Cameron–Storvick theorem associated with Gaussian paths on function space

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Abstract: The purpose of this paper is to provide a more general Cameron–Storvick theorem for the generalized analytic Feynman integral associated with Gaussian process $\mathcal{Z}_k$ on a very general Wiener space $C_{a,b}[0,T]$. The general Wiener space $C_{a,b}[0,T]$ can be considered as the set of all continuous sample paths of the generalized Brownian motion process determined by continuous functions $a(t)$ and $b(t)$ on $[0,T]$. As an interesting application, we apply this theorem to evaluate the generalized analytic Feynman integral of certain monomials in terms of Paley–Wiener–Zygmund stochastic integrals.

Key words: Cameron–Storvick theorem, generalized analytic Feynman integral, Gaussian process, generalized Brownian motion process, Paley–Wiener–Zygmund stochastic integral

1. Introduction

Let $(C_0[0,T], \mathcal{W}, \mathfrak{m})$ denote the classical Wiener space, where $C_0[0,T]$ is the set of all $\mathbb{R}$-valued continuous functions x on $[0,T]$ with $x(0) = 0$, $\mathcal{W}$ denotes the complete σ-field of all Wiener measurable subsets of $C_0[0,T]$, and $\mathfrak{m}$ denotes the Wiener measure characterized by

$$\mathfrak{m}(\{x : x(t) \le \tau\}) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\tau} \exp\left[-\frac{u^2}{2t} \right] du.$$

Using the Kolmogorov's extension theorem (for instance, see [13, 22]), the Wiener space $C_0[0,T]$ can be illustrated as the set of all (continuous) sample paths of the standard Brownian motion process (SBMP). In [1], Cameron provided an integration by parts formula for functionals on the classical Wiener space $C_0[0,T]$. In [2], Cameron and Storvick developed the parts formula for the analytic Feynman integral of functionals on $C_0[0,T]$. They also applied their result to establish the evaluation formula for the analytic Feynman integral of unbounded functionals on $C_0[0,T]$. The parts formula on $C_0[0,T]$ introduced in [1] also was developed in [17, 20] to establish various parts formulas for the analytic Feynman integral. The parts formula for the analytic Feynman integral is now called the Cameron–Storvick theorem.

On the other hand, the concept of the generalized Wiener integral and the generalized analytic Feynman integral on $C_0[0,T]$ were introduced in [11], and further developed and used in [3, 18, 19]. In [3, 11, 18, 19],

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the generalized Wiener integral was defined by the Wiener integral

$$\int_{C_0[0,T]} F(\mathscr{Z}_h(x,\cdot))d\mathfrak{m}(x),$$

where $\mathscr{Z}_h(x,\cdot)$ is a Gaussian path defined by the Paley–Wiener–Zygmund (PWZ) stochastic integral [15, 16] as follows:

$$\mathscr{Z}_h(x,t) = \int_0^t h(s)dx(s) \text{ with } h \in L_2[0,T].$$

The parts formula on the function space $C_{a,b}[0,T]$, which is a generalization of the Cameron–Storvick theorem was provided by Chang and Skoug in [8] and further developed in [6]. The function space $C_{a,b}[0,T]$ can be considered as the set of continuous sample paths of the generalized Brownian motion process (GBMP) determined by continuous functions $a(t)$ and $b(t)$ on $[0,T]$. A GBMP on a probability space $(\Omega, \mathcal{F}, P)$ and a time interval $[0,T]$ is a Gaussian process $Y \equiv \{Y_t\}_{t\in[0,T]}$ such that $Y_0 = 0$ almost surely, and for any cylinder set $I_{t_1,\ldots,t_n,B}$ having the form

$$I_{t_1,\ldots,t_n,B} = \big\{\omega \in \Omega : (Y(t_1,\omega), \cdots, Y(t_n,\omega)) \in B\big\}$$

with a set of time moments $0 = t_0 < t_1 < \cdots < t_n \leq T$ and a Borel set $B \subset \mathbb{R}^n$, the measure $P(I_{t_1,\ldots,t_n,B})$ of $I_{t_1,\ldots,t_n,B}$ is equal to

$$\left((2\pi)^n \prod_{j=1}^n \big(b(t_j) - b(t_{j-1})\big)\right)^{-1/2} \int_B \exp\left[-\frac{1}{2}\sum_{j=1}^n \frac{[(u_j - a(t_j)) - (u_{j-1} - a(t_{j-1}))]^2}{b(t_j) - b(t_{j-1})}\right]du_1 \cdots du_n$$

where $u_0 = 0$, $a(t)$ is a continuous real-valued function on $[0,T]$, and $b(t)$ is an increasing continuous real-valued function on $[0,T]$. For more details, see [21, 22]. Note that choosing $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, one can see that the GBMP reduces a SBMP (or, Wiener process).

The aim of this paper is to provide a more general Cameron–Storvick theorem for the generalized analytic Feynman integral associated with Gaussian paths on the function space $C_{a,b}[0,T]$. As an application, we apply our general Cameron–Storvick theorem to evaluate the generalized analytic Feynman integral of certain monomials in terms of PWZ stochastic integrals.

In order to present our assertions, we assume that $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ such that $a(0) = 0$, $a'(t) \in L^2[0,T]$, and

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty,$$

where $|a|(\cdot)$ denotes the total variation function of the function $a(\cdot)$, and $b(t)$ is an increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$. We also assume familiarity with [6, 8] and adopt the notation and terminologies of those papers. The basic concepts and definitions of the function space $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$, which forms a complete probability space, the concept of the scale-invariant measurability on $C_{a,b}[0,T]$, the Cameron–Martin space $C'_{a,b}[0,T]$ and the PWZ stochastic integral on $C_{a,b}[0,T]$ may also be found in [4, 5]. In particular, we refer to the reference [7] for the definition and the

properties of the Gaussian processes $\mathscr{Z}_k$ used in this paper. However, in order to propose our assertions in this paper, we shall introduce the following terminologies:

(i) The Hilbert space: Let

$$L_{a,b}^2[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < +\infty \text{ and } \int_0^T v^2(s)d|a|(s) < +\infty \right\}.$$

Then $L_{a,b}^2[0,T]$ is a separable Hilbert space with the inner product given by

$$(u,v)_{a,b} = \int_0^T u(t)v(t)dm_{|a|,b}(t) \equiv \int_0^T u(t)v(t)d[b(t) + |a|(t)],$$

where $m_{|a|,b}$ denotes the Lebesgue–Stieltjes measure induced by $|a|(\cdot)$ and $b(\cdot)$.

(ii) The Cameron–Martin space in $C_{a,b}[0,T]$: Let

$$C_{a,b}'[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L_{a,b}^2[0,T] \right\}.$$

Then $C_{a,b}' \equiv C_{a,b}'[0,T]$ with the inner product

$$(w_1, w_2)_{C_{a,b}'} = \int_0^T Dw_1(t)Dw_2(t)db(t)$$

is a separable Hilbert space, where the (homeomorphic) operator $D : C_{a,b}'[0,T] \to L_{a,b}^2[0,T]$ is given by

$$Dw(t) = z(t) = \frac{w'(t)}{b'(t)}.$$

(iii) The PWZ stochastic integral: Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal set in $(C_{a,b}'[0,T], \|\cdot\|_{C_{a,b}'})$ such that the De_n's are of bounded variation on $[0,T]$. Then for $w \in C_{a,b}'[0,T]$ and $x \in C_{a,b}[0,T]$, we define the PWZ stochastic integral $(w,x)^\sim$ as follows:

$$(w,x)^\sim = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (w, e_j)_{C_{a,b}'} De_j(t)dx(t)$$

if the limit exists. For each $w \in C_{a,b}'[0,T]$, the PWZ stochastic integral $(w,x)^\sim$ exists for μ-a.e. $x \in C_{a,b}[0,T]$. For each $w \in C_{a,b}'[0,T]$, the PWZ stochastic integral $(w,x)^\sim$ is a Gaussian random variable with mean $(w,a)_{C_{a,b}'}$ and variance $\|w\|_{C_{a,b}'}^2$. Furthermore, if $Dw = z \in L_{a,b}^2[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w,x)^\sim$ equals the Riemann–Stieltjes integral $\int_0^T z(t)dx(t)$. Also we note that for $w, x \in C_{a,b}'[0,T]$, $(w,x)^\sim = (w,x)_{C_{a,b}'}$.

2. Gaussian processes on $C_{a,b}[0,T]$

In order to present our Cameron–Storvick theorem on the function space $C_{a,b}[0,T]$, we follow the exposition of [4, 5, 7].

Let $C_{a,b}^*[0,T]$ be the set of functions k in $C_{a,b}'[0,T]$ such that Dk is continuous except for a finite number of finite jump discontinuities and is of bounded variation on $[0,T]$. For any $w \in C_{a,b}'[0,T]$ and $k \in C_{a,b}^*[0,T]$, let the operation $\odot$ between $C_{a,b}'[0,T]$ and $C_{a,b}^*[0,T]$ be defined by

$$w \odot k = D^{-1}(DwDk), \text{ i.e. } D(w \odot k) = DwDk,$$

where $DwDk$ denotes the pointwise multiplication of the functions Dw and Dk. Then $(C_{a,b}^*[0,T], \odot)$ is a commutative algebra with the identity b.

For each $t \in [0,T]$, let $\Phi_t(\tau) = D^{-1}\chi_{[0,t]}(\tau) = \int_0^\tau \chi_{[0,t]}(u)db(u)$, $\tau \in [0,T]$, and for $k \in C_{a,b}'[0,T]$ with $Dk \neq 0$ m_L-a.e. on $[0,T]$ (m_L denotes the Lebesgue measure on $[0,T]$), let $\mathcal{Z}_k(x,t)$ be the PWZ stochastic integral

$$\mathcal{Z}_k(x,t) = (k \odot \Phi_t, x)^{\sim}. \tag{2.1}$$

Let $\gamma_k(t) = \int_0^t Dk(u)da(u)$ and let $\beta_k(t) = \int_0^t (Dk(u))^2 db(u)$. Then the stochastic process $\mathcal{Z}_k : C_{a,b}[0,T] \times [0,T] \to \mathbb{R}$ is Gaussian with mean function

$$\int_{C_{a,b}[0,T]} \mathcal{Z}_k(x,t)d\mu(x) = \int_0^t h(u)da(u) = \gamma_k(t)$$

and covariance function

$$\int_{C_{a,b}[0,T]} \big(\mathcal{Z}_k(x,s) - \gamma_k(s)\big)\big(\mathcal{Z}_k(x,t) - \gamma_k(t)\big)d\mu(x) = \int_0^{\min\{s,t\}} \{Dk(u)\}^2(u)db(u) = \beta_k(\min\{s,t\}).$$

In addition, by [22, Theorem 21.1], $\mathcal{Z}_k(\cdot,t)$ is stochastically continuous in t on $[0,T]$. If Dk is of bounded variation on $[0,T]$, then, for all $x \in C_{a,b}[0,T]$, $\mathcal{Z}_k(x,t)$ is continuous in t. Of course if $k(t) \equiv b(t)$, then $\mathcal{Z}_b(x,t) = x(t)$, the continuous sample paths of the GBMP Y, which consist the function space $C_{a,b}[0,T]$. Furthermore, if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$ and the Gaussian process (2.1) with $k(t) \equiv t$ is a SBMP.

Given any $w \in C_{a,b}'[0,T]$ and $k \in C_{a,b}^*[0,T]$, it follows that

$$(w, \mathcal{Z}_k(x,\cdot))^{\sim} = (w \odot k, x)^{\sim} \tag{2.2}$$

for μ-a.e $x \in C_{a,b}[0,T]$.

In order to establish our Cameron–Storvick theorem for functionals on $C_{a,b}[0,T]$, we define a class $\mathrm{Supp}_{C_{a,b}^*}[0,T]$ as follows:

$$\mathrm{Supp}_{C_{a,b}^*}[0,T] = \{k \in C_{a,b}^*[0,T] : Dk \neq 0 \ m_L\text{-a.e. on } [0,T]\}.$$

Remark 2.1 *(i) The space $(\mathrm{Supp}_{C_{a,b}^*}[0,T], \odot)$ forms a monoid. The variance function $b(\cdot)$ of the GBMP Y is the identity in the space $(\mathrm{Supp}_{C_{a,b}^*}[0,T], \odot)$.*

(ii) Given a function k in $\mathrm{Supp}_{C_{a,b}^*}[0,T]$, the process $\mathcal{Z}_k$ on $C_{a,b}[0,T] \times [0,T]$ is the GBMP determined by the functions γ_k and β_k.

For any $k \in \mathrm{Supp}_{C^*_{a,b}}[0,T]$, the Lebesgue–Stieltjes integrals

$$\|w \odot k\|^2_{C'_{a,b}} = \int_0^T (Dw(t))^2 (Dk(t))^2 db(t),$$

and

$$(w \odot k, a)_{C'_{a,b}} = \int_0^T Dw(t) Dk(t) Da(t) db(t) = \int_0^T Dw(t) Dk(t) da(t)$$

exist for all $w \in C'_{a,b}[0,T]$. Using equation (2.2), one can see that the PWZ stochastic integral $(w, \mathcal{Z}_k(x,\cdot))^\sim$ is normally distributed with

$$N\Big((w \odot k, a)_{C'_{a,b}}, \|w \odot k\|^2_{C'_{a,b}}\Big).$$

Throughout the remainder of this paper, we thus require k to be in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$ for the process $\mathcal{Z}_k$.

3. Parts formula for functionals in Gaussian paths

In this section, we establish an integration by parts formula for functionals in Gaussian paths on the function space $C_{a,b}[0,T]$. To do this, we first provide the definition of the first variation (a kind of Gâteaux derivative) of functionals on the function space $C_{a,b}[0,T]$. The following definition is due to [1, 2].

Definition 3.1 *Let F be a $\mathcal{W}(C_{a,b}[0,T])$-measurable functional on $C_{a,b}[0,T]$ and let $w \in C_{a,b}[0,T]$. Then given two functions k_1 and k_2 in $C_{a,b}[0,T]$,*

$$\delta_{k_1,k_2} F(x|w) \equiv \delta F(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(w,\cdot)) = \frac{\partial}{\partial \alpha} F(\mathcal{Z}_{k_1}(x,\cdot) + \alpha \mathcal{Z}_{k_2}(w,\cdot))\Big|_{\alpha=0} \tag{3.1}$$

(if it exists) is called the first variation of F in the direction w.

Remark 3.2 *Setting $k_1 = k_2 \equiv b$ on $[0,T]$, our definition of the first variation reduces to the first variation studied in [6, 8]. That is,*

$$\delta_{b,b} F(x|w) = \delta F(x|w).$$

Let $\mathcal{Z}_k$ be the Gaussian process given by (2.1) on $C_{a,b}[0,T] \times [0,T]$. We define the $\mathcal{Z}_k$-function space integral (namely, the function space integral associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$) for functionals F on $C_{a,b}[0,T]$ by the formula

$$E_x[F(\mathcal{Z}_k(x,\cdot))] = \int_{C_{a,b}[0,T]} F(\mathcal{Z}_k(x,\cdot)) d\mu(x)$$

whenever the integral exists.

In order to establish an integration by parts formula for the function space integral associated with Gaussian paths on $C_{a,b}[0,T]$, we need a translation theorem for the function space integral. The following translation theorem is due to Chang and Choi [4].

Theorem 3.3 *Let k_1 be a function in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$ and let F be a functional on $C_{a,b}[0,T]$ such that $F(\mathcal{Z}_{k_1}(x,\cdot))$ is μ-integrable over $C_{a,b}[0,T]$. Then for any $\theta \in C'_{a,b}[0,T]$ and $k_2 \in \mathrm{Supp}_{C^*_{a,b}}[0,T]$,*

$$
\begin{aligned}
&E_x\big[F(\mathcal{Z}_{k_1}(x,\cdot) + \mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))\big] \\
&= \exp\left[-\frac{1}{2}\|\theta \odot k_2\|^2_{C'_{a,b}} - (\theta \odot k_2, a)_{C'_{a,b}}\right] E_x\Big[F(\mathcal{Z}_{k_1}(x,\cdot))\exp\big[(\theta, \mathcal{Z}_{k_2}(x,\cdot))^\sim\big]\Big].
\end{aligned}
\tag{3.2}
$$

We are now ready to present our integration by parts formula for functionals in Gaussian paths on $C_{a,b}[0,T]$.

Theorem 3.4 *Let k_1 and k_2 be functions in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$, let θ be a function in $C'_{a,b}[0,T]$, and let F be a functional on $C_{a,b}[0,T]$ such that $F(\mathcal{Z}_{k_1}(x,\cdot))$ is μ-integrable over $C_{a,b}[0,T]$. Furthermore assume that*

$$
E_x\big[\big|\delta F(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))\big|\big] < +\infty.
\tag{3.3}
$$

Then

$$
\begin{aligned}
&E_x\big[\delta F(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))\big] \\
&= E_x\big[(\theta, \mathcal{Z}_{k_2}(x,\cdot))^\sim F(\mathcal{Z}_{k_1}(x,\cdot))\big] - (\theta \odot k_2, a)_{C'_{a,b}} E_x\big[F(\mathcal{Z}_{k_1}(x,\cdot))\big].
\end{aligned}
\tag{3.4}
$$

Proof By using (3.1) and (3.2), it follows that

$$
\begin{aligned}
&E_x\big[\delta F(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))\big] \\
&= E_x\left[\frac{\partial}{\partial \alpha} F(\mathcal{Z}_{k_1}(x,\cdot) + \alpha \mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))\Big|_{\alpha=0}\right] \\
&= \frac{\partial}{\partial \alpha}\Big(E_x\big[F(\mathcal{Z}_{k_1}(x,\cdot) + \mathcal{Z}_{\alpha k_2}(\theta \odot k_1,\cdot))\big]\Big)\Big|_{\alpha=0} \\
&= \frac{\partial}{\partial \alpha}\left(\exp\left[-\frac{\alpha^2}{2}\|\theta \odot k_2\|_{C'_{a,b}} - \alpha(\theta \odot k_2, a)_{C'_{a,b}}\right] E_x\Big[F(\mathcal{Z}_{k_1}(x,\cdot))\exp\big[\alpha(\theta, \mathcal{Z}_{k_2}(x,\cdot))^\sim\big]\Big]\right)\Big|_{\alpha=0} \\
&= E_x\big[(\theta, \mathcal{Z}_{k_2}(x,\cdot))^\sim F(\mathcal{Z}_{k_1}(x,\cdot))\big] - (\theta \odot k_2, a)_{C'_{a,b}} E_x\big[F(\mathcal{Z}_{k_1}(x,\cdot))\big].
\end{aligned}
\tag{3.5}
$$

The second equality of (3.5) follows from (3.3) and Theorem 2.27 in [12]. □

4. Cameron–Storvick theorem for the generalized analytic Feynman integral associated with Gaussian paths

In this section, we establish the Cameron–Storvick theorem for the generalized analytic Feynman integral of functionals F on the function space $C_{a,b}[0,T]$. We begin this section with the definition of the generalized analytic Feynman integral associated with Gaussian process $\mathcal{Z}_k$ ($\mathcal{Z}_k$-Feynman integral) on $C_{a,b}[0,T]$.

Throughout the remainder of this paper, let $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers with positive real part, and nonzero complex numbers with nonnegative real part, respectively. For each $\lambda \in \mathbb{C}$, $\lambda^{1/2}$ denotes the principal square root of λ; i.e. $\lambda^{1/2}$ is always chosen to have nonnegative real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$ for all $\lambda \in \widetilde{\mathbb{C}}_+$.

Definition 4.1 *Given a function $k \in \mathrm{Supp}_{C^*_{a,b}}[0,T]$, let $\mathcal{Z}_k$ be the Gaussian process given by (2.1) and let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_{a,b}[0,T]$ such that the generalized $\mathcal{Z}_k$-function space integral (namely, the function space integral associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$)*

$$J_F(\mathcal{Z}_k;\lambda) = E_x[F(\lambda^{-1/2}\mathcal{Z}_k(x,\cdot))]$$

exists and is finite for all $\lambda > 0$. If there exists a function $J^*_F(\mathcal{Z}_k;\lambda)$ analytic on $\mathbb{C}_+$ such that $J^*_F(\mathcal{Z}_k;\lambda) = J_F(\mathcal{Z}_k;\lambda)$ for all $\lambda \in (0,+\infty)$, then $J^*_F(\mathcal{Z}_k;\lambda)$ is defined to be the analytic $\mathcal{Z}_k$-function space integral (namely, the analytic function space integral associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$) of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$E^{\mathrm{an}_\lambda}_x[F(\mathcal{Z}_k(x,\cdot))] \equiv \int^{\mathrm{an}_\lambda}_{C_{a,b}[0,T]} F(\mathcal{Z}_k(x,\cdot))d\mu(x) = J^*_F(\mathcal{Z}_k;\lambda).$$

Let q be a nonzero real number and let F be a scale-invariant measurable functional whose analytic $\mathcal{Z}_k$-function space integral, $E^{\mathrm{an}_\lambda}_x[F(\mathcal{Z}_k(x,\cdot))]$, exists for all λ in $\mathbb{C}_+$. If the following limit exists, we call it the generalized analytic $\mathcal{Z}_k$-Feynman integral of F with parameter q, and we write

$$E^{\mathrm{anf}_q}_x[F(\mathcal{Z}_k(x,\cdot))] \equiv \int^{\mathrm{anf}_q}_{C_{a,b}[0,T]} F(\mathcal{Z}_k(x,\cdot))d\mu(x) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} E^{\mathrm{an}_\lambda}_x[F(\mathcal{Z}_k(x,\cdot))]. \tag{4.1}$$

We are now ready to establish a Cameron–Storvick theorem for our generalized analytic Feynman integral. It will be helpful to establish the following lemma before giving the main theorem.

Lemma 4.2 *Let k_1, k_2, θ, and F be as in Theorem 3.4. For each $\rho > 0$, assume that $F(\rho\mathcal{Z}_{k_1}(x,\cdot))$ is μ-integrable over $C_{a,b}[0,T]$. Furthermore assume that for each $\rho > 0$,*

$$E_x\big[\big|\delta F(\rho\mathcal{Z}_{k_1}(x,\cdot)|\rho\mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))\big|\big] < +\infty.$$

Then

$$\begin{aligned}
&E_x\big[\delta F(\rho\mathcal{Z}_{k_1}(x,\cdot)|\rho\mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))\big] \\
&= E_x\big[(\theta, \mathcal{Z}_{k_2}(x,\cdot))^\sim F(\rho\mathcal{Z}_{k_1}(x,\cdot))\big] - (\theta \odot k_2, a)_{C'_{a,b}} E_x\big[F(\rho\mathcal{Z}_{k_1}(x,\cdot))\big].
\end{aligned} \tag{4.2}$$

Proof Let $G(x) = F(\rho x)$. Then

$$G(\mathcal{Z}_{k_1}(x,\cdot) + \alpha\mathcal{Z}_{k_2}(w,\cdot)) = F(\rho\mathcal{Z}_{k_1}(x,\cdot) + \rho\alpha\mathcal{Z}_{k_2}(w,\cdot))$$

and

$$\frac{\partial}{\partial\alpha}G(\mathcal{Z}_{k_1}(x,\cdot) + \alpha\mathcal{Z}_{k_2}(w,\cdot))\bigg|_{\alpha=0} = \frac{\partial}{\partial\alpha}F(\rho\mathcal{Z}_{k_1}(x,\cdot) + \rho\alpha\mathcal{Z}_{k_2}(w,\cdot))\bigg|_{\alpha=0}.$$

Thus $\delta G(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(\theta \odot k_1,\cdot)) = \delta F(\rho\mathcal{Z}_{k_1}(x,\cdot)|\rho\mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))$. Hence by Equation (3.4) with F replaced with G, we have

$$\begin{aligned}
E_x\big[\delta F(\rho\mathcal{Z}_{k_1}(x,\cdot)|\rho\mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))\big] &= E_x\big[\delta G(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(\theta \odot k_1,\cdot))\big] \\
&= E_x\big[(\theta, \mathcal{Z}_{k_2}(x,\cdot))^\sim G(\mathcal{Z}_{k_1}(x,\cdot))\big] - (\theta \odot k_2, a)_{C'_{a,b}} E_x\big[G(\mathcal{Z}_{k_1}(x,\cdot))\big] \\
&= E_x\big[(\theta, \mathcal{Z}_{k_2}(x,\cdot))^\sim F(\rho\mathcal{Z}_{k_1}(x,\cdot))\big] - (\theta \odot k_2, a)_{C'_{a,b}} E_x\big[F(\rho\mathcal{Z}_{k_1}(x,\cdot))\big]
\end{aligned}$$

which establishes (4.2). □

Next we provide the Cameron–Storvick theorem for the generalized analytic $\mathcal{Z}_k$-Feynman integral on the function space $C_{a,b}[0,T]$.

Theorem 4.3 *Let k_1, k_2, θ, and F be as in Lemma 4.2. Then if any two of the three generalized analytic Feynman integrals in the following equation exist, then the third one also exists, and equality holds:*

$$E_x^{\text{anf}_q}\big[\delta F(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(\theta \odot k_1, \cdot))\big]$$
$$= -iq E_x^{\text{anf}_q}\big[(\theta, \mathcal{Z}_{k_2}(x,\cdot))^{\sim} F(\mathcal{Z}_{k_1}(x,\cdot))\big] - (-iq)^{1/2}(\theta \odot k_2, a)_{C'_{a,b}} E_x^{\text{anf}_q}\big[F(\mathcal{Z}_{k_1}(x,\cdot))\big].$$

Proof Given $\rho > 0$ and $\theta \in C'_{a,b}[0,T]$, let $\theta_\rho = \frac{1}{\rho}\theta$. Then θ_ρ is a function in $C'_{a,b}[0,T]$, and $\theta \odot k_1 = \rho\theta_\rho \odot k_1$. By equation (4.2) with θ replaced with θ_ρ,

$$E_x\big[\delta F(\rho\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(\theta \odot k_1, \cdot))\big]$$
$$= E_x\big[\delta F(\rho\mathcal{Z}_{k_1}(x,\cdot)|\rho\mathcal{Z}_{k_2}(\theta_\rho \odot k_1, \cdot))\big]$$
$$= E_x\big[(\theta_\rho, \mathcal{Z}_{k_2}(x,\cdot))^{\sim} F(\rho\mathcal{Z}_{k_1}(x,\cdot))\big] - (\theta_\rho \odot k_2, a)_{C'_{a,b}} E_x\big[F(\rho\mathcal{Z}_{k_1}(x,\cdot))\big] \tag{4.3}$$
$$= \rho^{-2} E_x\big[(\theta, \rho\mathcal{Z}_{k_2}(x,\cdot))^{\sim} F(\rho\mathcal{Z}_{k_1}(x,\cdot))\big] - \rho^{-1}(\theta \odot k_2, a)_{C'_{a,b}} E_x\big[F(\rho\mathcal{Z}_{k_1}(x,\cdot))\big].$$

Now let $\rho = \lambda^{-1/2}$. Then Equation (4.3) becomes

$$E_x\big[\delta F(\lambda^{-1/2}\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(w,\cdot))\big]$$
$$= \lambda E_x\big[(\theta, \lambda^{-1/2}\mathcal{Z}_{k_2}(x,\cdot))^{\sim} F(\lambda^{-1/2}\mathcal{Z}_{k_1}(x,\cdot))\big] - \lambda^{1/2}(\theta \odot k_2, a)_{C'_{a,b}} E_x\big[F(\lambda^{-1/2}\mathcal{Z}_{k_1}(x,\cdot))\big]. \tag{4.4}$$

Since $\rho > 0$ was arbitrary, we have that equation (4.4) holds for all $\lambda > 0$. We now use Definition 4.1 to obtain our desired conclusions. □

Corollary 4.4 *Under the assumptions as given in Theorem 4.3, it follows that if any two of the three generalized analytic Feynman integrals in the following equation exist, then the third one also exists, and equality holds:*

$$E_x^{\text{anf}_q}\big[(\theta, \mathcal{Z}_{k_2}(x,\cdot))^{\sim} F(\mathcal{Z}_{k_1}(x,\cdot))\big]$$
$$= \frac{i}{q} E_x^{\text{anf}_q}\big[\delta F(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(\theta \odot k_1, \cdot))\big] + (-iq)^{-1/2}(\theta \odot k_2, a)_{C'_{a,b}} E_x^{\text{anf}_q}\big[F(\mathcal{Z}_{k_1}(x,\cdot))\big]. \tag{4.5}$$

Remark 4.5 *As commented in Section 2 above, if $k \equiv b$ on $[0,T]$, then $\mathcal{Z}_b(x,t) = x(t)$ for each $x \in C_{a,b}[0,T]$. In this case, the generalized analytic $\mathcal{Z}_b$-Feynman integral $E_x^{\text{anf}_q}[F(\mathcal{Z}_b(x,\cdot))]$ agrees with the previous definition of the generalized analytic Feynman integral $E_x^{\text{anf}_q}[F(x)]$, see [6, 8]. The proof of the Cameron–Storvick theorem for the generalized analytic Feynman integral $E_x^{\text{anf}_q}[F(x)]$ can be found in [8].*

In view of Remarks 3.2 and 4.5, we have the following corollary.

Corollary 4.6 ([6]) *Let θ be a function in $C'_{a,b}[0,T]$, and let F be a functional on $C_{a,b}[0,T]$ such that for each $\rho > 0$, $F(\rho x)$ is μ-integrable over $C_{a,b}[0,T]$. Furthermore assume that for each $\rho > 0$,*

$$E_x\big[\big|\delta F(\rho\mathcal{Z}_b(x,\cdot)|\rho\mathcal{Z}_b(\theta\odot b,\cdot))\big|\big] \equiv E_x\big[\big|\delta F(\rho x|\rho\theta)\big|\big] < +\infty.$$

Then if any two of the three generalized analytic Feynman integrals in the following equation exist, then the third one also exists, and equality holds:

$$E_x^{\mathrm{anf}_q}[\delta F(x|\theta)] = -iqE_x^{\mathrm{anf}_q}[(\theta,x)^\sim F(x)] - (-iq)^{1/2}(\theta,a)_{C'_{a,b}}E_x^{\mathrm{anf}_q}[F(x)].$$

The formulas and results in this paper are more complicated than the corresponding formulas and results in [1, 2, 17, 20] because the Gaussian process used in this paper is neither centered nor stationary in time. However, by choosing $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$, and so the expected results on $C_0[0,T]$ are immediate corollaries of the results in this paper.

5. Generalized analytic Feynman integral of monomials in terms of PWZ stochastic integrals

When we evaluate the following generalized analytic Feynman integral

$$E_x^{\mathrm{anf}_q}\left[\prod_{j=1}^m (\theta\odot k_j, x)^\sim\right], \tag{5.1}$$

we might not be able to use the change of variables theorem of the usual measure theory, because the set of Gaussian random variables $(\theta\odot k_j, x)^\sim$, $j = 1,\ldots,m$, are generally not independent. In this case, to apply the change of variables theorem for the calculation of (5.1), we might apply the Gram–Schmidt process for the set of functions $\{\theta\odot k_1,\ldots,\theta\odot k_m\}$.

Using Equation (4.5), we indeed see that the generalized Feynman integral of functionals having the form (5.1) can be calculated very explicitly. In this section, we present interesting examples to which Equation (4.5) can be applied.

Example 5.1 *Let k_1 and k_2 be functions in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$, and given a function θ in $C'_{a,b}[0,T]$, set $F(x) = (\theta,x)^\sim$ for $x \in C_{a,b}[0,T]$. Then using equations (3.1) and (2.2), it follows that for any w in $C'_{a,b}[0,T]$,*

$$\delta F(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(w,\cdot)) = \frac{\partial}{\partial\alpha}\big\{(\theta,\mathcal{Z}_{k_1}(x,\cdot))^\sim + \alpha(\theta,\mathcal{Z}_{k_2}(w,\cdot))^\sim\big\}\bigg|_{\alpha=0}$$

$$= (\theta,\mathcal{Z}_{k_2}(w,\cdot))^\sim = (\theta\odot k_2, w)^\sim = (\theta\odot k_2, w)_{C'_{a,b}}.$$

From this, we see that

$$\delta F(\mathcal{Z}_{k_1}(x,\cdot)|\mathcal{Z}_{k_2}(\theta\odot k_1,\cdot)) = (\theta\odot k_2, \theta\odot k_1)_{C'_{a,b}}. \tag{5.2}$$

Also using (4.1), it follows that

$$E_x^{\mathrm{anf}_q}\big[F(\mathcal{Z}_{k_1}(x,\cdot))\big] = E_x^{\mathrm{anf}_q}\big[(\theta,\mathcal{Z}_{k_1}(x,\cdot))^\sim\big] = E_x^{\mathrm{anf}_q}\big[(\theta\odot k_1, x)^\sim\big] = (-iq)^{-1/2}(\theta\odot k_1, a)_{C'_{a,b}}. \tag{5.3}$$

Next using Equations (2.2), (4.5), (5.2), *and* (5.3), *we obtain the formula*

$$
\begin{aligned}
&E_x^{\mathrm{anf}_q}\big[(\theta \odot k_2, x)^\sim (\theta \odot k_1, x))^\sim\big] \\
&\equiv E_x^{\mathrm{anf}_q}[(\theta, \mathcal{Z}_{k_2}(x, \cdot))^\sim (\theta, \mathcal{Z}_{k_1}(x, \cdot))^\sim] \\
&= E_x^{\mathrm{anf}_q}\big[(\theta, \mathcal{Z}_{k_2}(x, \cdot))^\sim F(\mathcal{Z}_{k_1}(x, \cdot))\big] \\
&= \frac{i}{q} E_x^{\mathrm{anf}_q}\big[\delta F(\mathcal{Z}_{k_1}(x, \cdot) | \mathcal{Z}_{k_2}(\theta \odot k_1, \cdot))\big] + (-iq)^{-1/2}(\theta \odot k_2, a)_{C'_{a,b}} E_x^{\mathrm{anf}_q}\big[F(\mathcal{Z}_{k_1}(x, \cdot))\big] \\
&= \frac{i}{q}(\theta \odot k_2, \theta \odot k_1)_{C'_{a,b}} + \frac{i}{q}(\theta \odot k_2, a)_{C'_{a,b}}(\theta \odot k_1, a)_{C'_{a,b}}.
\end{aligned}
\tag{5.4}
$$

In our next example, for any positive integer $m \in \{3, 4, \ldots\}$, we obtain a recurrence relation for the generalized analytic Feynman integral

$$
E_x^{\mathrm{anf}_q}\bigg[\prod_{j=1}^{m}(\theta \odot k_j, x)^\sim\bigg].
$$

Example 5.2 *For a positive integer $m \geq 3$, let $\{k_1, \ldots, k_{m-1}, k_m\}$ be a finite set of functions in* $\mathrm{Supp}_{C^*_{a,b}}[0, T]$, *and given a function $\theta \in C'_{a,b}[0, T]$, set*

$$
F(x) = \prod_{j=1}^{m-1}(\theta \odot k_j, x)^\sim = \prod_{j=1}^{m-1}(\theta \odot k_j, \mathcal{Z}_b(x, \cdot))^\sim.
$$

First, using Equation (3.1), *it follows that for all $w \in C'_{a,b}[0, T]$,*

$$
\begin{aligned}
\delta F(x | \mathcal{Z}_{k_m}(w, \cdot)) &= \delta F(\mathcal{Z}_b(x, \cdot) | \mathcal{Z}_{k_m}(w, \cdot)) \\
&= \frac{\partial}{\partial \alpha} \prod_{j=1}^{m-1}\big\{(\theta \odot k_j, \mathcal{Z}_b(x, \cdot))^\sim + \alpha(\theta \odot k_j, \mathcal{Z}_{k_m}(w, \cdot))^\sim\big\}\bigg|_{\alpha=0} \\
&= \sum_{l=1}^{m-1}\bigg(\prod_{\substack{j=1 \\ j \neq l}}^{m-1}(\theta \odot k_j, \mathcal{Z}_b(x, \cdot))^\sim\bigg)(\theta \odot k_l, \mathcal{Z}_{k_m}(w, \cdot))^\sim.
\end{aligned}
$$

Then, in particular, it follows that

$$
\begin{aligned}
\delta F(\mathcal{Z}_b(x, \cdot) | \mathcal{Z}_{k_m}(\theta \odot b, \cdot)) &= \delta F(x | \mathcal{Z}_{k_m}(\theta, \cdot)) \\
&= \sum_{l=1}^{m-1}\bigg(\prod_{\substack{j=1 \\ j \neq l}}^{m-1}(\theta \odot k_j, \mathcal{Z}_b(x, \cdot))^\sim\bigg)(\theta \odot k_l, \mathcal{Z}_{k_m}(\theta, \cdot))^\sim \\
&= \sum_{l=1}^{m-1}\bigg(\prod_{\substack{j=1 \\ j \neq l}}^{m-1}(\theta \odot k_j, \mathcal{Z}_b(x, \cdot))^\sim\bigg)(\theta \odot k_l, \theta \odot k_m)_{C'_{a,b}}.
\end{aligned}
\tag{5.5}
$$

Next, using (4.5) *with* k_1 *and* k_2 *replaced with* b *and* k_m, *respectively, and with*

$$F(x) = \prod_{j=1}^{m-1} (\theta \odot k_j, x)^{\sim},$$

(5.5), *and the equation* $\mathcal{Z}_b(x, \cdot) = x$, *it follows that*

$$E_x^{\mathrm{anf}_q}\left[\prod_{j=1}^{m} (\theta \odot k_j, x)^{\sim} \right]$$

$$= E_x^{\mathrm{anf}_q}\left[(\theta, \mathcal{Z}_{k_m}(x, \cdot))^{\sim} \prod_{j=1}^{m-1} (\theta \odot k_j, \mathcal{Z}_b(x, \cdot))^{\sim} \right]$$

$$= E_x^{\mathrm{anf}_q}\left[(\theta, \mathcal{Z}_{k_m}(x, \cdot))^{\sim} F(\mathcal{Z}_b(x, \cdot)) \right]$$

$$= \frac{i}{q} E_x^{\mathrm{anf}_q}\left[\delta F(\mathcal{Z}_b(x, \cdot) | \mathcal{Z}_{k_m}(\theta \odot b, \cdot)) \right] + (-iq)^{-1/2} (\theta \odot k_m, a)_{C'_{a,b}} E_x^{\mathrm{anf}_q}\left[F(\mathcal{Z}_b(x, \cdot)) \right]$$

$$= \frac{i}{q} \sum_{l=1}^{m-1} (\theta \odot k_l, \theta \odot k_m)_{C'_{a,b}} E_x^{\mathrm{anf}_q}\left[\prod_{\substack{j=1 \\ j \neq l}}^{m-1} (\theta \odot k_j, x)^{\sim} \right] + (-iq)^{-1/2} (\theta \odot k_m, a)_{C'_{a,b}} E_x^{\mathrm{anf}_q}\left[\prod_{j=1}^{m-1} (\theta \odot k_j, x)^{\sim} \right].$$

$$(5.6)$$

Remark 5.3 *Letting* $m = 3$ *in Equation* (5.6) *and applying Equations* (5.3) *and* (5.4) *allows us to easily and completely calculate the generalized Feynman integral*

$$E_x^{\mathrm{anf}_q}\left[(\theta \odot k_1, x)^{\sim} (\theta \odot k_2 x)^{\sim} (\theta \odot k_3, x)^{\sim} \right] = E_x^{\mathrm{anf}_q}\left[(\theta, \mathcal{Z}_{k_1}(x, \cdot))^{\sim} (\theta, \mathcal{Z}_{k_2}(x, \cdot))^{\sim} (\theta, \mathcal{Z}_{k_3}(x, \cdot))^{\sim} \right].$$

Then setting $m = 4$ *in Equation* (5.6) *allows us to completely evaluate the generalized Feynman integral*

$$E_x^{\mathrm{anf}_q}\left[(\theta \odot k_1, x)^{\sim} (\theta \odot k_2 x)^{\sim} (\theta \odot k_3, x)^{\sim} (\theta \odot k_4, x)^{\sim} \right]$$

$$= E_x^{\mathrm{anf}_q}\left[(\theta, \mathcal{Z}_{k_1}(x, \cdot))^{\sim} (\theta, \mathcal{Z}_{k_2}(x, \cdot))^{\sim} (\theta, \mathcal{Z}_{k_3}(x, \cdot))^{\sim} (\theta, \mathcal{Z}_{k_4}(x, \cdot))^{\sim} \right],$$

since we already have complete evaluation formulas for

$$E_x^{\mathrm{anf}_q}\left[\prod_{j=1}^{l} (\theta \odot k_j, x)^{\sim} \right], \ l = 1, 2, 3.$$

Then we can evaluate

$$E_x^{\mathrm{anf}_q}\left[\prod_{j=1}^{5} (\theta \odot k_j, x)^{\sim} \right],$$

since we have already evaluated

$$E_x^{\mathrm{anf}_q}\left[\prod_{j=1}^{l} (\theta \odot k_j, x)^{\sim} \right]$$

for $l = 1, 2, 3$ *and* 4*; etc.*

6. Concluding remark

In the theory of infinite dimensional analysis, the integration by parts formula is also one of the fundamental tools to analyze the integration of functionals on the infinite dimensional spaces. The first infinite dimensional integration by parts formula can be found in [1]. Since then the parts formulas of functionals on infinite dimensional spaces have been developed in a tremendous amount of papers and books in the literature by many mathematicians and physicists. As illustrated in Section 1 above, the study of the parts formulas also are concentrated on classical and abstract Wiener spaces. These fundamental structures have been very useful to me (as well as to other coauthors) in establishing many of the results in [6, 8, 17, 20]. I feel strongly that the fundamental results in this paper will prove to be very useful in future work by ourselves as well as other researchers in this area.

There have been several attempts to construct financial mathematical theories using the GBMP [9, 10, 14]. The framework and methods we used to obtain the results in [4–9, 14] and this article are very dependent upon results in the book [22] by Yeh concerning noncentered Gaussian processes, that is, GBMP.

References

[1] Cameron RH. The first variation of an indefinite Wiener integral. Proceedings of the American Mathematical Society 1951; 2: 914–924.

[2] Cameron RH, Storvick DA. Feynman integral of variations of functionals, in: Gaussian Random Fields (eds. Itô K, Hida T., 1990), Series on Probability and Statistics, Vol. 1, pp. 144–157, Singapore: World Scientific, 1991.

[3] Chang SJ, Choi JG. Classes of Fourier–Feynman transforms on Wiener space. Journal of Mathematical Analysis and Applications 2017; 449: 979–993.

[4] Chang SJ, Choi JG. Translation theorem for function space integral associated with Gaussian paths and applications. Bulletin of the Iranian Mathematical Society 2019; 45: 1367–1387.

[5] Chang SJ, Choi JG, Ko AY. Multiple generalized analytic Fourier–Feynman transform via rotation of Gaussian paths on function space. Banach Journal of Mathematical Analysis 2015; 9: 58–80.

[6] Chang SJ, Choi JG, Skoug D. Integration by parts formulas involving generalized Fourier-Feynman transforms on function space. Transactions of the American Mathematical Society 2003; 355: 2925–2948.

[7] Chang SJ, Choi JG, Skoug D. Rotation of Gaussian processes on function space. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matematicas 2019; 113: 2295–2308.

[8] Chang SJ, Skoug D. Generalized Fourier–Feynman transforms and a first variation on function space. Integral Transforms and Special Functions 2003; 14: 375–393.

[9] Chung DM, Lee JH, Generalized Brownian motions with application to finance. Journal of the Korean Mathematical Society 2006; 43: 357–371.

[10] Chung TS, Ji UC, Gaussian white noise and applications to finance, in: Quantum Information and Complexity (Hida T. Saitô K. Si S. Eds.), Proceedings of the Meijo Winter School 2003, Meijo University, Nagoya, Japan, 6 – 10 January 2003, pp.179–201, Singapore: World Scientific, 2004.

[11] Chung DM, Park C, Skoug D. Generalized Feynman integrals via conditional Feynman integrals. Michigan Mathematical Journal 1993; 40; 377–391.

[12] Folland GB. Real Analysis: Modern Techniques and Their Applications (Second edition). Hoboken, New Jersey, NJ, USA: Wiley and Sons, 1999.

[13] Kuo HH. Introduction to Stochastic Integration. Universitext, New York, NY, USA: Springer Science+Business Media Incorporated, 2006.

[14] Lee JH. The linear space of generalized Brownian motions with applications. Proceedings of the American Mathematical Society 2005; 133: 2147–2155.

[15] Paley, REAC, Wiener N, Zygmund A. Notes on random functions. Mathematische Zeitschrift 1933; 37: 647–668.

[16] Park C, Skoug D. A note on Paley–Wiener–Zygmund stochastic integrals. Proceedings of the American Mathematical Society 1988: 103; 591–601.

[17] Park C, Skoug D. Integration by parts formulas involving analytic Feynman integrals. PanAmerican Mathematical Journal 1998; 8: 1–11.

[18] Park C, Skoug D. A Kac–Feynman integral equation for conditional Wiener integrals. Journal of Integral Equations and Applications 1991; 3: 411–427.

[19] Park C, Skoug D. Generalized Feynman integrals: The $\mathcal{L}(L_2, L_2)$ theory. Rocky Mountain Journal of Mathematics 1995; 25: 739–756.

[20] Park C, Skoug D, Storvick D. Fourier–Feynman transforms and the first variation. Rendiconti del Circolo Matematico di Palermo, Second Series 1998: 47: 277–292.

[21] Yeh J. Singularity of Gaussian measures on function spaces induced by Brownian motion processes with nonstationary increments. llinois Journal of Mathematics 1971; 15: 37–46.

[22] Yeh J. Stochastic Processes and the Wiener Integral. New York, NY, USA: Marcel Dekker Incorporated, 1973.