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# A third-order $p$-laplacian boundary value problem on an unbounded domain

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**Abstract:** In this work, we apply Leray-Schauder continuation principle to establish the existence of at least one solution to the third order $p$-Laplacian boundary value problem on an unbounded domain of the form

$$(w(t)\varphi_p(u''(t)))' = K(t, u(t), u'(t), u''(t)), t \in (0, \infty)$$

$$u(0) = 0,\, u'(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t)dt,\ \lim_{t\to\infty}(w(t)\varphi_p(u''(t)) = 0$$

under the nonresonant condition $\sum_{i=1}^{m} \alpha_i \xi^2 \neq 2$.

**Key words:** Nonresonance, $p$-Laplacian, unbounded domain, third-order, boundary value problem

## 1. Introduction

The purpose of this paper is to obtain existence of at least one solution for the third order $p$-Laplacian boundary value problem

$$(w(t)\varphi_p(u''(t)))' = K(t, u(t), u'(t), u''(t)), t \in (0, \infty) \tag{1.1}$$

$$u'(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t)dt, u(0) = 0, \lim_{t\to\infty}(w(t)\varphi_p(u''(t))) = 0 \tag{1.2}$$

where
$\varphi_p(s) =| s |^{p-2} s$, $K : [0, \infty)\times\Re^3 \to \Re$ is a Caratheodory function with respect to $L^1[0, \infty)$, $\alpha_i \in \Re(1 \le i \le m)$, $0 < \xi_1 < \xi_2 < ... < \xi_m < 1$ . $w(t) > 0, t \in [0, \infty), w \in C[0, \infty) \cap C^1[0, \infty)$, $\dfrac{1}{w} \in L^1[0, \infty)$ and $\sum_{i=1}^{m} \alpha_i \xi^2 \neq 2$. The condition $\sum_{i=1}^{m} \alpha_i \xi^2 \neq 2$ is critical since we require a trivial kernel for the differential operator. The boundary value problem (1.1)–(1.2) is then said to be at nonresonance. In [6] we proved the existence of solutions for the boundary value problem.

$$(q(t)u''(t))' = f(t, u(t), u'(t), u''(t)), t \in (0, \infty) \tag{1.3}$$

$$u'(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t)dt, u(0) = 0, \lim_{t\to\infty} q(t)u''(t) = 0 \tag{1.4}$$

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under the resonant condition $\sum_{i=1}^{m} \alpha_i \xi_i^2 = 2$.
The method of investigation in [6] was based on coincidence degree arguments. In this work, we shall utilise topological degree methods based on the Leray-Schauder degree theory [10]. The contribution of this paper is to extend the results of (1.3)–(1.4) to a $p$ -Laplacian boundary value problem where the differential operator has trivial solutions.

Boundary value problems on an unbounded domain with integral boundary conditions have various applications. For example they are used to model, the steady flow of gas through a semi-infinite porous medium, theory of drain flows, blood flow models. Boundary value problems with $p$ - Laplacian operators also have applications in non-Newtonian mechanics, nonlinear elasticity, glaciology, etc.
For some recent works where Leray-Schauder principle has been applied for boundary value problems on unbounded domain or half-line see [1, 2, 4, 6–8] and references therein.

This paper is organized as follows. In Section 2 we present some background definitions, theorems and a-priori estimates that will be used in the proof of the main existence results. In Section 3 we prove the main existence result. Section 4 will be devoted to providing an example to demonstrate our results.

## 2. Preliminaries

In what follows, we shall use the following definition and lemmas.

**Definition 2.1** *The mapping $f : [0,\infty)\times\Re^n \to \Re$ is an $L^1[0,\infty)$ - Caratheodory if the following conditions hold*

*(i) for each $z \in R^n, f(t,z)$ is Lebesgue measurable;*

*(ii) for a.e $t \in (0,\infty)$, $f(t,z)$ is a continuous on $\Re^n$;*

*(iii) for each $r > 0$ there exists $\alpha_r \in (0,\infty)$ such that for a.e $t \in [0,\infty)$ and every $z$ such that $\mid z \mid\leq r$ we have $\mid f(t,z) \mid\leq \alpha_r(t)$*

Let $AC[0,\infty)$ denote the space of absolutely continuous functions on $[0,\infty)$. In what follows we shall utilise the following spaces

$$X = \Big\{ u \in C^2[0,\infty), (w\varphi_p(u'')))' \in AC[0,\infty), \\ \lim_{t\to\infty} e^{-t} \mid u^{(i)}(t) \mid \text{ exists for } 0 \leq i \leq 2, (w\varphi_p(u''))' \in L^1[0,\infty) \Big\}, \tag{2.1}$$

$$\parallel u \parallel= max \left[ \sup_{t\in[0,\infty)} e^{-t} \mid u(t) \mid, \sup_{t\in[0,\infty)} e^{-t} \mid u'(t) \mid, \sup_{t\in[0,\infty} e^{-t} \mid u''(t) \mid \right]. \tag{2.2}$$

Thus $X$ is a Banach Space . Let $Z = L^1[0,\infty)$ with the norm

$$\parallel y \parallel_1 = \int_0^\infty \mid y(t) \mid dt, \quad y \in Z \tag{2.3}$$

Define the mapping $N : X \to Z$ by

$$Nu(t) = K(t, u(t), u'(t), u''(t)), \quad t \in [0,\infty) \tag{2.4}$$

In this work, we shall utilise the Leray-Schauder continuation principle which has been applied in many papers in the literature. However since $[0,\infty)$ is not compact, the compactness principle on the bounded interval $[0,1]$ cannot be applied here. We shall rely on the following compactness criterion.

**Lemma 2.2** *[1] Let $H \subset X$. Then $H$ is relatively compact in $X$ if the following conditions are satisfied*

*(i) $H$ is bounded in $X$;*

*(ii) The family $W^i = \left\{\psi_i : \psi_i(t) = e^{-t}u^i(t), t \geq 0, u \in H\right\}$ is equicontinuous on any compact subinterval of $[0,\infty)$ for $i = 0, 1, ...n-1$;*

*(iii) The family $W^i = \left\{\psi_i : \psi_i(t) = e^{-t}u^i(t), t \geq 0, u \in H\right\}$ is equiconvergent at infinity for $i = 0, 1...n-1$.*

We shall need the following constants

$$A = e^{-1}\left\|\frac{1}{w}\right\|_1^{\frac{1}{p-1}} + e^{-1}\left\|\frac{1}{w}\right\|_1^{\frac{1}{p-1}} \frac{\sum_{i=1}^m \mid \alpha_i \mid \xi_i^2}{\mid 2-\sum_{i=1}^m \alpha_i \xi_i^2 \mid} \tag{2.5}$$

$$B = \left\|\frac{1}{w}\right\|_1^{\frac{1}{p-1}} + \left\|\frac{1}{w}\right\|_1^{\frac{1}{p-1}} \frac{\sum_{i=1}^m \mid \alpha_i \mid \xi_i^2}{\mid 2-\sum_{i=1}^m \mid \alpha_i \mid \xi_i^2 \mid} \tag{2.6}$$

$$C = \left\|\frac{1}{w}\right\|_\infty^{\frac{1}{p-1}}. \tag{2.7}$$

**Lemma 2.3** *Let $h \in Z$. Then the unique solution of the equation.*

$$(w(t)\varphi_p(u'')))' = h(t), t \in [0,\infty) \tag{2.8}$$

*subject to (1.2) is*

$$\begin{aligned} u(t) = &-\int_0^t \int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty h(\tau)d\tau\right)drds \\ &\frac{-2t}{2-\sum_{i=1}^m \alpha_i\xi_i^2}\sum_{i=1}^m \alpha_i \int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty h(\tau)d\tau\right)drdsdt. \end{aligned} \tag{2.9}$$

**Proof** From (2.8) and (1.2) we derive

$$\varphi_p(u''(t)) = -\frac{1}{w(t)}\int_t^\infty h(\tau)d\tau,$$

$$u'' = -\varphi_q\left(\frac{1}{w(t)}\right)\varphi_q\int_t^\infty h(\tau)d\tau$$

and hence

$$u(t) = -\int_0^t \int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty h(\tau)d\tau\right)drds + tu'(0).$$

Therefore from (1.2) we obtain

$$u'(0) = \frac{-2\sum_{i=1}^{m}\alpha_i}{2-\sum_{i=1}^{m}\alpha_i\xi_i^2}\int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty h(\tau)d\tau\right)drdsdt.$$

Hence (2.9) follows. $\square$

**Lemma 2.4** *Let $h \in Z$. Then the solution (2.9) satisfies*

$$e^{-t} \mid u(t) \mid \leq A \parallel h \parallel_1^{\frac{1}{p-1}}, \tag{2.10}$$

$$e^{-t} \mid u'(t) \mid \leq B \parallel h \parallel_1^{\frac{1}{p-1}} \tag{2.11}$$

*and*

$$e^{-t} \mid u''(t) \mid \leq C \parallel h \parallel_1^{\frac{1}{p-1}}. \tag{2.12}$$

*Hence,*

$$\parallel u \parallel \leq \max(A,B,C) \parallel h \parallel_1^{\frac{1}{p-1}}. \tag{2.13}$$

**Proof** From (2.9) we obtain

$$\begin{aligned}
e^{-t} \mid u(t) \mid &= e^{-t}\Bigg| -\int_0^t\int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty h(\tau)d\tau\right) \\
&\quad \frac{-2t\sum_{i=1}^{m}\alpha_i}{2-\sum_{i=1}^{m}\alpha_i\xi_i^2}\int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty h(\tau)d\tau\right)drdsdt\Bigg| \\
&\leq \sup_{t\in[0,\infty} e^{-t}\Bigg[\int_0^t\int_0^s \varphi_q\left(\frac{1}{\mid w(r)\mid}\right)\varphi_q\left(\int_r^\infty \mid h(\tau)\mid d\tau\right)drds \\
&\quad + \frac{2t\sum_{i=1}^{m}\mid\alpha_i\mid}{\mid 2-\sum_{i=1}\alpha_i\xi_i^2\mid}\int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{\mid w(r)\mid}\right)\varphi_q\left(\int_r^\infty \mid h(\tau)\mid d\tau\right)drdsdt\Bigg] \\
&\leq \Bigg[\sup_{t\in[0,\infty)} e^{-t}t \parallel \frac{1}{w} \parallel_1^{\frac{1}{p-1}} \\
&\quad + 2\sup_{t\in[0,\infty)} e^{-t}t\frac{\sum_{i=1}^{m}\mid\alpha_i\mid}{\mid 2-\sum_{i=1}^{m}\alpha_i\xi_i^2\mid}\int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{\mid w(r)\mid}\right)drdsdt\Bigg] \parallel h \parallel_1^{\frac{1}{p-1}} \\
&= \left[e^{-1} \parallel \frac{1}{w} \parallel_1^{\frac{1}{p-1}} + e^{-1} \parallel \frac{1}{w} \parallel_1^{\frac{1}{p-1}} \frac{\sum_{i=1}^{m}\mid\alpha_i\mid\xi^2}{\mid 2-\sum_{i=1}^{m}\alpha_i\xi_i^2\mid}\right] \parallel h \parallel_1^{\frac{1}{p-1}},
\end{aligned}$$

using (2.5), we get

$$e^{-t} \mid u(t) \mid = A \parallel h \parallel_1^{\frac{1}{p-1}} \tag{2.14}$$

$$\begin{aligned} e^{-t} \mid u'(t) \mid &= e^{-t} \Bigg| - \int_0^t \varphi_q \left( \frac{1}{w(r)} \right) \varphi_q \left( \int_r^\infty h(\tau) d\tau \right) dr, \\ &- \frac{2\sum_{i=1}^m \alpha_i}{2-\sum_{i=1}\alpha_i \xi_i^2} \int_0^{\xi_i} \int_0^t \int_0^s \varphi_q \left( \frac{1}{w(r)} \right) \varphi_q \left( \int_r^\infty h(\tau) d\tau \right) drdsdt \Bigg| \\ &\leq \sup_{t\in 0,\infty)} e^{-t} \Bigg[ \int_0^t \varphi_q \left( \frac{1}{\mid w(r) \mid} \right) \varphi_q \left( \int_r^\infty \mid h(\tau) \mid \right) d\tau) dr \\ &+ \frac{2\sum_{i=1}^m \mid \alpha_i \mid}{\mid 2-\sum_{i=1}\alpha_i \xi_i^2 \mid} \int_0^{\xi_i} \int_0^t \int_0^s \varphi_q \left( \frac{1}{\mid w(r) \mid} \right) \varphi_q \left( \int_r^\infty \mid h(\tau) d\tau \mid \right) drdsdt \Bigg] \\ &\leq \left[ \left\| \frac{1}{w} \right\|_1^{\frac{1}{p-1}} + \left\| \frac{1}{w} \right\|_1^{\frac{1}{p-1}} \frac{\sum_{i=1}^m \mid \alpha_i \mid \xi^2}{\mid 2-\sum_{i=1}\alpha_i \xi_i^2 \mid} \right] \parallel h \parallel_1^{\frac{1}{p-1}}, \end{aligned}$$

and from (2.6), we have

$$e^{-t} \mid u'(t) \mid = B \parallel h \parallel_1^{\frac{1}{p-1}}, \tag{2.15}$$

$$\begin{aligned} e^{-t} \mid u''(t) \mid &= e^{-t} \left| \varphi_q \left( \frac{1}{w(t)} \right) \varphi_q \left( \int_t^\infty h(\tau) d\tau d\tau \right) ds \right| \\ &\leq \sup_{t\in[0,\infty)} e^{-t} \mid \varphi_q \left( \frac{1}{\mid w(t) \mid} \right) \varphi_q \left( \int_t^\infty \mid h(\tau) d\tau \right) \\ &\leq \left\| \frac{1}{w} \right\|_\infty^{\frac{1}{p-1}} \parallel h \parallel_1^{\frac{1}{p-1}}, \end{aligned}$$

from (2.7), we have

$$e^{-t} \mid u''(t) \mid = C \parallel h \parallel_1^{\frac{1}{p-1}} . \tag{2.16}$$

Hence

$$\parallel u \parallel \leq \max(A,B,C) \parallel h \parallel_1^{\frac{1}{p-1}} = D \parallel h \parallel_1^{\frac{1}{p-1}} . \tag{2.17}$$

□

Define $M : Z \to X$ by

$$\begin{aligned} (Mh)(t) = &- \int_0^t \int_0^s \varphi_q \left( \frac{1}{w(r)} \right) \varphi_q \left( \int_r^\infty h(\tau) d\tau \right) drds \\ &\frac{-2t\sum_{i=1}^m \alpha_i}{2-\sum_{i=1}^m \alpha_i \xi_i^2} \int_0^{\xi_i} \int_0^t \int_0^s \varphi_q \left( \frac{1}{w(r)} \right) \varphi_q \left( \int_r^\infty h(\tau) d\tau \right) drdsdt. \end{aligned}$$

Then $M$ is well defined for all $h \in Z$ and is the unique solution of the differential equation

$$(w(t)\varphi_p(u''(t)))' = h(t) \text{ a. e } t \in (0, \infty),$$

subject to the boundary condition (1.2).
Let $T = MN : X \to X$ be defined by

$$Tu(t) = -\int_0^t \int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty h(\tau)d\tau\right)drds$$
$$\frac{-2t\sum_{i=1}^m \alpha_i}{2-\sum_{i=1}^m \alpha_i\xi_i^2}\int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty h(\tau)d\tau\right)drdsdt$$

for a.e $t \in [0, \infty)$. Then $T$ is well defined. We use the compactness criterion in Lemma 2.1 to prove that $T$ is compact.

**Lemma 2.5** *The mapping $T : X \to X$ is compact.*

**Proof** Let $H \subset X$ be bounded i.e there exists $r > 0$ such that $\mid u \mid < r$ for all $u \in H$. Since $K : [0, \infty) \times \Re^3 \to \Re$ is $L^1[0, \infty)$ Caratheodory there exists $\alpha_r \in Z$ such that $\mid Nu(t) \mid \leq \alpha_r(t)$ for all $t \in [0, \infty)$ and all $u \in H$. For $u \in H$, we have

$$e^{-t} \mid Tu(t) \mid = e^{-t}\Bigg|-\int_0^t\int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_0^\infty Nu(\tau)d\tau\right)drds$$
$$-\frac{2t\sum_{i=1}^m \alpha_i}{(2-\sum_{i=1}\alpha_i\xi_i^2)}\int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty Nu(\tau)d\tau\right)drdsdt\Bigg|$$
$$\leq \sup_{t\in[0,\infty)} e^{-t}\Bigg[\int_0^t\int_0^s \varphi_q\left(\frac{1}{\mid w(r)\mid}\right)\varphi_q\int_r^\infty (\mid Nu(\tau)\mid d\tau)\, drds$$
$$+\frac{2t\sum_{i=1}^m \mid \alpha_i \mid}{\mid 2-\sum_{i=1}^m \alpha_i\xi_i^2 \mid}\int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{\mid w(r)\mid}\right)\varphi_q\left(\int_r^\infty \mid Nu(\tau)d\tau\right)drdsdt\Bigg]$$
$$\leq A \parallel \alpha_r \parallel_1^{\frac{1}{p-1}}, \tag{2.18}$$

$$e^{-t}\left|(Tu)'(t)\right| = e^{-t}\,\Bigg|-\int_0^t \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty Nu(\tau)d\tau\right)dr$$
$$-\frac{2\sum_{i=1}^m \alpha_i}{(2-\sum_{i=1}^m \alpha_i\xi_i^2)}\int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{w(r)}\right)\varphi_q\left(\int_r^\infty Nu(\tau)d\tau\right)drdsdt\Bigg|$$
$$\leq \sup_{t\in[0,\infty)} e^{-t}\Bigg[\int_0^t \varphi_q\left(\frac{1}{\mid w(r)\mid}\right)\varphi_q\left(\int_r^\infty \mid Nu(\tau)\mid d\tau\right)dr$$
$$+\frac{2\sum_{i=1}^m \mid \alpha_i \mid}{\mid 2-\sum_{i=1}^m \alpha_i\xi_i^2 \mid}\int_0^{\xi_i}\int_0^t\int_0^s \varphi_q\left(\frac{1}{\mid w(r)\mid}\right)\varphi_q\left(\int_r^\infty \mid Nu(\tau)\mid d\tau\right)drdsdt\Bigg]$$
$$\leq B \parallel \alpha_r \parallel_1^{\frac{1}{p-1}}, \tag{2.19}$$

$$e^{-t}\left|(Tu)''(t)\right| = e^{-t}\left|-\varphi_q\left(\frac{1}{w(t)}\right)\varphi_q\left(\int_t^\infty Nu(\tau)d\tau\right)\right|$$
$$\leq \sup_{t\in[0,\infty)} e^{-t}\varphi_q\left(\frac{1}{\mid w(t)\mid}\right)\varphi_q\left(\int_t^\infty \mid Nu(\tau)\mid d\tau\right)$$
$$\leq C \parallel \alpha_r \parallel_1^{\frac{1}{p-1}}. \tag{2.20}$$

(2.18), (2.19) and (2.20) implies that

$$\parallel Tu \parallel \leq \max(A,B,C) \parallel \alpha_r \parallel_1^{\frac{1}{p-1}} = D \parallel \alpha_r \parallel_1^{\frac{1}{p-1}} \tag{2.21}$$

Thus $T(H)$ is bounded in $X$.

To prove equicontinuity of $T(H)$, let $t_1, t_2 \in [0,L]$ where $L \in (0,\infty)$ with $t_1 < t_2$ then

$$\left| e^{-t_2}(Tu)(t_2) - e^{-t_1}(Tu)(t_1)\right| = \left|\int_{t_1}^{t_2}\left[e^{-s}(Tu)(s)\right]'\right|$$
$$= \left|\int_{t_1}^{t_2} -e^{-s}\left[Tu\right](s)ds + \int_{t_1}^{t_2} e^{-s}\left[Tu\right]'(s)ds\right|$$
$$\leq \int_{t_1}^{t_2} e^{-s}\mid Tu\mid(s)ds + \int_{t_1}^{t_2} e^{-s}\mid\left[Tu\right]'(s)\mid ds$$
$$\leq 2(t_2-t_1)\parallel Tu\parallel \leq 2D(t_2-t_1)\parallel \alpha_r \parallel_1^{\frac{1}{p-1}} \to 0 \text{ as } t_1 \to t_2.$$

Similarly,

$$\mid e^{-t_2}(Tu)'(t_2) - e^{-t_1}(Tu)'(t_1)\mid = \left|\int_{t_1}^{t_2}\left[e^{-s}(Tu)'(s)\right]' ds\right|$$
$$= \left|\int_{t_1}^{t_2}\left[-e^{-s}(Tu)'(s) + e^{-s}(Tu)''(s)\right] ds\right|$$
$$\leq 2(t_2-t_1)\parallel Tu\parallel \leq 2(t_2-t_1)D\parallel \alpha_r \parallel_1^{\frac{1}{p-1}} \to 0 \text{ as } t_1 \to t_2,$$

and

$$\mid e^{-t_2}\varphi_p(Tu)''(t_2) - e^{-t_1}\varphi_p(Tu)''(t_1)\mid = \left| - e^{-t_2}\frac{1}{w(t_2)}\int_{t_2}^\infty Nu(\tau)d\tau + e^{-t_1}\frac{1}{w_{(t_1)}}\int_{t_1}^\infty Nu(\tau)d\tau\right|$$
$$= \left|\left(\frac{e^{-t_1}}{w(t_1)} - \frac{e^{-t_2}}{w(t_2)}\right)\int_{t_2}^\infty Nu(\tau)d\tau - \frac{e^{-t_1}}{w(t_1)}\int_{t_2}^\infty Nu(\tau)d\tau + \frac{e^{-t_1}}{w(t_1)}\int_{t_1}^\infty Nu(\tau)d\tau\right|$$
$$\leq \left|\frac{e^{-t_1}}{w(t_1)} - \frac{e^{-t_2}}{w(t_2)}\right| \parallel \alpha_r \parallel_1 + \frac{e^{-t_1}}{\mid w(t_1)\mid}\int_{t_1}^{t_2}\mid \alpha_r(\tau)\mid d\tau \to 0 \text{ as } t_1 \to t_2.$$

Therefore $\mid e^{-t_2}(Tu)''(t_2) - e^{-t_1}(Tu)''(t_1)\mid \to 0$ as $t_1 \to t_2$.

The above computations shows that the set $T(H)$ is equicontinuous on every compact subinterval of $[0,\infty)$

Next we show that $T(H)$ is equiconvergent at infinity. For $u \in H$, we have

$$
\begin{aligned}
e^{-t} \,|(Tu)(t)\,| \leq e^{-t} \Bigg[ & \int_0^t \int_0^s \varphi_q \left( \frac{1}{|\,w(r)\,|} \right) \varphi_q \left( \int_r^{\infty} |\,Nu(\tau)\,|\,d\tau \right) \\
& + 2t \frac{\sum_{i=1}^m |\,\alpha_i\,|}{|\,2 - \sum_{i=1}^m \alpha_i \xi_i^2\,|} \int_0^{\xi_i} \int_0^t \int_0^s \varphi_q \left( \frac{1}{|\,w(r)\,|} \varphi_q \left( \int_0^{\infty} |\,Nu(\tau)\,|\,d\tau \right) \right) drdsdt \Bigg] \\
& \leq e^{-t} t \Bigg[ \frac{1}{\|\,w\,\|_1^{\frac{1}{p-1}}} \|\,\alpha_r\,\|_1^{\frac{1}{p-1}} \\
& + 2 \,\|\,\alpha_r\,\|_1^{\frac{1}{p-1}} \frac{\sum_{i=1}^m |\,\alpha_i\,|}{|\,2 - \sum_{i=1}^m \alpha_i \xi_i^2\,|} \int_0^{\xi_i} \int_0^t \int_0^s \varphi_q \left( \frac{1}{|\,w(r)\,|} \right) drdsdt \Bigg] \to 0 \text{ as } t \to \infty;
\end{aligned}
$$

$$
\begin{aligned}
e^{-t} \,|(Tu)'(t)\,| \leq e^{-t} \Bigg[ & \int_0^t \varphi_q \left( \frac{1}{w(r)} \right) \varphi_q \left( \int_r^{\infty} |\,Nu(\tau)\,|\,d\tau \right) dr \\
& + 2 \frac{\sum_{i=1}^m |\,\alpha_i\,|}{|\,2 - \sum_{i=1}^m \alpha_i \xi_i^2\,|} \int_0^{\xi_i} \int_0^t \int_0^s \varphi_q \left( \frac{1}{|\,w(r)\,|)} \right) \varphi_q \left( \int_r^{\infty} |\,Nu(\tau)\,|\,d\tau \right) drdsdt \Bigg] \\
& \leq e^{-t} \Bigg[ \frac{1}{\|\,w\,\|_1^{\frac{1}{p-1}}} \|\,\alpha_r\,\|_1^{\frac{1}{p-1}} \\
& + 2 \,\|\,\alpha_r\,\|_1^{\frac{1}{p-1}} \frac{\sum_{i=1}^m |\,\alpha_i\,|}{|\,2 - \sum_{i=1}^m \alpha_i \xi_i^2\,|} \int_0^{\xi_i} \int_0^t \int_0^s \varphi_q \left( \frac{1}{|\,w(r)\,|} \right) drdsdt \Bigg] \to 0 \text{ as } t \to \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
e^{-t} \,|\,(Tu)''(t)\,| & \leq e^{-t} \Bigg[ \varphi_q \left( \frac{1}{|\,w(r)\,|} \right) \varphi_q \left( \int_r^{\infty} |\,Nu(\tau)\,|\,d\tau \right) \\
& \leq e^{-t} \,\|\,\varphi_q \left( \frac{1}{w} \right) \|_{\infty} \|\,\alpha_r\,\|_1^{\frac{1}{p-1}} \to 0 \text{ as } t \to \infty.
\end{aligned}
$$

This proves that $T(H)$ is equiconvergent at infinity. Therefore from lemma 2.1 we conclude that $T(H)$ is compact. The continuity of $T$ can be proven using the Lebesque dominated convergence theorem. This proves our lemma. □

## 3. Existence result

We are now ready to prove our main existence Theorem.

**Theorem 3.1** *Let $K : [0,\infty)\times\Re^3 \to \Re$ be an $L^1[0,\infty)$ Caratheodory function. Suppose that there exist positive functions $a_i, b : [0,\infty) \to [0,\infty), a_i, b \in L^1[0,\infty), i = 1,2,3$ such that*
$| K(t,x,y,z) |\leq e^{-t(p-1)} \left[a_1(t) \mid u(t) \mid^{p-1} + a_2(t) \mid u'(t) \mid^{p-1} + a_3(t) \mid u''(t) \mid^{p-1}\right] + b(t)$
*a.e $t \in [0,\infty)$ with $D^{p-1}\sum_{i=1}^{3} \| a_i \|_1 < 1$, where the constant $D$ is defined in (2.21). Then the boundary value problem (1.1) - (1.2) has at least one solution for every $b \in L^1[0,\infty)$*

**Proof** Let $u$ be any solution of the $\lambda$ dependent differential equation

$$(w(t)\varphi_p(u''(t))))' = \lambda K(t,u(t),u'(t),u''(t)) \text{ a.e } t \in [0,\infty), \lambda \in [0,1] \tag{3.1}$$

Subject to the boundary condition (1.2)
Then

$$\begin{aligned}\| (w(t)\varphi_p(u''(t)))' \|_1 &= \lambda \| Nu \|_1 \\ &\leq \| a_1 \|_1 \| u \|^{p-1} + \| a_2 \|_1 \| u \|^{p-1} + \| a_3 \|_1 \| u \|^{p-1} + \| b \|_1\end{aligned}$$

From (2.17) we obtain

$$\begin{aligned}&\leq D^{p-1}\sum_{i=1}^{3} \| a_i \|_1 \| h \|_1 + \| b \|_1 \\ &= D^{p-1}\sum_{i=1}^{3} \| a_i \|_1 \| (w(t)\varphi_p(u''(t))))' \|_1 + \| b \|_1\end{aligned}$$

which yields

$$\left(1 - \sum_{i=1}^{3} \| a_i \|_1 D^{p-1}\right) \| (w(t)\varphi_p(u''(t)))' \|_1 \leq \| b \|_1$$

or

$$\| (w(t)\varphi_p(u''(t)))' \|_1 \leq \frac{\| b \|_1}{1 - D^{p-1}\sum_{i=1}^{3} \| a_i \|_1}$$

and from (2.17) we obtain
$\| u \| \leq D \| (w(t)\varphi_p(u''(t)))' \|_1^{\frac{1}{p-1}} \leq D \left(\frac{\| b \|_1}{1 - D^{p-1}\sum_{i=1}^{3} \| a_i \|_1}\right)^{\frac{1}{p-1}}$. Therefore the set of solutions of (3.1)–(1.2) are a - priori bounded by a constant independent of $\lambda$ and of solutions. The theorem is therefore proved. □

## 4. Example

Consider the boundary value problem

$$[w(t)\varphi_p(u''(t))]' = e^{-2t}\left[\frac{| u(t) |^2}{(t+5)^2} + \frac{| u'(t) |^2}{10(t+1)^3} + cos^2 t\frac{| u''(t) |^2}{10(t+1)^4}\right] + b(t) \tag{4.1}$$

$$u'(0) = \sum_{i=1}^{2} \alpha_i \int_0^{\xi_i} u(t)dt, u(0) = 0, \lim_{t\to\infty}(w(t)\varphi_p(u''(t)) = 0 \tag{4.2}$$

$w(t) = e^t, p = 3, q = \dfrac{3}{2}$
where $b \in Z, b(t) > 0$

$$| K(t, u, u', u'') |= e^{-2t}\left[\frac{| u |}{(t+5)}, \frac{| u' |^2}{10(t+1)^3} + cos^2 t\frac{| u'' |^2}{10(t+1)^4}\right] + b(t)$$

$a_1(t) = \dfrac{1}{(t+5)^2}, a_2(t) = \dfrac{1}{10(t+1)^3}, a_3(t) = \dfrac{1}{10(t+1)^4}$

$$K(t, u, u', u'') \leq e^{-2t}\left[\frac{| u(t) |}{(t+5)} + \frac{| u' |^2}{10(t+1)^3} + \frac{| u'' |^2}{10(t+1)^4}\right] + b(t)$$

$\sum_{i=1}^{3} \parallel a_i \parallel_1 = \dfrac{1}{5} + \dfrac{1}{20} + \dfrac{1}{30} = \dfrac{17}{60}$

$\alpha_1 = 8, \alpha_2 = 7, \xi_1 = \dfrac{1}{\sqrt{32}}, \xi_2 = \dfrac{1}{\sqrt{84}}$

$\sum_{i=1}^{3} \alpha_i \xi_i^2 = \dfrac{1}{4} + \dfrac{1}{12} = \dfrac{1}{3} \neq 2$

$A = \dfrac{6}{5}e^{-1}, B = \dfrac{6}{5}, C = 1$

$D = max(A, B, C) = \dfrac{6}{5}$

$D^{p-1} \sum_{i=1}^{3} \parallel a_i \parallel_1 = (\dfrac{6}{5})^2 \times \dfrac{17}{60} = \dfrac{51}{125} < 1$

Therefore by Theorem 3.1 problem (4.1)–( 4.2) has at least one solution for every $b \in Z$.

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**Conflict of interest**

The authors proclaim that there are no contending interests regarding the publication of this research paper.

**Contribution of authors**

The first author conceived the idea and developed the theory. The second author did the literature review. The results were discussed by all authors who also contributed to the final manuscript.

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