 2$ or $k > 4$, we have

$$x^{-1} = (\overset{+}{k} \searrow \overset{+}{1}) \cdots (\overset{+}{(t-1)k} \searrow \overset{+}{(t-2)k+1})(\overset{+}{(t-1)k+\tfrac{k}{2}} \searrow \overset{+}{(t-1)k+1})\cdot$$

$$\cdot\, (\overset{+}{tk} \searrow \overset{+}{(t-1)k+\tfrac{k}{2}+1}) \text{ and}$$

$$x^{-1}y = (\overset{+}{1}, \overset{+}{k+2}, \overset{+}{k+3}, \overset{+}{k+1} \nearrow_k \overset{+}{(t-1)k+1}, \overset{-}{(t-1)k+\tfrac{k}{2}+1})(\overset{+}{\tfrac{k}{2}}, \overset{+}{n} \searrow_k \overset{+}{2k}, \overset{\varepsilon}{k}).$$

Thus, since $t+3$ and $t+1$ are odd, and hence coprime, we see that $(x^{-1}y)^{(t+3)(t+1)}$ consists of $t+3$ negative 1-cycles and $t(k-1)-3$ fixed points when $\varepsilon = +$, and of $2(t+2)$ negative 1-cycles and $2t-3$ fixed points when $\varepsilon = -$. Meanwhile, if $k=4$ and $t=2$, we have

$$x^{-1} = (\overset{+}{4}, \overset{+}{3}, \overset{+}{2}, \overset{+}{1})(\overset{+}{5}, \overset{+}{6})(\overset{+}{7}, \overset{+}{8}),$$

$$y = (\overset{+}{4}, \overset{+}{6}, \overset{+}{5}, \overset{+}{7})(\overset{-}{8}, \overset{+}{1})(\overset{+}{2}, \overset{\varepsilon}{3}) \text{ and}$$

$$x^{-1}y = (\overset{+}{1}, \overset{+}{6}, \overset{-}{7})(\overset{+}{2}, \overset{+}{8}, \overset{\varepsilon}{4}).$$

Hence, $(x^{-1}y)^3$ consists of three negative 1-cycles and five fixed points when $\varepsilon = +$, and of six negative 1-cycles and two fixed points when $\varepsilon = -$. Therefore, when $\varepsilon = +$ we have $(H \cap N) \not\subseteq N^+ \cup Z(G)$, meaning $H = G$. On the other hand, when $\varepsilon = -$ we have $H \leq G^+$ and $H \cap N^+ \not\leq Z(G^+)$, meaning $H = G^+$. Thus, $d_k(G) = d_k(G^+) = 2$.

Next, we consider the case $k \geq 6$ and $j = 1$. We use the same x as the case $j = 0$, but this time replace y with y_e, where

$$y_e = (\overset{+}{k}, \overset{+}{k+2}, \overset{+}{k+1}, \overset{+}{k+3} \nearrow \overset{+}{2k-1})(\overset{+}{2k} \nearrow \overset{+}{3k-1}) \cdots (\overset{+}{(t-1)k} \nearrow \overset{+}{tk-1}) \cdot$$
$$\cdot (\overset{-}{tk}, \overset{+}{n}, \overset{+}{2} \nearrow \overset{+}{\tfrac{k}{2}-1})(\overset{+}{\tfrac{k}{2}} \nearrow \overset{+}{k-2}, \overset{\varepsilon}{k-1}),$$

and let $H = \langle x, y_e \rangle$. Since $\hat{x} = a$ and $\hat{y}_e = b$, where a and b are the elements given in Lemma 3.5, we see that $\bar{H} = \mathrm{Sym}(n)$. Moreover,

$$x^{-1}y_e = (\overset{+}{1}, \overset{+}{k+2}, \overset{+}{k+3}, \overset{+}{k+1} \nearrow_k \overset{+}{(t-1)k+1}, \overset{-}{(t-1)k+\tfrac{k}{2}+1}, \overset{+}{n}, \overset{+}{2})(\overset{+}{\tfrac{k}{2}}, \overset{+}{tk} \searrow_k \overset{+}{2k}, \overset{\varepsilon}{k}).$$

When $\varepsilon = +$, we see that $x^{-1}y_e$ has a negative $(t{+}5)$-cycle and a positive $(t{+}1)$-cycle. Thus, $(x^{-1}y_e)^{(t+5)(t+1)} \in (H \cap N) \setminus (N^+ \cup Z(G))$, meaning that, once more, $H = G$. If $\varepsilon = -$, then $H \leq G^+$, and $(x^{-1}y_e)^{(t+5)(t+1)} \in (H \cap N^+) \setminus Z(G)$, which implies $H = G^+$. Thus, $d_k(G) = d_k(G^+) = 2$. $\square$

Lemma 3.7 *Suppose $n \geq 2k$, that $k \equiv 2 \mod 4$, and that $j = 2$. Then the following elements a and b generate* $\mathrm{Sym}(n)$.

$$a = (1,2)(\tfrac{k}{2} \nearrow k{-}1)(k{+}1 \nearrow 2k) \cdots ((t{-}1)k{+}1 \nearrow tk)(tk{+}1, tk{+}2)$$
$$b = (k{-}2, k{-}1)(k \nearrow 2k{-}1) \cdots ((t{-}1)k \nearrow tk{-}1)(tk, tk{+}1, tk{+}2, 1 \nearrow k{-}3)$$

Proof Let $H = \langle a, b \rangle$. Since $\frac{k}{2} \leq k-3$ it follows that H is a transitive group. We now show that H is primitive. Assume, by way of a contradiction, that H preserves a nontrivial block Δ containing k. Then $\Delta^a = \Delta$. For any $\lambda \in \Delta$, $\lambda^{\langle a \rangle} \subseteq \Delta$. Note that each cycle (other than fixed points) of a contains at least one point μ with the property that $\mu^a = \mu^b$. Therefore, if λ is not a fixed point of a, then there is some μ in $\lambda^{\langle a \rangle}$ such that μ^b is also contained in $\lambda^{\langle a \rangle}$. Therefore, $\Delta^b = \Delta$ and so $\Delta^H = \Delta$, contradicting the transitivity of H. Hence, $\Delta \subseteq \Omega \backslash \mathrm{supp}(a) = \{3, \ldots, \frac{k}{2}-1, k\}$ and so $|\Delta| \leq \frac{k}{2}-2$. If $k = 6$, this implies $|\Delta| = 1$, contradicting the fact that Δ is a nontrivial block. So assume that $k > 6$. If $\lambda \in \{3, \ldots, \frac{k}{2}-2\}$, then Δ^b

contains $\lambda^b = \lambda + 1$, which is fixed by a, meaning that $(\Delta^b)^a = \Delta^b$. However, $k+1 = k^b \in \Delta^b$. Hence, Δ^b contains $(k+1)^{\langle a \rangle} = \{k+1, \ldots, 2k\}$. Thus, $|\Delta^b| \geq k+1$, contradicting the fact that $|\Delta| \leq \frac{k}{2} - 2$. Therefore, $\Delta = \{k, \frac{k}{2} - 1\}$ and so $\Delta^{b^{-1}} = \{2k-1, \frac{k}{2} - 2\}$ and $(\Delta^{b^{-1}})^a = \{2k, \frac{k}{2} - 2\}$, another contradiction. Consequently, H is a primitive group.

Consider

$$a^{-1}b = (1, 3 \nearrow \tfrac{k}{2}, k-2, tk \searrow_{k} k, k+1 \nearrow_{k} tk+1)$$

which is a single cycle of length $2t + \frac{k}{2}$. Hence, $a^{-1}b$ fixes $kt + 2 - 2t - \frac{k}{2} = (k-2)(t - \frac{1}{2}) + 1$ points. Since $t \geq 2$ and $k \geq 6$, the cycle $a^{-1}b$ fixes at least seven points. Thus, by [5], $\mathrm{Alt}(n) \leq H$. Finally, as $a \in \mathrm{Sym}(n) \backslash \mathrm{Alt}(n)$, we conclude that $H = \mathrm{Sym}(n)$. ☐

Lemma 3.8 *Suppose $n \geq 2k$, that k and t are even, and that $j \notin \{0, 1, k-1\}$. Then $d_k(G) = d_k(G^+) = 2$.*

Proof Suppose $n \geq 2k$, k and t are even, and $j \notin \{0, 1, k-1\}$. Then the appropriate elements in Table 1 that generate $\mathrm{Sym}(n)$ are a_1 and b_3. We have that

$$\overset{+}{a}{}_1^{-1}\overset{+}{b}_3 = (\overset{+}{1}, \overset{+}{k+2}, \overset{+}{k+3}, \overset{+}{k+1} \nearrow_{k} \overset{+}{tk+1}, \overset{+}{tk+3} \nearrow \overset{+}{n})(\overset{+}{tk} \searrow_{k} \overset{+}{k}, \overset{+}{k-2} \searrow \overset{+}{k-j}).$$

Note that if $j = 2$, this reduces to $\overset{+}{a}{}_1^{-1}\overset{+}{b}_3 = (\overset{+}{1}, \overset{+}{k+2}, \overset{+}{k+3}, \overset{+}{k+1} \nearrow_{k} \overset{+}{tk+1})(\overset{+}{tk} \searrow_{k} \overset{+}{k}, \overset{+}{k-2})$. Let $x_1 = \overset{+}{a}_1(\overset{-}{k})(\overset{-}{1})$ and $y_1 = \overset{+}{b}_3$. Both x_1 and y_1 are then elements of G^+, and

$$\begin{aligned} x_1^{-1}y_1 &= (\overset{-}{k})(\overset{-}{1})\overset{+}{a}{}_1^{-1}\overset{+}{b}_3 \\ &= (\overset{-}{1}, \overset{+}{k+2}, \overset{+}{k+3}, \overset{+}{k+1} \nearrow_{k} \overset{+}{tk+1}, \overset{+}{tk+3} \nearrow \overset{+}{n})(\overset{+}{tk} \searrow_{k} \overset{+}{2k}, \overset{-}{k}, \overset{+}{k-2} \searrow \overset{+}{k-j}). \end{aligned}$$

Thus, $x_1^{-1}y_1$ is a product of a negative $(t+j+1)$-cycle and a negative $(t+j-1)$-cycle. Hence, $h_1 = (x_1^{-1}y_1)^{(t+j+1)(t+j-1)}$ consists of $2(t+j)$ negative 1-cycles and $n - 2(t+j)$ fixed points. Now $n = tk + j$, and we also know that $k \geq 4$ and $t \geq 2$. So

$$n - 2(t+j) = (k-2)t - j \geq 2k - 4 - j \geq k - j > 0.$$

Therefore, $h_1 \in N^+ \setminus Z(G)$, and so $\langle x_1, y_1 \rangle = G^+$, whence $d_k(G^+) = 2$.

To show that $d_k(G) = 2$ requires some more work. If j is even and $j > 2$, then let $x_2 = \overset{+}{a}_1(\overset{-}{n})$. Then, $x_2^{-1}y_1 = (\overset{-}{n})\overset{+}{a}{}_1^{-1}\overset{+}{b}_3$. On the other hand, if $j = 2$ and $k \equiv 0 \mod 4$, then set $x_2 = \overset{+}{a}_1(\overset{-}{tk+2})$. Then $x_2^{-1}y_1 = (\overset{-}{tk+1})\overset{+}{a}{}_1^{-1}\overset{+}{b}_3$. In both cases, $x_2^{-1}y_1$ consists of one negative cycle of length $t+j+1$ and one positive cycle of length $t+j-1$. Therefore, $h_1 = (x_1^{-1}y_1)^{(t+j+1)(t+j-1)}$ is contained in $H \cap N \setminus (N^+ \cup Z(G))$, which means $\langle x_1, y_1 \rangle = G$.

It remains to deal with the cases where $j = 2$ and $k \equiv 2 \mod 4$, and where j is odd. Suppose first that $j = 2$ and $k \equiv 2 \mod 4$. Let $x = \overset{+}{a}$ and $y = \overset{+}{b}$, where a and b are as give in Lemma 3.7, so that

$\langle a,b\rangle = \mathrm{Sym}(n)$. Let $x = \overset{+}{a}(\overset{-}{k})$ and $y = \overset{+}{b}$. Then $x^{-1}y = (\overset{-}{k})\overset{+}{a_1^{-1}}\overset{+}{b}$, so that

$$x^{-1}y = (\overset{+}{1}, \overset{+}{3} \nearrow \overset{+}{\tfrac{k}{2}}, \overset{+}{k-2}, \overset{+}{tk} \searrow_k \overset{+}{2k}, \overset{-}{k}, \overset{+}{k+1} \nearrow_k \overset{+}{tk+1}).$$

Therefore, $x^{-1}y$ is a negative cycle of length $2t+\frac{k}{2}$, which is odd. Hence, $(x^{-1}y)^{2t+\frac{k}{2}} \in N \setminus (N^+ \cup Z(G))$. Therefore, $\langle x,y\rangle = G$.

Finally, we suppose k is even, t is even and j is odd. Since $j \notin \{0,1,k-1\}$, this implies $k \geq 6$ and $j \geq 3$. Choose x and y to be

$$x = (\overset{+}{1} \nearrow \overset{+}{k})(\overset{+}{k+1} \nearrow \overset{+}{2k}) \cdots (\overset{+}{(t-1)k+1} \nearrow \overset{+}{tk})(\overset{+}{tk+1}, \overset{+}{tk+2}) \text{ and}$$

$$y = (\overset{+}{k}, \overset{+}{k+2}, \overset{+}{k+1}, \overset{+}{k+3} \nearrow \overset{+}{2k-1})(\overset{+}{2k} \nearrow \overset{+}{3k-1}) \cdots (\overset{+}{(t-1)k} \nearrow \overset{+}{tk-1})$$

$$\cdot (\overset{+}{tk} \nearrow \overset{+}{n}, \overset{+}{2} \nearrow \overset{+}{k-j-1})(\overset{-}{k-j}).$$

We first verify that $a = \hat{x}$ and $b = \hat{y}$ generate $\mathrm{Sym}(n)$. We claim that $\bar{H} = \langle a,b\rangle$ is a primitive group. Clearly $\bar{H}$ is transitive. Suppose, by way of a contradiction, that $\bar{H}$ preserves a nontrivial block Δ containing n. Hence, $\Delta^a = \Delta$, since a fixes n. For $\lambda \in \Delta$, $\lambda^{\langle a\rangle} \subseteq \Delta$. Note that each cycle (other than fixed points) of a contains at least one point μ with the property that $\mu^a = \mu^b$. Therefore, if $\lambda \in \mathrm{supp}(a)$, then there is some μ in $\lambda^{\langle a\rangle}$ such that μ^b is also contained in $\lambda^{\langle a\rangle}$. Thus, Δ contains both μ and μ^b. Hence, $\Delta^b = \Delta$ and $\Delta^H = \Delta$, contradicting the transitivity of H. Therefore, $\Delta \subseteq \Omega \backslash \mathrm{supp}(a) = \{tk+3, \ldots, n\}$. Consequently, $|\Delta| \leq j-2 < k$ and Δ contains a point $tk+i$ with $3 \leq i < j$. Thus, Δ^b contains 2 and $tk+i+1$. Since $tk+i+1 \notin \mathrm{supp}(a)$ it follows that Δ^b is left invariant by a and so $2^{\langle a\rangle} \cup \{tk+i+1\} \subseteq \Delta^b$. Therefore, $|\Delta^b| \geq k+1$, contradicting the bound on block size. Hence, $\bar{H}$ is primitive.

By direct calculation

$$x^{-1}y = (\overset{+}{1}, \overset{+}{k+2}, \overset{+}{k+3}, \overset{+}{k+1} \nearrow_k \overset{+}{tk+1}, \overset{+}{tk+3} \nearrow \overset{+}{n}, \overset{+}{2})(\overset{+}{k-j}, \overset{+}{tk} \searrow_k \overset{+}{k}, \overset{+}{k-1} \searrow \overset{+}{k-j+2}, \overset{-}{k-j+1}),$$

which is a product of a $(t+j+2)$ and $(t+j)$-cycle. Thus, $(a^{-1}b)^{t+j+2}$ is a $(t+j)$-cycle which fixes at least 3 points. Hence, by Theorem 1.3 of [5], $\mathrm{Alt}(n) \leq \bar{H}$. Since $a \in \mathrm{Sym}(n)\backslash\mathrm{Alt}(n)$ it follows that $\bar{H} = \mathrm{Sym}(n)$.

Moreover, $x^{-1}y$ consists of one negative cycle of length $(t+j)$ and one positive cycle of length $(t+j+2)$. Hence, $(x^{-1}y)^{(t+j)(t+j+2)}$ consists of $t+j$ negative 1-cycles and at least $t+j+2$ fixed points and so $(x^{-1}y)^{(t+j)(t+j+2)} \in (H \cap N) \setminus (N^+ \cup Z(G))$. Thus, by Lemma 2.1, $H = \langle x,y\rangle = G$. □

Lemma 3.9 *Suppose $n \geq 2k$, and that k is even and t is odd, with $j \neq k-1$.*

(i) If j is odd, then the following elements a and b generate $\mathrm{Sym}(n)$.

$$a = (1 \nearrow k)(k+1 \nearrow 2k) \cdots ((t-1)k+1 \nearrow tk);$$

$$b = (k, k+2, k+1, k+3 \nearrow 2k-1)(2k \nearrow 3k-1) \cdots ((t-1)k \nearrow tk-1) \cdot$$

$$\cdot (tk \nearrow n, 2 \nearrow k-j).$$

(ii) If $j=0$ and $k \equiv 2 \mod 4$, then the following elements c and d generate $\mathrm{Sym}(n)$.

$$c=(1 \nearrow k)(k+1 \nearrow 2k)\cdots((t-1)k+1 \nearrow tk);$$

$$d=(1,2)(\tfrac{k}{2} \nearrow k-1)(k \nearrow 2k-1)\cdots((t-1)k \nearrow tk-1).$$

Proof

(i) Suppose j is odd, let a and b be as in the statement of the lemma, and write $H=\langle a,b\rangle$. Clearly H is transitive. We claim that H is a primitive group. Assume, by way of a contradiction, that H preserves a non-trivial block Δ containing n. The point n is fixed by a, and so $\Delta^a=\Delta$. Suppose Δ contains a point $\lambda \in \mathrm{supp}(a)$. Then $\lambda^{\langle a\rangle} \subseteq \Delta$. If λ is in the first cycle of a then Δ contains 2, which is n^b, hence $\Delta^b=\Delta$. Thus, $\Delta^H=\Delta$, contradicting the transitivity of H. If λ is in the $(i+1)^{\text{st}}$ cycle of a for $1 \le i \le t-1$, then Δ contains $ik+1$ and $ik+2$. Since $(ik+1)^b=ik+2$ when $i>1$, and $(k+1)^{b^{-1}}=k+2$, it follows that $\Delta^b=\Delta$ and so $\Delta^H=\Delta$, contradicting the transitivity of H. If Δ contains $n-1$ then $\Delta^{b^{-1}}=\Delta$, again contradicting the transitivity of H. Consequently, $\Delta \subseteq \Omega\backslash(\mathrm{supp}(a)\cup\{n-1\})=\{tk+1 \nearrow n-2, n\}$. Hence, Δ contains a point $tk+i$ for $1 \le i < j-1$. Consider Δ^b containing $\{n, tk+i\}^b=\{2, tk+i+1\}$. Since $tk+i+1$ is fixed by a it follows that $(\Delta^b)^a=\Delta^b$. Hence, $2^{\langle a\rangle}=\{1,2,\ldots,k\} \subseteq \Delta^b$. In particular, $2,3 \in \Delta^b$. When $k-j<3$, $3^b=3$ and otherwise $2^b=3$. Hence, $(\Delta^b)b=\Delta^b$ and so $\Delta^H=\Delta$ contradicting the transitivity of H. Therefore, H is a primitive group.

By direct computation

$$a^{-1}b=(1,k+2,k+3,k+1 \nearrow_{k} (t-1)k+1, tk+1 \nearrow n, 2)(tk \searrow_{k} k, k-1 \searrow k-j+1).$$

is a product of a $(t+j+3)$-cycle and a $(t+j-1)$-cycle. Hence, $(a^{-1}b)^{t+j+3}$ is a $(t+j-1)$-cycle. Thus, by Theorem 1.3 of [5], $\mathrm{Alt}(n) \le H$. Since $a \in \mathrm{Sym}(n)\backslash\mathrm{Alt}(n)$ it follows that $H=\mathrm{Sym}(n)$ as required.

(ii) Let $H=\langle c,d\rangle$. Clearly H is a transitive group, and we now show it is a primitive group. Assume, by way of a contradiction, that H preserves a nontrivial block Δ containing $n=kt$. The point n is fixed by d and so $\Delta^d=\Delta$. Hence, for a point $\lambda \in \Delta$, $\lambda^{\langle d\rangle} \subseteq \Delta$. Every cycle of d contains a point μ with $\mu^d=\mu^c$. Thus, if $\lambda \in \mathrm{supp}(d)$, then $\lambda^{\langle d\rangle}$ contains two points μ, μ^c. Hence, $\Delta^c=\Delta$ and $\Delta^H=\Delta$, contradicting the transitivity of H. Therefore, $\Delta \subseteq \Omega\backslash\mathrm{supp}(d)$. When $k=6$, it follows that $\Delta \subseteq \{n\}$; hence, H is primitive. And when $k>6$ it follows that $\Delta \subseteq \{3,\ldots,\frac{k}{2}-1,n\}$. Suppose $\lambda \in \Delta \setminus \{3,n\}$. Then $\Delta^{c^{-1}}$ contains $tk-1$ and $\lambda^{c^{-1}} \in \{3,\ldots,\frac{k}{2}-2\}$. Since $\lambda^{c^{-1}} \notin \mathrm{supp}(d)$, $\Delta^{c^{-1}}$ is left invariant by d, hence contains $(tk-1)^{\langle d\rangle}$. In particular, $tk-2, tk-1 \in \Delta^{c^{-1}}$. Since $(tk-2)^c=tk-1$, it follows that $(\Delta^{c^{-1}})c=\Delta^{c^{-1}}$ and $\Delta^{c^{-1}H}=\Delta^{c^{-1}}$, contradicting the transitivity of H. Therefore, $\Delta=\{3,n\}$ and the block size is two. Consider $\Delta^{c^{-1}}=\{2,tk-1\}$. However, $2^{d^2}=2$ and $(tk-1)^{d^2} \neq tk-1$, contradicting the deduced block size. Hence, H is a primitive group.

Note that $c^{-1}d$ is a $(2t+\frac{k}{2}-2)$-cycle. Specifically,

$$c^{-1}d=(1 \nearrow_{k} (t-1)k+1, tk \searrow_{k} k, \tfrac{k}{2} \searrow 3).$$

Hence, $c^{-1}d$ has $tk-(2t+\frac{k}{2}-2)=(t-\frac{1}{2})(k-2)+1$ fixed points. Since $t \ge 1$ and $k \ge 6$, there are at least 3 fixed points. And so, by Theorem 1.3 of [5], H contains $\mathrm{Alt}(n)$. Since $c \in \mathrm{Sym}(n)\backslash\mathrm{Alt}(n)$, it follows that $H=\mathrm{Sym}(n)$.

□

Lemma 3.10 *Suppose $n \geq 2k$ and $j \neq k-1$. If k is odd, then $d_k(G') = 2$. If k is even and t is odd, then $d_k(G) = 2 = d_k(G^+)$.*

Proof The case $k = 3$ was covered by Lemma 3.1. Suppose from now on that $k > 3$, $2k \leq n$, and that either k is odd, or that k is even and t is odd. Let

$$x = (\overset{+}{1} \nearrow \overset{+}{k-2}, \overset{-}{k-1}, \overset{-}{k})(\overset{+}{k+1} \nearrow \overset{+}{2k}) \cdots (\overset{+}{(t-1)k+1} \nearrow \overset{+}{tk}) \text{ and}$$

$$y = (\overset{+}{k}, \overset{+}{k+2}, \overset{+}{k+1}, \overset{+}{k+3} \nearrow \overset{+}{2k-1})(\overset{+}{2k} \nearrow \overset{+}{3k-1}) \cdots (\overset{+}{(t-1)k} \nearrow \overset{+}{tk-1}) \cdot$$

$$\cdot (\overset{+}{tk} \nearrow \overset{+}{n}, \overset{+}{1} \nearrow \overset{+}{k-j-1}).$$

Then $\hat{x} = a_0$ and $\hat{y} = b_0$. Therefore, by Theorem 2.5, we have that $\langle \hat{x}, \hat{y} \rangle$ is $\mathrm{Alt}(n)$ when k is odd, and $\mathrm{Sym}(n)$ when k is even and t is odd. Let $H = \langle x, y \rangle$. Then $H \leq G^+$. We have

$$x^{-1} = (\overset{-}{1}, \overset{-}{k}, \overset{+}{k-1} \searrow \overset{+}{2})(\overset{+}{2k} \searrow \overset{+}{k+1}) \cdots (\overset{+}{tk} \searrow \overset{+}{(t-1)k+1}) \text{ and}$$

$$x^{-1}y = (\overset{-}{1}, \overset{+}{k+2}, \overset{+}{k+3}, \overset{+}{k+1} \underset{k}{\nearrow} \overset{+}{(t-1)k+1}, \overset{+}{tk+1} \nearrow \overset{+}{n})(\overset{+}{tk} \underset{k}{\searrow} \overset{+}{2k}, \overset{-}{k}, \overset{+}{k-1} \searrow \overset{+}{k-j}).$$

Thus, $x^{-1}y$ consists of one negative cycle of length $t+j+2$, one negative cycle of length $t+j$, and $t(k-2)-j-2$ fixed points. In particular, $2k-1$ is fixed. Let m be the least common multiple of $t+j$ and $t+j+2$, and set $h = (x_1^{-1}y)^m$. If m is odd, then h consists of $2(t+j+1)$ negative 1-cycles, with the remaining cycles being fixed points. If m is even, then exactly one of $t+j$ and $t+j+2$ is divisible by 4, and h therefore contains either $t+j$ or $t+j+2$ negative 1-cycles, with the remaining cycles being fixed points. Thus, $h \in (H \cap N^+) \setminus Z(G)$, which implies, when k is odd, that $H = G'$, and, when k is even and t is odd, that $H = G^+$. Therefore, $d_k(G') = 2$ when k is odd, and $d_k(G^+) = 2$ when k is even and t is odd.

Suppose for the rest of the proof that k is even and t is odd. It remains to show that $d_k(G) = 2$ in this case. If j is even and $j > 0$, then write

$$x_e = (\overset{+}{1} \nearrow \overset{+}{k})(\overset{+}{k+1} \nearrow \overset{+}{2k}) \cdots (\overset{+}{(t-1)k+1} \nearrow \overset{+}{tk})(\overset{-}{n})$$

and set $H_e = \langle x_e, y \rangle$. Since $\hat{x}_e = \hat{x}$, it follows from Theorem 2.5 that $\bar{H}_e = \mathrm{Sym}(n)$. Now

$$x_e^{-1}y = (\overset{+}{1}, \overset{+}{k+2}, \overset{+}{k+3}, \overset{+}{k+1} \underset{k}{\nearrow} \overset{+}{(t-1)k+1}, \overset{+}{tk+1} \nearrow \overset{+}{n-1}, \overset{-}{n})(\overset{+}{tk} \underset{k}{\searrow} \overset{+}{k}, \overset{+}{k-1} \searrow \overset{+}{k-j}).$$

Thus, $x_e^{-1}y$ consists of one negative cycle of length $t+j+2$ and one positive cycle of length $t+j$. Since t is odd, and j is even, $x_e^{-1}y^{(t+j+2)(t+j)} \in (H_e \cap N) \setminus (N^+ \cup Z(G))$. Hence, $H_e = G$.

Next, if k is even, t is odd and j is odd, then set

$$y_e = (\overset{+}{k}, \overset{+}{k+2}, \overset{+}{k+1}, \overset{+}{k+3} \nearrow \overset{+}{2k-1})(\overset{+}{2k} \nearrow \overset{+}{3k-1}) \cdots (\overset{+}{(t-1)k} \nearrow \overset{+}{tk-1}) \cdot$$

$$\cdot (\overset{+}{tk} \nearrow \overset{+}{n}, \overset{+}{2} \nearrow \overset{+}{k-j}).$$

Write $K = \langle x_e, y_e \rangle$. This time, $\hat{x}_e = a$ and $\hat{y}_e = b$, where a and b are the elements defined in Lemma 3.9(i), and so $\bar{K} = \mathrm{Sym}(n)$. We have

$$x_e^{-1} y_e = (\overset{+}{1}, \overset{+}{k+2}, \overset{+}{k+3}, \overset{+}{k+1} \nearrow_k \overset{+}{(t-1)k+1}, \overset{+}{tk+1} \nearrow \overset{+}{n-1}, \overset{-}{n}, \overset{+}{2})(\overset{+}{tk} \searrow_k \overset{+}{k}, \overset{+}{k-1} \searrow \overset{+}{k-j+1}).$$

In this case, $x_e^{-1} y_e$ consists of one negative cycle of length $t+j+3$ and one positive cycle of length $t+j-1$. Hence, since t is odd and j is odd, we see that $(x_e^{-1} y_e)^{(t+j+3)(t+j-1)} \in (K \cap N) \setminus (N^+ \cup Z(G))$. Therefore, $K = G$.

Finally, suppose k is even, t is odd and $j = 0$. That is, $n = tk$. We must split this last case further, and consider first the case when $k \equiv 2 \mod 4$. Redefine x and y as follows.

$$x = (\overset{+}{1} \nearrow \overset{+}{k})(\overset{+}{k+1} \nearrow \overset{+}{2k}) \cdots (\overset{+}{(t-1)k+1} \nearrow \overset{+}{tk})$$

$$y = (\overset{+}{1}, \overset{+}{2})(\overset{+}{\tfrac{k}{2}} \nearrow \overset{+}{k-1})(\overset{+}{k} \nearrow \overset{+}{2k-1}) \cdots (\overset{+}{(t-1)k} \nearrow \overset{+}{tk-1})(\overset{-}{tk})$$

Since $\hat{x} = c$ and $\hat{y} = d$, where c and d are the elements described in Lemma 3.9(ii), we have $\bar{H} = \langle c, d \rangle = \mathrm{Sym}(n)$. Moreover,

$$x^{-1} y = (\overset{+}{1} \nearrow_k \overset{+}{(t-2)k+1}, \overset{-}{(t-1)k+1}, \overset{+}{tk} \searrow_k \overset{+}{k}, \overset{+}{\tfrac{k}{2}} \searrow \overset{+}{3}).$$

Thus, $x^{-1} y$ is a $(2t + \frac{k}{2} - 2)$-cycle, which, since $\frac{k}{2}$ is odd, is a cycle of odd length. Therefore, $(x^{-1} y)^{(2t + \frac{k}{2} - 2)} \in (H \cap N) \setminus (N^+ \cup Z(G))$, and so $H = G$.

The last subcase to consider is when k is even, t is odd, $j = 0$ and $k \equiv 0 \mod 4$. In this case, we apply Lemma 3.6, with k replaced by $\frac{k}{2}$ and t replaced by $2t$. Note that $(\overset{-}{1})(\overset{+}{k} \searrow \overset{+}{1})(\overset{-}{k}) = (\overset{+}{k} \searrow \overset{+}{1})$. Let $u = x(\overset{-}{1})$ and $v = (\overset{-}{k})y$, where x and y are the elements used in Lemma 3.6 for the case $j = 0$, setting $\varepsilon = +$. That is,

$$u = (\overset{+}{1} \nearrow \overset{+}{k-1}, \overset{-}{k}) \cdots (\overset{+}{(t-2)k+1} \nearrow \overset{+}{(t-1)k})(\overset{+}{(t-1)k+1} \nearrow \overset{+}{(t-1)k+\tfrac{k}{2}})(\overset{+}{(t-1)k+\tfrac{k}{2}+1} \nearrow \overset{+}{tk}) \text{ and}$$

$$v = (\overset{-}{k}, \overset{+}{k+2}, \overset{+}{k+1}, \overset{+}{k+3} \nearrow \overset{+}{2k-1})(\overset{+}{2k} \nearrow \overset{+}{3k-1}) \cdots (\overset{+}{(t-1)k} \nearrow \overset{+}{n-1})(\overset{-}{tk}, \overset{+}{1} \nearrow \overset{+}{\tfrac{k}{2}-1})(\overset{+}{\tfrac{k}{2}} \nearrow \overset{+}{k-1}).$$

Then u and v both contain a negative $(\frac{k}{2})$-cycle, and so have order k. Since $\hat{u} = \hat{x}$ and $\hat{v} = \hat{y}$, we have $\langle \hat{u}, \hat{v} \rangle = \mathrm{Sym}(n)$. Moreover, $u^{-1} v = (\overset{-}{1}) x^{-1} (\overset{-}{k}) y = x^{-1} y$, which from the proof of Lemma 3.6 is an element of $N \setminus (N^+ \cup Z(G))$. Hence, $\langle u, v \rangle = G$, as required. $\square$

4. The case $n < 2k$

The work when $n < 2k$ splits into three parts: $n = 2k - 1$, $k < n < 2k - 1$, and $n = k$. We begin with $n = 2k - 1$.

Lemma 4.1 *Suppose $n = 2k - 1$, with k odd, and suppose that x and y are signed k-cycles of G such that x and y have order k and that $\mathrm{Alt}(n) = \langle \hat{x}, \hat{y} \rangle$. Then $\langle x, y \rangle \cong \mathrm{Alt}(n)$.*

Proof If $|\mathrm{supp}(\hat{x}) \cap \mathrm{supp}(\hat{y})| > 1$, then there is at least one point fixed by both x and y, contradicting the fact that $\langle \hat{x}, \hat{y} \rangle = \mathrm{Alt}(n)$. Therefore, $|\mathrm{supp}(\hat{x}) \cap \mathrm{supp}(\hat{y})| = 1$. Since $\mathrm{Sym}(n)$ acts transitively on the set of pairs of k-cycles in $\mathrm{Sym}(n)$ intersecting in one point, we may assume without loss of generality that we have $\hat{x} = (1, 2, \cdots, k)$ and $\hat{y} = (1, k+1, k+2, \cdots, n)$. This pair of k-cycles do indeed generate $\mathrm{Alt}(n)$, as observed in [3]. We may now write

$$x = (\overset{\varepsilon_1}{1}, \overset{\varepsilon_2}{2}, \cdots, \overset{\varepsilon_k}{k}) \text{ and}$$

$$y = (\overset{\nu_1}{1}, \overset{\nu_{k+1}}{k+1}, \cdots, \overset{\nu_n}{n})$$

where, because x and y have order k, $\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k = 1$ and $\nu_1 \nu_{k+1} \cdots \nu_n = 1$. Then the $\langle x, y \rangle$-orbit of 1 is

$$\Pi = \{1, \varepsilon_1 2, \varepsilon_1 \varepsilon_2 3, \ldots, (\varepsilon_1 \ldots \varepsilon_{k-1})k, \nu_1(k+1), \nu_1 \nu_{k+1}(k+2), \ldots, (\nu_1 \nu_{k+1} \nu_{k+2} \cdots \nu_{n-1})n\}.$$

Hence, $\langle x, y \rangle = \mathrm{Alt}(\Pi) \cong \mathrm{Alt}(n)$. □

Lemma 4.2 *Suppose $n = 2k-1$, with k odd and $k \geq 3$. If k is composite, then the following elements a and b of order k generate $\mathrm{Alt}(n)$, where p is any prime factor of k.*

$$a = (1 \nearrow k)(k+1, k+2, k+4 \nearrow k+p+1),$$
$$b = (1, k+1 \nearrow n).$$

Proof Let a and b be the elements given. Then a has cycle type $k^1 \cdot p^1 \cdot 1^{k-p-1}$ and b has cycle type $k^1 \cdot 1^{k-1}$. Since $p \mid k$, it follows that both a and b have order k.

Clearly, H is transitive. We claim that $H = \langle a, b \rangle$ is also primitive. Suppose for a contradiction that H preserves a nontrivial block Δ containing $k+3$. Then $\Delta^a = \Delta$ since $k+3$ is fixed by a. Suppose Δ contains $\lambda \in \mathrm{supp}(a)$. Then $\lambda^{\langle a \rangle} \subseteq \Delta$. If $\lambda \in \{1, 2, \ldots, k\}$, then $\{1, 2, \ldots, k\} \subseteq \Delta$ and so $|\Delta| \geq k+1 > \frac{n}{2}$, a contradiction to the fact that $|\Delta|$ divides n. If $\lambda \in \{k+1, k+2, k+4, \ldots, k+p+1\}$, then $\{k+1, k+2, k+4, \ldots, k+p+1\} \subseteq \Delta$. In particular, $k+1, k+2 \in \Delta$. Since $(k+1)^b = k+2$ it follows that $\Delta^b = \Delta$. Hence, $\Delta^H = \Delta$, contradicting the transitivity of H. Hence, it follows that $\Delta \subseteq \mathrm{supp}(b)$. Therefore, $|\Delta|$ divides $|\mathrm{supp}(b)| = k$ and, since Δ is a block, $|\Delta|$ divides $n = 2k-1$. Thus, we reach a contradiction, because k and $2k-1$ are coprime.

Hence, H is a primitive group of $\mathrm{Sym}(n)$ containing b, a cycle of length k. Since k is composite, $k \geq 9$, and hence b has at least eight fixed points. Hence, by Theorem 1.3 of [5], H contains $\mathrm{Alt}(n)$ and therefore $H = \mathrm{Alt}(n)$. □

Lemma 4.3 *Suppose $n = 2k-1$, with k odd and $k \geq 3$. If k is prime, then $d_k(G') = 3$, whereas if k is not prime, then $d_k(G') = 2$.*

Proof By Lemma 4.1, G' cannot be generated by two k-cycles. If k is prime, then these are the only kinds of elements of order k; therefore, we cannot generate G' with two elements of order k. However, we can choose three elements of order k to generate G'. Choose x, y and z as follows.

$$x = (\overset{+}{1} \nearrow \overset{+}{k});$$
$$y = (\overset{+}{1}, \overset{+}{k+1} \nearrow \overset{+}{n});$$
$$z = (\overset{-}{1}, \overset{+}{3} \nearrow \overset{+}{k}, \overset{-}{k+1}).$$

Write $H = \langle x, y, z \rangle$. By Theorem 2.5, $\langle \hat{x}, \hat{y} \rangle = \mathrm{Alt}(n)$. Since $x^{-1}z = (\overset{+}{1}, \overset{-}{k+1})(\overset{-}{2}, \overset{+}{3})$, it follows that $(x^{-1}z)^2 = (\overset{-}{1})(\overset{-}{2})(\overset{-}{3})(\overset{-}{k+1}) \in (H \cap N_e) \backslash Z(G)$. Now, Lemma 2.1 implies that $H = G'$.

Suppose from now on that k is composite and let p be a prime divisor of k. Let x and y be the following elements of G, with $H = \langle x, y \rangle$.

$$x = (\overset{+}{1} \nearrow \overset{+}{k})(\overset{+}{k+1}, \overset{-}{k+2}, \overset{+}{k+4} \nearrow \overset{+}{k+p}, \overset{-}{k+p+1});$$
$$y = (\overset{+}{1}, \overset{+}{k+1} \nearrow \overset{+}{n}).$$

By Lemma 4.2, we have that $\bar{H} = \langle a, b \rangle \cong \mathrm{Alt}(n)$. Hence, to show that $H = G'$, it suffices, by Lemma 2.1, to find a nontrivial element of $H \cap N^+$. By direct computation we find

$$x^{-1}y = (\overset{+}{1}, \overset{+}{k} \searrow \overset{+}{2}, \overset{-}{k+1}, \overset{+}{k+p+2} \nearrow \overset{+}{n})(\overset{+}{k+3}, \overset{-}{k+4}).$$

We see that $x^{-1}y$ consists of a negative $(2k-p-1)$-cycle, a negative 2-cycle, and some fixed points. Let $h = (x^{-1}y)^{2k-p-1}$. Then $h \in N^+$, and h contains least $2k-p-1$ negative 1-cycles. Hence, h is a nonidentity element of $H \cap N^+$. Therefore, by Lemma 2.1, we have that $H = G'$. □

Lemma 4.4 *Suppose $n = 2k-1$ with k even and $k \geq 8$. Then the following elements a and b generate* $\mathrm{Sym}(n)$.

$$a = (1 \nearrow \tfrac{k}{2})(\tfrac{k}{2}+1 \nearrow k)(k+1, k+2);$$
$$b = (1, k+1 \nearrow n)(2, k)(\tfrac{k}{2}, \tfrac{k}{2}+1).$$

Proof Let $H = \langle a, b \rangle$. Since $1^{\langle a \rangle} = \{1, 2, \ldots, \frac{k}{2}\}$ and $1^{\langle b \rangle} = \{1, k+1, k+2, \ldots, n\}$, it follows that $\{1, 2, \ldots, \frac{k}{2}, k+1, k+2, \ldots, n\} \subseteq 1^H$. Also, $(\frac{k}{2})^b = \frac{k}{2}+1$ and so

$$\{1, 2, \ldots, \tfrac{k}{2}+1, k+1, k+2, \ldots, n\} \subseteq 1^H.$$

Finally, since $(\frac{k}{2}+1)^{\langle a \rangle} = \{\frac{k}{2}+1, \frac{k}{2}+2, \ldots, k\}$, it follows that $\Omega \subseteq 1^H$. Thus, H is transitive.

We claim that H is a primitive group. Assume, by way of a contradiction, that H preserves a nontrivial block Δ which contains n. Then $\Delta^a = \Delta$, because n is fixed by a. Let $\lambda \in \Delta \setminus \{n\}$, and observe that $\lambda^{\langle a \rangle} \subseteq \Delta$. If $\lambda \in \{1, 2, \ldots, \frac{k}{2}\}$, then Δ contains $\frac{k}{2}$, and if $\lambda \in \{\frac{k}{2}+1, \frac{k}{2}+2, \ldots, k\}$, then Δ contains $\frac{k}{2}+1$. Since $\frac{k}{2}$ and $\frac{k}{2}+1$ are fixed by b^2 it follows that $\Delta^{b^2} = \Delta$. Hence, $n^{b^2} = k+1$ is contained in

Δ, forcing $(k+1)^a = k+2 \in \Delta$. Therefore, $\Delta^b = \Delta$, because $(k+1)^b = k+2$. Consequently, $\Delta^H = \Delta$, contradicting the transitivity of H. Therefore, $\lambda > k$. If $\lambda \in \{k+1, k+2\}$, then, since $\Delta^{\langle a \rangle} = \Delta$, we have $\{k+1, k+2\} \subseteq \Delta$. However, $(k+1)^b = k+2$ and so $\Delta^b = \Delta$, meaning that $\Delta^H = \Delta$, another contradiction. Hence, $\Delta \subseteq \Omega \backslash \mathrm{supp}(a) = \{k+3, \dots, n\}$. Therefore, Δ is a subset of the first cycle of b. Thus, $|\Delta|$ divides k. However, since Δ is a block, $|\Delta|$ divides $n = 2k-1$, a contradiction since k and $2k-1$ are coprime. Therefore, H is a primitive group. The element

$$a^{-1}b = (1, \tfrac{k}{2}+1, 2, k+1, k+3 \nearrow n)(3, k, k-1 \searrow \tfrac{k}{2}+2, \tfrac{k}{2} \searrow 4)$$

is a product of a $(k+1)$-cycle and a $(k-3)$-cycle, and so $(a^{-1}b)^{k+1}$ is a single cycle fixing at least $k+1 > 3$ points. Thus, by Theorem 1.3 of [5], H contains $\mathrm{Alt}(n)$. Finally, as $a \in \mathrm{Sym}(n) \backslash \mathrm{Alt}(n)$, $H = \mathrm{Sym}(n)$. □

Lemma 4.5 *Suppose $n = 2k-1$. If k is even, then $d_k(G) = 2$.*

Proof Let k be even, and let $n = 2k-1$. A calculation in MAGMA verifies that when $k = 4$ or $k = 6$, we have $d_k(G) = d_k(G^+) = 2$. Assume from now on that $k \geq 8$. Let

$$x = (\overset{+}{1} \nearrow \overset{+}{\tfrac{k}{2}-1}, \overset{-}{\tfrac{k}{2}})(\overset{+}{\tfrac{k}{2}+1} \nearrow \overset{+}{k-2}, \overset{\varepsilon}{k-1}, \overset{+}{k})(\overset{+}{k+1}, \overset{+}{k+2}) \text{ and}$$

$$y = (\overset{+}{1}, \overset{+}{k+1} \nearrow \overset{+}{n})(\overset{+}{2}, \overset{+}{k})(\overset{+}{\tfrac{k}{2}}, \overset{+}{\tfrac{k}{2}+1}),$$

where $\varepsilon \in \{+,-\}$. Write $H = \langle x, y \rangle$. Since $\hat{x} = a$ and $\hat{y} = b$, where a and b are as given in Lemma 4.4, we have $\bar{H} = \mathrm{Sym}(n)$. Now

$$x^{-1} = (\overset{-}{1}, \overset{+}{\tfrac{k}{2}} \searrow \overset{+}{2})(\overset{\varepsilon}{k}, \overset{+}{k-1} \searrow \overset{+}{\tfrac{k}{2}+1})(\overset{+}{k+1}, \overset{+}{k+2}) \text{ and}$$

$$x^{-1}y = (\overset{-}{1}, \overset{+}{\tfrac{k}{2}+1}, \overset{+}{2}, \overset{+}{k+1}, \overset{+}{k+3} \nearrow \overset{+}{n})(\overset{+}{3}, \overset{\varepsilon}{k}, \overset{+}{k-1} \searrow \overset{+}{\tfrac{k}{2}+2}, \overset{+}{\tfrac{k}{2}} \searrow \overset{+}{4}).$$

Then $x^{-1}y$ contains a $(k+1)$-cycle, a $(k-3)$-cycle, and one fixed point. Let $h = (x^{-1}y)^{(k+1)(k-3)}$. If $\varepsilon = +$, then $h \in N \setminus (N^+ \cup Z(G))$ and hence, by Lemma 2.1, $H = G$. If $\varepsilon = -$, then $H \leq G^+$, and $h \in N^+ \setminus Z(G)$, which implies that $H = G^+$, and the proof is complete. □

Lemma 4.6 *Suppose $3 \leq k < n < 2k-1$.*

(i) If $k = 3$, then $n = 4$ and $d_3(G') = 3$.

(ii) If k is odd and $k > 3$, then $d_k(G') = 2$.

(iii) If k is even, then $d_k(G) = d_k(G^+) = 2$.

Proof Suppose $3 \leq k < n < 2k-1$. Then $n = k+j$ where $0 < j < k-1$. The relevant elements from Definition 2.4 in this situation are

$$a_4 = (1 \nearrow k);$$

$$b_6 = (1 \nearrow k-j, k+1 \nearrow n).$$

In each case we will define elements x, y of G such that $\hat{x} = a_4$ and $\hat{y} = b_6$. Writing $\langle x, y\rangle = H$, Theorem 2.5 will then imply that, unless $(n,k) = (6,4)$, we have $\bar{H} = \mathrm{Sym}(n)$ when k is even, and $\bar{H} = \mathrm{Alt}(n)$ when k is odd.

(i) If $k = 3$, then $n = 4$. A quick MAGMA calculation verifies that no two elements of order 3 in G generate G'. In fact, the biggest subgroup of G generated by two elements of order 3 has index 16 — it is $\mathrm{Alt}(4) \times Z(G)$. However, G' can be generated by three elements of order 3. Therefore, $d_3(G') = 3$.

(ii) Suppose k is odd, with $k > 3$. Let

$$x = (\overset{-}{1}, \overset{+}{2} \nearrow \overset{+}{k-1}, \overset{-}{k}) \text{ and}$$
$$y = (\overset{+}{1} \nearrow \overset{+}{k-j}, \overset{+}{k+1} \nearrow \overset{+}{n}).$$

Since $\hat{x} = a_4$ and $\hat{y} = b_6$, we have $\bar{H} = \mathrm{Alt}(n)$. Now,

$$x^{-1}y = (\overset{-}{1}, \overset{+}{k} \searrow \overset{+}{3}, \overset{-}{2})(\overset{+}{1} \nearrow \overset{+}{k-j}, \overset{+}{k+1} \nearrow \overset{+}{n}) = (\overset{-}{1}, \overset{+}{k} \searrow \overset{+}{k-j+1}, \overset{+}{k+1} \nearrow \overset{+}{n})(\overset{-}{2}).$$

Therefore, $x^{-1}y$ consists of one negative cycle of length $2j+1$, along with one negative 1-cycle and $k-j-2$ fixed points. Consequently, $(x^{-1}y)^{2j+1}$ consists of negative 1-cycles along with at least one fixed point, as long as $j < k-2$. Hence, when $j < k-2$, we have that $(x^{-1}y)^{2j+1} \in (H \cap N^+) \setminus Z(G)$. Therefore, when $j < k-2$, Lemma 2.1 implies that $H = G'$.

It remains to deal with $j = k-2$. We can keep the same x and y, so that

$$x = (\overset{-}{1}, \overset{+}{2} \nearrow \overset{+}{k-1}, \overset{-}{k}), \qquad y = (\overset{+}{1}, \overset{+}{2}, \overset{+}{k+1} \nearrow \overset{+}{n}),$$

and $\bar{H} = \mathrm{Alt}(n)$. We have

$$[x,y] = x^{-1}x^y = (\overset{-}{1}, \overset{+}{k} \searrow \overset{+}{3}, \overset{-}{2})(\overset{-}{2}, \overset{+}{k+1}, \overset{+}{3} \nearrow \overset{+}{k-1}, \overset{-}{k}) = (\overset{+}{1}, \overset{-}{2})(\overset{-}{3}, \overset{+}{k+1}).$$

Therefore, since $k > 3$, we see that $[x,y]^2 \in H \cap N^+ \setminus Z(G)$. Hence, $H = G'$.

(iii) Suppose k is even. A quick check in MAGMA confirms that if $n = 6$ and $k = 4$, then $d_4(G) = d_4(G^+) = 2$. Assume then that $(n,k) \neq (6,4)$. To generate G^+, the same x, y used in the case k odd have the property that $\langle \hat{x}, \hat{y}\rangle = \mathrm{Sym}(n)$, but since x and y are both contained in G^+, and $(H \cap N^+) \not\leq Z(G)$, we have that $H = G^+$ by Lemma 2.1. Thus, $d_k(G^+) = 2$. It only remains to prove that $d_k(G) = 2$ in this case. To that end, set

$$x = (\overset{+}{1} \nearrow \overset{+}{k})(\overset{-}{n}) \text{ and}$$
$$y = (\overset{+}{1} \nearrow \overset{+}{k-j}, \overset{+}{k+1} \nearrow \overset{+}{n}).$$

Then $\hat{x} = a_4$ and $\hat{y} = b_6$. Hence, $\bar{H} = \mathrm{Sym}(n)$. Now $x^{-1} = (\overset{+}{k} \searrow \overset{+}{1})(\overset{-}{n})$. Hence,

$$x^{-1}y = (\overset{+}{1}, \overset{+}{k} \searrow \overset{+}{k-j+1}, \overset{+}{k+1} \nearrow \overset{+}{n-1}, \overset{-}{n}).$$

Therefore, $x^{-1}y$ consists of one negative $(2j+1)$-cycle and $(k-1)-j$ fixed points. As $j < k-1$, this means $(x^{-1}y)^{2j+1}$ consists of $2j+1$ negative 1-cycles, along with at least one fixed point. Thus, $(x^{-1}y)^{2j+1} \in (H \cap N) \setminus (N^{+} \cup Z(G))$. Hence, by Lemma 2.1, $H = G$.

□

When $n = k$, if we were to choose x and y to be k-cycles, then they would have to be positive k-cycles, or else the orders of x and y would be $2k$, but then $\langle x, y\rangle \leq G^{+}$. Therefore, in order to generate all of G when k is even, we must use elements other than k-cycles.

Lemma 4.7 *If* $n = k$ *and* $k \equiv 2 \mod 4$*, then* $d_k(G) = 2$*.*

Proof Let $x = (\overset{-}{2} \nearrow_2 \overset{-}{n})$ and $y = (\overset{+}{1}, \overset{+}{2}, \overset{+}{4}, \overset{+}{3}, \overset{+}{5} \nearrow \overset{+}{n})$, with $H = \langle x, y\rangle$. Write $a = \hat{x}$ and $b = \hat{y}$, so that $\bar{H} = \langle a, b\rangle$. We begin by showing that $\bar{H} = \mathrm{Sym}(n)$. This is easy to check for $n = 6$, so we assume $n \geq 10$. We claim first that $\bar{H}$ is primitive. If not, then $\bar{H}$ preserves a nontrivial block Δ containing 1. Then $\Delta^{a} = \Delta$. Thus, if $\lambda \in \Delta$, then $\lambda^{\langle a\rangle} \subseteq \Delta$. If λ were even, this would imply that $|\Delta| \geq \frac{n}{2} + 1$, a contradiction. Hence, Δ contains only odd numbers. If $\lambda = 3$, then $\Delta^{b^3} = \Delta$ since $1^{b^3} = 3$. Hence, $3^{b^3} \in \Delta$. However, $3^{b^3} = 10$, contradicting the fact that Δ contains only odd numbers. Hence, Δ contains 1 and at least one odd number λ with $\lambda > 3$. Now consider the block Δ^{b^2}, which contains $1^{b^2} = 4$ along with λ^{b^2}. Since λ^{b^2} is odd, it follows that Δ^{b^2} is left invariant by a. Hence, $4^{\langle a\rangle} \cup \lambda^{b^2} \subseteq \Delta^{b^2}$, but this forces $|\Delta^{b^2}| \geq \frac{n}{2} + 1$, a contradiction.

Thus, $\bar{H}$ is primitive group. The fact that $\bar{H}$ contains a (which is a cycle that fixes at least 3 points) now implies, by [5], that $\mathrm{Alt}(n) \leq \bar{H}$. As $b \in \mathrm{Sym}(n) \setminus \mathrm{Alt}(n)$ it follows that $\bar{H} = \mathrm{Sym}(n)$. Now, the element $x^{\frac{n}{2}}$ of H is a product of $\frac{n}{2}$ negative 1-cycles, an odd number. Thus, $x^{\frac{n}{2}} \in (H \cap N) \setminus (N^{+} \cup Z(G))$. Hence, $\langle x, y\rangle = G$. □

Lemma 4.8 *Suppose* $n = k \geq 3$*. If* k *is odd, then* $d_k(G') = 2$*. If* k *is even, then* $d_k(G) = d_k(G^{+}) = 2$*.*

Proof Suppose $n = k$. Let

$$x = (\overset{+}{1} \nearrow \overset{+}{n-2}, \overset{-}{n-1}, \overset{-}{n}),$$

$$y = (\overset{+}{1}, \overset{+}{3} \nearrow \overset{+}{n}, \overset{+}{2}),$$

and write $H = \langle x, y\rangle$. Since $\hat{x} = a_4$ and $\hat{y} = b_7$, we have, by Theorem 2.5, that $\bar{H} = \mathrm{Alt}(n)$ when k is odd, and $\bar{H} = \mathrm{Sym}(n)$ when k is even. We also note that $\langle x, y\rangle \leq G^{+}$. If $k > 3$, then

$$x^{-1}y = (\overset{-}{1}, \overset{-}{n}, \overset{+}{n-1} \searrow \overset{+}{2})(\overset{+}{1}, \overset{+}{3} \nearrow \overset{+}{n}, \overset{+}{2}) = (\overset{-}{1}, \overset{+}{2}, \overset{+}{3})(\overset{-}{n}),$$

so that $(x^{-1}y)^{3} \in (H \cap N^{+}) \setminus Z(G)$. If $k = 3$, then $xy = (\overset{+}{1}, \overset{-}{2}, \overset{-}{3})(\overset{+}{1}, \overset{+}{3}, \overset{+}{2}) = (\overset{-}{2})(\overset{-}{3})$, and $xy \in (H \cap N^{+}) \setminus Z(G)$. Therefore, Lemma 2.1 implies that $H = G'$ when k is odd, and that $H = G^{+}$ when k is even, as required.

Next we suppose that k is even. It remains to show that $d_k(G) = 2$. When $k \equiv 2 \mod 4$, this is just

Lemma 4.7. So, suppose $k \equiv 0 \mod 4$, and write $k = 4m$. Assume $k \geq 12$. Let

$$x = (\overset{+}{1} \nearrow \overset{+}{m-2}, \overset{-}{m-1}, \overset{+}{m} \nearrow \overset{+}{2m})(\overset{+}{2m+1} \nearrow \overset{+}{3m})(\overset{+}{3m+1} \nearrow \overset{+}{n}),$$

$$y = (\overset{+}{2m}, \overset{+}{2m+2}, \overset{+}{2m+1}, \overset{+}{2m+3} \nearrow \overset{+}{n-2}, \overset{-}{n-1})(\overset{+}{n}, \overset{+}{1} \nearrow \overset{+}{m-1})(\overset{+}{m} \nearrow \overset{+}{2m-2}, \overset{-}{2m-1}),$$

and set $H = \langle x, y \rangle$. Since x and y both contain a negative $\frac{k}{2}$-cycle, each of these elements has order k. However, $\hat{x}$ and $\hat{y}$ are elements of order $2m$. Thus, since $2m > 2$, we may use Definitions 2.3 and 2.4 to find an appropriate a and b of order $2m$ that generate $\mathrm{Sym}(n)$. The relevant elements from Table 1 are then a_2 and b_4 (where we are using $2m$ instead of k). One can check that indeed $\hat{x} = a_2$ and $\hat{y} = b_4$. Therefore, $\bar{H} = \mathrm{Sym}(n)$. Now

$$x^{-1} = (\overset{+}{2m} \searrow \overset{+}{m+1}, \overset{-}{m}, \overset{+}{m-1} \searrow \overset{+}{1})(\overset{+}{3m} \searrow \overset{+}{2m+1})(\overset{+}{n} \searrow \overset{+}{3m+1}), \text{ so}$$

$$x^{-1}y = (\overset{+}{1}, \overset{+}{2m+2}, \overset{+}{2m+3}, \overset{+}{2m+1}, \overset{+}{3m+1})(\overset{-}{m}, \overset{-}{n}, \overset{-}{2m}).$$

Therefore, $(x^{-1}y)^{15} \in (H \cap N) \setminus (N^{+} \cup Z(G))$. Hence, $H = G$ whenever $k \geq 12$. We have checked using Magma that $d_k(G) = 2$ when $n = k \in \{4, 8\}$. Therefore, whenever $k \equiv 0 \mod 4$, we have $d_k(G) = 2$. $\square$

Proof of Theorem 1.1 The case $k = 3$ is resolved by Lemma 3.1. Thus, we now assume that $k > 3$. For the case $n \geq 2k$, we have the subdivision $j = k-1$ and $j \neq k-1$. The former case is dealt with in Lemmas 3.2 and 3.3. Lemma 3.2 examines the subcase k odd, or k even and t odd, while Lemma 3.3 considers the situation of k and t both being even. For the subdivision $j \neq k-1$, the case when either k is odd, or k is even and t is odd, is covered by Lemma 3.10. Then the remaining case of k and t both being even is analysed in Lemmas 3.4, 3.6, and 3.8. This deals with the case $n \geq 2k$. Moving onto the case where $n < 2k$, we use Lemmas 4.3, 4.7, and 4.8 to settle the cases $n = 2k-1$ and $n = k$. Finally, the case when $n < 2k$ with $3 < k < 2k-1$ is dealt with in Lemma 4.6, so completing the proof of the theorem.

5. Exceptional groups

Obviously, $I_2(m)$ cannot be generated with two elements of order k with $k \geq 3$. That leaves E_6, E_7, E_8, F_4, H_3, and H_4, the results for which are shown in Table 2.

Acknowledgement

This work was done during the author's visit to the Mathematisches Forschungsinstitut Oberwolfach in February 2020, as part of the Research in Pairs programme. They would like to express their thanks to the Institute for providing such a stimulating environment.

Table 2. Values of $d_k(G)$ and $d_k(G')$ for exceptional groups.

G	$d_k(G)$	values of k
E_6	2	$4, 6, 8, 10, 12$
	0	$3, 5, 9$
E_6'	3	3
	2	$4, 5, 6, 9, 12$
	0	$8, 10$
E_7	2	$4, 6, 8, 10, 12, 14, 18, 30$
	0	$3, 5, 7, 9, 15$
E_7'	2	$3, 4, 5, 6, 7, 8, 9, 10, 12, 15$
	0	$14, 18, 30$
E_8	2	$4, 6, 8, 10, 12, 14, 18, 20, 24, 30$
	0	$3, 5, 7, 9, 15$
E_8'	2	$3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30$
F_4	3	4
	2	6
	0	$3, 8, 12$
F_4'	2	$3, 6, 12$
	0	$4, 8$
H_3	2	$6, 10$
	0	$3, 5$
H_3'	2	$3, 5$
	0	$6, 10$
H_4	2	$4, 6, 10$
	0	$3, 5, 12, 15, 20, 30$
H_4'	6	4
	3	$3, 6$
	2	$5, 10, 12, 15, 20, 30$

References

[1] Annin S, Maglione J. Economical generating sets for the symmetric and alternating groups consisting of cycles of a fixed length. Journal of Algebra and its Applications 2012; 11 (6): 8.

[2] Bosma W, Cannon J, Playoust C. The MAGMA algebra system. I. The user language. Journal of Symbolic Computation 1997; 24: 235-265.

[3] Garzoni D. Generating alternating and symmetric groups with two elements of fixed order. arXiv:1802.06213

[4] Humphreys JE. Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, UK, 1990.

[5] Jones G. Primitive permutation groups containing a cycle. Bulletin of the Australian Mathematical Society 2014; 89 (1): 159-165.

[6] Jordan C. Traité des substitutions et des équations algébriques. Gauthier-Villars 1870 (in French).

[7] Lanier J. Generating mapping class groups with elements of fixed finite order. Journal of Algebra 2018; 511: 455-470.

[8] Miller GA. On the groups generated by two operators. Bulletin of the American Mathematical Society 1901; 7 (10): 424-426.

[9] Miller GA. Possible orders of two generators of the alternating and of the symmetric group. Transactions of the American Mathematical Society 1928; 30 (1): 24-32.

[10] Yu RWT. On the generation of Coxeter groups and their alternating subgroups by involutions. Journal of Group Theory 2019; 22: 1001-1013.