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A general double-series identity and its application in hypergeometric reduction formulas

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Abstract: In this paper, we obtain a general double-series identity involving the bounded sequence of arbitrary complex numbers. As application of our double-series identity, we establish some reduction formulas for Srivastava–Daoust double hypergeometric function and Gaussian generalized hypergeometric function ${}_4F_3$. As special cases of our reduction formula for ${}_4F_3$ lead to some corollaries involving Clausen hypergeometric functions ${}_3F_2$. Making suitable adjustment of parameters in reduction formulas for ${}_4F_3$ and ${}_3F_2$, we obtain some results in terms of elementary functions and some special functions like Lerch generalized zeta function and incomplete beta function.

Key words: Srivastava–Daoust double hypergeometric function, hypergeometric summation theorem, hypergeometric transformation and reduction formula, bounded sequence, series rearrangement technique

1. Introduction, definitions and preliminaries

In our investigation here, we use the following standard notations:
$\mathbb{N} := \{1,2,3,\cdots\}$, $\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}$.
Also, as usual, the symbols $\mathbb{C},\mathbb{R},\mathbb{N}$ and $\mathbb{Z}$ denote the sets of complex numbers, real numbers, natural numbers and integers, respectively.
The general Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$ is defined by [13, p.22 Eq.(1), p.32 Q.N.(8) and Q.N.(9)], see also [19, p.23, Eq.(22) and Eq.(23)]:

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\backslash\{0\}) \\ \prod_{j=0}^{n-1}(\lambda+j) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}) \\ \frac{(-1)^k n!}{(n-k)!} & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ 0\leqq k\leqq n) \\ 0 & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ k>n) \\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu=-k;\ k\in\mathbb{N};\ \lambda\in\mathbb{C}\backslash\mathbb{Z}), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the Gamma quotient exists.
A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

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$$
{}_pF_q\left[\begin{array}{ccc}(\alpha_p); & & \\ & & z \\ (\beta_q); & & \end{array}\right] = {}_pF_q\left[\begin{array}{ccc}\alpha_1,\alpha_2,\ldots,\alpha_p; & & \\ & & z \\ \beta_1,\beta_2,\ldots,\beta_q; & & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\ \frac{z^n}{n!}\ , \tag{1.1}
$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the complex variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$
\beta_j \ \neq\ 0,-1,-2,\ldots\ ;\quad j=1,2,\ldots,q.
$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by Equation (1.1), is meaningful when

(i) domain of convergence $|z|<\infty$, if $p\leq q$,

(ii) domain of convergence $|z|<1$, if $p=q+1$,

(iii) diverges for all z, $z\neq 0$, if $p>q+1$,

(iv) converges absolutely for $|z|=1$, if $p=q+1$ and $\Re(\omega)>0$,

(v) converges conditionally for $|z|=1(z\neq 1)$, if $p=q+1$ and $-1<\Re(\omega)\leqq 0$,

(vi) diverges for $|z|=1$, if $p=q+1$ and $\Re(\omega)\leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and parametric excess

$$
\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j, \tag{1.2}
$$

$\Re(\omega)$ being the real part of complex number ω.
Each of the following results will be needed in our present study.

In an earlier paper [15, p.199] Srivastava and Daoust defined a generalization of the Kampé de Fériet function [2, p.150] by means of the double hypergeometric series (see also [16] and [17]):

$$
F^{A:\ B;\ B'}_{C:\ D;\ D'}\left(\begin{array}{cc}[(a_A)\ :\ \vartheta,\ \varphi]:[(b_B):\psi];[(b'_{B'}):\psi']; & \\ & x,\ y \\ [(c_C)\ :\ \delta,\ \varepsilon]:[(d_D):\eta];[(d'_{D'}):\eta']; & \end{array}\right)
$$

$$
= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m\vartheta_j+n\varphi_j}\prod_{j=1}^{B}(b_j)_{m\psi_j}\prod_{j=1}^{B'}(b'_j)_{n\psi'_j}}{\prod_{j=1}^{C}(c_j)_{m\delta_j+n\varepsilon_j}\prod_{j=1}^{D}(d_j)_{m\eta_j}\prod_{j=1}^{D'}(d'_j)_{n\eta'_j}}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.3}
$$

where the coefficients

$$
\left\{\begin{array}{l}\vartheta_1,...,\vartheta_A;\ \varphi_1,...,\varphi_A;\ \psi_1,...,\psi_B;\ \psi'_1,...,\psi'_{B'};\ \delta_1,...,\delta_C;\\ \varepsilon_1,...,\varepsilon_C;\ \eta_1,...,\eta_D;\ \eta'_1,...,\eta'_{D'}\end{array}\right. \tag{1.4}
$$

are real and positive; and for the sake of brevity, (a_A) is taken to denote the sequence of A parameters $a_1, a_2 \cdots a_A$, with similar interpretations for $(b_B), (b'_{B'})$, and so on.
Let

$$\varDelta_1 = 1 + \sum_{j=1}^{C} \delta_j + \sum_{j=1}^{D} \eta_j - \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{B} \psi_j \tag{1.5}$$

and

$$\varDelta_2 = 1 + \sum_{j=1}^{C} \varepsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^{A} \varphi_j - \sum_{j=1}^{B'} \psi'_j. \tag{1.6}$$

Case I. The double power series in (1.3) converges for all complex values of x and y when $\varDelta_1 > 0$ and $\varDelta_2 > 0$;
Case II. The double power series in (1.3) is convergent for suitably constrained values of $|x|$ and $|y|$ when $\varDelta_1 = 0$ and $\varDelta_2 = 0$ (for more details of this case, see [17]);
Case III. The double power series in (1.3) would diverge except when, trivially, $x = y = 0$ when $\varDelta_1 < 0$ and $\varDelta_2 < 0$.
A terminating summation theorem [12, p.555, Entry (20)]:

$${}_4F_3\left[\begin{array}{c} -p,\ \alpha,\ \beta,\ \gamma; \\ \\ 1+\alpha,\ \beta-\ell,\ \gamma-m; \end{array}\ 1\right] = \frac{(p)!\ (1+\alpha-\beta)_\ell\ (1+\alpha-\gamma)_m}{(1+\alpha)_p\ (1-\beta)_\ell\ (1-\gamma)_m}, \tag{1.7}$$

where $p \geq \ell + m;\ p,\ \ell,\ m \in \mathbb{N}_0, \alpha, \beta, \gamma, \beta - \ell, \gamma - m \in \mathbb{C}\backslash\mathbb{Z}_0^-$. We have verified the result (1.7) numerically by using Mathematica software.
Decomposition series identity:

$$\sum_{n=0}^{\infty} \Delta(n) = \sum_{n=0}^{\infty} \Delta(2n) + \sum_{n=0}^{\infty} \Delta(2n+1), \tag{1.8}$$

Cauchy's double series identity:

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \Theta(n, r) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \Theta(n-r, r), \tag{1.9}$$

provided that both sides of identities (1.8) and (1.9) are absolutely convergent.
The **Hurwitz–Lerch zeta function** $\Phi(z, s, a)$ is defined by (see, e.g., [14, p. 194])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.10}$$

$$\left(a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1\right).$$

The **Hurwitz–Lerch zeta function** $\Phi(z, s, a)$ is easily found to be related to ${}_2F_1$ as follows (see, e.g., [12, p. 462, Entry 7.2.4.-122])

$${}_2F_1(1,\ \lambda\ ;\ \lambda+1\ ;\ z) = \lambda\, \Phi(z, 1, \lambda). \tag{1.11}$$

The **incomplete beta function** $B_x(\alpha,\beta)$ is defined by (see, e.g., [14, p. 10, Eq. (61)])

$$B_x(\alpha,\beta) := \int_0^x t^{\alpha-1}\,(1-t)^{\beta-1}\,dt \quad (\Re(\alpha)>0);\ 0<x<1 \tag{1.12}$$

The **incomplete beta function** $B_z(\alpha,\beta)$ is connected with ${}_2F_1$ as follows: (see, e.g., [14, p. 68, Eq. (32)])

$$B_z(\alpha,\beta) = \alpha^{-1}\,z^{\alpha}\,{}_2F_1(\alpha,\,1-\beta\,;\,\alpha+1\,;\,z). \tag{1.13}$$

By means of the linearization method, Li and Chu[8] established analytical formulae for a class of nonterminating ${}_3F_2$-series with unit argument. Several closed formulae are presented as applications. Shpot and Srivastava[22] derived three term summation formulas for Clausenian hypergeometric series ${}_3F_2(1)$ with negative integral parameter differences. Tremblay[23] give a list of twenty-four presumed new transformations involving the Gauss hypergeometric function with quadratic rational arguments. They are obtained from known transformation formulas of the hypergeometric function, for the most part in Goursat's Thesis[5]. Also Tremblay [24] add twelve new transformations formulas for the Gauss hypergeometric function having higher-order rational arguments. Choi and Rathie[4] established 176 interesting summation formulas for the Kampé de Fériet function in the form of 16 general summation formulas based on the transformation formulas due to Liu and Wang.The results are derived with the help of generalizations of Kummer's summation theorem, Gauss' second summation theorem and Bailey's summation theorem established earlier by Lavoie et al.

Motivated by the work done by [1, 7, 10, 11], we derive a general double-series identity by using series rearrangement technique in Section 2. In Section 3, this double-series identity is used to derive a reduction formula for the Srivastava–Daoust double hypergeometric function with arguments $\left(z, \frac{-z}{4}\right)$ and also obtain a reduction formula ${}_4F_3\left(\frac{-27z}{4(1-z)^3}\right)$ in Section 4. Five corollaries of our result ${}_4F_3\left(\frac{-27z}{4(1-z)^3}\right)$ enables few reduction formulas for Clausen hypergeometric functions ${}_3F_2\left(\frac{-27z}{4(1-z)^3}\right)$. Finally, in Sections 5 and 6, certain applications of reduction formula ${}_4F_3\left(\frac{-27z}{4(1-z)^3}\right)$ are also given in terms of elementary functions and special functions .

Remark 1.1 Any values of parameters and arguments in Sections 2 to 6, leading to the results which do not make sense, are tacitly excluded.

2. General double-series identity

In this section, we present an identity which transforms a double series into a single series, in the following theorem.

Theorem 2.1 *Let us assume that* $\{\Phi(\mu)\}_{\mu=1}^{\infty}$ *be a bounded sequence of essentially arbitrary complex numbers or real numbers such that* $\Phi(0)\neq 0$*. Then, the following general double-series identity holds true:*

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Phi(n+r)\frac{(3a)_{n+3r}\ (b)_r\ (-1)^r\ z^{n+r}}{(3a)_{2r}\ (b+1)_r\ n!\ r!} = \sum_{n=0}^{\infty}\Phi(n)\frac{(3a-2b)_n\ z^n}{(1+b)_n}, \tag{2.1}$$

$$\left(3a,\ b+1,\ 3a-2b\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ z\in\mathbb{C}\right)$$

provided that infinite series occurring on both sides of Equation (2.1) are absolutely convergent. For the convergence condition on complex variable z *in Equation (2.1), it is necessary that actual form of sequence*

$\{\Phi(\mu)\}_{\mu=1}^{\infty}$ *should be known. In this connection, for the convergence condition on complex variable* z *see the results (3.1) and (4.1), where value of* $\Phi(\mu)$ *is known.*

Proof of the assertion (2.1):

Let

$$\Psi(z) := \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Phi(n+r)\frac{(3a)_{n+3r}\ (b)_r\ (-1)^r\ z^{n+r}}{(3a)_{2r}\ (b+1)_r\ n!\ r!}. \tag{2.2}$$

Replacing n by $n-r$ in the right hand side of Equation (2.2) and also using Cauchy's double series identity (1.9), we get

$$\Psi(z) = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\Phi(n)\frac{(3a)_{n+2r}\ (b)_r\ (-1)^r\ z^{n}}{(3a)_{2r}\ (b+1)_r\ (n-r)!\ r!}$$

$$= \sum_{n=0}^{\infty}\Phi(n)\frac{(3a)_n\ z^n}{n!}\sum_{r=0}^{n}\frac{(3a+n)_{2r}\ (b)_r\ (-n)_r}{(3a)_{2r}\ (b+1)_r\ r!}$$

$$= \sum_{n=0}^{\infty}\Phi(n)\frac{(3a)_n\ z^n}{n!}\sum_{r=0}^{n}\frac{(\frac{3a+n}{2})_r\ (\frac{3a+n+1}{2})_r\ (b)_r\ (-n)_r}{(\frac{3a}{2})_r\ (\frac{3a+1}{2})_r\ (b+1)_r\ r!}$$

$$= \sum_{n=0}^{\infty}\left\{\Phi(n)\frac{(3a)_n\ z^n}{n!}\ {}_4F_3\left[\begin{matrix} -n,\ \frac{3a+n}{2},\ \frac{3a+n+1}{2},\ b; \\ \frac{3a}{2},\ \frac{3a+1}{2},\ b+1; \end{matrix}\ 1\right]\right\}. \tag{2.3}$$

Now using decomposition series identity (1.8) in Equation (2.3), we obtain

$$\Psi(z) = \sum_{n=0}^{\infty}\Phi(2n)\frac{(3a)_{2n}\ z^{2n}}{2n!}\ {}_4F_3\left[\begin{matrix} -2n,\ \frac{3a}{2}+n,\ \frac{3a+1}{2}+n,\ b; \\ \frac{3a}{2},\ \frac{3a+1}{2},\ b+1; \end{matrix}\ 1\right]+$$

$$+\sum_{n=0}^{\infty}\Phi(2n+1)\frac{(3a)_{2n+1}\ z^{2n+1}}{(2n+1)!}\ {}_4F_3\left[\begin{matrix} -(2n+1),\ \frac{3a+1}{2}+n,\ \frac{3a+2}{2}+n,\ b; \\ \frac{3a}{2},\ \frac{3a+1}{2},\ b+1; \end{matrix}\ 1\right]. \tag{2.4}$$

Now using terminating summation theorem (1.7) in two members ${}_4F_3(1)$ of the right hand side of Equation (2.4), we have

$$\Psi(z) = \sum_{n=0}^{\infty}\Phi(2n)\frac{(3a)_{2n}\ z^{2n}}{(1+b)_{2n}}\left\{\frac{\left(\frac{2+2b-3a}{2}-n\right)_n\left(\frac{1+2b-3a}{2}-n\right)_n}{\left(1-\frac{3a}{2}-n\right)_n\left(1-\frac{3a+1}{2}-n\right)_n}\right\}+$$

$$+\sum_{n=0}^{\infty}\Phi(2n+1)\frac{(3a)(3a+1)_{2n}\ z^{2n+1}}{(1+b)_{2n+1}}\left\{\frac{\left(\frac{1+2b-3a}{2}-n\right)_n\left(\frac{2b-3a}{2}-n\right)_{n+1}}{\left(1-\frac{3a+1}{2}-n\right)_n\left(1-\frac{3a+2}{2}-n\right)_{n+1}}\right\}. \tag{2.5}$$

Now applying some algebraic properties of Pochhammer symbols in the right hand side of Equation (2.5), after some simplification, we obtain:

$$\Psi(z) = \sum_{n=0}^{\infty}\Phi(2n)\frac{(3a-2b)_{2n}\ z^{2n}}{(1+b)_{2n}}+\sum_{n=0}^{\infty}\Phi(2n+1)\frac{(3a-2b)_{2n+1}\ z^{2n+1}}{(1+b)_{2n+1}}. \tag{2.6}$$

Again using the decomposition series identity (1.8) in Equation (2.6), we get the right hand side of assertion (2.1).

3. First application of double series identity

Here we establish a result for reducibility of Srivastava–Daoust double hypergeometric function (1.3) as in the following theorem.

Theorem 3.1 *The following reduction formula holds true:*

$$F^{D+1:0;1}_{G:0;3}\left(\begin{array}{lll} [(d_D){:}1,1],[3a{:}1,3] : - ; & [b{:}1] ; & \\ & & z,-\frac{z}{4} \\ [(g_G){:}1,1] : - ; & [\frac{3a}{2}:1],[\frac{3a+1}{2}:1],[b+1:1]; & \end{array}\right)$$

$$= {}_{D+2}F_{G+1}\left[\begin{array}{ll} (d_D),\ 3a-2b,\ 1; & \\ & z \\ (g_G),\quad 1+b\ ; & \end{array}\right], \tag{3.1}$$

where $g_1, g_2, ..., g_G,\ 1+b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ *and* (d_D) *denotes the sequence of* D *number of parameters given by* $d_1, d_2, d_3, ..., d_D$ *with similar interpretation for others.*

When $D \leq G-1$ *then both sides of Equation (3.1) are convergent for* $|z| < \infty$ *and when* $D = G$ *then both sides of Equation (3.1) are convergent for some suitably constrained values of* $|z|$.

Proof of the assertion (3.1):

Put $\Phi(\mu) = \frac{(d_1)_\mu (d_2)_\mu \cdots (d_D)_\mu}{(g_1)_\mu (g_2)_\mu \cdots (g_G)_\mu}$ $(\mu \in \mathbb{N}_0)$ in both sides of the double series identity (2.1), we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(d_1)_{n+r}(d_2)_{n+r}\cdots(d_D)_{n+r}\ (3a)_{n+3r}\ (b)_r\ (-1)^r\ z^{n+r}}{(g_1)_{n+r}(g_2)_{n+r}\cdots(g_G)_{n+r}\ (3a)_{2r}\ (b+1)_r\ n!\ r!}$$

$$= \sum_{n=0}^{\infty} \frac{(d_1)_n(d_2)_n\cdots(d_D)_n\ (3a-2b)_n\ z^n}{(g_1)_n(g_2)_n\cdots(g_G)_n\ (1+b)_n}.$$

Therefore,

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(d_1)_{n+r}(d_2)_{n+r}\cdots(d_D)_{n+r}\ (3a)_{n+3r}\ (b)_r\ z^n\ (-z)^r}{(g_1)_{n+r}(g_2)_{n+r}\cdots(g_G)_{n+r}\left(\frac{3a}{2}\right)_r\ \left(\frac{3a+1}{2}\right)_r\ (b+1)_r\ n!\ 2^{2r}\ r!}$$

$$= \sum_{n=0}^{\infty} \frac{(d_1)_n(d_2)_n\cdots(d_D)_n\ (3a-2b)_n\ (1)_n\ z^n}{(g_1)_n(g_2)_n\cdots(g_G)_n\ (1+b)_n\ n!}. \tag{3.2}$$

Now using the definition of double hypergeometric function (1.3) of Srivastava–Daoust in the left hand side of equation (3.2) and also using the definition of generalized hypergeometric function (1.1) in the right hand side of Equation (3.2), then we get the required result (3.1).

4. Second application of double series identity

In this section, we establish a relation between generalized Gaussian hypergeometric functions of one variable in the following form:

Theorem 4.1 *The following reduction formula holds true:*

$$(1-z)^{-3a}\ {}_4F_3\left[\begin{array}{c} a,\ a+\frac{1}{3},\ a+\frac{2}{3},\ b; \\ \frac{3a}{2},\ \frac{3a+1}{2},\ b+1; \end{array} \frac{-27z}{4(1-z)^3}\right] = {}_2F_1\left[\begin{array}{c} 1,\ 3a-2b; \\ b+1; \end{array} z\right]. \tag{4.1}$$

$\big($*In general* $|\arg(1-z)|<\pi,\ |z|<1$ *and* $27|z|<4|1-z|^3;\ \frac{3a}{2}, \frac{3a+1}{2}, b+1 \in \mathbb{C}\backslash\mathbb{Z}_0^-\big)$

We have verified the above reduction formula (4.1) numerically by using Mathematica software under the condition $-\frac{1}{2} \leq \Re(z) < 1$ *and* $\Im(z) = 0$.

Proof of the assertion (4.1):

When we put $\Phi(\mu) = 1$ in both sides of double-series identity (2.1), then we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(3a)_{n+3r}\ (b)_r\ (-1)^r\ z^{n+r}}{(3a)_{2r}\ (b+1)_r\ n!\ r!} = \sum_{n=0}^{\infty}\frac{(3a-2b)_n\ z^n}{(1+b)_n}$$

$$\sum_{r=0}^{\infty}\frac{(3a)_{3r}\ (b)_r\ (-1)^r\ z^r}{(3a)_{2r}\ (b+1)_r\ r!}\sum_{n=0}^{\infty}\frac{(3a+3r)_n\ z^n}{n!} = \sum_{n=0}^{\infty}\frac{(3a-2b)_n\ (1)_n\ z^n}{(1+b)_n\ n!}$$

$$\sum_{r=0}^{\infty}\frac{(3a)_{3r}\ (b)_r\ (-1)^r\ z^r\ (1-z)^{-(3a+3r)}}{(3a)_{2r}\ (b+1)_r\ r!} = \sum_{n=0}^{\infty}\frac{(3a-2b)_n\ (1)_n\ z^n}{(1+b)_n\ n!}$$

$$(1-z)^{-3a}\sum_{r=0}^{\infty}\frac{3^{3r}(a)_r(a+\frac{1}{3})_r(a+\frac{2}{3})_r\ (b)_r\ (-z)^r}{2^{2r}(\frac{3a}{2})_r(\frac{3a+1}{2})_r\ (b+1)_r\ \ (1-z)^{3r}\ r!} = \sum_{n=0}^{\infty}\frac{(3a-2b)_n\ (1)_n\ z^n}{(1+b)_n\ n!}. \tag{4.2}$$

Now using the definition (1.1) in both sides of Equation (4.2), we get the required result (4.1). The reduction formula (4.1) was derived by Gessel–Stanton (see [6, p.304, Eq. (5.13)]; see also[3, p.601, Entry 8.1.3.1]) using a different method (partial fraction expansion method).

Under the same convergence condition stated in theorem (4.1), we obtain the following corollaries:

Corollary 4.1: Put $b = \frac{3a}{2}$ in both sides of the reduction formula (4.1), after simplification we get

$${}_3F_2\left[\begin{array}{c} a,\ a+\frac{1}{3},\ a+\frac{2}{3}; \\ \frac{3a+1}{2},\ \frac{3a+2}{2}; \end{array} \frac{-27z}{4(1-z)^3}\right] = (1-z)^{3a},$$

which is the known result recorded in Prudnikov et al. [12, p.499, Entry(29)]. Gessel-Stanton [6, p.304, Eq.(5.10)] proved corollary (4.1) by using Lagrange inversion formula.

Corollary 4.2: Substitute $b = \frac{3a+1}{2}$ in both sides of the reduction formula (4.1), after simplification we obtain

$${}_3F_2\left[\begin{array}{c} a,\ a+\frac{1}{3},\ a+\frac{2}{3}; \\ \frac{3a}{2},\ \frac{3a+3}{2}; \end{array} \frac{-27z}{4(1-z)^3}\right] = (1-z)^{3a}\sum_{r=0}^{1}\frac{(-1)_r(1)_r\ z^r}{(\frac{3a+3}{2})_r\ r!}$$

$$= (1-z)^{3a}\left[1 - \frac{2z}{3(a+1)}\right] .$$

Corollary 4.3: Substitute $b = a-1$ in both sides of the reduction formula (4.1), after simplification we find

$${}_3F_2\left[\begin{array}{c} a+\frac{1}{3},\ a+\frac{2}{3},\ a-1; \\ \frac{3a}{2},\ \frac{3a+1}{2}; \end{array} \frac{-27z}{4(1-z)^3}\right] = (1-z)^{3a}\sum_{r=0}^{\infty}\frac{(1)_r\ (a+2)_r\ z^r}{(a)_r\ r!}$$

$$= (1-z)^{3a}\left[(1-z)^{-1} + \frac{2z}{a}(1-z)^{-2} + \frac{2z^2}{a(a+1)}(1-z)^{-3}\right]$$

$$= (1-z)^{3a-3}\left[1 + 2\left(\frac{1-a}{a}\right)z + \left(\frac{a-1}{a+1}\right)z^2\right].$$

Corollary 4.4: Put $b = a - \frac{2}{3}$ in both sides of the reduction formula (4.1), after simplification we have

$${}_3F_2\left[\begin{array}{c} a,\ a+\frac{2}{3},\ a-\frac{2}{3}; \\ \frac{3a}{2},\ \frac{3a+1}{2}; \end{array} \frac{-27z}{4(1-z)^3}\right] = (1-z)^{3a}\sum_{r=0}^{\infty}\frac{(1)_r\ (a+\frac{1}{3}+1)_r\ z^r}{(a+\frac{1}{3})_r\ r!}$$

$$= (1-z)^{3a}\left[(1-z)^{-1} + \frac{3z}{3a+1}(1-z)^{-2}\right]$$

$$= (1-z)^{3a-2}\left[1 + \left(\frac{2-3a}{1+3a}\right)z\right].$$

Corollary 4.5: Put $b = a - \frac{1}{3}$ in both sides of the reduction formula (4.1), after simplification we get

$${}_3F_2\left[\begin{array}{c} a,\ a+\frac{1}{3},\ a-\frac{1}{3}; \\ \frac{3a}{2},\ \frac{3a+1}{2}; \end{array} \frac{-27z}{4(1-z)^3}\right] = (1-z)^{3a}\ {}_1F_0\left[\begin{array}{c} 1; \\ -; \end{array} z\right]$$

$$= (1-z)^{3a}\left[(1-z)^{-1}\right]$$

$$= (1-z)^{3a-1}.$$

Remark 4.1 We have verified all five corollaries numerically by using Mathematica software.

5. Further application of reduction formula (4.1) in terms of elementary functions

In this section, we consider some application of reduction formula (4.1) in terms of elementary functions with convergence condition $|\arg(1-z)| < \pi,\ |\arg(z)| < \pi,\ |z| < 1$ and $27|z| < 4|1-z|^3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ as in the following cases:

(1) Put $a = b = 1$ in both sides of the reduction formula (4.1) and using the result [12, p.476, Entry (148)], after simplification we obtain

$${}_4F_3\left[\begin{array}{c} 1,\ 1,\ \frac{4}{3},\ \frac{5}{3}; \\ \frac{3}{2},\ 2,\ 2; \end{array} \frac{-27z}{4(1-z)^3}\right] = -\frac{(1-z)^3\ \ell n(1-z)}{z}.$$

(2) Put $a = b = \frac{1}{2}$ in both sides of the reduction formula (4.1) and using the result [12, p.473, Entry (83)], we get

$$ {}_4F_3\left[\begin{array}{ccc} \frac{1}{2},\ \frac{1}{2},\ \frac{5}{6},\ \frac{7}{6}; & & \\ & \frac{-27z}{4(1-z)^3} & \\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2}; & & \end{array}\right] = \frac{(1-z)^{\frac{3}{2}}\ \tanh^{-1}\sqrt{(z)}}{\sqrt{(z)}} = \frac{(1-z)^{\frac{3}{2}}}{2\sqrt{(z)}}\ \ell n\left(\frac{1+\sqrt{(z)}}{1-\sqrt{(z)}}\right). $$

(3) Put $a = b = \frac{3}{4}$ in both sides of the reduction formula (4.1) and using the result [12, p.475, Entry (134)], we find

$$ {}_4F_3\left[\begin{array}{ccc} \frac{3}{4},\ \frac{3}{4},\ \frac{13}{12},\ \frac{17}{12}; & & \\ & \frac{-27z}{4(1-z)^3} & \\ \frac{9}{8},\ \frac{13}{8},\ \frac{7}{4}; & & \end{array}\right] = \frac{3(1-z)^{\frac{9}{4}}}{4x^3}\left[\ell n\left(\frac{1+x}{1-x}\right) - 2\tan^{-1}x\right], $$

where $x = z^{\frac{1}{4}}$.

(4) Put $a = 1, b = \frac{1}{2}$ in both sides of the reduction formula (4.1) and using the result [12, p.477, Entry (163)], we have

$$ {}_4F_3\left[\begin{array}{ccc} 1,\ \frac{4}{3},\ \frac{5}{3},\ \frac{1}{2}; & & \\ & \frac{-27z}{4(1-z)^3} & \\ \frac{3}{2},\ \frac{3}{2},\ 2; & & \end{array}\right] = \frac{(1-z)^2}{2}\left[1 + \frac{\sin^{-1}\sqrt{(z)}}{\sqrt{z(1-z)}}\right]. $$

(5) Put $a = b = -\frac{1}{2}$ in both sides of the reduction formula (4.1) and using the result [12, p.469, Entry (9)], we get

$$ {}_4F_3\left[\begin{array}{ccc} -\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{6},\ \frac{1}{6}; & & \\ & \frac{-27z}{4(1-z)^3} & \\ -\frac{3}{4},\ -\frac{1}{4},\ \frac{1}{2}; & & \end{array}\right] = (1-z)^{-\frac{3}{2}}\left[1 - \sqrt{(z)}\ \tanh^{-1}\sqrt{(z)}\right]. $$

(6) Put $a = \frac{1}{6}$ and $b = \frac{1}{2}$ in both sides of the main reduction formula (4.1) and using the result [12, p.469, Entry (10)], we find

$$ {}_4F_3\left[\begin{array}{ccc} \frac{1}{2},\ \frac{1}{2},\ \frac{1}{6},\ \frac{5}{6}; & & \\ & \frac{-27z}{4(1-z)^3} & \\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{2}; & & \end{array}\right] = \frac{(1-z)^{\frac{1}{2}}}{2}\left[1 + (1-z)\frac{\tanh^{-1}\sqrt{(z)}}{\sqrt{(z)}}\right]. $$

(7) Put $a = \frac{1}{2}$ and $b = 1$ in both sides of the reduction formula (4.1) and using the result [12, p.469, Entry (11)], we have

$$ {}_4F_3\left[\begin{array}{ccc} 1,\ \frac{1}{2},\ \frac{5}{6},\ \frac{7}{6}; & & \\ & \frac{-27z}{4(1-z)^3} & \\ \frac{3}{4},\ \frac{5}{4},\ 2; & & \end{array}\right] = \frac{2(1-z)^{\frac{3}{2}}}{3z}[1 - (1-z)^{\frac{3}{2}}]. $$

(8) Put $a = b = \frac{1}{4}$ in both sides of the reduction formula (4.1) and using the result [12, p.472, Entry (66)], we get

$$ {}_4F_3\left[\begin{array}{ccc} \frac{1}{4},\ \frac{1}{4},\ \frac{7}{12},\ \frac{11}{12}; & & \\ & \frac{-27z}{4(1-z)^3} & \\ \frac{3}{8},\ \frac{7}{8},\ \frac{5}{4}; & & \end{array}\right] = \frac{(1-z)^{\frac{3}{4}}}{4x}\left[\ell n\left(\frac{1+x}{1-x}\right) + 2\tan^{-1}(x)\right], $$

where $x = z^{\frac{1}{4}}$.

(9) Put $a = \frac{2}{3}$ and $b = \frac{1}{2}$ in both sides of the reduction formula (4.1) and using the result [12, p.476, Entry

(147)], we obtain

$$ {}_3F_2\left[\begin{array}{ccc} \frac{1}{2}, \frac{2}{3}, \frac{4}{3}; & & \frac{-27z}{4(1-z)^3} \\ & \frac{3}{2}, \frac{3}{2}; & \end{array}\right] = \frac{(1-z)^2 \sin^{-1}\sqrt{(z)}}{\sqrt{z(1-z)}}. $$

(10) Put $a=b=\frac{1}{2}$ and replacing z by $-z$ in both sides of the reduction formula (4.1) and using the result [12, p.488, Entry (8)], after simplification we find

$$ {}_4F_3\left[\begin{array}{ccc} \frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}; & & \frac{27z}{4(1+z)^3} \\ & \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; & \end{array}\right] = \frac{(1+z)^{\frac{3}{2}}\ \tan^{-1}\sqrt{(z)}}{\sqrt{(z)}}. $$

(11) Put $a=\frac{8}{3}$ and $b=2$ in both sides of the reduction formula (4.1) and using the result [12, p.479, Entry (195)], after simplification we have

$$ {}_3F_2\left[\begin{array}{ccc} \frac{8}{3}, \frac{10}{3}, 2; & & \frac{-27z}{4(1-z)^3} \\ & \frac{8}{2}, \frac{9}{2}; & \end{array}\right] = \frac{(1-z)^6(3-2z)}{3}. $$

(12) Put $a=\frac{13}{6}$ and $b=\frac{3}{2}$ in both sides of the reduction formula (4.1) and using the result [12, p.478, Entry (186)], after simplification we get

$$ {}_3F_2\left[\begin{array}{ccc} \frac{13}{6}, \frac{17}{6}, \frac{3}{2}; & & \frac{-27z}{4(1-z)^3} \\ & \frac{13}{4}, \frac{15}{4}; & \end{array}\right] = \frac{(1-z)^{\frac{9}{2}}(5-3z)}{5}. $$

(13) Put $a=b=3$ in both sides of the reduction formula (4.1) and using the result [12, p.478, Entry (181)], after simplification we obtain

$$ {}_4F_3\left[\begin{array}{ccc} 3, 3, \frac{10}{3}, \frac{11}{3}; & & \frac{-27z}{4(1-z)^3} \\ & \frac{9}{2}, 5, 4; & \end{array}\right] = -\frac{3(1-z)^9}{2z^3}\left[z(z+2)+2\ell n(1-z)\right]. $$

(14) Put $a=\frac{5}{3}$ and $b=\frac{3}{2}$ in both sides of the reduction formula (4.1) and using the result [12, p.477, Entry (164)], we find

$$ {}_4F_3\left[\begin{array}{ccc} \frac{5}{3}, 2, \frac{7}{3}, \frac{3}{2}; & & \frac{-27z}{4(1-z)^3} \\ & \frac{5}{2}, 3, \frac{5}{2}; & \end{array}\right] = \frac{3(1-z)^5}{2z}\left[\frac{\sin^{-1}\sqrt{(z)}}{\sqrt{z(1-z)}}-1\right]. $$

(15) Put $a=b=2$ in both sides of the reduction formula (4.1) and using the result [12, p.477, Entry (165)], we have

$$ {}_4F_3\left[\begin{array}{ccc} 2, 2, \frac{7}{3}, \frac{8}{3}; & & \frac{-27z}{4(1-z)^3} \\ & 3, 3, \frac{7}{2}; & \end{array}\right] = -\frac{2(1-z)^6}{z^2}[z+\ell n(1-z)]. $$

(16) Put $a = \frac{5}{6}$ and $b = 1$ in both sides of the reduction formula (4.1) and using the result [12, p.473, Entry (84)], after simplification we get

$$ {}_4F_3\left[\begin{array}{c} 1,\ \frac{5}{6},\ \frac{7}{6},\ \frac{3}{2};\\ \\ 2,\ \frac{5}{4},\ \frac{7}{4}; \end{array} \frac{-27z}{4(1-z)^3}\right] = \frac{2(1-z)^{\frac{5}{2}}}{1+\sqrt{(1-z)}}. $$

(17) Put $a = \frac{5}{3}$ and $b = 1$ in both sides of the reduction formula (4.1) and using the result [12, p.478, Entry (178)], after simplification we obtain

$$ {}_3F_2\left[\begin{array}{c} 1,\ \frac{5}{3},\ \frac{7}{3};\\ \\ 3,\ \frac{5}{2}; \end{array} \frac{-27z}{4(1-z)^3}\right] = \frac{(1-z)^3(2-z)}{2}. $$

(18) Put $a = \frac{8}{3}$ and $b = 3$ in both sides of the reduction formula (4.1) and using the result [12, p.477, Entry (167)], we find

$$ {}_4F_3\left[\begin{array}{c} 3,\ 3,\ \frac{8}{3},\ \frac{10}{3};\\ \\ 4,\ \frac{8}{2},\ \frac{9}{2}; \end{array} \frac{-27z}{4(1-z)^3}\right] = \frac{3(1-z)^8}{z^3}[z(2-z)+2(1-z)\ell n(1-z)]. $$

(19) Put $a = \frac{5}{3}$ and $b = 2$ in both sides of the reduction formula (4.1) and using the result [12, p.477, Entry (150)], we have

$$ {}_4F_3\left[\begin{array}{c} \frac{5}{3},\ \frac{7}{3},\ 2,\ 2;\\ \\ \frac{5}{2},\ 3,\ 3; \end{array} \frac{-27z}{4(1-z)^3}\right] = \frac{2(1-z)^5}{z^2}[z+(1-z)\ell n(1-z)]. $$

(20) Put $a = \frac{11}{6}$ and $b = 2$ in both sides of the reduction formula (4.1) and using the result [12, p.477, Entry (158)], we get

$$ {}_4F_3\left[\begin{array}{c} \frac{11}{6},\ \frac{13}{6},\ \frac{5}{2},\ 2;\\ \\ \frac{11}{4},\ \frac{13}{4},\ 3; \end{array} \frac{-27z}{4(1-z)^3}\right] = \frac{4(1-z)^{\frac{11}{2}}}{(1+\sqrt{1-z})^2}. $$

6. More application of reduction formula (4.1) in some special functions

In this section, we consider two particular cases of (4.1), which are connected with other important special functions. Such functions are given by [see Equations (1.10), (1.11), (1.12) and 1.13].

(1) Put $a = b = \lambda$ in both sides of the reduction formula (4.1) and using the result [12, p.462, Entry (122)], after simplification we get

$$ {}_4F_3\left[\begin{array}{c} \lambda,\ \lambda,\ \frac{3\lambda+1}{3},\ \frac{3\lambda+2}{3};\\ \\ \frac{3\lambda}{2},\ \frac{3\lambda+1}{2},\ \lambda+1; \end{array} \frac{-27z}{4(1-z)^3}\right] = (1-z)^{3\lambda}\ [\lambda\Phi(z,1,\lambda)], $$

$$ \left(|\arg(1-z)| < \pi,\ |z| < 1 \text{ and } 27|z| < 4|1-z|^3 \in \mathbb{C}\backslash\mathbb{Z}_0^-; \tfrac{3\lambda}{2}, \tfrac{3\lambda+1}{2}, \lambda+1 \in \mathbb{C}\backslash\mathbb{Z}_0^-\right) $$

where $\Phi(z,1,\lambda)$ is called Lerch generalized zeta function.

(2) Put $a=\frac{\lambda+2\zeta-2}{3}$ and $b=\zeta-1$ in both sides of the reduction formula (4.1) and using the result [12, p.461, Entry (119)], we obtain

$$ {}_4F_3\left[\begin{array}{cc} \frac{\lambda+2\zeta-2}{3},\ \frac{\lambda+2\zeta-1}{3},\ \frac{\lambda+2\zeta}{3},\ \zeta-1; & \\ & \frac{-27z}{4(1-z)^3} \\ \frac{\lambda+2\zeta-2}{2},\ \frac{\lambda+2\zeta-1}{2},\ \zeta; & \end{array}\right] = (1-z)^{3(\zeta-1)}[z^{1-\zeta}(\zeta-1)B_z(\zeta-1,\lambda-\zeta+1)], $$

$$ \left(|\arg(1-z)|<\pi,\ |\arg(z)|<\pi,\ |z|<1 \text{ and } 27|z|<4|1-z|^3\in\mathbb{C}\backslash\mathbb{Z}_0^-;\frac{\lambda+2\zeta-2}{2},\frac{\lambda+2\zeta-1}{2},\zeta\in\mathbb{C}\backslash\mathbb{Z}_0^-\right) $$

where $B_z(\zeta-1,\lambda-\zeta+1)$ is called incomplete beta function.

Remark 6.1 All above results of Sections 4–6 are new and these results are not available in the important paper of Milgram[9] and Tremblay[23],[24] and results are different from the list of Milgram[9] and Tremblay[23],[24].

7. Concluding remarks

We conclude our present investigation by observing that the several interesting general double-series identities, reduction formulas for Srivastava–Daoust double hypergeometric function with arguments $(z,\frac{-z}{4})$ and generalized hypergeometric function having the argument $\left(\frac{-27z}{4(1-z)^3}\right)$, related elementary functions and some special functions like Lerch generalized zeta function and incomplete beta function, can be obtained in an analogous manner. The hypergeometric functions are useful in constructing so-called conformal mappings of polygonal regions whose sides are circular arcs. Therefore, the results derived in this paper provide a significant step which can yield some potential applications in the field of applied mathematics, mathematical physics and engineering (see, for example, [18]; see also the recent works [20], [21] and [22] dealing extensively with the Gauss and Clausen hypergeometric functions).

Conflicts of interest:

The authors declare that they have no conflicts of interest.

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References

[1] Andrews GE, Askey R, Roy R. Special Functions. Cambridge, UK: Cambridge University Press, 1999.

[2] Appell P, Kampé de Fériet J. Fonctions Hypergéométriques et Hypersphériques: Polynômes d'Hermite. Paris, France: Gauthier-Villars, 1926 (in French).

[3] Brychkov YA. Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas, Taylor and Fancis Group, Boca Raton, London, New York, NY, USA: CRC Press, 2008.

[4] Choi J, Rathie AK. General summation formulas for the Kampé de Fériet function. Montes Taurus Journal of Pure and Applied Mathematics 2019; 1(1): 107-128.

[5] Goursat E. Sur l' équation différentielle linéaire qui admet pour intégrale la série hypergéométrique, Annales scientifique de l' É.N.S. 2e série tome 1881; 10(2): S3–S142. doi:10.24033/ASENS.207

[6] Gessel I, Stanton D. Strange evaluations of hypergeometric series. SIAM Journal on Mathematical Analysis 1982; 13 (2): 295-308. doi: 10.1137/0513021

[7] Lebedev NN. Special Functions and Their Applications; Translated from the Russian by Richard A. Silverman, Englewood Cliffs, New Jersey, NJ: Prentice-Hall, Incorporated, 1965.

[8] Li NN, Chu W. Non-terminating ${}_3F_2$-series with unit argument. Integral Transforms and Special Functions 2018; 29 (6): 450-469. doi:10.1080/10652469.2018.1454447

[9] Milgram M. On hypergeometric ${}_3F_2(1)$- a review. 2010; arXiv. math.CA:0603096

[10] Milovanović GV, Rathie AK. Four unified results for reducibility of Srivastava's triple hypergeometric series H_B. Montes Taurus Journal of Pure and Applied Mathematics 2021; 3 (3): 155–164.

[11] Olver FWJ, Lozier DW, Boisvert RF, Clark CW (Editors). NIST Handbook of Mathematical Functions. Cambridge, UK: Cambridge University Press, 2010.

[12] Prudnikov AP, Brychkov YuA, Marichev OI. Integrals and Series, Volume 3: More special functions. Nauka, Moscow, 1986; Translated from the Russian by G. G. Gould, New York, NY, USA: Gordon and Breach Science Publishers, 1990.

[13] Rainville ED. Special Functions. New York, NY, USA: The Macmillan Company, 1960; Reprinted by Bronx, New York, NY, USA: Chelsea Publishing Company, 1971.

[14] Srivastava HM, Choi J. Zeta and q-Zeta Functions and Associated Series and Integrals, Amsterdam, London and New York, NY, USA: Elsevier Science Publishers, 2012.

[15] Srivastava HM, Daoust MC. On Eulerian integrals associated with Kampé de Fériet's function. Publications De L'Institut Mathématique Nouvelle série, tome 1969; 9 (23): 199-202.

[16] Srivastava HM, Daoust MC. Certain generalized Neumann expansions associated with the Kampé de Fériet's function. Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series A, 72-Indagationes Mathematicae 1969; 31: 449-457.

[17] Srivastava HM, Daoust MC. A note on the convergence of Kampé de Fériet's double hypergeometric series. Mathematische Nachrichten 1972; 53: 151-159. doi: 10.1002/mana.19720530114

[18] Srivastava HM, Kashyap BRK. Special Functions in Queuing Theory and Related Stochastic Processes, New York and London: Academic Press, 1982.

[19] Srivastava HM, Manocha HL. A Treatise on Generating Functions. John Wiley and Sons, New York, Chichester, Brisbane and Toronto, UK: Halsted Press (Ellis Horwood Limited, Chichester), 1984.

[20] Srivastava HM, Qureshi MI, Jabee S. Some general series identities and summation theorems for the Gauss hypergeometric function with negative integer numerator and denominator parameters. Journal of Nonlinear and Convex Analysis; 21: 463-478.

[21] Srivastava HM, Qureshi MI, Jabee S. Some general series identities and summation theorems for Clausen's hypergeometric function with negative integer numerator and denominator parameters. Journal of Nonlinear and Convex Analysis 2020; 21: 805–819.

[22] Shpot MA, Srivastava HM. The Clausenian hypergeometric function ${}_3F_2$ with unit argument and negative integral parameter differences. Applied Mathematics and Computation 2015; 259: 819-827. doi: 10.1016/j.amc.2015.03.031

[23] Tremblay R. New quadratic transformations of hypergeometric functions and associated summation formulas obtained with the well-poised fractional calculus operator. Montes Taurus Journal of Pure and Applied Mathematics 2020; 2 (1): 36-62.

[24] Tremblay R. Using the well-poised fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$ to obtain transformations of the Gauss hypergeometric function with higher level arguments.Montes Taurus Journal of Pure and Applied Mathematics 2021; 3 (3): 260-283.